Neutrosophic Crisp Set Theory

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Abstract. The purpose of this paper is to introduce new types of neutrosophic crisp sets with three types 1, 2, 3. After given the fundamental definitions and operations, we obtain several properties, and discussed the relationship between neutrosophic crisp sets and others. Also, we introduce and study the neutrosophic crisp point and neutrosophic crisp relations. Possible applications to database are touched upon.

Keywords: Neutrosophic Set, Neutrosophic Crisp Sets; Neutrosophic Crisp Relations; Generalized Neutrosophic Sets; Intuitionistic Neutrosophic Sets.

1 Introduction
Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. The fundamental concepts of neutrosophic set, introduced by Smarandache in \cite{16, 17, 18} and Salama et al. in \cite{4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 19, 20, 21}, provides a natural foundation for treating mathematically the neutrosophic phenomena which exist pervasively in our real world and for building new branches of neutrosophic mathematics. Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts \cite{1, 2, 3, 4, 23} such as a neutrosophic set theory. In this paper we introduce new types of neutrosophic crisp set. After given the fundamental definitions and operations, we obtain several properties, and discussed the relationship between neutrosophic crisp sets and others. Also, we introduce and study the neutrosophic crisp points and relation between two new neutrosophic crisp notions. Finally, we introduce and study the notion of neutrosophic crisp relations.

2 Terminologies
We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in \cite{16, 17, 18}, and Salama et al. \cite{7, 11, 12, 20}. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $[0,1]$ is nonstandard unit interval.

Definition 2.1 \cite{7} A neutrosophic crisp set (NCS for short) $A = \{A_1, A_2, A_3\}$ can be identified to an ordered triple \{A_1, A_2, A_3\} are subsets on $X$ and every crisp set in $X$ is obviously a NCS having the form $\{A_i, A_j\}$.

Salama et al. constructed the tools for developed neutrosophic crisp set, and introduced the NCS $\phi_N, X_N$ in $X$ as follows: $\phi_N$ may be defined as four types:

1. Type 1: $\phi_N = \{\phi, \phi, X\}$, or
2. Type 2: $\phi_N = \{\phi, X, X\}$, or
3. Type 3: $\phi_N = \{\phi, X, \phi\}$, or
4. Type 4: $\phi_N = \{\phi, \phi, \phi\}$

1. $X_N$ may be defined as four types

1. Type 1: $X_N = \{X, \phi, \phi\}$, or
2. Type 2: $X_N = \{X, X, X\}$, or
3. Type 3: $X_N = \{X, X, \phi\}$, or
4. Type 4: $X_N = \{X, X, \phi\}$

Definition 2.2 \cite{6, 7} Let $A = \{A_1, A_2, A_3\}$ a NCS on $X$, then the complement of the set $A$ (A$^c$, for short) may be defined as three kinds

1. Type 1: $A^c = \{A_1^c, A_2^c, A_3^c\}$,
2. Type 2: $A^c = \{A_3^c, A_2^c, A_1^c\}$
Definition 2.3 [6, 7]
Let $X$ be a non-empty set, and NCSS $A$ and $B$ in the form $A = \{A_1, A_2, A_3\}$, $B = \{B_1, B_2, B_3\}$, then we may consider two possible definitions for subsets ($A \subseteq B$)

- Type 1: $A \subseteq B \iff A_1 \subseteq B_1, A_2 \subseteq B_2$, and $A_3 \supseteq B_3$.
- Type 2: $A \subseteq B \iff A_1 \subseteq B_1, A_2 \supseteq B_2$, and $A_3 \supseteq B_3$.

Definition 2.5 [6, 7]
Let $X$ be a non-empty set, and NCSs $A$ and $B$ in the form $A = \{A_1, A_2, A_3\}$, $B = \{B_1, B_2, B_3\}$ are NCSS Then

1) $A \cap B$ may be defined as two types:
   - Type 1: $A \cap B = \{A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3\}$
   - Type 2: $A \cap B = \{A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3\}$

2) $A \cup B$ may be defined as two types:
   - Type 1: $A \cup B = \{A_1 \cup B_1, A_2 \cup B_2, A_3 \cup B_3\}$
   - Type 2: $A \cup B = \{A_1 \cup B_1, A_2 \cup B_2, A_3 \cup B_3\}$

3) Some types of Neutrosophic Crisp Sets
We shall now consider some possible definitions for some types of neutrosophic crisp sets

Definition 3.1
The object having the form $A = \{A_1, A_2, A_3\}$ is called

1) (Neutrosophic Crisp Set with Type 1) If satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, and $A_2 \cap A_3 = \phi$.
   (NCS-Type1 for short).

