Neutrosophic crisp Sets via Neutrosophic crisp Topological Spaces

NCTS

Wadei Al-Omeri

1 Mathematical Sciences, Faculty of Science and Technology, Ajloun National University, P.O.Box 43, Ajloun, 26810, jordan. E-mail: wadeimoon1@hotmail.com

Abstract: In this paper, the structure of some classes of neutrosophic crisp nearly open sets are investigated via topology, and some applications are given. Finally, we generalize the crisp topological and neutrosophic crisp studies to the notion of neutrosophic crisp set.

Keywords: Set Theory, Topology, Neutrosophic crisp set theory, Neutrosophic crisp topology, Neutrosophic crisp α-open set, Neutrosophic crisp semi-open set, Neutrosophic crisp continuous function.

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their crisp and fuzzy counterparts, such as a neutrosophic set theory in [9, 11, 10]. It followed the introduction of the neutrosophic set concepts in [13, 12, 14, 15, 5, 7, 8, 16, 17] and the fundamental definitions of neutrosophic set operations. Smarandache [9, 11] and Salama et al. in [13, 18] provide a natural foundation for treating mathematically the neutrosophic phenomena which exist pervasively in our real world and for building new branches of neutrosophic mathematics.

In this paper, we introduce the concept of neutrosophic crisp sets. We investigate the properties of continuous, open and closed maps in the neutrosophic crisp topological spaces, also give relations between neutrosophic crisp pre-continuous mapping and neutrosophic crisp semi-precontinuous mapping and some other continuous mapping, and show that the category of intuitionistic fuzzy topological spaces is a bireflective full subcategory of neutrosophic crisp topological spaces.

2 Terminology

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [9, 11, 10], and Salama et al. [13, 12, 14, 15, 5, 7, 8, 16, 17, 6]. Smarandache introduced the neutrosophic components $T$, $I$, $F$ which represent the membership, indeterminacy, and non-membership values respectively, where $\tilde{0}, 1^*$ is a non-standard unit interval. Hanafy and Salama et al.[8, 16] considered some possible definitions for basic concepts of the neutrosophic crisp set and its operations.

Definition 1 [20] Let $X$ be a non-empty fixed set. A neutrosophic crisp set (NCS) $A$ is an object having the form $A = \{A_1, A_2, A_3\}$, where $A_1, A_2, A_3$ are subsets of $X$ satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, and $A_2 \cap A_3 = \phi$.

Remark 2 [20] Neutrosophic crisp set $A = \{A_1, A_2, A_3\}$ can be identified as an ordered triple $\{A_1, A_2, A_3\}$ where $A_1, A_2, A_3$ are subsets on $X$, and one can define several relations and operations between NCSs.

Types of NCSs $\phi_N$ and $X_N$ [20] in $X$ as follows:
1- $\phi_N$ may be defined in many ways as a NCS, as follows
1. $\phi_N = \langle \phi, \phi, X \rangle$ or
2. $\phi_N = \langle \phi, X, X \rangle$ or
3. $\phi_N = \langle \phi, X, \phi \rangle$ or
4. $\phi_N = \langle \phi, \phi, \phi \rangle$

2- $X_N$ may be defined in many ways as a NCS, as follows
1. $X_N = \langle X, \phi, \phi \rangle$ or
2. $X_N = \langle X, X, \phi \rangle$ or
3. $X_N = \langle X, X, X \rangle$ or

Definition 3 [20] Let $X$ is a non-empty set, and the NCSs $A$ and $B$ in the form $A = \{A_1, A_2, A_3\}$, $B = \{B_1, B_2, B_3\}$, then we may consider two possible definition for subsets $A \subseteq B$, may defined in two ways:
1. $A \subseteq B \iff A_1 \subseteq B_1$, $A_2 \subseteq B_2$, and $A_3 \supseteq B_3$ or
2. $A \subseteq B \iff A_1 \subseteq B_1$, $A_2 \supseteq B_2$, and $A_3 \supseteq B_3$

