Abstract. Recently, the concept of NeutroAlgebraic and AntiAlgebraic Structures were introduced and analyzed by Florentin Smarandache. His new approach to the study of Neutrosophic Structures presents a more robust tool needed for managing uncertainty, incompleteness, indeterminate and imprecise information. In this paper, we introduce for the first time the concept of NeutroVectorSpaces. Specifically, we study a particular class of the NeutroVectorSpaces called of type 4$S$ and their elementarily properties are presented. It is shown that the NeutroVectorSpaces of type 4$S$ may contain NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type 4$S$ are not NeutroSubspaces. Also, it is shown that if $NV$ is a NeutroVector Space of a particular type and $NW$ is a NeutroSubspace of $NV$, the NeutroQuotientSpace $NV/NW$ does not necessarily belong to the same type as $NV$.

Keywords: Neutrosophy; Vector Space; NeutroField; weak NeutroVectorSpace; strong NeutroVectorSpace; weak AntiVectorSpace; strong AntiVectorSpace; NeutroSubspace; weak NeutroQuotientSpace; strong NeutroQuotientSpace.

1. Introduction

As an extension of his work in [15], Florentine Smarandache in [12] introduced a new way of handling uncertainty, incompleteness, indeterminate and imprecise information. He studied and presented the concept of NeutroAlgebraicStructures and AntiAlgebraicStructures, which can be generated from a classical algebraic structure by a process called neutro-sophication and anti-sophication respectively. The emergence of these processes has given birth to a new field of research in the theory of neutrosophic algebraic structures. More details on neutrosophic algebraic structures can be found in [4]-[10].

Smarandache in [13] recalled, improved and extended several definitions and properties of NeutroAlgebras and AntiAlgebras given in [12]. This new concept was examined by Agboola et al. in [1] viz-a-viz the classical number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$. In [2], Agboola formally presented the notion of NeutroGroups by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In addition, he showed that generally, Langrange’s
theorem and 1st isomorphism theorem of the classical groups do not hold in the class of NeutroGroups considered. Also in [3], Agboola studied NeutroRing, NeutroSubring, NeutroIdeal, NeutroQuotientRings and he showed that the 1st isomorphism theorem of the classical rings holds in this class of NeutroRing. Recently, Rezaei and Smarandache [11] introduced the concept of Neutro-BE-algebras and Anti-BE-algebras and they showed that given any classical algebra $S$ with $n$ operations (laws and axioms) where $n \geq 1$ we can generate $(2^n - 1)$ NeutroStructures and $(3^n - 2^n)$ AntiStructures. For comprehensive review of new trends in neutrosophic theory readers should see [4–6,8–10].

The present paper will be concerned with the introduction of NeutroVectorSpaces. Specifically in the paper, we will introduce and study a class of NeutroVectorSpaces called NeutroVectorSpaces of type $4S$ (i.e., 4 of its scalar multiplication axioms are NeutroAxioms) and we will present some of their elementarily properties. It will be shown that the NeutroVectorSpaces of type $4S$ may contain NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type $4S$ are not NeutroSubspaces. Also, it will be shown that if $NV$ is a NeutroVectorSpace of a particular type and $NW$ is a NeutroSubspace of $NV$, then the NeutroQuotientSpace $NV/NW$ does not necessarily belong to the same type as $NV$.

2. Preliminaries

In this section, we will give some definitions, examples and results that will be used in the sequel.

**Definition 2.1.** [14]

(i) A ClassicalOperation is an operation well-defined for all the set’s elements while a Neutro-Operation is an operation partially well-defined, partially indeterminate, and partially outer defined on the given set. An AntiOperation is an operation that is outer defined for all the set’s elements.

(ii) A classicalLaw/Axiom defined on a nonempty set is a law/axiom that is totally true for all the set’s elements while a NeutroLaw/Axiom defined on a nonempty set is a law/axiom that is true for some set’s element, indeterminate for other set’s elements, or false for the other set’s elements. An AntiLaw/Axiom defined on a nonempty set is a law/axiom that is false for all set’s elements.

(iii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom while an AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

**Theorem 2.2.** [11] Let $U$ be a nonempty finite or infinite universe of discourse and let $S$ be a finite or infinite subset of $U$. If $n$ classical operations (laws and axioms) are defined on $S$ where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebraicStructures and $(3^n - 2^n)$ AntiAlgebraicStructures.
**Definition 2.3.** [Classical group]

Let $G$ be a nonempty set and let $*: G \times G \to G$ be a binary operation on $G$. The couple $(G, *)$ is called a classical group if the following conditions hold:

1. (G1) $x * y \in G \ \forall x, y \in G$ [closure law].
2. (G2) $x * (y * z) = (x * y) * z \ \forall x, y, z \in G$ [axiom of associativity].
3. (G3) There exists $e \in G$ such that $x * e = e * x = x \ \forall x \in G$ [axiom of existence of neutral element].
4. (G4) There exists $y \in G$ such that $x * y = y * x = e \ \forall x \in G$ [axiom of existence of inverse element] where $e$ is the neutral element of $G$.
5. (G5) $x * y = y * x$, then $(G, *)$ is called an abelian group.

**Definition 2.4.** [NeutroSophication of the law and axioms of the classical group]

1. (NG1) There exist at least three duplets $(x, y), (u, v), (p, q), \in G$ such that $x * y \in G$ (inner-defined with degree of truth T) and $[u * v = \text{indeterminate (with degree of indeterminacy I)}$ or $p * q \notin G$ (outer-defined/falsehood with degree of falsehood F)] [NeutroClosureLaw].
2. (NG2) There exist at least three triplets $(x, y, z), (p, q, r), (u, v, w) \in G$ such that $x * (y * z) = (x * y) * z$ (inner-defined with degree of truth T) and $[(p * (q * r)) \text{or} ((p * q) * r) = \text{indeterminate (with degree of indeterminacy I)}$ or $u * (v * w) \neq (u * v) * w$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of associativity (NeutroAssociativity)].
3. (NG3) There exists an element $e \in G$ such that $x * e = e * x = x$ (inner-defined with degree of truth T) and $[(x * e) \text{or}(e * x) = \text{indeterminate (with degree of indeterminacy I)}$ or $x * e \neq x \neq e * x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in G$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].
4. (NG4) There exists an element $u \in G$ such that $x * u = u * x = e$ (inner-defined with degree of truth T) and $[[(x * u) \text{or}(u * x) = \text{indeterminate (with degree of indeterminacy I)}$ or $x * u \neq e \neq u * x$ (outer-defined/falsehood with degree of falsehood F)] for at least one $x \in G$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)] where $e$ is a NeutroNeutralElement in $G$.
5. (NG5) There exist at least three duplets $(x, y), (u, v), (p, q) \in G$ such that $x * y = y * x$ (inner-defined with degree of truth T) and $[[(u * v) \text{or}(u * u) = \text{indeterminate (with degree of indeterminacy I)}$ or $p * q \neq q * p$ (outer-defined/falsehood with degree of falsehood F)] [NeutroAxiom of commutativity (NeutroCommutativity)].

**Definition 2.5.** A NeutroGroup $NG$ is an alternative to the classical group $G$ that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ with no AntiLaw or AntiAxiom.

**Definition 2.6.** A NeutroAbelianGroup $NG$ is an alternative to the classical abelian group $G$ that has at least one NeutroLaw or at least one of $\{NG1, NG2, NG3, NG4\}$ and $NG5$ with no AntiLaw or AntiAxiom.
Example 2.7. Let \( NG = \mathbb{N} = \{1, 2, 3, 4, \ldots\} \). Then \( (NG, .) \) is a finite NeutroGroup where "." is the binary operation of ordinary multiplication.