2) (Neutrosophic Crisp Set with Type 2) If satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, and $A_2 \cap A_3 = \phi$ and $A_1 \cup A_2 \cup A_3 = \phi$.
   (NCS-Type2 for short).

3) (Neutrosophic Crisp Set with Type 3) If satisfying $A_1 \cap A_2 \cap A_3 = \phi$ and $A_1 \cup A_2 \cup A_3 = \phi$.
   (NCS-Type3 for short).

Definition 3.3
1) (Neutrosophic Set [9, 16, 17]): Let $X$ be a non-empty fixed set. A neutrosophic set (NS for short) $A$ is an object having the form $A = \{\mu_A(x), \sigma_A(x), \nu_A(x)\}$ where $\mu_A(x)$, $\sigma_A(x)$, and $\nu_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set $A$ where $0 \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1$ and $0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3$.

2) (Generalized Neutrosophic Set [8]): Let $X$ be a non-empty fixed set. A generalized neutrosophic (GNS for short) set $A$ is an object having the form $A = \{x, \mu_A(x), \sigma_A(x), \nu_A(x)\}$ where $\mu_A(x)$, $\sigma_A(x)$, and $\nu_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set $A$ where $0 \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1$ and $0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3$.

3) (Intuitionistic Neutrosophic Set [22]): Let $X$ be a non-empty fixed set. An intuitionistic neutrosophic set $A$ (INS for short) is an object having the form $A = \{\mu_A(x), \sigma_A(x), \nu_A(x)\}$ where $\mu_A(x)$, $\sigma_A(x)$, and $\nu_A(x)$ which represent the degree of member ship function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set $A$ where $0 \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1$ and $0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3$.

Remark 3.1
1) The neutrosophic set not to be generalized neutrosophic set in general.
2) The generalized neutrosophic set in general not intuitionistic NS but the intuitionistic NS is generalized NS.

Intuitionistic NS $\rightarrow$ Generalized NS $\rightarrow$ NS
Corollary 3.1
Let X non-empty fixed set and $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ be INS on X Then:
1) Type1- $A^c$ of INS be a GNS,
2) Type2- $A^c$ of INS be a INS.
3) Type3- $A^c$ of INS be a GINS.

Proof
Since A INS then $\mu_A(x), \sigma_A(x), \nu_A(x)$, and
$\mu_A(x) \wedge \sigma_A(x) \leq 0.5, \nu_A(x) \wedge \mu_A(x) \leq 0.5$
$\nu_A(x) \wedge \sigma_A(x) \leq 0.5$ Implies
$\mu^c_A(x), \sigma^c_A(x), \nu^c_A(x) \leq 0.5$ then is not to be Type1- $A^c$

INS. On other hand the Type 2- $A^c$,
$A^c = \langle \nu_A(x), \sigma_A(x), \mu_A(x) \rangle$ be INS and Type3- $A^c$,
$A^c = \langle \nu_A(x), \sigma_A(x), \mu_A(x) \rangle$ and $\sigma^c_A(x) \leq 0.5$ implies to
$A^c = \langle \nu_A(x), \sigma_A(x), \mu_A(x) \rangle$ GNS and not to be INS

Example 3.1
Let $X = \{a, b, c\}$, and $A, B, C$ are neutrosophic sets on $X$,
$A = \{0.7, 0.9, 0.8\} \cup \{0.6, 0.7, 0.6\} \cup \{0.9, 0.7, 0.8\}$
$B = \{0.7, 0.9, 0.5\} \cup \{0.6, 0.4, 0.5\} \cup \{0.9, 0.5, 0.8\}$
$C = \{0.7, 0.9, 0.5\} \cup \{0.6, 0.8, 0.5\} \cup \{0.9, 0.5, 0.8\}$
By the Definition 3.3 no.3 $\mu_A(x) \wedge \sigma_A(x) \vee \nu_A(x) \geq 0.5$, A be not
GNS and INS,
$B = \{0.7, 0.9, 0.5\} \cup \{0.6, 0.4, 0.5\} \cup \{0.9, 0.5, 0.8\}$ not INS,
where $\sigma_A(b) = 0.4 < 0.5$. Since
$\mu_A(x) \wedge \sigma_A(x) \vee \nu_A(x) \leq 0.5$ then $B$ is a GNS but not INS.
$A^c = \{0.3, 0.1, 0.2\} \cup \{0.4, 0.3, 0.4\} \cup \{0.1, 0.3, 0.2\}$
Be a GNS, but not INS.