Definition 4 [20] Let $X$ is a non-empty set, and the NCSs $A$ and $B$ in the form $A = \{A_1, A_2, A_3\}$, $B = \{B_1, B_2, B_3\}$. Then
1. $A \cap B$ may be defined in two ways:
   i) $A \cap B = \{A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3\}$
   ii) $A \cap B = \{A_1 \cap B_1, A_2 \cap B_2, A_3 \cap B_3\}$
2. $A \cup B$ may be defined in two ways:
   i) $A \cup B = \{A_1 \cup B_1, A_2 \cup B_2, A_3 \cup B_3\}$
   ii) $A \cup B = \{A_1 \cup B_1, A_2 \cup B_2, A_3 \cup B_3\}$
3. $\langle A \rangle = \{A_1, A_2, A_3\}$
4. $<> A = \{A_1^c, A_2, A_3\}$

**Definition 5** [20] A neutrosophic crisp topology (NCT) on a non-empty set $X$ is a family $\Gamma$ of neutrosophic crisp subsets in $X$ satisfying the following axioms:

1. $\phi_N, X_N \in \Gamma$.
2. $A_1 \cap A_2 \in \Gamma$, for any $A_1$ and $A_2 \in \Gamma$.
3. $\cup A_j \in \Gamma$, $\forall \{A_j : j \in J\} \subseteq \Gamma$.

In this case the pair $(X, \Gamma)$ is said to be a neutrosophic crisp topological space (NCTS) in $X$. The elements in $\Gamma$ are said to be neutrosophic crisp open sets (NCONS) in $X$. A neutrosophic crisp set $F$ is closed (NCCS) if and only if its complement $F^c$ is an open neutrosophic crisp set.

**Remark 6** [20] Neutrosophic crisp topological spaces are very natural generalizations of topological spaces and intuitionistic topological spaces, and they allow more general functions to be members of topology:

\[ TS \Rightarrow ITS \Rightarrow NCTS \]

**Definition 7** [20] Let $(X, \Gamma)$ be NCTS and $A = \{A_1, A_2, A_3\}$ be a NCS in $X$. Then the neutrosophic crisp closure of $A$ (NCcl$(A)$ for short) and neutrosophic crisp interior (NCint$(A)$ for short) of $A$ are defined by

\[ NCcl(A) = \cap \{K : \text{is a NCCS in } X \text{ and } A \subseteq K\} \]
\[ NCint(A) = \cup \{G : G \text{ is a NCOS in } X \text{ and } G \subseteq A\}, \]

where $X$ is a neutrosophic crisp open set, and NCOS is a neutrosophic crisp open set. Note that for any NCS in $(X, \Gamma)$, we have

1. $NCcl(A) \cap A = NCcl(A)$,
2. $NCint(A) \cap A = NCint(A)$

It can be also shown that NCcl$(A)$ is NCCS (neutrosophic crisp closed set) and NCint$(A)$ is a NCOS in $X$.

1. $A$ is in $X$ if and only if $NCcl(A) \supseteq A$.
2. $A$ is a NCOS in $X$ if and only if $NCint(A) = A$.

**Definition 8** [20] Let $(X, \Gamma)$ be a NCTS and $A, B$ be a NCS in $X$, then the following properties hold:

1. $NCint(A) \subseteq A$,
2. $A \subseteq NCcl(A)$.
3. $A \subseteq B \Rightarrow NCint(A) \subseteq NCint(B)$.
4. $A \subseteq B \Rightarrow NCcl(A) \subseteq NCcl(B)$.
5. $NCint(A \cap B) = NCint(A) \cap NCint(B)$.
6. $NCint(A \cup B) = NCint(A) \cup NCint(B)$.
7. $NCint(X_N) = X_N$, $NCcl(\phi_N) = \phi_N$

**Definition 9** [21] Let $(X, \Gamma)$ be a NCTS and $A = \{A_1, A_2, A_3\}$ be a NCS in $X$, then $A$ is said to be

1. Neutrosophic crisp $\alpha$-open set (NCaOS) iff $A \subseteq NCint(NCcl(NCint(A)))$.
2. Neutrosophic crisp semi-open set (NCSOS) iff $A \subseteq NCcl(NCint(A))$.
3. Neutrosophic crisp pre-open set (NCPOS) iff $A \subseteq NCint(NCcl(A))$.

The class of all neutrosophic crisp $\alpha$-open sets NCT$^{\alpha}$ which is finer than NCT, the class of all neutrosophic crisp semi-open sets $NCT^{\sigma}$, and the class of all neutrosophic crisp pre-open sets $NCT^{p}$.

