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NeutroVectorSpaces I

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◇ In commemoration of the 60th birthday of the second author

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Abstract. Recently, the concept of NeutroAlgebraic and AntiAlgebraic Structures were introduced and analyzed by Florentin Smarandache. His new approach to the study of Neutrosophic Structures presents a more robust tool needed for managing uncertainty, incompleteness, indeterminate and imprecise information. In this paper, we introduce for the first time the concept of NeutroVectorSpaces. Specifically, we study a particular class of the NeutroVectorSpaces called of type $4S$ and their elementarily properties are presented. It is shown that the NeutroVectorSpaces of type $4S$ may contain NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type $4S$ are not NeutroSubspaces. Also, it is shown that if NV is a NeutroVector Space of a particular type and NW is a NeutroSubspace of NV, the NeutroQuotientSpace NV/NW does not necessarily belong to the same type as NV.

Keywords: Neutrosophy; Vector Space; NeutroField; weak NeutroVectorSpace; strong NeutroVectorSpace; weak AntiVectorSpace; strong AntiVectorSpace; NeutroSubspace; weak NeutroQuotientSpace; strong NeutroQuotientSpace.

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1. **Introduction**

As an extension of his work in [15], Florentine Smarandache in [12] introduced a new way of handling uncertainty, incompleteness, indeterminate and imprecise information. He studied and presented the concept of NeutroAlgebraicStructures and AntiAlgebraicStructures, which can be generated from a classical algebraic structure by a process called neutro-sophication and anti-sophication respectively. The emergence of these processes has given birth to a new field of research in the theory of neutrosophic algebraic structures. More details on neutrosophic algebraic structures can be found in [4]- [10].

Smarandache in [13] recalled, improved and extended several definitions and properties of NeutroAlgebras and AntiAlgebras given in [12]. This new concept was examined by Agboola et al. in [1] viz-a-viz the classical number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$. In [2], Agboola formally presented the notion of NeutroGroups by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In addition, he showed that generally, Langrange's theorem and 1st isomorphism theorem of the classical groups do not hold in the class of NeutroGroups considered. Also in [3], Agboola studied NeutroRing, NeutroSubring, NeutroIdeal, NeutroQuotientRings and he showed that the 1st isomorphism theorem of the classical rings holds in this class of NeutroRing. Recently, Rezaei and Smarandache [11] introduced the concept of Neutro-BE-algebras and Anti-BE-algebras and they showed that given any classical algebra S with n operations (laws and axioms) where $n \geq 1$ we can generate $(2^n - 1)$ NeutroStructures and $(3^n - 2^n)$ AntiStructures. For comprehensive review of new trends in neutrosophic theory readers should see [4–6, 8–10].

The present paper will be concerned with the introduction of NeutroVectorSpaces. Specifically in the paper, we will introduce and study a class of NeutroVectorSpaces called NeutroVectorSpaces of type $4S$ (i.e., 4 of its scalar multiplication axioms are NeutroAxioms) and we will present some of their elementarily properties. It will be shown that the NeutroVectorSpaces of type $4S$ may contain NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type $4S$ are not NeutroSubspaces. Also, it will be shown that if NV is a NeutroVectorSpace of a particular type and NW is a NeutroSubspace of NV, then the NeutroQuotientSpace NV/NW does not necessarily belong to the same type as NV.

2. **Preliminaries**

In this section, we will give some definitions, examples and results that will be used in the sequel.

Definition 2.1. [14]

(i) A ClassicalOperation is an operation well-defined for all the set's elements while a NeutroOperation is an operation partially well-defined, partially indeterminate, and partially outer defined on the given set. An AntiOperation is an operation that is outer defined for all the set's elements.

(ii) A classicalLaw/Axiom defined on a nonempty set is a law/axiom that is totally true for all the set's elements while a NeutroLaw/Axiom defined on a nonempty set is a law/axiom that is true for some set's element, indeterminate for other set's elements, or false for the other set's elements. An AntiLaw/Axiom defined on a nonempty set is a law/axiom that is false for all set's elements.

(iii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom while an AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Theorem 2.2. *[11] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebraicStructures and $(3^n - 2^n)$ AntiAlgebraicStructures.*

Definition 2.3. [Classical group]

Let G be a nonempty set and let $* : G \times G \to G$ be a binary operation on G. The couple $(G, *)$ is called a classical group if the following conditions hold:

(G1) $x * y \in G \; \forall x, y \in G$ [closure law].

(G2) $x * (y * z) = (x * y) * z \; \forall x, y, z \in G$ [axiom of associativity].

(G3) There exists $e \in G$ such that $x * e = e * x = x \; \forall x \in G$ [axiom of existence of neutral element].

(G4) There exists $y \in G$ such that $x * y = y * x = e \; \forall x \in G$ [axiom of existence of inverse element] where e is the neutral element of G.

If in addition $\forall x, y \in G$, we have

(G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.

Definition 2.4. [NeutroSophication of the law and axioms of the classical group]

(NG1) There exist at least three duplets $(x, y), (u, v), (p, q), \in G$ such that $x * y \in G$ (inner-defined with degree of truth T) and [$u*v =$ indeterminate (with degree of indeterminacy I) or $p*q \notin G$ (outer-defined/falsehood with degree of falsehood F)] [NeutroClosureLaw].

(NG2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in G$ such that $x*(y*z) = (x*y)*z$ (inner-defined with degree of truth T) and [[$p * (q * r)$]or [$(p * q) * r$] = indeterminate (with degree of indeterminacy I) or $u * (v * w) \neq (u * v) * w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NG3) There exists an element $e \in G$ such that $x * e = e * x = x$ (inner-defined with degree of truth T) and [[$x * e$]or[$e * x$] = indeterminate (with degree of indeterminacy I) or $x * e \neq x \neq e * x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in G$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NG4) There exists an element $u \in G$ such that $x * u = u * x = e$ (inner-defined with degree of truth T) and [[$x * u$]or[$u * x$)] = indeterminate (with degree of indeterminacy I) or $x * u \neq e \neq u * x$ (outer-defined/falsehood with degre of falsehood F)] for at least one $x \in G$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)] where e is a NeutroNeutralElement in G.

(NG5) There exist at least three duplets $(x, y), (u, v), (p, q) \in G$ such that $x * y = y * x$ (inner-defined with degree of truth T) and [[$u * v$]or[$v * u$] = indeterminate (with degree of indeterminacy I) or $p * q \neq q * p$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.5. A NeutroGroup NG is an alternative to the classical group G that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ with no AntiLaw or AntiAxiom.

Definition 2.6. A NeutroAbelianGroup NG is an alternative to the classical abelian group G that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ and $NG5$ with no AntiLaw or AntiAxiom.

Example 2.7. Let $NG = \mathbb{N} = \{1, 2, 3, 4 \cdots, \}$. Then $(NG, .)$ is a finite NeutroGroup where $''.''$ is the binary operation of ordinary multiplication.

Definition 2.8. [Classical ring] Let R be a nonempty set and let $+, . : R \times R \to R$ be binary operations of the usual addition and multiplication respectively defined on R. The triple $(R, +, .)$ is called a classical ring if the following conditions $(R1 - R9)$ hold:

(R1) $x + y \in R \ \forall x, y \in R$ [closure law of addition].

(R2) $x + (y + z) = (x + y) + z \ \forall x, y, z \in R$ [axiom of associativity].

(R3) There exists $e \in R$ such that $x + e = e + x = x \ \forall x \in R$ [axiom of existence of neutral element].

(R4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e \ \forall x \in G$ [axiom of existence of inverse element]

(R5) $x + y = y + x \ \forall x, y \in R$ [axiom of commutativity].

(R6) $x.y \in R \ \forall x, y \in R$ [closure law of multiplication].

(R7) $x.(y.z) = (x.y).z \ \forall x, y, z \in R$ [axiom of associativity].