Every crisp set be NCS.

Definition 3.3
A NCS-Type2, $\phi_{N_2}, X_{N_2}$ in X as follows:
1) $\phi_{N_2}$ may be defined as two types:
   a) Type1: $\phi_{N_2} = \langle \phi, \phi, X \rangle$, or
   b) Type2: $\phi_{N_2} = \langle \phi, X, \phi \rangle$, or
   c) Type3: $\phi_{N_2} = \langle \phi, \phi, \phi \rangle$.
2) $X_{N_2}$ may be defined as one type

Corollary 3.2
In general
1- Every NCS-Type 1, 2, 3 are NCS,
2- Every NCS-Type 1 not to be NCS-Type2, 3.
3- Every NCS-Type 2 not to be NCS-Type1, 3.
4- Every NCS-Type 3 not to be NCS-Type2, 1, 2.
5- Every crisp set be NCS.

Example 3.2
Let $X = \{a, b, c, d, e, f\}$,
$A = \{\{a, b, c, d\}, \{e\}, \{f\}\}$,
$D = \{\{a, b\}, \{e, c\}, \{f, d\}\}$ be a NCS-Type 2.
\( B = \{(a, b, c), (d, e)\} \) be a NCT-Type 1 but not NCS-Type 2, 3. \( C = \{(a, b, c, d, e)\} \) be a NCS-Type 3 but not NCS-Type 1, 2.

**Definition 3.5**

Let \( X \) be a non-empty set, \( A = \{A_1, A_2, A_3\} \)

1) If \( A \) be a NCS-Type 1 on \( X \), then the complement of the set \( A \) (\( A^c \), for short) may be defined as one kind of complement Type 1: \( A^c = \{A_1, A_2, A_3\} \).

2) If \( A \) be a NCS-Type 2 on \( X \), then the complement of the set \( A \) (\( A^c \), for short) may be defined as one kind of complement Type 2: \( A^c = \{A_3, A_2, A_1\} \).

3) If \( A \) be NCS-Type 3 on \( X \), then the complement of the set \( A \) (\( A^c \), for short) may be defined as one kind of complement defined as three kinds of complements

\[ (C_i) \text{ Type1: } A^c = \{A_1, A_2, A^c_3\}, \]
\[ (C_i) \text{ Type2: } A^c = \{A_3, A_2, A_1\}, \]
\[ (C_i) \text{ Type3: } A^c = \{A_3, A_2, A_1\} \]

**Example 3.3**

Let \( X = \{a, b, c, d, e, f\} \), \( A = \{a, b, c, d, e\} \) be a NCS-Type 2, \( B = \{a, b, c, \phi, d, e\} \) be a NCS-Type 1., \( C = \{a, b, c, \phi, d, e\} \) be a NCS-Type 3, then the complement \( A = \{a, b, c, d, e\} \), \( A^c = \{\phi\} \)

\( C^c = \{a, b, c, d, e\} \) NCS-Type 2, the complement of \( B = \{a, b, c, \phi, d, e\} \), \( B^c = \{d, e, \phi\} \)

\( C \) NCS-Type 1. The complement of \( C = \{a, b, c, d, e\} \) may be defined as three types:

Type 1: \( C^c = \{a, b, c, d, e\} \), \( \{a, b, c, d, e\} \).

Type 2: \( C^c = \{a, b, c, d, e\} \).

Type 3: \( C^c = \{a, b, c, d, e\} \).

**Proposition 3.1**

Let \( \{A_j : j \in J\} \) be arbitrary family of neutrosophic crisp subsets on \( X \), then

1) \( \cap A_j \) may be defined two types as:
   i) Type 1: \( \cap A_j = \{\cap A_{j_1} \cap A_{j_2} \cap A_{j_3}\} \).
   ii) Type 2: \( \cap A_j = \{\cap A_{j_1} \cup A_{j_2} \cup A_{j_3}\} \).