Definition 2.8. [Classical ring] Let \( R \) be a nonempty set and let \(+, \cdot : R \times R \to R\) be binary operations of the usual addition and multiplication respectively defined on \( R \). The triple \((R, +, \cdot)\) is called a classical ring if the following conditions \((R1 - R9)\) hold:

1. \( x + y \in R \forall x, y \in R \) [closure law of addition].
2. \( x + (y + z) = (x + y) + z \forall x, y, z \in R \) [axiom of associativity].
3. There exists \( e \in R \) such that \( x + e = e + x = x \forall x \in R \) [axiom of existence of neutral element].
4. There exists \(-x \in R\) such that \( x + (-x) = (-x) + x = e \forall x \in G \) [axiom of existence of inverse element]
5. \( x + y = y + x \forall x, y \in R \) [axiom of commutativity].
6. \( x.y \in R \forall x, y \in R \) [closure law of multiplication].
7. \( x.(y.z) = (x.y).z \forall x, y, z \in R \) [axiom of associativity].
8. \( x.(y + z) = (x.y) + (x.z) \forall x, y, z \in R \) [axiom of left distributivity].
9. \((y + z).x = (y.x) + (z.x) \forall x, y, z \in R \) [axiom of right distributivity].

If in addition we have,

10. \( x.y = y.x \forall x, y \in R \) [axiom of commutativity],

then \((R, +, \cdot)\) is called a commutative ring.

Definition 2.9. [NeutroSophication of the laws and axioms of the classical ring]

1. There exist at least three duplets \((x, y), (u, v), (p, q) \in R\) such that \(x+y \in R\) (inner-defined with degree of truth \(T\)) and \([u+v = \text{indeterminate with degree of indeterminacy} I\) or \(p+q \notin R\) (outer-defined/falsehood with degree of falsehood \(F\))] [NeutroClosure law of addition].
2. There exist at least three triplets \((x, y, z), (p, q, r), (u, v, w) \in R\) such that \(x+(y+z) = (x+y)+z\) (inner-defined with degree of truth \(T\)) and \([p + (q + r)]\text{or}[(p + q) + r]\) = indeterminate (with degree of indeterminacy \(I\)) or \((u + (v + w)) \neq (u + v) + w\) (outer-defined/falsehood with degree of falsehood \(F\)) for at least one \(x \in R\) [NeutroAxiom of associativity (NeutroAssociativity)].
3. There exists an element \(e \in R\) such that \(x + e = x + e = x\) (inner-defined with degree of truth \(T\)) and \([x + e] = \text{indeterminate with degree of indeterminacy} I\) or \(e + x \neq e + x\) (outer-defined/falsehood with degree of falsehood \(F\)) for at least one \(x \in R\) [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].
4. There exists \(-x \in R\) such that \(x + (-x) = (-x) + x = e\) (inner-defined with degree of truth \(T\)) and \([-x + x] = \text{indeterminate with degree of indeterminacy} I\) or \(-x + x \neq e \neq x + (-x)\) (outer-defined/falsehood with degree of falsehood \(F\)) for at least one \(x \in R\) [NeutroAxiom of existence of inverse element (NeutroInverseElement)].
Definition 2.10. A NeutroRing $NR$ is an alternative to the classical ring $R$ that has at least one NeutroLaw or at least one of \{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\} with no AntiLaw or AntiAxiom.

Definition 2.11. A NeutroNoncommutativeRing $NR$ is an alternative to the classical noncommutative ring $R$ that has at least one NeutroLaw or at least one of \{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\} and $NR10$ with no AntiLaw or AntiAxiom.

Example 2.12. (i) Let $NR = \mathbb{Z}$ and let $\oplus$ be a binary operation of ordinary addition and for all $x, y \in NR$, let $\odot$ be a binary operation defined on $NR$ as $x \odot y = \sqrt{xy}$. Then $(NR, \oplus, \odot)$ is a NeutroRing.

(ii) Let $NR = \mathbb{Q}$ and let $\oplus$ be a binary operation of ordinary addition and for all $x, y \in NR$, let $\odot$ be a binary operation defined on $NR$ as $x \odot y = x/y$. Then $(NR, \oplus, \odot)$ is a NeutroRing.
3. Formulation of NeutroVectorSpaces

In this section, we present the concept of NeutroVectorSpaces and study their elementary properties.

**Definition 3.1.** [Classical Vector Space]

A vector space consists of a nonempty set $V$ of objects (called vectors) that can be added, that can be multiplied by a real or complex number (called a scalar in this context), and for which the following laws and axioms hold:

**The Law and Axioms for vector addition**

(A1) If $u$ and $v$ are in $V$, then $u + v$ is in $V$.

(A2) $u + (v + w) = (u + v) + w$ for all $u, v$, and $w$ in $V$.

(A3) An element 0 in $V$ exist such that $v + 0 = v = 0 + v$ for every $v$ in $V$.

(A4) For each $v$ in $V$, an element $-v$ in $V$ exist such that $-v + v = 0$ and $v + (-v) = 0$.

(A5) $u + v = v + u$ for all $u$ and $v$ in $V$.

**The Law and Axioms for scalar multiplication**

(S1) If $v$ is in $V$, then $av$ is in $V$ for all $a$ in $\mathbb{R}$.

(S2) $a(v + w) = av + aw$ for all $v$ and $w$ in $V$ and all $a \in \mathbb{R}$.

(S3) $(a + b)v = av + bv$ for all $v$ in $V$ and all $a$ and $b \in \mathbb{R}$.

(S4) $a(bv) = (ab)v$ for all $v$ in $V$ and all $a$ and $b$ in $\mathbb{R}$.

(S5) $1v = v$ for all $v$ in $V$.

**Definition 3.2.** [NeutroSophication of the law and axioms of the classical vector space]

**NeutroSophication of the law and axioms for vector addition**

(NA1) There exist at least three duplets $(u, v), (w, x), (y, z) \in V$ such that $u + v \in V$ (inner-defined with degree of truth $T$) and $w + x = \text{indeterminate}$ (with degree of indeterminacy $I$) or $y + z \notin V$ (outer-defined/falsehood with degree of falsehood $F$).

(NA2) There exist at least three triplets $(u, v, w), (x, y, z), (p, q, r) \in V$ such that $u + (v + w) = (u + v) + w$ (inner-defined with degree of truth $T$) and $[x + (y + z)] = \text{indeterminate}$ (with degree of indeterminacy $I$) or $p + (q + r) \neq (p + q) + r$ (outer-defined/falsehood with degree of falsehood $F$).

(NA3) There exists an element $e \in V$ such that $v + e = e + v = v$ (inner-defined with degree of truth $T$) and $[v + e] = \text{indeterminate}$ (with degree of indeterminacy $I$) or $v + e \neq v \neq e + v$ (outer-defined/falsehood with degree of falsehood $F$) for at least one $v \in V$.

(NA4) There exists $-v \in V$ such that $v + (-v) = (-v) + v = e$ (inner-defined with degree of truth $T$) and $[-v + v] = \text{indeterminate}$ (with degree of indeterminacy $I$) or $-v + v \neq e \neq v + (-v)$ (outer-defined/falsehood with degree of falsehood $F$).
Definition 3.3. [AntiSophication of the law and axioms of the classical vector space]

**AntiSophication of the law and axioms for vector addition**

(AA1) For all the duplets \((u, v) \in V\), \(u + v \notin V\).

(AA2) For all the triplets \((u, v, w) \in V\), \(u + (v + w) \neq (u + v) + w\).

(AA3) There does not exist an element \(e\) in \(V\) such that \(v + e = v = e + v\) for every \(v\) in \(V\).

(AA4) There does not exist \(-v\) in \(V\) such that \(v + (−v) = (−v) + v = e\) for all \(v \in V\) where \(e\) is a AntiNeutralElement in \(V\).