(R8) $x.(y + z) = (x.y) + (x.z) \ \forall x, y, z \in R$ [axiom of left distributivity].

(R9) $(y + z).x = (y.x) + (z.x) \ \forall x, y, z \in R$ [axiom of right distributivity].

If in addition we have,

(R10) $x.y = y.x \ \forall x, y \in R$ [axiom of commutativity],

then $(R, +, .)$ is called a commutative ring.

Definition 2.9. [NeutroSophication of the laws and axioms of the classical ring]

(NR1) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x+y \in R$ (inner-defined with degree of truth T) and $[u + v =$ indeterminate (with degree of indeterminacy I) or $p + q \notin R$ (outer-defined/falsehood with degree of falsehood F)] [NeutroClosure law of addition].

(NR2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x+(y+z) = (x+y)+z$ (inner-defned with degree of truth T) and $[[p + (q + r)]$or$[(p + q) + r] =$ indeterminate (with degree of indeterminacy I) or $u + (v + w) \neq (u + v) + w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NR3) There exists an element $e \in R$ such that $x + e = x + e = x$ (inner-defined with degree of truth T) and $[[x + e]$or$[e + x] =$ indeterminate (with degree of indeterminate I) or $x + e \neq x \neq e + x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in R$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NR4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e$ (inner-defined with degree of truth T) and $[[-x + x]$or$[x + (-x)] =$ indeterminate (with the degree of indeterminate I) or $-x + x \neq e \neq x + (-x)$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in R$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)].

(NR5) There exist at least three duplets $(x, y), (u, v), (p, q) \in R$ such that $x + y = y + x$ (inner-defined with degree of truth T) and $[[p + q]\text{or}[q + p]$ = indeterminate (with degree of indeterminacy I) or $u + v \neq v + u$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

(NR6) There exist at least three duplets $(x, y), (p, q), (u, v) \in R$ such that $x.y \in R$ (inner-defined with degree of truth T) and $[u.v$ = indeterminate (with degree of indeterminacy I) or $p.q \notin R$ (outer-defined/falsehood with degree of falsehood F)] NeutroClosure law of multiplication].

(NR7) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x.(y.z) = (x.y).z$ (inner-defined with degree of truth T) and $[[p.(q.r)]\text{or}[(p.q).r]$ = indeterminate (with degree of indeterminacy I) or $u.(v.w) \neq (u.v).w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].

(NR8) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $x.(y + z) = (x.y) + (x.z)$ (inner-defined with degree of truth T) and $[[p.(q + r)]\text{or}[(p.q) + (p.r)]$ = indeterminate (with degree of indeterminacy I) or $u.(v + w) \neq (u.v) + (u.w)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of left distributivity (NeutroLeftDistributivity)].

(NR9) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in R$ such that $(y + z).x = (y.x) + (z.x)$ (inner-defined with degree of truth T) and $[[(v + w).u]\text{or}[(v.u) + (w.u)]$ = indeterminate (with degree of indeterminacy I) or $(v + w).u \neq (v.u) + (w.u)$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of right distributivity (NeutroRightDistributivity)].

(NR10) There exist at least three duplets $(x, y), (p, q), (u, v) \in R$ such that $x.y = y.x$ (inner-defined with degree of truth T) and $[[p.q]\text{or}[q.p]$ = indeterminate (with degree of indeterminacy I) or $u.v \neq v.u$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.10. A NeutroRing NR is an alternative to the classical ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ with no AntiLaw or AntiAxiom.

Definition 2.11. A NeutroNoncommutativeRing NR is an alternative to the classical noncommutative ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ and $NR10$ with no AntiLaw or AntiAxiom.

Example 2.12. (i) Let $NR = \mathbb{Z}$ and let $\oplus$ be a binary operation of ordinary addition and for all $x, y \in NR$, let $\odot$ be a binary operation defined on NR as $x \odot y = \sqrt{xy}$. Then $(NR, \oplus, \odot)$ is a NeutroRing.

(ii) Let $NR = \mathbb{Q}$ and let $\oplus$ be a binary operation of ordinary addition and for all $x, y \in NR$, let $\odot$ be a binary operation defined on NR as $x \odot y = x/y$. Then $(NR, \oplus, \odot)$ is a NeutroRing.

3. **Formulation of NeutroVectorSpaces**

In this section, we present the concept of NeutroVectorSpaces and study their elementary properties.

Definition 3.1. [7] [Classical Vector Space]
A vector space consists of a nonempty set V of objects (called vectors) that can be added, that can be multiplied by a real or complex number (called a scalar in this context), and for which the following laws and axioms hold:

The Law and Axioms for vector addition

(A1) If u and v are in V, then $u+v$ is in V.

(A2) $u+(v+w)=(u+v)+w$ for all u,v, and w in V.

(A3) An element 0 in V exist such that $v+0=v=0+v$ for every v in V.

(A4) For each v in V, an element $-v$ in V exist such that $-v+v=0$ and $v+(-v)=0$.

(A5) $u+v=v+u$ for all u and v in V.

The Law and Axioms for scalar multiplication

(S1) If v is in V, then av is in V for all a in $\mathbb{R}$.

(S2) $a(v+w)=av+aw$ for all v and w in V and all $a\in\mathbb{R}$.

(S3) $(a+b)v=av+bv$ for all v in V and all a and $b\in\mathbb{R}$.

(S4) $a(bv)=(ab)v$ for all v in V and all a and b in $\mathbb{R}$.

(S5) $1v=v$ for all v in V.

Definition 3.2. [NeutroSophication of the law and axioms of the classical vector space]
NeutroSophication of the law and axioms for vector addition

(NA1) There exist at least three duplets $(u,v),(w,x),(y,z)\in V$ such that $u+v\in V$ (inner-defined with degree of truth T) and [$w+x=$ indeterminate (with degree of indeterminacy I) or $y+z\notin V$ (outer-defined/falsehood with degree of falsehood F)].

(NA2) There exist at least three triplets $(u,v,w),(x,y,z),(p,q,r)\in V$ such that $u+(v+w)=(u+v)+w$ (inner-defined with degree of truth T) and [$[x+(y+z)]$or$[(x+y)+z]=$ indeterminate (with degree of indeterminacy I) or $p+(q+r)\neq(p+q)+r$ (outer-defined/falsehood with degree of falsehood F)].

(NA3) There exists an element $e\in V$ such that $v+e=e+v=v$ (inner-defined with degree of truth T) and [$[v+e]$or$[e+v]=$ indeterminate (with degree of indeterminacy I) or $v+e\neq v\neq e+v$ (outer-defined/falsehood with degree of falsehood F)] for at least one $v\in V$.

(NA4) There exists $-v\in V$ such that $v+(-v)=(-v)+v=e$ (inner-defined with degree of truth T) and [$[-v+v]$or$[v+(-v)]=$ indeterminate (with degree of indeterminacy I) or $[-v+v\neq e\neq v+(-v)$ (outer-defined/falsehood with degree of falsehood F)]

(NA5) There exist at least three duplets $(u, v), (x, y), (w, z) \in V$ such that $u+v = v+u$ (inner-defined with degree of truth T) and [[$x + y$]or[$y + x$] = indeterminate (with degree of indeterminacy I) or $w + z \neq z + w$ (outer-defined/falsehood with degree of falsehood F)].

NeutroSophication of the law and axioms for scalar multiplication

(NS1) There exist at least three duplets $(a, v), (b, u), (c, x)$ with $a, b, c \in K$ and $v, u, x \in V$ such that $av \in V$ (inner-defined with degree of truth T) and [bu = indeterminate (with degree of indeterminacy I) or $cx \notin V$ (degree of falsehood F)].

(NS2) There exist at least three triplets $(k, x, y), (m, u, v), (n, w, z)$ with $k, m, n \in K$ and $x, y, u, v, w, z \in X$ such that $k(x + y) = kx + ky$ (inner-defined with degree of truth T) and [[$m(u+v)$]or[$mu+mv$] = indeterminate (with degree of indeterminacy I) or $n(w+z) \neq nw+nz$ (outer-defined/falsehood with degree of falsehood F)].