2) \( \cup A_j \) may be defined two types as:

**Definition 3.6**

(a) If \( B = \{B_1, B_2, B_3\} \) is a NCS in \( Y \), then the preimage of \( B \) under \( f \), denoted by \( f^{-1}(B) \), is a NCS in \( X \) defined by \( f^{-1}(B) = \{f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3)\} \)

(b) If \( A = \{A_1, A_2, A_3\} \) is a NCS in \( Y \), then the image of \( A \) under \( f \), denoted by \( f(A) \), is the a NCS in \( Y \) defined by \( f(A) = \{f(A_1), f(A_2), f(A_3)\} \).

Here we introduce the properties of images and preimages some of which we shall frequently use in the following.

**Corollary 3.3**

Let \( A \), \( B_j : j \in J \) be a family of NCS in \( X \), and \( B \), \( B_j : j \in K \) NCS in \( Y \), and \( f : X \rightarrow Y \) a function. Then

(a) \( A \subseteq A \iff f(A) \subseteq f(A) \), \( B_j \subseteq B_j \iff f^{-1}(B_j) \subseteq f^{-1}(B_j) \),

(b) \( A \subseteq f^{-1}(f(A)) \) and if \( f \) is injective, then \( A = f^{-1}(f(A)) \),

(c) \( f^{-1}(f(B)) \subseteq B \) and if \( f \) is surjective, then \( f^{-1}(f(B)) = B \),

(d) \( f^{-1}(\cap B_j) = \cap f^{-1}(B_j), f^{-1}(\cup B_j) = \cup f^{-1}(B_j) \),

(e) \( f(\cup A_j) = \cup f(A_j), f(\cap A_j) \subseteq \cap f(A_j) \), and if \( f \) is injective, then \( f(\cap A_j) = \cap f(A_j) \),

(f) \( f^{-1}(y_j) = x_j, f^{-1}(\phi_j) = \phi_j \),

(g) \( f(\phi_j) = \phi_j, f(\mathcal{X}_j) = \mathcal{X}_j \) if \( f \) is subjective.

**Proof**

Obvious

4 Neutrosophic Crisp Points

One can easily define a nature neutrosophic crisp set in \( X \), called "neutrosophic crisp point" in \( X \), corresponding to an element \( X \):

**Definition 4.1**

Let \( A = \{A_1, A_2, A_3\} \) be a neutrosophic crisp set on \( A \) set \( X \), then \( p = (p_1, p_2, p_3) \), \( p_1 \neq p_2 \neq p_3 \in X \) is called a neutrosophic crisp point on \( A \).
A NCP \( p = \{\{p_1\}, \{p_2\}, \{p_3\}\} \), is said to be belong to a neutrosophic crisp set \( A = \{A_1, A_2, A_3\} \), of \( X \), denoted by \( p \in A \), if may be defined by two types

Type 1: \( \{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2 \) and \( \{p_3\} \subseteq A_3 \) or

Type 2: \( \{p_1\} \subseteq A_1, \{p_2\} \supseteq A_2 \) and \( \{p_3\} \subseteq A_{31} \)

**Theorem 4.1**

Let \( A = \{A_1, A_2, A_3\} \) and \( B = \{B_1, B_2, B_3\} \) be neutrosophic crisp subsets of \( X \). Then \( A \subseteq B \) iff \( p \in A \) implies \( p \in B \) for any neutrosophic crisp point \( p \) in \( X \).

**Proof**

Let \( A \subseteq B \) and \( p \in A \), Type 1: \( \{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2 \) and \( \{p_3\} \subseteq A_3 \) or Type 2: \( \{p_1\} \subseteq A_1, \{p_2\} \supseteq A_2 \) and \( \{p_3\} \subseteq A_{31} \) Thus \( p \in B \). Conversely, take any point in \( X \). Let \( p_1 \in A_{11} \) and \( p_2 \in A_{22} \) and \( p_3 \in A_{33} \). Then \( p \) is a neutrosophic crisp point in \( X \). and \( p \in A \). By the hypothesis \( p \in B \).

Thus \( p_1 \in B_1 \) or Type1: \( \{p_1\} \subseteq B_1, \{p_2\} \subseteq B_2 \) and \( \{p_3\} \subseteq B_{31} \) or Type 2: \( \{p_1\} \subseteq B_1, \{p_2\} \supseteq B_2 \) and \( \{p_3\} \subseteq B_{31} \). Hence \( A \subseteq B \).

