Abstract. The purpose of this paper is to define the so called "neutrosophic crisp points" and "neutrosophic crisp ideals", and obtain their fundamental properties. Possible application to GIS topology rules are touched upon.

Keywords: Neutrosophic Crisp Point, Neutrosophic Crisp Ideal.

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their crisp and fuzzy counterparts. The idea of "neutrosophic set" was first given by Smarandache [12, 13]. In 2012 neutrosophic operations have been investigated by Salama et al. [4 - 10]. The fuzzy set was introduced by Zadeh [13]. The intuitionistic fuzzy set was introduced by Atanassov [1, 2, 3]. Salama et al. [9] defined intuitionistic fuzzy ideal for a set and generalized the concept of fuzzy ideal concepts, first initiated by Sarker [11]. Here we shall present the crisp version of these concepts.

2 Terminologies

We recollect some relevant basic preliminaries, and in particular the work of Smarandache in [12, 13], and Salama at el. [4 -10].

3 Neutrosophic Crisp Points

One can easily define a natural type of neutrosophic crisp set in X, called "neutrosophic crisp point" in X, corresponding to an element \( p \in X \):

3.1 Definition

Let \( X \) be a nonempty set and \( p \in X \). Then the neutrosophic crisp point \( p_N \) defined by

\[
\{ p, \phi, \{ p \}' \}
\]

is called a neutrosophic crisp point (NCP for short) in \( X \), where NCP is a triple (\{only one element in \( X \)}, \{the empty set\}, \{the complement of the same element in \( X \})

Neutrosophic crisp points in \( X \) can sometimes be inconvenient when expressing a neutrosophic crisp set in \( X \) in terms of neutrosophic crisp points. This situation will occur if \( A = \{ A_1, A_2, A_3 \} \), and \( p \notin A_1 \), where \( A_1, A_2, A_3 \) are three subsets such that \( A_1 \cap A_2 = \phi \), \( A_1 \cap A_3 = \phi \), \( A_2 \cap A_3 = \phi \). Therefore we define the vanishing neutrosophic crisp points as follows:

3.2 Definition

Let \( X \) be a nonempty set, and \( p \in X \) a fixed element in \( X \). Then the neutrosophic crisp set

\[
\{ p, \phi, \{ p \}' \}
\]

is called "vanishing neutrosophic crisp point“ (VNCP for short) in \( X \), where VNCP is a triple (\{only one element in \( X \)}, \{the empty set\}, \{the complement of the same element in \( X \})

3.1 Example

Let \( X = \{ a, b, c, d \} \) and \( p = b \in X \). Then

\[
\{ b, \phi, \{ a, c, d \}' \}
\]

Now we shall present some types of inclusions of a neutrosophic crisp point to a neutrosophic crisp set:

3.3 Definition

Let \( \{ p, \phi, \{ p \}' \} \) be a NCP in \( X \) and \( A = \{ A_1, A_2, A_3 \} \) a neutrosophic crisp set in \( X \).

(a) \( p_N \) is said to be contained in \( A \) \( ( p_N \in A \) for short) iff \( p \in A_1 \).

(b) Let \( p_N \) be a VNCP in \( X \), and \( A = \{ A_1, A_2, A_3 \} \) a neutrosophic crisp set in \( X \). Then \( p_N \) is said to be contained in \( A \) \( ( p_N \in A \) for short) iff \( p \notin A_3 \).

3.1 Proposition

Let \( \{ p, \phi, \{ p \}' \} \) be a NCP in \( X \) and \( A = \{ A_1, A_2, A_3 \} \) a neutrosophic crisp set in \( X \).

(a) \( p_N \in \bigcap_{j \in J} D_j \) iff \( p_N \in D_j \) for each \( j \in J \).

(b) \( p_N \in \bigcap_{j \in J} D_j \) iff \( p_N \notin D_j \) for each \( j \in J \).

(c) \( p_N \in \bigcup_{j \in J} D_j \) iff \( \exists j \in J \) such that \( p_N \notin D_j \).
\((b_2)\) \(p_{n_x} \in \bigcap_{j \in J} D_j\) iff \(\exists j \in J\) such that \(p_{n_x} \in D_j\).

**Proof**

Straightforward.

### 3.2 Proposition

Let \(A = \{A_1, A_2, A_3\}\) and \(B = \{B_1, B_2, B_3\}\) be two neutrosophic crisp sets in \(X\). Then

a) \(A \subseteq B\) iff for each \(p_N \in A\) we have \(p_N \in B\) and for each \(p_{NN} \in B\) we have \(p_N \in A \Rightarrow p_{NN} \in B\).

b) \(A = B\) iff for each \(p_N \in A\) we have \(p_N \in B\) and for each \(p_{NN} \in B\) we have \(p_{NN} \in A \Rightarrow p_N \in B\).

**Proof**

Obvious.