(NS3) There exist at least three triplets (k, m, x),(p,q,y),(r,s,z) with $k, m, p, q, r, s \in K$ and $x, y, z \in X$ such that $(k+m)x = kx+mx$ (inner-defined with degree of truth T) and [[$(p+q)y$]or[$py+qy$] = indeterminate (with degree of indeterminacy I) or $(r + s)z \neq rz + sz$ (outer-defined/falsehood with degree of falsehood F)].

(NS4) There exist at least three triplets $(k, m, x), (p, q, y), (r, s, z)$ with $k, m, p, q, r, s \in K$ and $x, y, z \in X$ such that $k(mx) = (km)x = (mk)x$ (inner-defined with degree of truth T) and [[$p(qy)$]or[$q(py)$]or[$(pq)y$] = indeterminate (with degree of indeterminacy I) or $r(sz) \neq (rs)z$ (outer-defined/falsehood with degree of falsehood F)].

(NS5) There exists an element $k \in K$ such that $kv = v$ (inner-defined with degree of truth T) and [kv = indeterminate (with degree of indeterminacy I) or $kv \neq v$ (outer-defined/falsehood with degree of falsehood F)] for at least one $v \in V$.

Definition 3.3. [AntiSophication of the law and axioms of the classical vector space]

AntiSophication of the law and axioms for vector addition

(AA1) For all the duplets $(u, v) \in V$, $u + v \notin V$.

(AA2) For all the triplets $(u, v, w) \in V$, $u + (v + w) \neq (u + v) + w$.

(AA3) There does not exist an element e in V such that $v + e = v = e + v$ for every v in V.

(AA4) There does not exist $-v$ in V such that $v + (-v) = (-v) + v = e$ for all $v \in V$ where e is a AntiNeutralElement in V.

(AA5) For all the duplets $(u, v) \in V$, $u + v \neq v + u$.

AntiSophication of the law and axioms for scalar multiplication

(AS1) For all $v \in V$ and $a \in \mathbb{R}$, $av \notin V$.

(AS2) For all $u, v \in V$ and $a \in \mathbb{R}$, $a(u + v) \neq au + av$.

(AS3) For all $v \in V$ and $a, b \in \mathbb{R}$, $(a + b)v \neq av + bv$.

(AS4) For all $v \in V$ and $a, b \in \mathbb{R}$, $a(bv) \neq (ab)v$.

(AS5) For all $v \in V$, $1v \neq v$.

Definition 3.4. Let $(K, +, \cdot)$ be a field. A NeutroField $(NK, +, \cdot)$ is an alternative to the classical field $(K, +, \cdot)$ that has at least one NeutroLaw or at least one NeutroAxiom with no Antilaw or AntiAxiom.

Definition 3.5. Let $(K, +, \cdot)$ be a field. An AntiField $(AK, +, \cdot)$ is an alternative to the classical field $(K, +, \cdot)$ that has at least one AntiLaw or at least one AntiAxiom.

Definition 3.6. Let " + " be addition of vectors, " · " be multiplication of vector by scalars and let K be a Neutro/classical field. A NeutroVectorSpace $(NV, +, \cdot)$ is an alternative to the classical vector space $(V, +, \cdot)$ that has at least one NeutroLaw or at least one of $\{NA1 - NS5\}$ with no Antilaw or AntiAxiom.

If K is a classical field, then the quadruple $(NV, +, \cdot, K)$ is called a weak NeutroVectorSpace over K. And the quadruple $(NV, +, \cdot, K)$ is called a strong NeutroVectorSpace if K is a NeutroField (i.e., $K = NK$).

Definition 3.7. Let " + " be addition of vectors, " · " be multiplication of vectors by scalars and let K be a Anti/classical field. An AntiVectorSpace $(AV, +, \cdot)$ is an alternative to the classical vector space $(V, +, \cdot)$ that has at least one AntiLaw or at least one of $\{AA1 - AS5\}$.

If K is a classical field, then the quadruple $(AV, +, \cdot, K)$ is called a weak AntiVectorSpace over K. And the quadruple $(AV, +, \cdot, K)$ is called a strong AntiVectorSpace if K is a AntiField (i.e., $K = AK$).

Theorem 3.8. *Let $(V, +, \cdot)$ be a classical vector space over a field K. Then,*

(1) *there are 1023 classes of NeutroVector Spaces.*
(2) *there are 58025 classes of AntiVector Spaces.*

Proof. The proof follows easily from Theorem 2.2. □

Theorem 3.8 shows that there are many classes of NeutroVector Spaces. The trivial cases from the 1023 classes are the cases where $NA1 - NS5$ hold. Examples of weak and strong NeutroVectorSpaces for the trivial cases are given in Example 3.9.

Example 3.9. Let $V = \mathbb{Z}_{12}$ and $K = \mathbb{R}$. Define addition and scalar multiplication by

$$x \oplus y = \frac{2x + 3y}{2} \text{ and } k \odot a = ka^2$$

where $\oplus$ is addition modulo 12. Then $(V, \oplus, \odot)$ is a weak NeutroVectorSpace over a field K.

To see this:

(1) We will show that $(V, \oplus)$ is a NeutroAbelianGroup.

(a) There exist at least $x, y \in V$ such that $x \oplus y \in V$ and at least $a, b \in V$ such that $a \oplus b \notin V$. For instance, if we take $(x, y) = (1, 2)$ and $(a, b) = (2, 1)$, we will see that $NV1$ holds. Therefore $\oplus$ is NeutroClosed.

(b) Let $x, y, z \in V$. Then
$x \oplus (y \oplus z) = \frac{4x+6y+9z}{4}$ and $(x \oplus y) \oplus z = \frac{4x+6y+6z}{4}$, equating these we have
$4x + 6y + 9z = 4x + 6y + 6z$ which gives $9z = 6z$,
$\therefore 3z = 0$ this implies that $z = 0, 4$ and 8.
Thus, only the triplets $(x, y, 0), (x, y, 4),$ and $(x, y, 8)$ can verify the associativity of $\oplus$ and therefore, $\oplus$ is NeutroAssociative.

(c) Let $e \in V$ such that $x \oplus e = \frac{2x+3e}{2} = x$ and $e \oplus x = \frac{2e+3x}{2} = x$.
Then $\frac{2x+3e}{2} = \frac{2e+3x}{2}$ from which we obtain $e = x$.
The elements of V that satisfy $x \oplus x = x$ are $0, 8$. This shows that V has NeutroNeutral element.

(d) Considering each NeutroNeutral element in (b) we can show that V has NeutroInverse element.

(e) Let $x, y \in V$, $x \oplus y = \frac{2x+3y}{2}$ and $y \oplus x = \frac{2y+3x}{2}$.
If $''\oplus''$ is commutative, we will have $\frac{2x+3y}{2} = \frac{2y+3x}{2}$ from which we obtain $x = y$. This shows that only the duplet (x, x) can verify commutativity of $\oplus$.
Thus, $\oplus$ is NeutroCommutative. Hence, $(V, \oplus)$ is a NeutroAbelianGroup.

(2) We wish to find at least a triplet (k, m, u) with $u \in V$ and $k, m \in K$, such that $k \odot (m \odot u) = (km) \odot u$.
Now, consider $(km) \odot u = (km)u^2 = kmu^2$ and $k \odot (m \odot u) = k \odot (mu^2) = k(mu^2)^2 = km^2u^4$.
Equating these we have

$$kmu^2 = km^2u^4,$$

which gives

$$mu^2 = 1.$$

Since we need at least a triplet, take $m = 1$, then elements of V that will satisfy $mu^2 = 1$ are $5, 7, 11$.
So, $k \odot (m \odot u) = (km) \odot u$ for at least the triplets $(k, 1, 5), (k, 1, 7)$ and $(k, 1, 11)$.