**Theorem 4.2**

Let \( A = \{A_1, A_2, A_3\} \), be a neutrosophic crisp subset of \( X \). Then \( A = \bigcup\{p : p \in A\} \).

**Proof**

Obvious.

**Proposition 4.1**

Let \( \{A_j : j \in J\} \) is a family of NCSs in \( X \). Then

(a) \( p = \{\{p_1\}, \{p_2\}, \{p_3\}\} \in \bigcap_{j \in J} A_j \) iff \( p \in A_j \) for each \( j \in J \).

(b) \( p \in \bigcup_{j \in J} A_j \) iff \( \exists j \in J \) such that \( p \in A_j \).

**Proposition 4.2**

Let \( A = \{A_1, A_2, A_3\} \) and \( B = \{B_1, B_2, B_3\} \) be two neutrosophic crisp sets in \( X \). Then \( A \subseteq B \) iff for each \( p \) we have \( p \in A \Rightarrow p \in B \) and for each \( p \) we have \( p \in A \Rightarrow p \in B \). if \( A = B \) for each \( p \) we have \( p \in A \Rightarrow p \in B \). and for each \( p \) we have \( p \in A \Rightarrow p \in B \).

**Proposition 4.3**

Let \( A = \{A_1, A_2, A_3\} \) be a neutrosophic crisp set in \( X \). Then \( A = \bigcup \{p_1 : p_1 \in A_1\}, \{p_2 : p_2 \in A_2\}, \{p_3 : p_3 \in A_3\} \).

**Definition 4.2**

Let \( f : X \to Y \) be a function and \( p \) be a neutrosophic crisp point in \( X \). Then the image of \( p \) under \( f \), denoted by \( f(p) \), is defined by \( f(p) = \{b | f(p) = b\} \), where \( q_1 = f(p_1), q_2 = f(p_2) \) and \( q_3 = f(p_3) \). It is easy to see that \( f(p) \) is indeed a NCP in \( Y \), namely \( f(p) = q \), where \( q = f(p) \), and it is exactly the same meaning of the image of a NCP under the function \( f \).

**Definition 4.3**

Let \( X \) be a nonempty set and \( p \in X \). Then the neutrosophic crisp point \( p_N \) defined by \( p_N = \{p, \phi, \{p\}^c\} \) is called a neutrosophic crisp point (NCP for short) in \( X \). where \( NCP \) is a triple (empty set, the complement of the same element in \( X \)). Neutrosophic crisp points in \( X \) can sometimes be inconvenient when express neutrosophic crisp set in \( X \) in terms of neutrosophic crisp points. This situation will occur if \( A = \{A_1, A_2, A_3\} \) NCS-Type 1, \( p \notin A_1 \). Therefore we shall define “vanishing” neutrosophic crisp points as follows:

**Definition 4.4**

Let \( X \) be a nonempty set and \( p \in X \) a fixed element in \( X \). Then the neutrosophic crisp set \( p_N = \{p, \phi, \{p\}^c\} \) is called vanishing” neutrosophic crisp point (VNCP for short) in \( X \), where VNCP is a triple (empty set, the complement of the same element in \( X \)).

**Example 4.1**

Let \( X = \{a, b, c, d\} \) and \( p = b \in X \). Then

\[
\begin{align*}
  p_N = \langle b, \phi, \{a, c, d\} \rangle, \\
  p_{N_{\phi}} = \langle \phi, b, \{a, c, d\} \rangle, \\
  P = \langle \{b\}, \{a\}, \{d\} \rangle.
\end{align*}
\]

Now we shall present some types of inclusion of a neutrosophic crisp point to a neutrosophic crisp set:

**Definition 4.5**

Let \( p_N = \{p, \phi, \{p\}^c\} \) is a NCP in \( X \) and \( A = \{A_1, A_2, A_3\} \) a neutrosophic crisp set in \( X \).
(a) $p_N$ is said to be contained in $A$ ($p_N \in A$ for short) iff $p \in A_1$.

(b) $p_{NN}$ be VNCP in X and $A = \{A_1, A_2, A_3\}$ a neutrosophic crisp set in X. Then $p_{NN}$ is said to be contained in $A$ ($p_{NN} \in A$ for short) iff $p \notin A_1$.

Remark 4.2

$p_N$ and $p_{NN}$ are NCS-Type 1

Proposition 4.4

Let $\{A_j : j \in J\}$ is a family of NCSs in X. Then

(a1) $p_N \in \bigcap_{j \in J} A_j$ iff $p_N \in A_j$ for each $j \in J$.