**Definition 10** [20] Let $(X, \Gamma)$ be NCTS and $A = \{A_1, A_2, A_3\}$ be a NCS in $X$. Then the $\alpha$-neutrosophic crisp closure of $A$ (NCccl$(A)$ for short) and $\alpha$-neutrosophic crisp interior (NCcl$(A)$ for short) of $A$ are defined by

\[ \alpha NCcl(A) = \cap \{K : \text{is an NCaCS in } X \text{ and } A \subseteq K\} \]
\[ \alpha NCint(A) = \cup \{G : G \text{ is an NCaOS in } X \text{ and } G \subseteq A\} \]

**Proposition 11** [20] Let $(X, \Gamma)$ be NCTS and $A, B$ be two neutrosophic crisp sets in $X$. Then the following properties hold:

1. $NCint(A) \subseteq A$,
2. $A \subseteq NCcl(A)$,
3. $A \subseteq B \Rightarrow NCint(A) \subseteq NCint(B)$,
4. $A \subseteq B \Rightarrow NCcl(A) \subseteq NCcl(B)$,
5. $NCint(A \cap B) = NCint(A) \cap NCint(B)$,
6. $NCcl(A \cup B) = NCcl(A) \cup NCcl(B)$,
7. $NCint(X_N) = X_N$,
8. $NCcl(\phi_N) = \phi_N$

**Example 12** Let $X = \{a, b, c, d\}$, $\phi_N, X_N$ be any types of the universal and empty subsets, and $A, B$ two neutrosophic crisp subsets on $X$ defined by $A = \{\{a\}, \{b, d\}, \{c\}\}$, $B = \{\{a\}, \{b\}, \{c, d\}\}$ then the family $\Gamma = \{\phi_N, X_N, A, B\}$ is a neutrosophic crisp topology on $X$. 

Wadei Al-Omeri, Neutrosophic crisp Sets via Neutrosophic crisp Topological Spaces NCTS
3 Neutrosophic Crisp Open Set

In this section, we will present an equivalent definition to Neutrosophic crisp $\alpha$-open set and prove many special properties of it. Moreover, we will explain the relationship between different classes of neutrosophic crisp open sets by diagram.

**Definition 13** Let $(X, \Gamma)$ be a NCTS and $A = \{A_1, A_2, A_3\}$ be a NCS in $X$, then $A$ is said to be

1. Neutrosophic crisp feebly-open (NCFOS) if there is a Neutrosophic crisp open set $U$ such that $U \subseteq A \subseteq sNCcl(U)$, where $sNCcl(U)$ is denote neutrosophic closure with respect to $NC\Gamma^*$, is defined by the intersection of all Neutrosophic crisp semi closed sets containing $A$.

2. Neutrosophic crisp $\beta$-open set (NC$\beta$OS) iff $A \subseteq NC\Gamma(NC\Gamma(NC\Gamma(A)))$.

3. Neutrosophic crisp semipre-open set (NCSPOs) iff there exists a neutrosophic crisp preopen set $U$ such that $U \subseteq A \subseteq NC\Gamma(U)$.

4. Neutrosophic crisp regular-open set (NCROS) iff $A = NC\Gamma(NC\Gamma(A))$.

5. Neutrosophic crisp semio-open (NCS$\alpha$OS) iff there exists a Neutrosophic crisp $\alpha$-open set $U$ such that $U \subseteq A \subseteq NC\Gamma(U)$.

The class of all neutrosophic crisp feebly-open sets $NC\Gamma^{feebly}$, the calls all neutrosophic crisp $\beta$-open sets $NC\Gamma^{\beta}$, the class of all neutrosophic crisp semi-pre-open sets $NC\Gamma^{sp}$, the class of all neutrosophic crisp regular-open sets $NC\Gamma^{r}$, and the class of all neutrosophic crisp semi-open sets $NC\Gamma^{s}$.

A neutrosophic crisp $A$ is said to be a neutrosophic crisp semi-closed set, neutrosophic crisp $\alpha$-closed set, neutrosophic crisp preclosed set, and neutrosophic crisp regular closed set, Neutrosophic crisp feebly-open, Neutrosophic crisp $\beta$-open set, Neutrosophic crisp semi-pre-open set, Neutrosophic crisp semi-open respectively $(NC\Gamma\alpha CS, NC\Gamma\alpha OS, NC\Gamma\beta CS, NC\Gamma\beta OS, NC\Gamma\alpha OS, NC\Gamma\alpha OS, NC\Gamma\beta OS, NC\Gamma\beta OS, NC\Gamma\alpha OS)$, see the following table.