(AA5) For all the duplets \((u, v) \in V\), \(u + v \neq v + u\).

**AntiSophication of the law and axioms for scalar multiplication**

(AS1) For all \(v \in V\) and \(a \in \mathbb{R}\), \(av \notin V\).

(AS2) For all \(u, v \in V\) and \(a \in \mathbb{R}\), \(a(u + v) \neq au + av\).

(AS3) For all \(v \in V\) and \(a, b \in \mathbb{R}\), \((a + b)v \neq av + bv\).

(AS4) For all \(v \in V\) and \(a, b \in \mathbb{R}\), \(a(bv) \neq (ab)v\).
For all \( v \in V \), \( 1v \neq v \).

**Definition 3.4.** Let \((K,+,\cdot)\) be a field. A NeutroField \((NK,+,),\) is an alternative to the classical field \((K,+,\cdot)\) that has at least one NeutroLaw or at least one NeutroAxiom with no Antilaw or AntiAxiom.

**Definition 3.5.** Let \((K,+,\cdot)\) be a field. An AntiField \((AK,+,\cdot)\) is an alternative to the classical field \((K,+,\cdot)\) that has at least one AntiLaw or at least one AntiAxiom.

**Definition 3.6.** Let \( + \) be addition of vectors, \( \cdot \) be multiplication of vector by scalars and let \( K \) be a Neutro/classical field. A NeutroVectorSpace \((NV,+,\cdot)\) is an alternative to the classical vector space \((V,+,\cdot)\) that has at least one NeutroLaw or at least one of \{NA1 - NS5\} with no Antilaw or AntiAxiom.

If \( K \) is a classical field, then the quadruple \((NV,+,\cdot,K)\) is called a weak NeutroVectorSpace over \( K \). And the quadruple \((NV,+,\cdot,K)\) is called a strong NeutroVectorSpace if \( K \) is a NeutroField (i.e., \( K = NK \)).

**Definition 3.7.** Let \( + \) be addition of vectors, \( \cdot \) be multiplication of vectors by scalars and let \( K \) be a Anti/classical field. An AntiVectorSpace \((AV,+,\cdot)\) is an alternative to the classical vector space \((V,+,\cdot)\) that has at least one AntiLaw or at least one of \{AA1 - AS5\}.

If \( K \) is a classical field, then the quadruple \((AV,+,\cdot,K)\) is called a weak AntiVectorSpace over \( K \). And the quadruple \((AV,+,\cdot,K)\) is called a strong AntiVectorSpace if \( K \) is a AntiField (i.e., \( K = AK \)).

**Theorem 3.8.** Let \((V,+,\cdot)\) be a classical vector space over a field \( K \). Then,

1. there are 1023 classes of NeutroVectorSpaces.
2. there are 58025 classes of AntiVectorSpaces.

**Proof.** The proof follows easily from Theorem 2.2.

Theorem 3.8 shows that there are many classes of NeutroVectorSpaces. The trivial cases from the 1023 classes are the cases where \( NA1 - NS5 \) hold. Examples of weak and strong NeutroVectorSpaces for the trivial cases are given in Example 3.9.

**Example 3.9.** Let \( V = \mathbb{Z}_{12} \) and \( K = \mathbb{R} \). Define addition and scalar multiplication by

\[
x \oplus y = \frac{2x + 3y}{2} \quad \text{and} \quad k \odot a = ka^2
\]

where \( \oplus \) is addition modulo 12. Then \((V,\oplus,\odot)\) is a weak NeutroVectorSpace over a field \( K \).

To see this:

1. We will show that \((V,\oplus)\) is a NeutroAbelianGroup.
(a) There exist at least \(x, y \in V\) such that \(x \oplus y \in V\) and at least \(a, b \in V\) such that \(a \oplus b \notin V\). For instance, if we take \((x, y) = (1, 2)\) and \((a, b) = (2, 1)\), we will see that \(NV1\) holds. Therefore \(\oplus\) is NeutroClosed.

(b) Let \(x, y, z \in V\). Then
\[
x \oplus (y \oplus z) = \frac{4x + 6y + 9z}{4}
\]
equating these we have
\[
4x + 6y + 9z = 4x + 6y + 6z \text{ which gives } 9z = 6z,
\]
\[
\therefore 3z = 0 \text{ this implies that } z = 0, 4 \text{ and } 8.
\]
Thus, only the triplets \((x, y, 0), (x, y, 4), \) and \((x, y, 8)\) can verify the associativity of \(\oplus\) and therefore, \(\oplus\) is NeutroAssociative.

(c) Let \(e \in V\) such that
\[
x \oplus e = \frac{2x + 3e}{2} = x \text{ and } e \oplus x = \frac{2e + 3x}{2} = x.
\]
Then \(\frac{2x + 3e}{2} = \frac{2e + 3x}{2}\) from which we obtain \(e = x\).

The elements of \(V\) that satisfy \(x \oplus x = x\) are 0, 4, and 8. This shows that \(V\) has NeutroNeutral element.

(d) Considering each NeutroNeutral element in (b) we can show that \(V\) has NeutroInverse element.

(e) Let \(x, y \in V\), \(x \oplus y = \frac{2x + 3y}{2}\) and \(y \oplus x = \frac{2y + 3x}{2}\).

If "\(\oplus\)" is commutative, we will have \(\frac{2x + 3y}{2} = \frac{2y + 3x}{2}\) from which we obtain \(x = y\). This shows that only the duplet \((x, x)\) can verify commutativity of \(\oplus\).

Thus, \(\oplus\) is NeutroCommutative. Hence, \((V, \oplus)\) is a NeutroAbelianGroup.

(2) We wish to find at least a triplet \((k, m, u)\) with \(u \in V\) and \(k, m \in K\), such that \(k \odot (m \odot u) = (km) \odot u\).

Now, consider \((km) \odot u = (km)u^2 = kmu^2\) and \(k \odot (m \odot u) = k \odot (mu^2) = k(mu)^2 = km^2u^4\).

Equating these we have
\[
kmu^2 = km^2u^4,
\]
which gives
\[
mu^2 = 1.
\]

Since we need at least a triplet, take \(m = 1\), then elements of \(V\) that will satisfy \(mu^2 = 1\) are 5, 7, 11.

So, \(k \odot (m \odot u) = (km) \odot u\) for at least the triplets \((k, 1, 5), (k, 1, 7)\) and \((k, 1, 11)\).

(3) We want to show that, there exist at least a triplet \((k, m, u)\) with \(u \in V\) and \(k, m \in K\), such that \((k + m) \odot u = k \odot u \odot m \odot u\).

Consider, \((k + m) \odot u = (k + m)u^2 = ku^2 + mu^2\) and \(k \odot u \odot m \odot u = ku^2 \odot mu^2 = \frac{2ku^2 + 3mu^2}{2}\).

Equating these we have
\[
ku^2 + mu^2 = \frac{2ku^2 + 3mu^2}{2},
\]
which gives
\[ mu^2 = 0 \implies u^2 = 0 \]
\[ \therefore u = 0 \text{ and } 6. \]

This shows that only the triplets \((k, m, 0)\) and \((k, m, 6)\) can verify \((k + m) \odot u = k \odot u \oplus m \odot u\).

(4) We want to show that there exists at least a triplet \((k, u, v)\) with \(u, v \in V\) and \(k \in K\), such that \(k \odot (u \oplus v) = k \odot u \oplus k \odot v\).

Now, consider \(k \odot (u \oplus v) = k \odot \frac{(2u+3v)^2}{2} = k \odot \frac{(2u+3v)^2}{4} = \frac{4ku^2+12kuv+9kv^2}{4} = \frac{4ku^2+9kv^2}{2}\) and \(k \odot u \oplus k \odot v = ku^2 \oplus kv^2 = \frac{2ku^2+3kv^2}{2}\).