(3) We want to show that, there exist at least a triplet (k, m, u) with $u \in V$ and $k, m \in K$, such that $(k + m) \odot u = k \odot u \oplus m \odot u$.
Consider, $(k + m) \odot u = (k + m)u^2 = ku^2 + mu^2$ and
$k \odot u \oplus m \odot u = ku^2 \oplus mu^2 = \frac{2ku^2+3mu^2}{2}$.
Equating these we have

$$ku^2 + mu^2 = \frac{2ku^2 + 3mu^2}{2},$$

which gives

$$mu^2 = 0 \implies u^2 = 0$$

$$\therefore u = 0 \text{ and } 6.$$

This shows that only the triplets $(k, m, 0)$ and $(k, m, 6)$ can verify $(k + m) \odot u = k \odot u \oplus m \odot u$.

(4) We want to show that there exists at least a triplet (k, u, v) with $u, v \in V$ and $k \in K$, such that $k \odot (u \oplus v) = k \odot u \oplus k \odot v$.
Now, consider $k \odot (u \oplus v) = k \odot \frac{(2u+3v)}{2} = k\frac{(2u+3v)^2}{4} = \frac{4ku^2+12kuv+9kv^2}{4} = \frac{4ku^2+9kv^2}{4}$ and $k \odot u \oplus k \odot v = ku^2 \oplus kv^2 = \frac{2ku^2+3kv^2}{2}$.
Equating these we have

$$4ku^2 + 9kv^2 = 4ku^2 + 6kv^2,$$

which gives

$$9kv^2 = 6kv^2$$

$$3kv^2 = 0 \implies v^2 = 0.$$

So,

$$v = 0, 6.$$

This shows that only the triplets $(k, u, 0)$ and $(k, u, 6)$ can verify $k \odot (u \oplus v) = k \odot u \oplus k \odot v$.

(5) We want to show that there exists at least a $u \in V$ such that $1 \odot u = u$.
We have that the only elements of V that satisfy $1 \odot u = u^2 = u$ are 4 and 9.

Accordingly, $(V, \oplus, \odot)$ is a weak NeutroVectorSpace over a field $K = \mathbb{R}$.

Example 3.10. Let $X = \{a, b, c, d, e\}$ be a universe of discourse and let $\mathbb{K} = \{a, b, c, d\}$. Let $\oplus$ and $\odot$ be the binary operations defined on $\mathbb{K}$ as shown in the Cayley tables below.

TABLE 1. (a) Cayley table for the binary operation $''\oplus''$ and (b) Cayley table for the binary operation $''\odot''$

$\oplus$	a	b	c	d
a	a	c	a	c
b	b	d	b	d
c	c	a	c	a
d	d	b	d	b or d

(a)

$\odot$	a	b	c	d
a	a	c	a	c
b	b	d	b	d
c	a	c	a	c
d	b	d	b	d

(b)

(1) $(\mathbb{K}, \oplus, \odot)$ is a trivial NeutroField.

(2) $(\mathbb{K}, \oplus, \odot)$ taken over itself is a strong NeutroVector Space.

(1) To show that $(\mathbb{K}, \oplus, \odot)$ is a trivial NeutroField we proceed as follows:

(a) $(\mathbb{K}, \oplus)$ is a NeutroAbelianGroup. It is clear from the table that;

(i) $d \oplus d =$ b or d.

So, the composition $d \oplus d$ is indeterminate with 6.25% degree of indeterminacy and all other compositions are true with 93.75% degree of truth. Hence $''\oplus''$ is NeutroClosed.

(ii) $d \oplus (c \oplus a) = (d \oplus c) \oplus a = d$,

$a \oplus (c \oplus d) = a$, but $(a \oplus c) \oplus d = c \neq a$. Hence $''\oplus''$ is NeutroAssociative.

(iii) Since only the duplet $(x, x) \in \mathbb{K}$ verify commutativity, for $x = a, b, c \in \mathbb{K}$.

Hence $''\oplus''$ is NeutroCommutative.

(iv) Let N_x and I_x represent additive neutral and inverse element respectively with respect to any element $x \in \mathbb{K}$.

Then $N_a = a$, $N_c = c$ and N_b, N_d do not exist.

$I_a = a,\ I_c = c$ and $I_b,\ I_d$ do not exist.

Hence, $(\mathbb{K}, \oplus)$ is a NeutroAbelianGroup.

(b) $(\mathbb{K}, \odot)$ is a NeutroAbelianGroup. It is clear from the table that;

(i) $(a \odot b) \odot d = a \odot (b \odot d) = c$,

$(b \odot c) \odot d = d$ but $b \odot (c \odot d) = b \neq d$. Hence $''\odot''$ is NeutroAssociative.

(ii) $a \odot c = c \odot a = a$,

$a \odot b = c$ but $b \odot a = b$. Hence, $\odot$ is NeutroCommutative.

(iii) Let U_x and I_x represent multiplicative neutral and inverse element(s) respectively with respect to any element $x \in \mathbb{K}$.

Then, $U_a = a$ and c. $U_d = b$ and d. U_b and U_c do not exist.

$I_a = a$ and c. $I_d = b$ and d. I_c and I_b do not exist.

Hence, $(\mathbb{K}, \odot)$ is a NeutroAbelianGroup.

(c) Now, we show that $\odot$ is distributive over $\oplus$. It is clear from the table that ;

(i) $a \odot (b \oplus c) = a \odot b \oplus a \odot c = c$,

$b \odot (a \oplus b) = b$, but $b \odot a \oplus b \odot b = d \neq b$. So, $''\odot''$ is left NeutroDistributive over $''\oplus''$.

(ii) $(b \oplus c) \odot a = b \odot a \oplus c \odot a = b$,

$(c \oplus b) \odot d = c$, but $c \odot d \oplus b \odot d = a \neq c$. So, $''\odot''$ is right NeutroDistributive over $''\oplus''$. Hence, $''\odot''$ is NeutroDistributive over $''\oplus''$.

Accordingly, $(\mathbb{K}, \oplus, \odot)$ is a trivial NeutroField.

(2) That $(\mathbb{K}, \oplus, \odot)$ is a strong NeutroVector Space over itself, follows easily from all the properties established in solution of 1 above.

Proposition 3.11. *Every NeutroField taken over itself is a strong NeutroVectorSpace.*

Proof. The proof follows from Example 3.10 . □

4. **A Study of a Class of NeutroVectorSpaces**

In this section, we shall consider a particular class of NeutroVectorSpaces $(NV, +, \cdot)$ where

(1) $(NV, +)$ is a classical abelian group.
(2) $S1$ is totally true for all $v \in V$ and $a \in K$.
(3) $S2, S3, S4$ and $S5$ are either partially true or partially indeterminate or partially false for some elements of V and K.

We shall refer to this class of NeutroVectorSpace as NeutroVectorSpace of type $4S$ (i.e., 4 of its scalar multiplication axioms are NeutroAxioms).

Example 4.1. Let $K = \mathbb{Z}_p$ (where p is prime) and $NV = \mathbb{Z}_8$. Define $\oplus$ and $\odot$ by

$$a \oplus b = a + b \text{ and } k \odot a = a^2 + ka.$$

Where $''+''$ is addition modulo 8 .
Then $(NV, \oplus, \odot)$ is a weak NeutroVectorSpace of type $4S$ over the field $K = \mathbb{Z}_p$.

It is easy to show that $(NV, \oplus)$ is an abelian group. Also, it is easy to see that $S1$ holds. Now it remains to show that $NS2, NS3, NS4$ and $NS5$ hold.