### 3.4 Proposition

Let \(A = \{A_1, A_2, A_3\}\) be a neutrosophic crisp set in \(X\). Then

\[ A = (\bigcup \{p_N : p_N \in A\}) \cup (\bigcup \{p_{NN} : p_{NN} \in A\}). \]

**Proof**

It is sufficient to show the following equalities:

\(A_1 = (\bigcup \{p\}) \cup (\bigcup \{p_N \in A\}),\)

\(A_2 = \emptyset,\)

and \(A_3 = (\bigcap \{p \cap q : p \in A\}) \cap (\bigcap \{p \cap q : p_{NN} \in A\}),\)

which are fairly obvious.

### 3.4 Definition

Let \(f : X \rightarrow Y\) be a function.

(a) Let \(p_N\) be a neutrosophic crisp point in \(X\). Then the image of \(p_N\) under \(f\), denoted by \(f(p_N)\), is defined by \(f(p_N) = (\{q\} \cap \phi, \{q\} \cap \phi^c),\) where \(q = f(p)\).

(b) Let \(p_{NN}\) be a VNCP in \(X\). Then the image of \(p_{NN}\) under \(f\), denoted by \(f(p_{NN})\), is defined by \(f(p_{NN}) = (\phi, \{q\} \cap \phi^c),\) where \(q = f(p)\).

It is easy to see that \(f(p_N)\) is indeed a NCP in \(Y\), namely \(f(p_N) = q_N\), where \(q = f(p)\), and it is exactly the same meaning of the image of a NCP under the function \(f\).

\(f(p_{NN})\) is also a VNCP in \(Y\), namely \(f(p_{NN}) = q_{NN}\), where \(q = f(p)\).

### 3.4 Proposition

Any NCS \(A\) in \(X\) can be written in the form \(A = A \cup A \cup A\), where \(A = \bigcup_{N} \{p_N : p_N \in A\}\), \(A = \emptyset\) and \(A = \bigcup_{N} \{p_{NN} : p_{NN} \in A\}\). It is easy to show that, if \(A = \{A_1, A_2, A_3\}\), then \(A = \{x, A_1, \phi, A_3^c\}\) and \(A = \{x, \phi, A_2, A_3\}\).

### 3.5 Proposition

Let \(f : X \rightarrow Y\) be a function and \(A = \{A_1, A_2, A_3\}\) be a neutrosophic crisp set in \(X\). Then we have \(f(A) = f(A) \cup f(A) \cup f(A)\).

**Proof**

This is obvious from \(A = A \cup A \cup A\).

### 4 Neutrosophic Crisp Ideal Subsets

#### 4.1 Definition

Let \(X\) be non-empty set, and \(L\) a non-empty family of NCSs. We call \(L\) a neutrosophic crisp ideal (NCL for short) on \(X\) if

i. \(L \cup B \subseteq A \Rightarrow A \subseteq L\) [heredity],

ii. \(L \cup B \subseteq A \Rightarrow A \cap B \subseteq L\) [Finite additivity].

A neutrosophic crisp ideal \(L\) is called a \(\sigma\)-neutrosophic crisp ideal if \(\bigcup_{\{M_j : j \in J\}} \leq L\), implies \(\bigcup_{\{M_j : j \in J\}} \in L\) (countable additivity).

The smallest and largest neutrosophic crisp ideals on a non-empty set \(X\) are \(\{\emptyset\}\) and the NSs on \(X\). Also, \(NCL_f\), \(NCL_c\) are denoting the neutrosophic crisp ideals (NCL for short) of neutrosophic subsets having finite and countable support of \(X\) respectively. Moreover, if \(A\) is a nonempty NS in \(X\), then \(\{B \in NS : B \subseteq A\}\) is an NCL on \(X\). This is called the principal NCL of all NCSs, denoted by \(NCL(\{A\})\).
4.1 Remark

i. If \( X_N \notin L \), then L is called neutrosophic proper ideal.

ii. If \( X_N \in L \), then L is called neutrosophic improper ideal.

iii. \( \varnothing \in L \).

4.1 Example

Let \( \{a, b, c\}, A = \{\{a\}, \{a, b, c\}, \{c\}\} \), \( B = \{\{a\}, \{a, b\}, \{c\}\} \), \( C = \{\{a\}, \{b\}, \{c\}\} \), \( D = \{\{a\}, \{c\}\} \), \( E = \{\{a\}, \{a, b\}, \{c\}\} \), \( F = \{\{a\}, \{a, c\}, \{c\}\} \), \( G = \{\{a\}, \{b, c\}, \{c\}\} \).

Then the family \( L = \{\varnothing, A, B, D, E, F, G\} \) of NCSs is an NCL on X.

4.2 Definition

Let \( L_1 \) and \( L_2 \) be two NCLs on X. Then \( L_2 \) is said to be finer than \( L_1 \), or \( L_1 \) is coarser than \( L_2 \), if \( L_1 \subseteq L_2 \). If also \( L_1 \neq L_2 \), then \( L_2 \) is said to be strictly finer than \( L_1 \), or \( L_1 \) is strictly coarser than \( L_2 \).