(a2) $p_{NN} \in \bigcap_{j \in J} A_j$ iff $p_{NN} \in A_j$ for each $j \in J$.

(b1) $p_N \in \bigcup_{j \in J} A_j$ iff $\exists j \in J$ such that $p_N \in A_j$.

(b2) $p_{NN} \in \bigcup_{j \in J} A_j$ iff $\exists j \in J$ such that $p_{NN} \in A_j$.

Proof

Straightforward.

Proposition 4.5

Let $A = \{A_1, A_2, A_3\}$ and $B = \{B_1, B_2, B_3\}$ are two neutrosophic crisp sets in X. Then $A \subseteq B$ iff for each $p_N$ we have $p_N \in A \Rightarrow p_N \in B$ and for each $p_{NN}$ we have $p_N \in A \Rightarrow p_{NN} \in B$. $A = B$ iff for each $p_N$ we have $p_N \in A \Rightarrow p_N \in B$ and for each $p_{NN}$ we have $p_{NN} \in A \Rightarrow p_{NN} \in B$.

Proof

Obvious

Proposition 4.6

Let $A = \{A_1, A_2, A_3\}$ be a neutrosophic crisp set in X.

Then $A = (\bigcup \{p_N : p_N \in A\}) \cup (\bigcup \{p_{NN} : p_{NN} \in A\})$.

Proof

It is sufficient to show the following equalities:

$a_1 = \bigcup \{p_N : p_N \in A\}$, $A_1 = \phi$

and $a_3 = \bigcup \{p_{NN} : p_{NN} \in A\}$, $A_3 = \phi$

which are fairly obvious.

Definition 4.6

Let $f : X \rightarrow Y$ be a function and $p_N$ be a neutrosophic crisp point in X. Then the image of $p_N$ under $f$, denoted by $f(p_N)$ is defined by $f(p_N) = \{[q], \phi_q, [q]^c\}$ where $q = f(p)$.

Let $p_{NN}$ be a VNCP in X. Then the image of $p_{NN}$ under $f$, denoted by $f(p_{NN})$, is defined by $f(p_{NN}) = \{\phi_q, [q], [q]^c\}$ where $q = f(p)$.

It is easy to see that $f(p_N)$ is indeed a NCP in Y, namely $f(p_N) = q_N$ where $q = f(p)$, and it is exactly the same meaning of the image of a NCP under he function $f$.

Proposition 4.7

States that any NCS $A$ in X can be written in the form $A = A_1 \cup A_2 \cup A_3$, where $A = \cup \{p_N : p_N \in A\}$.

$A = \phi_1$ and $A = \cup \{p_{NN} : p_{NN} \in A\}$. It is easy to show that, if $A = \{A_1, A_2, A_3\}$, then $A = \{A_1, A_2, A_3\}$ and $A = \{A_1, A_2, A_3\}$.

Proposition 4.8

Let $f : X \rightarrow Y$ be a function and $A = \{A_1, A_2, A_3\}$ be a neutrosophic crisp set in X. Then we have $f(A) = f(A_1) \cup f(A_2) \cup f(A_3)$.

Proof

This is obvious from $A = A_1 \cup A_2 \cup A_3$.

Proposition 4.9

Let $A = \{A_1, A_2, A_3\}$ and $B = \{B_1, B_2, B_3\}$ be two neutrosophic crisp sets in X. Then

a) $A \subseteq B$ iff for each $p_N$ we have $p_N \in A \Rightarrow p_N \in B$ and for each $p_{NN}$ we have $p_{NN} \in A \Rightarrow p_{NN} \in B$.

b) $A = B$ iff for each $p_N$ we have $p_N \in A \Rightarrow p_N \in B$ and for each $p_{NN}$ we have $p_{NN} \in A \Rightarrow p_{NN} \in B$.

Proof

Obvious

Proposition 4.10

Let $A = \{A_1, A_2, A_3\}$ be a neutrosophic crisp set in X. Then $A = \{p_N : p_N \in A\}$.

Proof

Obvious
It is sufficient to show the following equalities:
\[ A_1 = \left( \bigcup \{ p : p \in A \} \right) \cup \left( \bigcup \{ \phi : p_{NN} \in A \} \right) = A \] and
\[ A_2 = \left( \bigcap \{ p : p \in A \} \right) \cap \left( \bigcap \{ \phi : p_{NN} \in A \} \right) = \phi , \]
which are fairly obvious.