(1) We want to show that there exists at least a triplet (k, x, y) with $k \in K$ and $x, y \in NV$ such that
$$k \odot (x \oplus y) = k \odot x \oplus k \odot y.$$
Now, $k \odot (x \oplus y) = k \odot (x + y) = (x + y)^2 + k(x + y) = x^2 + y^2 + 2xy + kx + ky.$
And $k \odot x \oplus k \odot y = (x^2 + kx) \oplus (y^2 + ky) = x^2 + y^2 + kx + ky.$
$$\therefore x^2 + y^2 + 2xy + kx + ky = x^2 + y^2 + kx + ky$$
$$\implies xy = 0.$$
Hence $x = 0$ or $y = 0, (x, y) = (2, 4), (x, y) = (4, 2), (x, y) = (4, 6)$ and $(x, y) = (6, 4)$.
This shows that only the triplets $(k, x, 0), (k, 0, y), (k, 2, 4), (k, 4, 2), (k, 4, 6)$ and $(k, 6, 4)$ can verify $NS2$.
(2) We want to show that there exists at least a triplet (k, m, u) with $k, m \in K$ and $u \in NV$ such that
$$(k + m) \odot u = k \odot u + m \odot u.$$
$(k + m) \odot u = u^2 + (k + m)u = u^2 + ku + mu$ and $k \odot u \oplus m \odot u = 2u^2 + ku + mu.$
Then, we have
$$u^2 + ku + mu = 2u^2 + ku + mu$$
$$\implies u^2 = 0.$$

$$\therefore u = 0 \text{ and } 4.$$

Hence, only the triplet $(k, m, 0)$ and $(k, m, 4)$ can verify $NS3$.

(3) We want to show that there exists at least a triplet (k, m, u) with $u \in NV$ and $k, m \in K$, such that $k \odot (m \odot u) = (km) \odot u$.

Now, consider $(km) \odot u = u^2 + (km)u = u^2 + kmu$ and

$$k \odot (m \odot u) = k \odot (u^2 + mu) = (u^2 + mu)^2 + k(u^2 + mu) = u^4 + 2mu^3 + m^2u^2 + ku^2 + kmu.$$

Equating these we have

$$u^2 + 2mu + m^2 + k = 1.$$

Since we need at least a triplet, take $k = 1$, then we have $u^2 + 2um + m^2 = 0$ and this gives $u = -m$.

Hence, at least the triplet $(1, m, -m)$ satisfies $NS4$.

(4) We want to show that there exists at least $v \in NV$ such that $1 \odot v = v$.

From definition of $\odot$ we have that the only elements of NV that satisfy

$$1 \odot u = v^2 + v = v \text{ are } 0 \text{ and } 4.$$

Hence $(NV, \oplus, \odot)$ is a weak NeutroVectorSpace of type $4S$ over the field $K = \mathbb{Z}_p$.

Example 4.2. Let $X = \{a, b, c, d, e\}$ be a universe of discourse. Let $\mathbb{K} = \{a, b, c, d\}$ be the Neutrofield defined in Example 3.10 and let $NV = \left\{v_1 = \frac{a}{e}, v_2 = \frac{b}{e}, v_3 = \frac{c}{e}, v_4 = \frac{d}{e}\right\}$.

Define on NV the binary operation $+'$ as in the table below and scalar multiplication $\star$ by

$$a \star v = \frac{a \odot x}{e},$$

here $\odot$ is the multiplication in $\mathbb{K}$ defined in Table 1 (b) for all elements in $\mathbb{K}$.

Table 2. Cayley table for the binary operation $+'$

$+'$	v_1	v_2	v_3	v_4
v_1	v_1	v_2	v_3	v_4
v_2	v_2	v_3	v_4	v_1
v_3	v_3	v_4	v_1	v_2
v_4	v_4	v_1	v_2	v_3

Then $(NV, +', \star)$ is a strong NeutroVectorSpace of type $4S$ over $\mathbb{K}$.

It is clear from Table 2 that $(NV, +')$ is an abelian group. Also, it is easy to see that $S1$ holds. Now it remains to show that $NS2, NS3, NS4$ and $NS5$ hold.

It can be seen from Table 1 (a), Table 1 (b) and Table 2 that ;

(1) for $c \in \mathbb{K}$ and $v_2, v_3 \in NV$,

$$\begin{aligned} c \star (v_2 +' v_3) &= c \star (v_2 +' v_3) \\ &= c \star v_4 \text{ from Table 2} \\ &= \tfrac{c \odot d}{e} \\ &= \tfrac{c}{e} \because c \odot d = c, \text{ from Table 1 (b)} \\ &= v_3. \end{aligned}$$

$$\begin{aligned} c \star v_2 +' c \star v_3 &= \tfrac{c \odot b}{e} +' \tfrac{c \odot c}{e} \\ &= \tfrac{c}{e} +' \tfrac{a}{e} \text{ from Table1 (b)} \\ &= v_3 +' v_1 \\ &= v_3. \end{aligned}$$

$$\implies c \star (v_2 +' v_3) = c \star v_2 +' c \star v_3 = v_3,$$

and for $b \in \mathbb{K}$ and $v_3, v_4 \in NV$,

$$b \star (v_3 +' v_4) = v_4 \text{ but } b \star v_3 +' b \star v_4 = v_1 \neq v_4.$$

This shows that $NS2$ holds.

(2) for $a, c \in \mathbb{K}$ and $v_2 \in NV$,

$$(a \oplus c) \star v_2 = a \star v_2 +' c \star v_2 = v_3,$$

and for $a, b \in \mathbb{K}$ and $v_4 \in NV$,

$$(a \oplus b) \star v_4 = v_3 \text{ but } a \star v_4 +' b \star v_4 = v_2 \neq v_3.$$

This shows that $NS3$ holds.

(3) for $a, b \in \mathbb{K}$ and $v_4 \in NV$,

$$(a \odot b) \star v_4 = a \star (b \star v_4) = v_3,$$

and for $b, c \in \mathbb{K}$ and $v_4 \in NV$

$$(b \odot c) \star v_4 = v_4 \text{ but } b \star (c \star v_4) = v_2 \neq v_4.$$

This shows that $NS4$ holds.

(4) We know from Table 1 that NeutroUnityElements in K are $U_a = a, c$ and $U_d = b, d$.
Now, suppose we consider the NeutroUnityElement $U_d = b$ only.
We have that $b \star v_4 = v_4$ and $b \star v_3 = v_2 \neq v_3$.
This shows that $NS5$ holds.

Hence, we have that $(NV, +', \star)$ is a strong NeutroVectorSpace of type $4S$ over the NeutroField $\mathbb{K}$. From now on, every weak(strong) NeutroVectorSpaces of type $4S$ over $K(NK)$ will simply be called a weak(strong) NeutroVectorSpace over $K(NK)$.

Proposition 4.3. *Let $(NV, +'_1, \star_1)$ and $(NH, +'_2, \star_2)$ be two weak NeutroVectorSpace over the field K and let*

$$NV \times NH = \{(v, h) : v \in NV \text{ and } h \in NH\},$$

for $x = (v_1, h_1)$, $y = (v_2, h_2) \in NV \times NH$ and $k \in K$ define :

$$x \oplus y = ((v_1 +'_1 v_2), (h_1 +'_2 h_2),$$

$$k \odot x = (k \star_1 v_1, k \star_2 v_2).$$

Then $(NV \times NH, \oplus, \odot)$ *is a weak NeutroVectorSpace over the field* K.

Proof. Since $(NV, +_1')$ and $(NH, +_2')$ are classical abelian groups, then it can be shown that $(NV \times NH, \oplus)$ is a classical abelian group. Also, it is easy to see that $S1$ is true in $(NV \times NH)$.
Now, it remains to show that $NS2 - NS5$ hold in $NV \times NH$.