Equating these we have
\[ 4ku^2 + 9kv^2 = 4ku^2 + 6kv^2, \]
which gives
\[ 9kv^2 = 6kv^2 \]
\[ 3kv^2 = 0 \implies v^2 = 0. \]

So,
\[ v = 0, 6. \]

This shows that only the triplets \((k, u, 0)\) and \((k, u, 6)\) can verify \(k \odot (u \oplus v) = k \odot u \oplus k \odot v\).

(5) We want to show that there exists at least a \(u \in V\) such that \(1 \odot u = u\).

We have that the only elements of \(V\) that satisfy \(1 \odot u = u\) are 4 and 9.

Accordingly, \((V, \oplus, \odot)\) is a weak NeutroVectorSpace over a field \(K = \mathbb{R}\).

**Example 3.10.** Let \(X = \{a, b, c, d, e\}\) be a universe of discourse and let \(K = \{a, b, c, d\}\). Let \(\oplus\) and \(\odot\) be the binary operations defined on \(K\) as shown in the Cayley tables below.

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<td>c</td>
<td>a</td>
<td>c</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>b</td>
<td>d</td>
<td>b</td>
</tr>
</tbody>
</table>

(b)

(1) \((K, \oplus, \odot)\) is a trivial NeutroField.

(2) \((K, \oplus, \odot)\) taken over itself is a strong NeutroVector Space.
(1) To show that \((K, \oplus, \odot)\) is a trivial NeutroField we proceed as follows:

(a) \((K, \oplus)\) is a NeutroAbelianGroup. It is clear from the table that;

(i) \(d \oplus d = b \) or \(d\).

So, the composition \(d \oplus d\) is indeterminate with 6.25\% degree of indeterminacy and all other compositions are true with 93.75\% degree of truth. Hence "\(\oplus\)" is NeutroClosed.

(ii) \(d \oplus (c \oplus a) = (d \oplus c) \oplus a = d\),

\(a \oplus (c \oplus d) = a\), but \((a \oplus c) \oplus d = c \neq a\). Hence "\(\oplus\)" is NeutroAssociative.

(iii) Since only the duplet \((x, x) \in K\) verify commutativity, for \(x = a, b, c \in K\).

Hence "\(\oplus\)" is NeutroCommutative.

(iv) Let \(N_x\) and \(I_x\) represent additive neutral and inverse element respectively with respect to any element \(x \in K\).

Then \(N_a = a, N_c = c\) and \(N_b, N_d\) do not exist.

\(I_a = a, I_c = c\) and \(I_b, I_d\) do not exist.

Hence, \((K, \oplus)\) is a NeutroAbelianGroup.

(b) \((K, \odot)\) is a NeutroAbelianGroup. It is clear from the table that;

(i) \((a \odot b) \odot d = a \odot (b \odot d) = c\),

\((b \odot c) \odot d = d\) but \(b \odot (c \odot d) = b \neq d\). Hence "\(\odot\)" is NeutroAssociative.

(ii) \(a \odot c = c \odot a = a\),

\(a \odot b = c\) but \(b \odot a = b\). Hence, \(\odot\) is NeutroCommutative.

(iii) Let \(U_x\) and \(I_x\) represent multiplicative neutral and inverse element(s) respectively with respect to any element \(x \in K\).

Then, \(U_a = a\) and \(c. U_d = b\) and \(d. U_b\) and \(U_c\) do not exist.

\(I_a = a\) and \(c. I_d = b\) and \(d. I_c\) and \(I_b\) do not exist.

Hence, \((K, \odot)\) is a NeutroAbelianGroup.

(c) Now, we show that \(\odot\) is distributive over \(\oplus\). It is clear from the table that :

(i) \(a \odot (b \oplus c) = a \odot b \oplus a \odot c = c\),

\(b \odot (a \oplus b) = b\), but \(b \odot a \oplus b \odot b = d \neq b\). So, "\(\odot\)" is left NeutroDistributive over "\(\oplus\)".

(ii) \((b \oplus c) \odot a = b \odot a \odot c \odot a = b\),

\((c \oplus b) \odot d = c\), but \(c \odot d \oplus b \odot d = a \neq c\). So, "\(\odot\)" is right NeutroDistributive over "\(\oplus\)". Hence, "\(\odot\)" is NeutroDistributive over "\(\oplus\)".

Accordingly, \((K, \oplus, \odot)\) is a trivial NeutroField.

(2) That \((K, \oplus, \odot)\) is a strong NeutroVector Space over itself, follows easily from all the properties established in solution of 1 above.

Proposition 3.11. Every NeutroField taken over itself is a strong NeutroVectorSpace.
Proof. The proof follows from Example 3.10.

4. A Study of a Class of NeutroVectorSpaces

In this section, we shall consider a particular class of NeutroVectorSpaces \((NV, +, \cdot)\) where

1. \((NV, +)\) is a classical abelian group.
2. \(S1\) is totally true for all \(v \in V\) and \(a \in K\).
3. \(S2, S3, S4\) and \(S5\) are either partially true or partially indeterminate or partially false for some elements of \(V\) and \(K\).

We shall refer to this class of NeutroVectorSpace as NeutroVectorSpace of type 4 (i.e., 4 of its scalar multiplication axioms are NeutroAxioms).

Example 4.1. Let \(K = \mathbb{Z}_p\) (where \(p\) is prime) and \(NV = \mathbb{Z}_8\). Define \(\oplus\) and \(\odot\) by

\[
a \oplus b = a + b \quad \text{and} \quad k \odot a = a^2 + ka.
\]

Where “+” is addition modulo 8.

Then \((NV, \oplus, \odot)\) is a weak NeutroVectorSpace of type 4 over the field \(K = \mathbb{Z}_p\).

It is easy to show that \((NV, \oplus)\) is an abelian group. Also, it is easy to see that \(S1\) holds.

Now it remains to show that \(NS2, NS3, NS4\) and \(NS5\) hold.

1. We want to show that there exists at least a triplet \((k, x, y)\) with \(k \in K\) and \(x, y \in NV\) such that

\[
k \odot (x \oplus y) = k \odot x \oplus k \odot y.
\]

Now, \(k \odot (x \oplus y) = k \odot (x + y) = (x + y)^2 + k(x + y) = x^2 + y^2 + 2xy + kx + ky\).

And \(k \odot x \oplus k \odot y = (x^2 + kx) \oplus (y^2 + ky) = x^2 + y^2 + kx + ky\).

\[
\therefore x^2 + y^2 + 2xy + kx + ky = x^2 + y^2 + kx + ky
\]

\[
\Rightarrow xy = 0.
\]

Hence \(x = 0\) or \(y = 0\), \((x, y) = (2, 4), (x, y) = (4, 2), (x, y) = (4, 6)\) and \((x, y) = (6, 4)\).

This shows that only the triplets \((k, x, 0)\), \((k, 0, y)\), \((k, 2, 4)\), \((k, 4, 2)\), \((k, 4, 6)\) and \((k, 6, 4)\) can verify \(NS2\).

2. We want to show that there exists at least a triplet \((k, m, u)\) with \(k, m \in K\) and \(u \in NV\) such that

\[
(k + m) \odot u = k \odot u + m \odot u.
\]

\[
(k + m) \odot u = u^2 + (k + m)u = u^2 + ku + mu \quad \text{and} \quad k \odot u \oplus m \odot u = 2u^2 + ku + mu.
\]

Then, we have

\[
u^2 + ku + mu = 2u^2 + ku + mu
\]

\[
\Rightarrow u^2 = 0.
\]
\[ u = 0 \text{ and } 4. \]

Hence, only the triplet \((k, m, 0)\) and \((k, m, 4)\) can verify NS3.