<table>
<thead>
<tr>
<th>Table of Abbreviations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Abbreviations</strong></td>
</tr>
<tr>
<td>NCFOS</td>
</tr>
<tr>
<td>NC$\beta$OS</td>
</tr>
<tr>
<td>NCSPOs</td>
</tr>
<tr>
<td>NCROS</td>
</tr>
<tr>
<td>NCS$\alpha$OS</td>
</tr>
<tr>
<td>NC$\alpha$OS</td>
</tr>
<tr>
<td>NCSOS</td>
</tr>
<tr>
<td>NCPOS</td>
</tr>
</tbody>
</table>

**Remark 14** From above the following implication and none of these implications is reversible as shown by examples given below

![Diagram](image.png)

**Example 15** Let $X = \{a, b, c, d\}$, $\phi_N, X_N$ be any types of the universal and empty subset, and $A_1 = \{\{a\}, \{b\}, \{c\} \} A_2 = \{\{a\}, \{b, d\}, \{c\}\}$, then the family $\Gamma = \{\phi_N, X_N, A_1, A_2\}$ is a neutrosophic crisp topology on $X$. The NCS $A_1$ & and $A_2$ are neutrosophic crisp open (NCPOS), then its neutrosophic crisp $\alpha$-open sets i.e. $(A \subseteq NC\Gamma(NC\Gamma(NC\Gamma(A))))$ neutrosophic crisp pre-open sets i.e. $(A \subseteq NC\Gamma(NC\Gamma(NC\Gamma(A))))$, neutrosophic crisp semi-open sets i.e. $(A \subseteq NC\Gamma(NC\Gamma(NC\Gamma(A))))$. Also $A_2$ is neutrosophic crisp $\beta$-open sets, hence its Neutrosophic crisp semi-pre-open set.

If $A_3 = \{\{a\}, \{d\}, \{c\}\}$, then its clear $A_3$ is neutrosophic crisp $\alpha$-open set but not neutrosophic crisp open set.

If $A_4 = \{\{a, b\}, \{c\}, \{d\}\}$, then $A_4$ is neutrosophic crisp pre-open set but not neutrosophic crisp regular-open set, and we can see also that $A_4$ is neutrosophic crisp $\beta$-open but not neutrosophic crisp semi-open set.

**Theorem 16** An neutrosophic crisp $A$ in a NCTS $(X, \Gamma)$ is a NC$\alpha$OS if and only if it is both a (NCSOS) and a (NCPOS).

**Proof.** Necessity follows from the diagram given above. Suppose that $A$ is both a (NCSOS) and a (NCPOS). Then
A ⊆ cl(int(A)), and so

\[ \text{cl}(A) \subseteq \text{cl(cl(int(A)))} = \text{cl}(\text{int}(A)). \]

It follows that \( A \subseteq \text{int(cl(A))} \subseteq \text{int(cl(int(A)))} \), so that \( A \) is a \((NCoAS)\). We give condition(s) for a NCS to be a \((NCoAS)\).

**Theorem 17** Let \( A \) be a NCS in a NCTS \((X, \Gamma)\). If \( B \) is a \((NCSOS)\) such that \( B \subseteq A \subseteq \text{int}(B) \), then \( A \) is a \((NCoAS)\).

**Proof:** Since \( B \) is a \((NCSOS)\), we have \( B \subseteq \text{cl}(\text{int}(B)) \). Thus, \( A \subseteq \text{int}(\text{cl}(B)) \subseteq \text{int}(\text{cl}(\text{int}(B))) \subseteq \text{int}(\text{cl}(B)) \). and so \( A \) is a \((NCoAS)\).

**Lemma 18** Any union of \((NCoAS)\) (resp., \((NPOS)\)) is a \((NCoAS)\) (resp., \((NPOS)\)).

**Proof.** The proof is straightforward.

**Definition 19** Let \( (a_1,a_2,a_3) \subseteq X \). A neutrosophic crisp point \((NCP)\) \( p(a_1,a_2,a_3) \) of \( X \) is a NCS of \( X \) defined by \( a_1 \cap a_2 = 0 \), \( a_1 \cap a_3 = 0 \), \( a_2 \cap a_3 = 0 \). Let \( p(a_1,a_2,a_3) \) be a NCP of a NCTS \((X, \Gamma)\). An NCS \( A \) of \( X \) is said to be a neutrosophic crisp neighborhood \((NC\\Gamma)\) of \( p(a_1,a_2,a_3) \) if there exists a NCS \( B \) in \( X \) such that \( p(a_1,a_2,a_3) \in B \subseteq A \).