**Definition 4.7**

Let \( f : X \rightarrow Y \) be a function.

(a) Let \( p_N \) be a neutrosophic crisp point in X. Then the image of \( p_N \) under \( f \), denoted by \( f(p_N) \), is defined by
\[ f(p_N) = \left\{ q \in Y : \left[ \left( \left\{ p : p \in A \right\} \cup \left\{ \phi : p_{NN} \in A \right\} \right) \subseteq \left\{ q : q \in B \right\} \right] \right\} , \]
where \( q = f(p) \).

(b) Let \( p_{NN} \) be a VNCP in X. Then the image of \( p_{NN} \) under \( f \), denoted by \( f(p_{NN}) \), is defined by
\[ f(p_{NN}) = \left\{ \phi, q \in Y : \left[ \left( \left\{ \phi : \phi \in A \right\} \cup \left\{ q : q \in B \right\} \right) \subseteq \left\{ \phi, q : \phi, q \in A \right\} \right] \right\} , \]
where \( q = f(p) \). It is easy to see that \( f(p_{NN}) \) is indeed a NCP in \( Y \), namely \( f(p_{NN}) = q_{NN} \), where \( q = f(p) \), and it is exactly the same meaning of the image of a NCP under the function \( f \). \( f(p_{NN}) \) is also a VNCP in \( Y \), namely \( f(p_{NN}) = q_{NN} \), where \( q = f(p) \).

**Proposition 4.11**

Any NCS \( A \) in \( X \) can be written in the form \( A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \), where \( A_1 = \left( \bigcup \{ p : p \in A \} \right) \cup \left( \bigcup \{ \phi : p_{NN} \in A \} \right) \), \( A_2 = \phi \), and \( A_3 = \left( \bigcap \{ p : p \in A \} \right) \cap \left( \bigcap \{ \phi : p_{NN} \in A \} \right) = \phi \). It is easy to show that, if \( A = \{ A_1, A_2, A_3 \} \), then \( A = \{ x, A_1, \phi, A_5 \} \) and \( A = \{ x, \phi, A_2, A_4 \} \).

**Proposition 4.12**

Let \( f : X \rightarrow Y \) be a function and \( A = \{ A_1, A_2, A_3 \} \) be a neutrosophic crisp set in \( X \). Then we have
\[ f(A) = f(A_1) \cup f(A_2) \cup f(A_3) . \]

**Proof**

This is obvious from \( A = A_1 \cup A_2 \cup A_3 \).

5 Neutrosophic Crisp Set Relations

Here we give the definition relation on neutrosophic crisp sets and study its properties.

Let \( X, Y, Z \) be three crisp nonempty sets

**Definition 5.1**

Let \( X, Y, Z \) be two non-empty crisp sets and NCSs \( A, B, C \) in \( X, Y \), then
\[ A \times B = \{ A_1 \times B_1, A_2 \times B_2, A_3 \times B_3 \} \] on \( X \times Y \).

ii) We will call a neutrosophic crisp relation \( R \subseteq A \times B \) on the direct product \( X \times Y \).

The collection of all neutrosophic crisp relations on \( X \times Y \) is denoted as \( NCR(X \times Y) \).

**Definition 5.2**

Let \( R \) be a neutrosophic crisp relation on \( X \times Y \), then the inverse of \( R \) is denoted by \( R^{-1} \) where \( R^{-1} \subseteq B \times A \).

**Example 5.1**

Let \( X = \{ a, b, c, d \} \), \( A = \{ \{a, b\}, \{c\}, \{d\} \} \) and \( B = \{ \{a\}, \{c\}, \{d, b\} \} \) then the product of two neutrosophic crisp sets given by
\[ A \times B = \{ \{a, b\}, \{c\}, \{d, b\} \} \] and
\[ B \times A = \{ \{a\}, \{c\}, \{d\} \} \].