(3) We want to show that there exists at least a triplet \((k, m, u)\) with \(u \in NV\) and \(k, m \in K\), such that \(k \odot (m \odot u) = (km) \odot u\).

Now, consider \((km) \odot u = u^2 + (km)u = u^2 + kmu\) and
\[
k \odot (m \odot u) = k \odot (u^2 + mu) = (u^2 + mu)^2 + k(u^2 + mu) = u^4 + 2mu^3 + m^2u^2 + ku^2 + kmu.
\]

Equating these we have
\[ u^2 + 2mu + m^2 + k = 1. \]

Since we need at least a triplet, take \(k = 1\), then we have \(u^2 + 2um + m^2 = 0\) and this gives \(u = -m\).

Hence, at least the triplet \((1, m, -m)\) satisfies NS4.

(4) We want to show that there exists at least \(v \in NV\) such that \(1 \odot v = v\).

From definition of \(\odot\) we have that the only elements of \(NV\) that satisfy \(1 \odot u = v\) are 0 and 4.

Hence \((NV, \odot, \odot)\) is a weak NeutroVectorSpace of type 4S over the field \(K = \mathbb{Z}_p\).

**Example 4.2.** Let \(X = \{a, b, c, d, e\}\) be a universe of discourse. Let \(K = \{a, b, c, d\}\) be the Neutrofield defined in Example 3.10 and let \(NV = \{v_1 = \frac{a}{e}, v_2 = \frac{b}{e}, v_3 = \frac{c}{e}, v_4 = \frac{d}{e}\}\).

Define on \(NV\) the binary operation \(+^\prime\) as in the table below and scalar multiplication \(*\) by
\[ a * v = \frac{a \odot x}{e}, \]
here \(\odot\) is the multiplication in \(K\) defined in Table 1 (b) for all elements in \(K\).

**Table 2.** Cayley table for the binary operation \(+^\prime\)

<table>
<thead>
<tr>
<th>+'</th>
<th>v1</th>
<th>v2</th>
<th>v3</th>
<th>v4</th>
</tr>
</thead>
<tbody>
<tr>
<td>v1</td>
<td>v1</td>
<td>v2</td>
<td>v3</td>
<td>v4</td>
</tr>
<tr>
<td>v2</td>
<td>v2</td>
<td>v3</td>
<td>v4</td>
<td>v1</td>
</tr>
<tr>
<td>v3</td>
<td>v3</td>
<td>v4</td>
<td>v1</td>
<td>v2</td>
</tr>
<tr>
<td>v4</td>
<td>v4</td>
<td>v1</td>
<td>v2</td>
<td>v3</td>
</tr>
</tbody>
</table>

Then \((NV, +', *)\) is a strong NeutroVectorSpace of type 4S over \(K\).

It is clear from Table 2 that \((NV, +')\) is an abelian group. Also, it is easy to see that S1 holds.

Now it remains to show that NS2, NS3, NS4 and NS5 hold.

It can be seen from Table 1 (a), Table 1 (b) and Table 2 that:
(1) for \( c \in \mathbb{K} \) and \( v_2, v_3 \in NV \),
\[
\begin{align*}
    c \ast (v_2 +' v_3) &= c \ast (v_2 +' v_3) \\
    &= c \ast v_4 \text{ from Table 2} \\
    &= \circ \odot e \\
    &= \frac{e}{e} \ast c \odot d = c, \text{ from Table 1 (b)} \\
    &= v_3.
\end{align*}
\]
\[
\Rightarrow c \ast (v_2 +' v_3) = c \ast v_2 +' c \ast v_3 = v_3,
\]
and for \( b \in \mathbb{K} \) and \( v_3, v_4 \in NV \),
\[
b \ast (v_3 +' v_4) = v_4 \text{ but } b \ast v_3 +' b \ast v_4 = v_1 \neq v_4.
\]
This shows that NS2 holds.

(2) for \( a, c \in \mathbb{K} \) and \( v_2 \in NV \),
\[
(a \oplus c) \ast v_2 = a \ast v_2 +' c \ast v_2 = v_3,
\]
and for \( a, b \in \mathbb{K} \) and \( v_4 \in NV \),
\[
(a \oplus b) \ast v_4 = v_3 \text{ but } a \ast v_4 +' b \ast v_4 = v_2 \neq v_3.
\]
This shows that NS3 holds.

(3) for \( a, b \in \mathbb{K} \) and \( v_4 \in NV \),
\[
(a \ominus b) \ast v_4 = a \ast (b \ast v_4) = v_3,
\]
and for \( b, c \in \mathbb{K} \) and \( v_4 \in NV \)
\[
(b \ominus c) \ast v_4 = v_4 \text{ but } b \ast (c \ast v_4) = v_2 \neq v_1.
\]
This shows that NS4 holds.

(4) We know from Table 1 that NeutroUnityElements in \( K \) are \( U_a = a, c \) and \( U_d = b, d \).
Now, suppose we consider the NeutroUnityElement \( U_d = b \) only.
We have that \( b \ast v_4 = v_4 \) and \( b \ast v_3 = v_2 \neq v_3 \).
This shows that NS5 holds.

Hence, we have that \((NV, +', \ast)\) is a strong NeutroVectorSpace of type 4S over the NeutroField \( \mathbb{K} \).
From now on, every weak(strong) NeutroVectorSpaces of type 4S over \( K(NK) \) will simply be called a weak(strong) NeutroVectorSpace over \( K(NK) \).

**Proposition 4.3.** Let \((NV, +'_1, \ast_1)\) and \((NH, +'_2, \ast_2)\) be two weak NeutroVectorSpace over the field \( K \) and let
\[
NV \times NH = \{(v, h) : v \in NV \text{ and } h \in NH\},
\]
for \( x = (v_1, h_1), y = (v_2, h_2) \in NV \times NH \) and \( k \in K \) define :
\[
x \oplus y = ((v_1 +'_1 v_2), (h_1 +'_2 h_2)),
\]

Author(s), Paper’s title
Then \((NV \times NH, \oplus, \odot)\) is a weak NeutroVectorSpace over the field \(K\).

**Proof.** Since \((NV, +'_1)\) and \((NH, +'_2)\) are classical abelian groups, then it can be shown that \((NV \times NH, \oplus)\) is a classical abelian group. Also, it is easy to see that \(S1\) is true in \((NV \times NH)\).

Now, it remains to show that \(NS2 - NS5\) hold in \(NV \times NH\).

1. There exists at least a triplet \((k, (v_1, h_1), (v_2, h_2))\) with \((v_1, h_1), (v_2, h_2) \in NV \times NH\) and \(k \in K\), such that

\[
k \odot ((v_1, h_1) \oplus (v_2, h_2)) = k \odot (v_1 +'_1 v_2, h_1 +'_2 h_2) = (k \ast_1 (v_1 +'_1 v_2), k \ast_2 (h_1 +'_2 h_2)) = (k \ast_1 v_1 +'_1 k \ast_1 v_2, k \ast_2 h_1 +'_2 k \ast_2 h_2) \therefore NS2 holds in NV and NH.
\]

Also, there exists at least a triplet \((m, (a_1, b_1), (a_2, b_2))\) with \((a_1, b_1), (a_2, b_2) \in NV \times NH\) and \(m \in K\), such that

\[
m \odot ((a_1, b_1) \oplus (a_2, b_2)) = m \odot (a_1 +'_1 a_2, b_1 +'_2 b_2) = (m \ast_1 (a_1 +'_1 a_2), m \ast_2 (b_1 +'_2 b_2)) \\
\neq (m \ast_1 a_1 +'_1 m \ast_1 a_2, m \ast_2 b_1 +'_2 m \ast_2 b_2) \therefore NS2 holds in NV and NH.
\]

\[
= (m \ast_1 a_1, m \ast_2 b_1) \oplus (m \ast_1 a_2, m \ast_2 b_2)
\]

Hence, \(NS2\) holds in \(NV \times NH\).