(1) There exists at least a triplet $(k, (v_1, h_1), (v_2, h_2))$ with $(v_1, h_1), (v_2, h_2) \in NV \times NH$ and $k \in K$, such that

$$\begin{aligned} k \odot ((v_1, h_1) \oplus (v_2, h_2)) &= k \odot (v_1 +_1' v_2,\ h_1 +_2' h_2) \\ &= (k \star_1 (v_1 +_1' v_2),\ k \star_2 (h_1 +_2' h_2)) \\ &= (k \star_1 v_1 +_1' k \star_1 v_2,\ k \star_2 h_1 +_2' k \star_2 h_2) \because NS2 \text{ holds in } NV \text{and } NH. \\ &= (k \star_1 v_1, k \star_2 h_1) \oplus (k \star_1 v_2, k \star_2 h_2) \\ &= k \odot (v_1, h_1) \oplus k \odot (v_2, h_2). \end{aligned}$$

Also, there exists at least a triplet $(m, (a_1, b_1), (a_2, b_2))$ with $(a_1, b_1), (a_2, b_2) \in NV \times NH$ and $m \in K$, such that

$$\begin{aligned} m \odot ((a_1, b_1) \oplus (a_2, b_2)) &= m \odot (a_1 +_1' a_2,\ b_1 +_2' b_2) \\ &= (m \star_1 (a_1 +_1' a_2),\ m \star_2 (b_1 +_2' b_2)) \\ &\neq (m \star_1 a_1 +_1' m \star_1 a_2,\ m \star_2 b_1 +_2' m \star_2 b_2) \because NS2 \text{ holds in } NV \text{and } NH. \\ &= (m \star_1 a_1, m \star_2 b_1) \oplus (m \star_1 a_2, m \star_2 b_2) \\ &= m \odot (a_1, b_1) \oplus m \odot (a_2, b_2). \end{aligned}$$

Hence, $NS2$ holds in $NV \times NH$.

(2) There exists at least a triplet $(k, m, (v, h))$ with $k, m \in K$ and $(v, h) \in NV \times NH$ such that

$$\begin{aligned} (k + m) \odot (v, h) &= ((k + m) \star_1 v,\ (k + m) \star_2 h) \\ &= ((k \star_1 v +_1' m \star_1 v),\ (k \star_2 h +_2' m \star_2 h)) \because NS3 \text{ holds in } NV \text{and } NH. \\ &= ((k \star_1 v,\ k \star_2 h) \oplus (m \star_1 v, m \star_2 h)) \\ &= k \odot (v, h) \oplus m \odot (v, h). \end{aligned}$$

Also, there exists at least a triplet $(p, q, (a, b))$ with $p, q \in K$ and $(a, b) \in NV \times NH$ such that

$$\begin{aligned} (p + q) \odot (a, b) &= ((p + q) \star_1 a,\ (p + q) \star_2 b) \\ &\neq ((p \star_1 a +_1' q \star_1 a),\ (p \star_2 b +_2' q \star_2 b)) \because NS3 \text{ holds in } NV \text{and } NH. \\ &= ((p \star_1 a,\ p \star_2 b) \oplus (q \star_1 a, q \star_2 b)) \\ &= p \odot (a, b) \oplus q \odot (a, b). \end{aligned}$$

Hence, $NS3$ holds in $NV \times NH$.

(3) There exists at least a triplet $(k, m, (v, h))$ with $k, m \in K$ and $(v, h) \in NV \times NH$ such that

$$\begin{aligned} (km) \odot (v, h) &= ((km) \star_1 v, (km) \star_2 h) \\ &= (k \star_1 (m \star_1 v), k \star_2 (m \star_2 h)) \because NS4 \text{ holds in } NV \text{and } NH. \\ &= k \odot ((m \star_1 v), (m \star_2 h)) \\ &= k \odot (m \odot (v, h)). \end{aligned}$$

Also, there exists at least a triplet $(p, q, (a, b))$ with $p, q \in K$ and $(a, b) \in NV \times NH$ such that

$$\begin{aligned} (pq) \odot (a, b) &= ((pq) \star_1 a, (pq) \star_2 b) \\ &\neq (p \star_1 (q \star_1 a), p \star_2 (q \star_2 b)) \ \because NS4 \text{ holds in } NV \text{and } NH. \\ &= p \odot ((q \star_1 a), (p \star_2 b)) \\ &= p \odot (q \odot (v, h)). \end{aligned}$$

Hence, $NS4$ holds in $NV \times NH$.

(4) There exists $(v, h) \in NV \times NH$ such that

$$\begin{aligned} 1 \odot (v, h) &= (1 \star v, 1 \star h) \\ &= (v, h). \ \because NS5 \text{ holds in } NV \text{and } NH. \end{aligned}$$

Also, there exists $(a, b) \in NV \times NH$ such that

$$\begin{aligned} 1 \odot (a, b) &= (1 \star_1 a, 1 \star_2 b) \\ &\neq (a, b). \ \because NS5 \text{ holds in } NV \text{and } NH. \end{aligned}$$

Accordingly, $(NV \times NH, \oplus, \odot)$ is a weak NeutroVectorSpace over the field K. □

Proposition 4.4. *Let $(NV, +_1', \star_1)$ be a weak NeutroVectorSpace over the field K and let $(H, +, \cdot)$ be a classical vector space over the same field K and let*

$$NV \times H = \{(v, h) : v \in NV \text{ and } h \in H\}$$

and for $x = (v_1, h_1)$, $y = (v_2, h_2) \in NV \times H$ and $k \in K$ define :

$$x \oplus y = ((v_1 +_1' v_2), (h_1 + h_2) \text{ and } k \odot x = (k \star v_1, k \cdot v_2).$$

Then $(NV \times H, \oplus, \cdot)$ is a weak NeutroVectorSpace over the field K.

Proof. The proof is similar to the proof of Proposition 4.3 . □

Proposition 4.5. *Let $(NV, +_1', \star_1)$ and $(NH, +_2', \star_2)$ be two strong NeutroVectorSpaces over the NeutroField NK and let*

$$NV \times NH = \{(v, h) : v \in NV \text{ and } h \in NH\}$$

and for $x = (v_1, h_1)$, $y = (v_2, h_2) \in NV \times NH$ and $k \in NK$ define :

$$x \oplus y = ((v_1 +_1' v_2), (h_1 +_2' h_2) \text{ and } k \odot x = (k \star_1 v_1, k \star_2 v_2).$$

Then $(NV \times NH, \oplus, \odot)$ is a strong NeutroVectorSpace over the NeutroField NK.

Proof. The proof follows similar approach as the proof of Proposition 4.3. □

Definition 4.6. Let NV be a NeutroVectorSpace. Then NW is a NeutroSubspace of NV if and only if NW is a subset of NV, and NW is itself a NeutroVectorSpace with the same operations as in NV.

Example 4.7. Let $(NV, \oplus, \odot)$ be a weak NeutroVectorSpace of Example 4.1 and let $NW = 2\mathbb{Z}_8$ be a subset of NV. Following the approach in Example 4.1, it can be shown that $(NW, \oplus, \odot)$ is a weak NeutroVectorSpace over the field $\mathbb{Z}_p$. Hence NW is a weak NeutroSubspace of NV.

Example 4.8. Let $(NV, +', \star)$ be the strong NeutroVectorSpace of Example 4.2 , NV is the only strong NeutroSubspace of NV.

Remark 4.9. It should be noted that a NeutroVectorSpace NV of a particular class may contain a NeutroSubspace NW which belongs to another class.

We will illustrate Remark 4.9 with Example 4.10 .

Example 4.10. Let $(NV, +', \star)$ be the strong NeutroVectorSpace of Example 4.2 and let $NW = \{v_1, v_3\}$ be a subset of NV. Then $(NW, +', \star)$ is a NeutroVectorSpace of a class other than the class of NV.

We can see from Table 2 that $(NW, +')$ is an abelian group. Now, it can be seen from Table 1 (a), Table 1 (b) and Table 2 that ;

(1) $S1$ fails to hold. Since $\star$ is not true for all $a \in \mathbb{K}$ and $v \in NW$.
For instance, take $b \in \mathbb{K}$ and $v_3 \in NW$, then

$$b \star v_3 = \frac{b \cdot c}{e} = \frac{b}{e} = v_2 \notin V_3.$$

But if we take $a \in \mathbb{K}$ then for all $v \in NW$ we will have that $a \star v \in NW$.
Hence, $NS1$ holds in NW.