**Theorem 20** Let \((X, \Gamma)\) be a NCTS. Then a neutrosophic crisp \( A \) of \( X \) is a neutrosophic crisp \( \sigma \)-open (resp., neutrosophic crisp \( \sigma \)-pre-open) if and only if for every \((NCP)\) \( p(a_1,a_2,a_3) \in A \), there exists a \((NCoAS)\) (resp., \((NPOS)\)) \( B_{p(a_1,a_2,a_3)} \) such that \( p(a_1,a_2,a_3) \subseteq B_{p(a_1,a_2,a_3)} \subseteq A \).

**Proof.** If \( A \) is a \((NCoAS)\) (resp., \((NPOS)\)), then we may take \( B_{p(a_1,a_2,a_3)} = A \) for every \( p(a_1,a_2,a_3) \in A \). Conversely assume that for every \((NPOS)\) \( p(a_1,a_2,a_3) \in A \), there exists a \((NCoAS)\) (resp., \((NPOS)\)) \( B_{p(a_1,a_2,a_3)} \) such that \( p(a_1,a_2,a_3) \subseteq B_{p(a_1,a_2,a_3)} \subseteq A \).

Then,

\[ A = \bigcup \{ p(a_1,a_2,a_3) | p(a_1,a_2,a_3) \in A \} \subseteq \bigcup \{ B_{p(a_1,a_2,a_3)} | p(a_1,a_2,a_3) \in A \} \subseteq A \]

**Theorem 21** Let \((X, \Gamma)\) be a NCTS.

1. If \( V \in NC\\Gamma SO\\Gamma(X) \) and \( A \in NC\\Gamma AS(X) \), then \( V \cap A \in NC\\Gamma SO\\Gamma(X) \).

2. If \( V \in NC\\Gamma POS(X) \) and \( A \in NC\\Gamma AS(X) \), then \( V \cap A \in NC\\Gamma POS(X) \).

**Proof.** (1) Let \( V \in NC\\Gamma SO\\Gamma(X) \) and \( A \in NC\\Gamma AS(X) \). Then we obtain,

\[ V \cap A \subseteq NC\\Gamma(\text{int}(V)) \cap NC\\Gamma(\text{int}(A)) \]

\[ \supseteq NC\\Gamma(\text{int}(V) \cap NC\\Gamma(\text{int}(A))) \]

This shows that \( V \cap A \in NC\\Gamma SO\\Gamma(X) \).

(2) Let \( V \in NC\\Gamma POS(X) \) and \( A \in NC\\Gamma AS(X) \). Then we obtain,

\[ V \cap A \subseteq NC\\Gamma(\text{int}(V)) \cap NC\\Gamma(\text{int}(\text{int}(A))) \]

\[ \subseteq NC\\Gamma(\text{int}(V) \cap NC\\Gamma(\text{int}(A))) \]

This shows that \( V \cap A \in NC\\Gamma POS(X) \).

**Theorem 22** Let \( A \) be a subset of a neutrosophic crisp topological space \((X, \Gamma)\). Then the following properties hold:

1. A subset \( A \) of \( X \) is \( NC\\Gamma AS \) if and only if it is \( NC\\Gamma POS \) and \( NC\\Gamma SO\\Gamma \).

2. If \( A \) is \( NC\\Gamma AS \), then \( A \) is \( NC\\Gamma OS \).

3. If \( A \) is \( NC\\Gamma POS \), then \( A \) is \( NC\\Gamma OS \).

**Proof.** (1) Necessity: This is obvious.

Sufficiency: Let \( A \) be \( NC\\Gamma SO\\Gamma \) and \( NC\\Gamma POS \). Then we have

\[ A \subseteq NC\\Gamma(\text{int}(\text{cl}(A))) \]

\[ \subseteq NC\\Gamma(\text{int}(\text{cl}(\text{int}(A)))) \]

This shows that \( A \) is \( NC\\Gamma AS \).