2. There exists at least a triplet \((k, (v, h))\) with \(k, m \in K\) and \((v, h) \in NV \times NH\) such that

\[
(k + m) \odot (v, h) = ((k + m) \ast_1 v, (k + m) \ast_2 h) = (k \ast_1 v +'_1 m \ast_1 v, (k \ast_2 h +'_2 m \ast_2 h)) \therefore NS3 holds in NV and NH.
\]

Also, there exists at least a triplet \((p, q, (a, b))\) with \(p, q \in K\) and \((a, b) \in NV \times NH\) such that

\[
(p + q) \odot (a, b) = ((p + q) \ast_1 a, (p + q) \ast_2 b) \\
\neq ((p \ast_1 a +'_1 q \ast_1 a), (p \ast_2 b +'_2 q \ast_2 b)) \therefore NS3 holds in NV and NH.
\]

\[
= ((p \ast_1 a, p \ast_2 b) \oplus (q \ast_1 a, q \ast_2 b))
\]

Hence, \(NS3\) holds in \(NV \times NH\).

3. There exists at least a triplet \((k, (v, h))\) with \(k, m \in K\) and \((v, h) \in NV \times NH\) such that

\[
(km) \odot (v, h) = ((km) \ast_1 v, (km) \ast_2 h) = (k \ast_1 (m \ast_1 v), k \ast_2 (m \ast_2 h)) \therefore NS4 holds in NV and NH.
\]

\[
= k \odot ((m \ast_1 v, (m \ast_2 h))
\]

\[
= k \odot (m \odot (v, h)).
\]
Also, there exists at least a triplet \((p, q, (a, b))\) with \(p, q \in K\) and \((a, b) \in NV \times NH\) such that

\[
(pq) \odot (a, b) = ((pq) \ast_1 a, (pq) \ast_2 b)
\]

\[
\neq (p \ast_1 (q \ast_1 a, p \ast_2 (q \ast_2 b)) \quad \therefore NS4 \text{ holds in } NV \text{ and } NH.
\]

\[
= p \odot ((q \ast_1 a, (p \ast_2 b))
\]

\[
= p \odot (q \odot (v, h))
\]

Hence, \(NS4\) holds in \(NV \times NH\).

(4) There exists \((v, h) \in NV \times NH\) such that

\[
1 \odot (v, h) = (1 \ast v, 1 \ast h)
\]

\[
= (v, h). \quad \therefore NS5 \text{ holds in } NV \text{ and } NH.
\]

Also, there exists \((a, b) \in NV \times NH\) such that

\[
1 \odot (a, b) = (1 \ast_1 a, 1 \ast_2 b)
\]

\[
\neq (a, b). \quad \therefore NS5 \text{ holds in } NV \text{ and } NH.
\]

Accordingly, \((NV \times NH, \oplus, \odot)\) is a weak NeutroVectorSpace over the field \(K\).

**Proposition 4.4.** Let \((NV, +', \ast_1)\) be a weak NeutroVectorSpace over the field \(K\) and let \((H, +, \cdot)\) be a classical vector space over the same field \(K\) and let

\[
NV \times H = \{ (v, h) : v \in NV \text{ and } h \in H \}
\]

and for \(x = (v_1, h_1), y = (v_2, h_2) \in NV \times H\) and \(k \in K\) define :

\[
x \oplus y = ((v_1 +_1 v_2), (h_1 + h_2)) \text{ and } k \odot x = (k \ast v_1, k \cdot v_2).
\]

Then \((NV \times H, \oplus, \cdot)\) is a weak NeutroVectorSpace over the field \(K\).

*Proof.* The proof is similar to the proof of Proposition 4.3.

**Proposition 4.5.** Let \((NV, +', \ast_1)\) and \((NH, +', \ast_2)\) be two strong NeutroVectorSpaces over the NeutroField \(NK\) and let

\[
NV \times NH = \{ (v, h) : v \in NV \text{ and } h \in NH \}
\]

and for \(x = (v_1, h_1), y = (v_2, h_2) \in NV \times NH\) and \(k \in NK\) define :

\[
x \oplus y = ((v_1 +'_1 v_2), (h_1 +'_2 h_2)) \text{ and } k \odot x = (k \ast_1 v_1, k \ast_2 v_2).
\]

Then \((NV \times NH, \oplus, \odot)\) is a strong NeutroVectorSpace over the NeutroField \(NK\).

*Proof.* The proof follows similar approach as the proof of Proposition 4.3.

**Definition 4.6.** Let \(NV\) be a NeutroVectorSpace. Then \(NW\) is a NeutroSubspace of \(NV\) if and only if \(NW\) is a subset of \(NV\), and \(NW\) is itself a NeutroVectorSpace with the same operations as in \(NV\).
Example 4.7. Let \((NV, \oplus, \odot)\) be a weak NeutroVectorSpace of Example 4.1 and let \(NW = 2\mathbb{Z}_8\) be a subset of \(NV\). Following the approach in Example 4.1, it can be shown that \((NW, \oplus, \odot)\) is a weak NeutroVectorSpace over the field \(\mathbb{Z}_p\). Hence \(NW\) is a weak NeutroSubspace of \(NV\).

Example 4.8. Let \((NV, +', \ast)\) be the strong NeutroVectorSpace of Example 4.2, \(NV\) is the only strong NeutroSubspace of \(NV\).

Remark 4.9. It should be noted that a NeutroVectorSpace \(NV\) of a particular class may contain a NeutroSubspace \(NW\) which belongs to another class.

We will illustrate Remark 4.9 with Example 4.10.

Example 4.10. Let \((NV, +', \ast)\) be the strong NeutroVectorSpace of Example 4.2 and let \(NW = \{v_1, v_3\}\) be a subset of \(NV\). Then \((NW, +', \ast)\) is a NeutroVectorSpace of a class other than the class of \(NV\).

We can see from Table 2 that \((NW, +')\) is an abelian group. Now, it can be seen from Table 1(a), Table 1(b) and Table 2 that:

1. \(S1\) fails to hold. Since \(\ast\) is not true for all \(a \in \mathbb{K}\) and \(v \in NW\).
   
   For instance, take \(b \in \mathbb{K}\) and \(v_3 \in NW\), then
   \[b \ast v_3 = \frac{b \cdot c}{e} = \frac{b}{e} = v_2 \notin V_3.\]
   
   But if we take \(a \in \mathbb{K}\) then for all \(v \in NW\) we will have that \(a \ast v \in NW\).
   
   Hence, \(NS1\) holds in \(NW\).

2. for \(a, b \in \mathbb{K}\) and \(v_1, v_3 \in NW\), we have
   
   \[a \ast (v_1 +' v_3) = a \ast v_1 +' a \ast v_3 = v_1,\]
   
   and \(b \ast (v_1 +' v_3) = v_2\) but \(b \ast v_1 +' b \ast v_3 = v_4 \neq v_2\).
   
   This shows that \(NS2\) holds in \(NW\).

3. for \(a, c \in \mathbb{K}\) and \(v_3 \in NW\),
   
   \[(a \oplus c) \ast v_3 = a \ast v_3 +' c \ast v_3 = v_1,\]
   
   and for \(a, b \in \mathbb{K}\) and \(v_3 \in NW\),
   
   \[(a \oplus b) \ast v_3 = v_1\] but \(a \ast v_3 +' b \ast v_3 = v_3 \neq v_1.\]
   
   This shows that \(NS3\) holds.