(2) for $a, b \in \mathbb{K}$ and $v_1, v_3 \in NW$, we have

$$a \star (v_1 +' v_3) = a \star v_1 +' a \star v_3 = v_1,$$

and $b \star (v_1 +' v_3) = v_2$ but $b \star v_1 +' b \star v_3 = v_4 \neq v_2$.
This shows that $NS2$ holds in NW.

(3) for $a, c \in \mathbb{K}$ and $v_3 \in NW$,

$$(a \oplus c) \star v_3 = a \star v_3 +' c \star v_3 = v_1,$$

and for $a, b \in \mathbb{K}$ and $v_3 \in NW$,

$$(a \oplus b) \star v_3 = v_1 \text{ but } a \star v_3 +' b \star v_3 = v_3 \neq v_1.$$

This shows that $NS3$ holds.

(4) for $a, c \in \mathbb{K}$ and $v_3 \in NW$,

$$(a \odot c) \star v_3 = a \star (c \star v_3) = v_1,$$

and for $a, b \in \mathbb{K}$ and $v_3 \in NW$,

$$(a \odot b) \star v_3 = v_1 \text{ but } a \star (b \star v_3) = v_3 \neq v_1.$$

This shows that $NS4$ holds.

(5) We know from Table 1 that NeutroUnityElements in K are $U_a = a, c$ and $U_d = b, d$.
Now, suppose we consider the NeutroUnityElement $U_a = c$ only.
We have that $c \star v_1 = v_1$ and $c \star v_3 = v_1 \neq v_3$.
This shows that $NS5$ holds.

Hence, we have that $(NW, +', \star)$ is a strong NeutroSubpace of type $5S$ over the NeutroField $\mathbb{K}$. This implies that the NeutroSubspace NW does not belong to the same class as NV.

Example 4.11. Let $NV = \mathbb{Z}_{12}$ and $K = \mathbb{Z}_p$. Define $\oplus$ and $\odot$ for all $u, v \in V$ and $k \in K$ by

$$u \oplus v = u + v \text{ and } k \odot v = v^2 + kv.$$

Where $''+''$ is addition mod 12.
Following the approach of Example 4.1 it can be shown that $(NV, \oplus, \odot)$ is a weak NeutroVectorSpace of type $4S$ over the field K.
Let $NW = 2\mathbb{Z}_{12}$ and $NH = 3\mathbb{Z}_{12}$ be two subsets of NV. Also, by following similar approach as in Example 4.1 it can be shown that $(NW, \oplus, \odot)$ and $(NH, \oplus, \odot)$ are weak NeutroSubspaces of NV.

Now consider the following :

(1) $NW + NH = \{0, 1, 2, \cdots, 11\} = NV$.
(2) $NW \cup NH = \{0, 2, 3, 4, 6, 8, 9, 10\}$.
(3) $NW \cap NH = \{0, 6\}$.

These show that $NW + NH$ is a NeutroSubspace of NV but $NW \cup NH$ and $NW \cap NH$ are not NeutroSubspaces of NV.
These observations are recorded in Proposition 4.12 .

Proposition 4.12. *Let NW and NH be any two weak NeutroSubspaces of a NeutroVectorSpace NV over a field K. Then*

(1) $NW + NH = \bigcup\{(w + h) : w \in NW \text{ and } h \in NU\}$ *is a NeutroSubspace of NV.*
(2) $NW \cap NU$ *is not necessarily a NeutroSubspace of NV.*
(3) $NW \cup NU$ *is not necessarily a NeutroSubspace of NV.*

Definition 4.13. Let NW be a weak(strong) NeutroSubspace of a weak(strong) NeutroVectorSpace NV over a field (NeutroField) $K(NK)$. The quotient NV/NW is defined by the set

$$\{v + NW : v \in NV\}.$$

Proposition 4.14. *Let $(NV, +', \star)$ be a weak NeutroVectorSpace and $(NW, +', \star)$ be a weak NeutroSubspace of NV. The quotient NV/NW is a weak NeutroVectorSpace over a field K if addition and multiplication are defined for all $\bar{u} = u + NW$, $\bar{v} = v + NW \in NV/NW$ and $k \in K$ as follows:*

$$\bar{u} \oplus \bar{v} = (u + NW) \oplus (v + NW) = (u +' v) + NW,$$

and

$$\alpha \odot \bar{u} = \alpha \odot (u + NW) = (\alpha \star u) + NW.$$

This weak NeutroVectorSpace $(NV/NW, \oplus, \odot)$ over a field K is called a weak NeutroQuotientSpace.

Proof. We can easily show that $\oplus$ and $\odot$ are well defined.

The proof that $(NV/NW, \oplus)$ is an abelian group follows similar approach as the proof in classical case. Now it remains to show that $NS2, NS3, NS4$ and $NS5$ all hold.

(1) Since $NS2$ holds in NV, then there exist at least the triplets (k, u, v) and (m, a, b) with $u, v, a, b \in NV$ and $k, m \in K$ such that $k \star (u +' v) = k \star u +' k \star v$ and $m \star (a +' b) \neq m \star a +' m \star b$. Let $\bar{u}, \bar{v}, \bar{a}, \bar{b} \in NV/NW$ and $k, m \in K(NK)$. Then

$$\begin{aligned} k \odot (\bar{u} \oplus \bar{v}) &= k \odot ((u +' v) + NW) \\ &= k \star (u +' v) + NW \\ &= (k \star u +' k \star v) + NW \\ &= (k \star u) + NW \oplus (k \star v) + NW \\ &= k \odot (u + NW) \oplus k \odot (v + NW) \\ &= k \odot \bar{u} \oplus k \odot \bar{v}. \end{aligned}$$

So, it implies $k \odot (\bar{u} \oplus \bar{v}) = k \odot \bar{u} \oplus k \odot \bar{v}$.

And also,

$$\begin{aligned} m \odot (\bar{a} \oplus \bar{b}) &= m \odot ((a +' b) + NW) \\ &= m \star (a +' b) + NW \\ &\neq (m \star a +' m \star b) + NW \\ &= (m \star a) + NW \oplus (m \star b) + NW \\ &= m \odot (a + NW) \oplus m \odot (b + NW) \\ &= m \odot \bar{a} \oplus m \odot \bar{b}. \end{aligned}$$

This implies $m \odot (\bar{a} \oplus \bar{b}) \neq m \odot \bar{a} \oplus m \odot \bar{b}$. Hence, we can conclude that $NS2$ holds in NV/NW.

(2) Since $NS3$ holds in NV, then there exist at least the triplets (k, m, u) and (p, q, v) with $u, v \in NV$ and $k, m, p, q \in K$ such that $(k + m) \star u = k \star u +' m \star u$ and $(p + q) \star v \neq p \star v +' q \star v$. Let $\bar{u}, \bar{v} \in NV/NW$ and $k, m, p, q \in K(NK)$. Then

$$\begin{aligned} (k + m) \odot \bar{u} &= (k + m) \odot (u + NW) \\ &= (k + m) \star u + NW \\ &= (k \star u +' m \star u) + NW \\ &= (k \star u) + NW \oplus (m \star u) + NW \\ &= k \odot (u + NW) \oplus m \odot (u + NW) \\ &= k \odot \bar{u} \oplus m \odot \bar{u}. \end{aligned}$$

So, it implies $(k+m)\odot\bar{u} = k\odot\bar{u}\oplus m\odot\bar{u}$.