(2) Since \( A \) is \( NC\\Gamma SO\\Gamma \), we have

\[ A \subseteq NC\\Gamma(\text{int}(\text{cl}(A))) \]

\[ \subseteq NC\\Gamma(\text{int}(\text{cl}(\text{int}(A)))) \]

This shows that \( A \) is \( NC\\Gamma OS \).

(3) The proof is obvious.

**Definition 23** Let \((X, \Gamma)\) be \( NC\\Gamma SO\\Gamma \) and \( A = \{A_1, A_2, A_3\} \) be a \( NC\\Gamma SO\\Gamma \) in \( X \). Then the *-neutrosophic crisp closure of \( A \) (\( * - NC\\Gamma cl(A) \) for short) and *-neutrosophic crisp interior (\( * - NC\\Gamma Int(A) \) for short) of \( A \) are defined by

1. \( pNC\\Gamma cl(A) = \cap \{K : \text{is a \( NC\\Gamma PCS \) in \( X \) and \( A \subseteq K \)}\} \)

2. \( pNC\\Gamma Int(A) = \cup \{G : G \text{ is a \( NC\\Gamma POS \) in \( X \) and \( G \subseteq A \)}\} \)

3. \( sNC\\Gamma cl(A) = \cap \{K : \text{is a \( NC\\Gamma CS \) in \( X \) and \( A \subseteq K \)}\} \)

4. \( sNC\\Gamma Int(A) = \cup \{G : G \text{ is a \( NC\\Gamma SO\\Gamma \) in \( X \) and \( G \subseteq A \)}\} \)

5. \( \beta NC\\Gamma cl(A) = \cap \{K : \text{is a \( NC\\Gamma OS \) in \( X \) and \( A \subseteq K \)}\} \)

6. \( \beta NC\\Gamma Int(A) = \cup \{G : G \text{ is a \( NC\\Gamma OS \) in \( X \) and \( G \subseteq A \)}\} \)

Wadei Al-Omeri, Neutrosophic crisp Sets via Neutrosophic crisp Topological Spaces NCTS
7. \( r\text{NCcl}(A) = \cap \{ K : \text{is a NCRCS in } X \text{ and } A \subseteq K \} \),
8. \( r\text{NCint}(A) = \cup \{ G : G \text{ is a NCROS in } X \text{ and } G \subseteq A \} \).

**Theorem 24** For any neutrosophic crisp subset \( A \) of \( NCT_1(X) \).

\( A \) is said to be neutrosophic crisp \( \alpha \)-open set if and only if there exists a neutrosophic crisp open set \( G \) such that \( G \subseteq A \subseteq \text{NCint}(\text{NCcl}(G)) \).

**Proof.** Necessity: If \( A \) be a neutrosophic crisp \( \alpha \)-open set \( \implies A \subseteq \text{NCint}(\text{NCcl}(A)) \). Hence \( G \subseteq A \subseteq \text{NCint}(\text{NCcl}(G)) \), where \( G = \text{NCint}(A) \)

Sufficiency: obvious.

This completes the proof of the theorem.

**Theorem 25** For any neutrosophic crisp subset of \( X \), the following properties are equivalent:

1. \( A \in \text{NCos}(X) \).
2. There exists a neutrosophic crisp open set \( G \) such that \( G \subseteq A \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(G))) \).
3. \( A \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(A))) \)
4. \( \text{NCcl}(A) = \text{NCcl}(\text{NCint}(\text{NCcl}(A))) \)

**Proof.** (1) \( \implies \) (2). Let \( A \in \text{NCos}(X) \), there exists a neutrosophic crisp \( \alpha \)-open set \( U \) in \( X \) such that \( U \subseteq A \subseteq \text{NCcl}(U) \). Hence there exists \( G \) neutrosophic crisp open set such that \( G \subseteq U \subseteq \text{NCint}(\text{NCcl}(G)) \) (by Theorem 24). Therefore \( \text{NCcl}(G) \subseteq \text{NCcl}(U) \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(G))) \). Then \( G \subseteq U \subseteq A \subseteq \text{NCcl}(U) \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(G))) \). Therefore \( G \subseteq A \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(G))) \) for some \( G \) neutrosophic crisp open sets.