4. for \(a, c \in \mathbb{K}\) and \(v_3 \in NW\),
   
   \[(a \circ c) \ast v_3 = a \ast (c \ast v_3) = v_1.\]
and for \( a, b \in K \) and \( v_3 \in NW \),

\[
(a \odot b) * v_3 = v_1 \text{ but } a \star (b \star v_3) = v_3 \neq v_1.
\]

This shows that \( NS4 \) holds.

(5) We know from Table 1 that NeutroUnityElements in \( K \) are \( U_a = a, c \) and \( U_d = b, d \).

Now, suppose we consider the NeutroUnityElement \( U_a = c \) only.

We have that \( c \star v_1 = v_1 \) and \( c \star v_3 = v_1 \neq v_3 \).

This shows that \( NS5 \) holds.

Hence, we have that \((NW, +', \star)\) is a strong NeutroSubpace of type 5S over the NeutroField \( K \). This implies that the NeutroSubspace \( NW \) does not belong to the same class as \( NV \).

**Example 4.11.** Let \( NV = \mathbb{Z}_{12} \) and \( K = \mathbb{Z}_p \). Define \( \oplus \) and \( \odot \) for all \( u, v \in V \) and \( k \in K \) by

\[
 u \oplus v = u + v \text{ and } k \odot v = v^2 + kv.
\]

Where “+” is addition mod 12.

Following the approach of Example 4.1 it can be shown that \((NV, \oplus, \odot)\) is a weak NeutroVectorSpace of type 4S over the field \( K \).

Let \( NW = 2\mathbb{Z}_{12} \) and \( NH = 3\mathbb{Z}_{12} \) be two subsets of \( NV \). Also, by following similar approach as in Example 4.1 it can be shown that \((NW, \oplus, \odot)\) and \((NH, \oplus, \odot)\) are weak NeutroSubspaces of \( NV \).

Now consider the following :

1. \( NW + NH = \{0, 1, 2, \cdots, 11\} = NV \).
2. \( NW \cup NH = \{0, 2, 3, 4, 6, 8, 9, 10\} \).
3. \( NW \cap NH = \{0, 6\} \).

These show that \( NW + NH \) is a NeutroSubspace of \( NV \) but \( NW \cup NH \) and \( NW \cap NH \) are not NeutroSubspaces of \( NV \).

These observations are recorded in Proposition 4.12.

**Proposition 4.12.** Let \( NW \) and \( NH \) be any two weak NeutroSubspaces of a NeutroVectorSpace \( NV \) over a field \( K \). Then

1. \( NW + NH = \bigcup\{(w + h) : w \in NW \text{ and } h \in NU\} \) is a NeutroSubspace of \( NV \).
2. \( NW \cap NU \) is not necessarily a NeutroSubspace of \( NV \).
3. \( NW \cup NU \) is not necessarily a NeutroSubspace of \( NV \).

**Definition 4.13.** Let \( NW \) be a weak(strong) NeutroSubspace of a weak(strong) NeutroVectorSpace \( NV \) over a field (NeutroField) \( K(NK) \). The quotient \( NV/NW \) is defined by the set

\[
\{v + NW : v \in NV\}.
\]
Proposition 4.14. Let \((NV, +', \cdot)\) be a weak NeutroVectorSpace and \((NW, +', \cdot)\) be a weak Neutro-Subspace of \(NV\). The quotient \(NV/NW\) is a weak NeutroVectorSpace over a field \(K\) if addition and multiplication are defined for all \( \bar{u} = u + NW,\ \bar{v} = v + NW \in NV/NW\) and \(k \in K\) as follows:

\[
\bar{u} \oplus \bar{v} = (u + NW) \oplus (v + NW) = (u +' v) + NW,
\]

and

\[
\alpha \odot \bar{u} = \alpha \odot (u + NW) = (\alpha \ast u) + NW.
\]

This weak NeutroVectorSpace \((NV/NW, \oplus, \odot)\) over a field \(K\) is called a weak NeutroQuotientSpace.

Proof. We can easily show that \(\oplus\) and \(\odot\) are well defined. The proof that \((NV/NW, \oplus)\) is an abelian group follows similar approach as the proof in classical case. Now it remains to show that \(NS2,NS3,NS4\) and \(NS5\) all hold.

(1) Since \(NS2\) holds in \(NV\), then there exist at least the triplets \((k, u, v)\) and \((m, a, b)\) with \(u, v, a, b \in NV\) and \(k, m \in K\) such that \(k \ast (u +' v) = k \ast u +' k \ast v\) and \(m \ast (a +' b) \neq m \ast a +' m \ast b\).

Let \(\bar{u}, \bar{v}, \bar{a}, \bar{b} \in NV/NW\) and \(k, m \in K(K)\). Then

\[
k \odot (\bar{u} \oplus \bar{v}) = k \odot ((u +' v) + NW) = k \ast (u +' v) + NW = (k \ast u +' k \ast v) + NW = (k \ast u) + NW \oplus (k \ast v) + NW = k \odot (u + NW) \oplus k \odot (v + NW) = k \odot \bar{u} \oplus k \odot \bar{v}.
\]

So, it implies \(k \odot (\bar{u} \oplus \bar{v}) = k \odot \bar{u} \oplus k \odot \bar{v}\).

And also,

\[
m \odot (\bar{a} \oplus \bar{b}) = m \odot ((a +' b) + NW) = m \ast (a +' b) + NW \neq (m \ast a +' m \ast b) + NW = (m \ast a) + NW \oplus (m \ast b) + NW = m \odot (a + NW) \oplus m \odot (b + NW) = m \odot \bar{a} \oplus m \odot \bar{b}.
\]

This implies \(m \odot (\bar{a} \oplus \bar{b}) \neq m \odot \bar{a} \oplus m \odot \bar{b}\). Hence, we can conclude that \(NS2\) holds in \(NV/NW\).

(2) Since \(NS3\) holds in \(NV\), then there exist at least the triplets \((k, m, u)\) and \((p, q, v)\) with \(u, v \in NV\) and \(k, m, p, q \in K\) such that \((k + m) \ast u = k \ast u +' m \ast u\) and \((p + q) \ast v \neq p \ast v +' q \ast v\).

Let \(\bar{u}, \bar{v} \in NV/NW\) and \(k, m, p, q \in K(K)\). Then

\[
(k + m) \odot \bar{u} = (k + m) \odot (u + NW) = (k + m) \ast u + NW = (k \ast u +' m \ast u) + NW = (k \ast u) + NW \oplus (m \ast u) + NW = k \odot (u + NW) \oplus m \odot (u + NW) = k \odot \bar{u} \oplus m \odot \bar{u}.
\]
So, it implies \((k + m) \odot \bar{u} = k \odot \bar{u} \oplus m \odot \bar{u}\).

And also,
\[
(p + q) \odot \bar{v} = (p + q) \odot (v + NW) \\
= (p + q) \star v + NW \\
\neq (p \star v + q \star v) + NW \\
= (p \star v) + NW \oplus (q \star v) + NW \\
= p \odot (v + NW) \oplus q \odot (v + NW) \\
= p \odot \bar{v} \oplus q \odot \bar{v}.
\]

So, it implies \((p + q) \odot \bar{v} \neq p \odot \bar{v} \oplus q \odot \bar{v}\).

Hence, we can conclude that \(NS3\) holds in \(NV/NW\).

(3) Since \(NS4\) holds in \(NV\), then there exist at least the triplets \((k, m, u)\) and \((p, q, v)\) with \(u, v \in NV\) and \(k, m, p, q \in K\) such that \((km) \star u = k \star (m \star u)\) and \((pq) \star v \neq p \star (q \star v)\).