Two NCLs said to be comparable, if one is finer than the other. The set of all NCLs on X is ordered by the relation: \( L_1 \) is coarser than \( L_2 \); this relation is induced the inclusion in NCSs.

The next Proposition is considered as one of the useful result in this sequel, whose proof is clear. \( L_j = \{A_{j,1}, A_{j,2}, A_{j,3}\} \).

4.1 Proposition

Let \( \{L_j : j \in J\} \) be any non-empty family of neutrosophic crisp ideals on a set X. Then \( \bigcap_{j \in J} L_j \) and \( \bigcup_{j \in J} L_j \) are neutrosophic crisp ideals on X, where

\[
\bigcap_{j \in J} L_j = \left( \bigcap_{j \in J} A_{j,1} \right) \cap \left( \bigcap_{j \in J} A_{j,2} \right) \cap \left( \bigcap_{j \in J} A_{j,3} \right)
\]

or

\[
\bigcap_{j \in J} L_j = \left( \bigcap_{j \in J} A_{j,1} \right) \cap \left( \bigcap_{j \in J} A_{j,2} \right) \cup \left( \bigcap_{j \in J} A_{j,3} \right)
\]

and

\[
\bigcup_{j \in J} L_j = \left( \bigcup_{j \in J} A_{j,1} \right) \cup \left( \bigcup_{j \in J} A_{j,2} \right) \cup \left( \bigcup_{j \in J} A_{j,3} \right)
\]

or

\[
\bigcup_{j \in J} L_j = \left( \bigcup_{j \in J} A_{j,1} \right) \cap \left( \bigcup_{j \in J} A_{j,2} \right) \cap \left( \bigcup_{j \in J} A_{j,3} \right)
\]

In fact, \( L \) is the smallest upper bound of the sets of the \( L_j \) in the ordered set of all neutrosophic crisp ideals on X.

4.2 Remark

The neutrosophic crisp ideal defined by the single neutrosophic set \( \varnothing \) is the smallest element of the ordered set of all neutrosophic crisp ideals on X.

4.2 Proposition

A neutrosophic crisp set \( A = \{A_1, A_2, A_3\} \) in the neutrosophic crisp ideal L on X is a base of L iff every member of L is contained in A.

Proof

(Necessity) Suppose A is a base of L. Then clearly every member of L is contained in A.

(Sufficiency) Suppose the necessary condition holds. Then the set of neutrosophic crisp subsets in X contained in A coincides with L by the Definition 4.3.

4.3 Proposition

A neutrosophic crisp ideal L, with base \( A = \{A_1, A_2, A_3\} \), is finer than a fuzzy ideal L with base \( B = \{B_1, B_2, B_3\} \), iff every member of B is contained in A.

Proof

Immediate consequence of the definitions.

4.1 Corollary

Two neutrosophic crisp ideals bases \( A, B, \) on X, are equivalent iff every member of A is contained in B and vice versa.

4.1 Theorem

Let \( \eta = \{A_1, A_2, A_3\} : j \in J \) be a non-empty collection of neutrosophic crisp subsets of X. Then there exists a neutrosophic crisp ideal \( L(\eta) = \{A \in \text{NCS} : A \subseteq \bigcup_{j \in J} A_j\} \) on X for some finite collection \( \{A_j : j = 1, 2, ..., n \in \eta\} \).

Proof

It’s clear.

4.3 Remark

The neutrosophic crisp ideal \( L(\eta) \) defined above is said to be generated by \( \eta \) and \( \eta \) is called sub-base of \( L(\eta) \).
4.2 Corollary
Let L₁ be a neutrosophic crisp ideal on X and A ∈ NCSs, then there is a neutrosophic crisp ideal L₂ which is finer than L₁ and such that A ∈ L₂ iff A ∪ B ∈ L₂ for each B ∈ L₁.

Proof
It’s clear.

4.2 Theorem
If L = \{\phi_N, \{A_1, A_2, A_3\}\} is a neutrosophic crisp ideals on X, then:

i) \[ L = \{\phi_N, \{A_1, A_2, A_3^c\}\} \]

is a neutrosophic crisp ideals on X.

ii) \( \{\phi_N, \{A_3, A_2, A_1^c\}\} \)

is a neutrosophic crisp ideals on X.

Proof
Obvious.

4.3 Theorem
Let A = \{A_1, A_2, A_3\} ∈ L₁, and

B = \{B_1, B_2, B_3\} ∈ L₂, where L₁ and L₂ are neutrosophic crisp ideals on X, then A*B is a neutrosophic crisp set:

\[ A*B = \{A_1*B_1, A_2*B_2, A_3*B_3\} \]

where

\[ A_1*B_1 = \bigcup\{A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3\} \]

\[ A_2*B_2 = \bigcap\{A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3\} \]

and

\[ A_3*B_3 = \bigcap\{A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3\} \].

References