(2) \( \implies \) (3). Let there exists a neutrosophic crisp open set \( G \) such that \( G \subseteq A \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(G))) \). Hence \( \text{NCcl}(G) \subseteq \text{NCcl}(\text{NCint}(A)) \), then \( \text{NCint}(\text{NCcl}(G)) \subseteq \text{NCint}(\text{NCcl}(A)) \).

Therefore, \( \text{NCcl}(\text{NCint}(\text{NCcl}(G))) \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(A))) \), then (by hypothesis) \( A \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(A))) \).

(3) \( \implies \) (4). Obvious.

(4) \( \implies \) (1). Let \( \text{NCcl}(A) = \text{NCcl}(\text{NCint}(\text{NCcl}(A))) \). Then \( A \subseteq \text{NCcl}(\text{NCint}(\text{NCcl}(A))) \). To prove \( A \in \text{NCos}(X) \). Since \( \text{NCint}(\text{NCcl}(A)) \subseteq \text{NCint}(\text{NCcl}(A)) \), therefore \( \text{NCcl}(\text{NCint}(\text{NCcl}(A))) \subseteq \text{NCcl}(\text{NCcl}(A)) \). \( \therefore A \subseteq \text{NCcl}(\text{NCcl}(A)) \), use \( U = \text{NCint}(A) \) Hence there exists a neutrosophic crisp open set \( U \) such that \( U \subseteq A \subseteq \text{NCcl}(U) \).

On other hand, \( U \) is neutrosophic crisp \( \alpha \)-open set. Hence \( A \in \text{NCos}(X) \).

**Proposition 26** Let \( (X, \Gamma) \) be a \( NCTS \), then arbitrary union of neutrosophic crisp \( \alpha \)-open set is a neutrosophic crisp \( \alpha \)-open set and arbitrary intersection neutrosophic crisp \( \alpha \)-open set is neutrosophic crisp \( \alpha \)-closed set.

**Proof.** Let \( A = \{ A_i, i \in \Lambda \} \) be a collection of neutrosophic crisp \( \alpha \)-open sets. Then, for each \( i \in \Lambda \), \( A_i \subseteq \text{NCint}(\text{NCcl}(\text{NCint}(A_i))) \). It follows that

\[
\bigcup A_i \subseteq \text{NCint}(\bigcup \text{NCcl}(\text{NCint}(A_i)))
\]

\[
\subseteq \text{NCint}(\text{NCcl}(\bigcup \text{NCint}(A_i)))
\]

\[
= \text{NCint}(\text{NCcl}(\bigcup \text{NCint}(A_i)))
\]

Hence \( \bigcup A_i \) is a neutrosophic crisp \( \alpha \)-open set. The second part follows immediately from the first part by taking complements.

Having shown that arbitrary union of neutrosophic crisp \( \alpha \)-open sets is a neutrosophic crisp \( \alpha \)-open set, it is natural to consider whether or not the intersection of neutrosophic crisp \( \alpha \)-open sets is a neutrosophic crisp \( \alpha \)-open set, and the following example shows that the intersection of neutrosophic crisp \( \alpha \)-open sets is not a neutrosophic crisp \( \alpha \)-open set.

**Example 27** Let \( X = \{ a, b, c, d \} \), \( \phi_N \), \( X_N \) be any types of the universal and empty subset, and \( A_1 = \{ \{a\}, \{b\}, \{c\} \} \), \( A_2 = \{ \{a\}, \{a\}, \{c\} \} \), then the family \( \Gamma = \{ \phi_N, X_N, A_1, A_2 \} \) is a neutrosophic crisp topology on \( X \). Let \( A_3 = \{ \{b\}, \{c\}, \{d\} \} \) The \( NC_1 \) & \( NC_2 \) are neutrosophic crisp open \( (NCOS) \), then its sets neutrosophic crisp \( \alpha \)-open sets i.e \( A \subseteq \text{NCint}(\text{NCcl}(\text{NCint}(A))) \). In fact, \( A_1 \cap A_3 = A_3 \) is a \( NC_1 \) on \( X \) given by \( A_1 \cap A_3 = \{ \phi, \phi, \{d, c\} \} \) or \( A_2 \cap A_3 = \{ \phi, \{a\}, \{d, c\} \} \) and so \( A_2 \cap A_3 \subseteq \text{NCint}(\text{NCcl}(\text{NCint}(A_2 \cap A_3))) \) and hence the intersection is not neutrosophic crisp \( \alpha \)-open set.