Let \(\bar{u}, \bar{v} \in NV/NW\) and \(k, m, p, q \in K\). Then
\[
(km) \odot \bar{u} = (km) \odot (u + NW) \\
= (km) \star u + NW \\
= (k \star (m \star u)) + NW \\
= k \odot (m \odot (u + NW)) \\
= k \odot (m \odot \bar{u}).
\]

So, it implies \((km) \odot \bar{u} = k \odot (m \odot \bar{u})\).

And also,
\[
(pq) \odot \bar{v} = (pq) \odot (v + NW) \\
= (pq) \star v + NW \\
\neq (p \star (q \star v)) + NW \\
= p \odot ((q \star v) + NW) \\
= p \odot (q \odot (v + NW)) \\
= p \odot (q \odot \bar{v}).
\]

So, it implies that \((pq) \odot \bar{v} \neq p \odot (q \odot \bar{v})\).

Hence, we can conclude that \(NS4\) holds in \(NV/NW\).

(4) In \(NV\) we have at least \(u\) and \(v\) such that \(1 \star u = u\) and \(1 \star v \neq v\).

So, in \(NV/NW\) there exist \(\bar{u}\) and \(\bar{v}\) such that
\[
1 \odot \bar{u} = 1 \odot (u + NW) = (1 \star u) + NW = u + NW = \bar{u}
\]

and
\[
1 \odot \bar{v} = 1 \odot (v + NW) = (1 \star v) + NW \neq v + NW = \bar{v}.
\]

So, it implies \(1 \odot \bar{u} = \bar{u}\) and \(1 \odot \bar{v} = \bar{v}\).

Hence, we can conclude that \(NS5\) holds in \(NV/NW\).

Accordingly, \((NV/NW, \oplus, \odot)\) is a weak NeutroVectorSpace over the field \(K\).
**Remark 4.15.** Let \( NV \) be weak(strong) NeutroVectorSpace of type 4\( S \) over a field(NeutroField) \( K(NK) \) and let \( NW \) be a NeutroSubspace of \( NV \). Then, the weak(strong) NeutroQuotient Space \( NV/NW \) over \( K(NK) \) is not necessarily of type 4\( S \).

We illustrate Remark 4.15 by Example 4.16.

**Example 4.16.** Let \((NV = \mathbb{Z}_{12}, +', \star)\) be a weak NeutroVectorSpace of type 4\( S \) over \( K = \mathbb{Z}_p \) and let \((NW = 2\mathbb{Z}_{12}, +', \star)\) be a NeutroSubspace of \( NV \). Where +’ is addition mod 12 and \( \star \) is defined as

\[
k \star v = v^2 +' kv
\]

for all \( v \in NV \) and \( k \in K \).

Then for all \( \bar{u}, \bar{v} \in NV/NW \) and \( k \in K \) define the operation \( \oplus \) and \( \odot \) by

\[
\bar{u} \oplus \bar{v} = (u +' v) + NW
\]

and

\[
k \odot \bar{u} = (k \star u) + NW.
\]

Then \((NV/NW, \oplus, \odot)\) is a weak NeutroVectorSpace over \( K \) of type other than 4\( S \).

We know that \( NV = \{0, 1, 2, \cdots, 11\} \) and \( NW = \{0, 2, 4, 6, 8, 10\} \) then we have

\[
NV/NW = \{NW, 1 + NW\}.
\]

**Table 3.** Cayley table for the binary operation \( \oplus \)

<table>
<thead>
<tr>
<th></th>
<th>NW</th>
<th>1 + NW</th>
</tr>
</thead>
<tbody>
<tr>
<td>NW</td>
<td>NW</td>
<td>1 + NW</td>
</tr>
<tr>
<td>1 + NW</td>
<td>1 + NW</td>
<td>NW</td>
</tr>
</tbody>
</table>

From Table 3 it is clear that \((NV/NW, \oplus)\) is an abelian group.

Now,

1. \( NS2 \) fails to hold since for any triplet \((k, \bar{u}, \bar{v})\) we pick, with \( k \in K \) and \( \bar{u}, \bar{v} \in NV/NW \),

\[
k \odot (\bar{u} \oplus \bar{v}) = k \odot \bar{u} \oplus k \odot \bar{v}
\]

is always satisfied. This implies that \( S2 \) is totally true in \( NV/NW \).

2. There exists at least a triplet \((k, m, \bar{v})\) with \( k, m \in K \) and \( \bar{v} \in NV/NW \) such that

\[
(k + m) \odot (\bar{v}) = k \odot \bar{v} \oplus m \odot \bar{v}.
\]

Now,

\[
(k + m) \odot (\bar{v}) = ((k + m) \star \bar{v}) + NW = (v^2 +' kv +' mv) + NW
\]
and
\[ k \odot \bar{v} \oplus m \odot \bar{v} = ((k \star v) + NW) \oplus ((m \star v) + NW) = (2v^2 + 'kvu + 'mv) + NW. \]

Equating these we have that \( v^2 + NW = NW \) which implies \( v^2 \in NW \).
So, only the triple \((k, m, NW)\) satisfies \((k + m) \odot (\bar{v}) = k \odot \bar{v} \oplus m \odot \bar{v} \).
Hence, \( NS3 \) holds in \( NV/NW \).

(3) There exists at least a triplet \((k, m, \bar{v})\) with \( k, m \in K \) and \( \bar{v} \in NV/NW \) such that
\[ (km) \odot (\bar{v}) = k \odot (m \odot \bar{v}). \]

Now,
\[ (km) \odot (\bar{v}) = ((km) \star v) + NW = (v^2 + 'kmv) + NW \]

and
\[ k \odot (m \odot \bar{v}) = k \odot ((m \star v) + NW) = (k \star (v^2 + 'mv)) + NW = (v^4 + '2v^3m + 'm^2v^2 + kv^2 + 'kmv) + NW. \]

Equating these we have that \((v^4 + '2v^3m + 'm^2v^2 + kv^2) + NW = v^2 + NW \) which implies \((v^2 + '2vm + 'm^2 + 'k) + NW = 1 + NW \).
Since we needed at least a triplet, take \( k = 1 \), then we have
\[ (v^2 + '2vm + 'm^2 + '1) + NW = 1 + NW \] which gives \((v^2 + '2vm + 'm^2) + NW = NW \). So, we have that \((v^2 + '2vm + 'm^2) \in NW \). Then, at least the triplet \((1, m, NW)\) satisfies \((km) \odot (\bar{v}) = k \odot (m \odot \bar{v}) \). Hence, \( NS4 \) holds in \( NV/NW \).

(4) We can easily see that \( 1 \odot NW = NW \) and
\[ 1 \odot (1 + NW) = (1 \star 1) + NW = 2 + NW = NW \neq 1 + NW. \]
Hence, \( NS5 \) holds in \( NV/NW \).

Accordingly, we have that \((NV/NW, \oplus, \odot)\) is a weak NeutroVectorSpace of type 3S over \( K \).
This implies that the NeutroQuotient Space \((NV/NW, \oplus, \odot)\) does not belong to the class of NeutroVectorSpace \( NV \).

5. Conclusions

In this paper, we have for the first time introduced the concept of NeutroVectorSpaces. Specifically, a class of NeutroVectorSpaces called of type 4S was investigated and some of their elementary properties and examples were presented. It was shown that NeutroVectorSpaces of type 4S contained NeutroSubspaces of other types and that the intersections of NeutroSubspaces of type 4S are not necessarily NeutroSubspaces. Also, it was shown that if \( NV \) is a NeutroVectorSpace of a particular type and \( NW \) is a NeutroSubspace of \( NV \), the NeutroQuotientSpace \( NV/NW \) does not necessarily belong to the same type as \( NV \). We hope to continue this work in our next paper to be titled “NeutroVectorSpaces II”.

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