**Example 5.2**

Let \( X = \{ a, b, c, d, e, f \} \), \( A = \{ \{a, b, c, d\}, \{e\}, \{f\} \} \) and \( D = \{ \{a, b\}, \{e, c\}, \{f, d\} \} \) be a NCS-Type 2,
\[ B = \{ \{a, b, c\}, \{d\}, \{e, f\} \} \] be a NCS-Type 1.
\[ C = \{ \{a, b\}, \{c, d\}, \{e, f\} \} \] be a NCS-Type 3. Then
\[ A \times D = \{ \{a, b\}, \{a, b\}, \{b, e\}, \{b, f\}, \{e, c\}, \{c, d\}, \{d, a\}, \{d, b\}, \{e, c\}, \{e, f\}, \{f, d\} \} \]
\[ D \times C = \{ \{a, b\}, \{b, e\}, \{b, f\}, \{e, c\}, \{e, f\}, \{f, d\}, \{d, e\}, \{d, f\}, \{e, c\}, \{e, f\}, \{f, d\} \} \]
We can construct many types of relations on products.

We can define the operations of neutrosophic crisp relation.

**Definition 5.3**

Let \( R \) and \( S \) be two neutrosophic crisp relations between \( X \) and \( Y \) for every \((x, y) \in X \times Y \) and NCSs \( A \) and \( B \) in \( X \) and \( Y \) then we can defined the following operations
i) \( R \subseteq S \) may be defined as two types
a) Type1: \( R \subseteq S \) if \( A_1 \subseteq B_1, A_2 \subseteq B_2, A_3 \subseteq B_3 \)

b) Type2: \( R \subseteq S \) if \( A_1 \subseteq B_1, A_2 \subseteq B_2, A_3 \subseteq B_3, A_4 \subseteq B_4 \),

ii) \( R \cup S \) may be defined as two types
a) Type1: \( R \cup S \)

b) Type2: \( R \cup S \)
b) Type 2:
\[ R \cup S = \{ A_{1R} \cap B_{1S}, A_{2R} \cap B_{2S}, A_{3R} \cap B_{3S} \} \].
ii) \[ R \cap S \] may be defined as two types
a) Type 1: \[ R \cap S = \{ A_{1R} \cap B_{1S}, A_{2R} \cap B_{2S}, A_{3R} \cap B_{3S} \} \],
b) Type 2: \[ R \cap S = \{ A_{1R} \cap B_{1S}, A_{2R} \cap B_{2S}, A_{3R} \cap B_{3S} \} \].

**Theorem 5.1**

Let \( R \), \( S \) and \( Q \) be three neutrosophic crisp relations between \( X \) and \( Y \) for every \((x, y) \in X \times Y\), then
\[
i) \ R \subseteq S \Rightarrow R^{-1} \subseteq S^{-1}.
\]
\[
ii) \ (R \cup S)^{-1} \Rightarrow R^{-1} \cup S^{-1}.
\]
\[
iii) \ (R \cap S)^{-1} \Rightarrow R^{-1} \cap S^{-1}.
\]
\[
iv) \ (R^T)^{-1} = R.
\]
\[
v) \ R \cap (S \cup Q) = (R \cap S) \cup (R \cap Q).
\]
\[
vi) \ R \cup (S \cap Q) = (R \cup S) \cap (R \cup Q).
\]
\[
vii) \text{If } S \subseteq R, \ Q \subseteq R, \ \text{then } S \cup Q \subseteq R
\]

**Proof**

Clear

**Definition 5.4**

The neutrosophic crisp relation \( I \in NCR(X \times X) \), the neutrosophic crisp relation of identity may be defined as two types
\[
i) \ \text{Type1: } I = \{ (A \times A), (A \times A), \phi \}.
\]
\[
ii) \ \text{Type2: } I = \{ (A \times A), (A \times A), \phi \}.
\]

Now we define two composite relations of neutrosophic crisp sets.

**Definition 5.5**

Let \( R \) be a neutrosophic crisp relation in \( X \times Y \), and \( S \) be a neutrosophic crisp relation in \( Y \times Z \). Then the composition of \( R \) and \( S \), \( R \circ S \) be a neutrosophic crisp relation in \( X \times Z \) as a definition may be defined as two types
\[
i) \ \text{Type 1: } R \circ S = \{ (A \times B) \cap (A \times B) \cap (A \times B) \}.
\]
\[
ii) \ \text{Type 2: } R \circ S = \{ (A \times B) \cap (A \times B) \cap (A \times B) \}.
\]

**Example 5.3**

Let \( X = \{a, b, c, d\} \), \( A = \{a, b, c, d\} \), and \( B = \{a, b, c, d\} \) then the product of two events given by \( A \times B = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\} \), and
\[
B \times A = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\}.
\]

**References**


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Received: July 30, 2014. Accepted: August 20, 2014.