And also,

$$\begin{aligned}(p+q)\odot\bar{v} &= (p+q)\odot(v+NW)\\ &= (p+q)\star v+NW\\ &\neq (p\star v+' q\star v)+NW\\ &= (p\star v)+NW\oplus(q\star v)+NW\\ &= p\odot(v+NW)\oplus q\odot(v+NW)\\ &= p\odot\bar{v}\oplus q\odot\bar{v}.\end{aligned}$$

So, it implies $(p+q)\odot\bar{v}\neq p\odot\bar{v}\oplus q\odot\bar{v}$.

Hence, we can conclude that $NS3$ holds in NV/NW.

(3) Since $NS4$ holds in NV, then there exist at least the triplets (k,m,u) and (p,q,v) with $u,v\in NV$ and $k,m,p,q\in K$ such that $(km)\star u = k\star(m\star u)$ and $(pq)\star v\neq p\star(q\star v)$.

Let $\bar{u},\bar{v}\in NV/NW$ and $k,m,p,q\in K(NK)$. Then

$$\begin{aligned}(km)\odot\bar{u} &= (km)\odot(u+NW)\\ &= (km)\star u+NW\\ &= (k\star(m\star u))+NW\\ &= k\odot((m\star u)+NW)\\ &= k\odot(m\odot(u+NW))\\ &= k\odot(m\odot\bar{u}).\end{aligned}$$

So, it implies $(km)\odot\bar{u} = k\odot(m\odot\bar{u})$.

And also,

$$\begin{aligned}(pq)\odot\bar{v} &= (pq)\odot(v+NW)\\ &= (pq)\star v+NW\\ &\neq (p\star(q\star v))+NW\\ &= p\odot((q\star v)+NW)\\ &= p\odot(q\odot(v+NW))\\ &= p\odot(q\odot\bar{v}).\end{aligned}$$

So, it implies that $(pq)\odot\bar{v}\neq p\odot(q\odot\bar{v})$.

Hence, we can conclude that $NS4$ holds in NV/NW.

(4) In NV we have at least u and v such that $1\star u = u$ and $1\star v\neq v$.

So, in NV/NW there exist $\bar{u}$ and $\bar{v}$ such that

$$1\odot\bar{u} = 1\odot(u+NW) = (1\star u)+NW = u+NW = \bar{u}$$

and

$$1\odot\bar{v} = 1\odot(v+NW) = (1\star v)+NW \neq v+NW = \bar{v}.$$

So, it implies $1\odot\bar{u} = \bar{u}$ and $1\odot\bar{v} = \bar{v}$.

Hence, we can conclude that $NS5$ holds in NV/NW.

Accordingly, $(NV/NW,\oplus,\odot)$ is a weak NeutroVectorSpace over the field K. □

Remark 4.15. Let NV be weak(strong) NeutroVectorSpace of type $4S$ over a field(NeutroField) $K(NK)$ and let NW be a NeutroSubspace of NV. Then, the weak(strong) NeutroQuotient Space NV/NW over $K(NK)$ is not necessarily of type $4S$.

We illustrate Remark 4.15 by Example 4.16 .

Example 4.16. Let $(NV = \mathbb{Z}_{12}, +', \star)$ be a weak NeutroVectorSpace of type $4S$ over $K = \mathbb{Z}_p$ and let $(NW = 2\mathbb{Z}_{12}, +', \star)$ be a NeutroSubspace of NV. Where $+'$ is addition mod 12 and $\star$ is defined as

$$k \star v = v^2 +' kv$$

for all $v \in NV$ and $k \in K$.
Then for all $\bar{u}, \bar{v} \in NV/NW$ and $k \in K$ define the operation $\oplus$ and $\odot$ by

$$\bar{u} \oplus \bar{v} = (u +' v) + NW$$

and

$$k \odot \bar{u} = (k \star u) + NW.$$

Then $(NV/NW, \oplus, \odot)$ is a weak NeutroVectorSpace over K of type other than $4S$.

We know that $NV = \{0, 1, 2, \cdots, 11\}$ and $NW = \{0, 2, 4, 6, 8, 10\}$ then we have

$$NV/NW = \{NW, 1 + NW\}.$$

Table 3. Cayley table for the binary operation $\oplus$

$\oplus$	NW	$1 + NW$
NW	NW	$1 + NW$
$1 + NW$	$1 + NW$	NW

From Table 3 it is clear that $(NV/NW, \oplus)$ is an abelian group.
Now,

(1) $NS2$ fails to hold since for any triplet $(k, \bar{u}, \bar{v})$ we pick, with $k \in K$ and $\bar{u}, \bar{v} \in NV/NW$,

$$k \odot (\bar{u} \oplus \bar{v}) = k \odot \bar{u} \oplus k \odot \bar{v}$$

is always satisfied. This implies that $S2$ is totally true in NV/NW.

(2) There exists at least a triplet $(k, m, \bar{v})$ with $k, m \in K$ and $\bar{v} \in NV/NW$ such that

$$(k + m) \odot (\bar{v}) = k \odot \bar{v} \oplus m \odot \bar{v}.$$

Now,

$$(k + m) \odot (\bar{v}) = ((k + m) \star v) + NW = (v^2 +' kv +' mv) + NW$$

and

$$k \odot \bar{v} \oplus m \odot \bar{v} = ((k \star v) + NW) \oplus ((m \star v) + NW) = (2v^2 +' ku +' mv) + NW.$$

Equating these we have that $v^2 + NW = NW$ which implies $v^2 \in NW$.

So, only the triple (k, m, NW) satisfies $(k + m) \odot (\bar{v}) = k \odot \bar{v} \oplus m \odot \bar{v}$.

Hence, $NS3$ holds in NV/NW.

(3) There exists at least a triplet $(k, m, \bar{v})$ with $k, m \in K$ and $\bar{v} \in NV/NW$ such that

$$(km) \odot (\bar{v}) = k \odot (m \odot \bar{v}).$$

Now,

$$(km) \odot (\bar{v}) = ((km) \star v) + NW = (v^2 +' kmv) + NW$$

and

$$k \odot (m \odot \bar{v}) = k \odot ((m \star v) + NW) = (k \star (v^2 +' mv)) + NW = (v^4 +' 2v^3m +' m^2v^2 + kv^2 +' kmv) + NW.$$

Equating these we have that $(v^4 +' 2v^3m +' m^2v^2 + kv^2) + NW = v^2 + NW$ which implies $(v^2 +' 2vm +' m^2 +' k) + NW = 1 + NW$.

Since we needed at least a triplet, take $k = 1$, then we have $(v^2 +' 2vm +' m^2 +' 1) + NW = 1 + NW$ which gives $(v^2 +' 2vm +' m^2) + NW = NW$. So, we have that $(v^2 +' 2vm +' m^2) \in NW$. Then, at least the triplet $(1, m, NW)$ satisfies $(km) \odot (\bar{v}) = k \odot (m \odot \bar{v})$. Hence, $NS4$ holds in NV/NW.

(4) We can easily see that $1 \odot NW = NW$ and

$$1 \odot (1 + NW) = (1 \star 1) + NW = 2 + NW = NW \neq 1 + NW.$$

Hence, $NS5$ holds in NV/NW.

Accordingly, we have that $(NV/NW, \oplus, \odot)$ is a weak NeutroVectorSpace of type $3S$ over K. This implies that the NeutroQuotient Space $(NV/NW, \oplus, \odot)$ does not belong to the class of NeutroVectorSpace NV.

5. **Conclusions**

In this paper, we have for the first time introduced the concept of NeutroVectorSpaces. Specifically, a class of NeutroVectorSpaces called of type $4S$ was investigated and some of their elementary properties and examples were presented. It was shown that NeutroVectorSpaces of type $4S$ contained NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type $4S$ are not necessarily NeutroSubspaces. Also, it was shown that if NV is a NeutroVectorSpace of a particular type and NW is a NeutroSubspace of NV, the NeutroQuotientSpace NV/NW does not necessarily belong to the same type as NV. We hope to continue this work in our next paper to be titled "NeutroVectorSpaces II".

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