**Proposition 28** In a \( NCTS(X, \Gamma) \), a \( NC_A \) is neutrosophic crisp \( \alpha \)-closed if and only if \( A = \alpha\text{NCcl}(A) \).

**Proof.** Assume that \( A \) is a neutrosophic crisp \( \alpha \)-closed set. Obviously, \( A \in \{ B_1 | B_1 \text{ is a neutrosophic crisp } \alpha \text{-closed set and } B \subseteq B_1 \} = \text{NCcl}(A) \).

Conversely, suppose that \( A = \alpha\text{NCcl}(A) \), which shows that \( A \in \{ B_1 | B_1 \text{ is a neutrosophic crisp } \alpha \text{-closed set and } B \subseteq B_1 \} \).

Hence \( A \) is a neutrosophic crisp \( \alpha \)-closed set.

4 Neutrosophic Crisp Continuity

**Definition 29** Let \( (X, \Gamma_1) \) and \( (Y, \Gamma_2) \) be two \( NCTS \) and let \( f : X \to Y \) be a function then \( f \) is said to be

- (1) Continuous [20] if the preimage of each \( NC \) in \( \Gamma_2 \) is a \( NC \) in \( \Gamma_1 \). i.e \( f^{-1}(B) \) is neutrosophic crisp open set in \( X \) for each neutrosophic crisp open set \( B \) in \( Y \) where \( B = \{ B_1, B_2, B_3 \} \), then the preimage of \( B \) under \( f \), denoted by \( f^{-1}(B) \), is neutrosophic crisp open in \( X \) defined by \( f^{-1}(B) = \{ f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3) \} \).
(2) Open [20], iff the image of each NCS in $\Gamma_1$ is a NCS in $\Gamma_2$. i.e, if $A = \{A_1, A_2, A_3\}$ is NCS in $X$, then the image of A under $f$ denoted by $f(A)$ is NCS in $Y$ defined by $f(A) = \{f(A_1), f(A_2), f(A_3)\}$.

**Corollary 30** [20] Let $A = \{A_i, i \in J\}$, be neutrosophic crisp sets in $X$, and $B = \{B_j, j \in K\}$ neutrosophic crisp sets in $Y$, and $: X \rightarrow Y$ be a function. Then

1. $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$, and $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,
2. $f^{-1}(\bigcup B_i) = \bigcup f^{-1}(B_i)$, $f^{-1}(\bigcap B_i) = \bigcap f^{-1}(B_i)$,
3. $f^{-1}(Y_N) = X_N, f^{-1}(\phi_N) = \phi_N$,
4. $A \subseteq B \Rightarrow NCcl(A) \subseteq NCcl(B)$,
5. $A \subseteq f^{-1}(f(A))$, and if $f$ is surjective, then $A \subseteq f^{-1}(f(A))$.

**Definition 31** Let $f : X \rightarrow Y$ be a function from a NCTS $(X, \Gamma_1)$ into a NCTS $(Y, \Gamma_2)$ is said to be

1. neutrosophic crisp $\alpha$-continuous if $f^{-1}(B)$ is a neutrosophic crisp open set $B$ in $X$ for each neutrosophic crisp open set $B$ in $Y$.
2. neutrosophic crisp pre-continuous if $f^{-1}(B)$ is neutrosophic crisp pre-open set in $X$ for each neutrosophic crisp open set in $Y$.
3. neutrosophic crisp semi-continuous if $f^{-1}(B)$ is neutrosophic crisp semi-open set in $X$ for each neutrosophic crisp open set in $Y$.
4. neutrosophic crisp semipre-continuous if $f^{-1}(B)$ is neutrosophic crisp semi-open set in $X$ for each neutrosophic crisp open set in $Y$.
5. neutrosophic crisp $\beta$-continuous if $f^{-1}(B)$ is neutrosophic crisp semi-open set in $X$ for each neutrosophic crisp open set in $Y$.

**Theorem 32** For a mapping $f$ from a NCTS $(X, \Gamma_1)$ to a NCTS $(Y, \Gamma_2)$, the following are equivalent.

1. $f$ is neutrosophic crisp pre-continuous.
2. $f^{-1}(B)$ is a NCPCS in $X$ for every NCCS $B$ in $Y$.
3. $NCcl(NCint(f^{-1}(A))) \subseteq f^{-1}(NCcl(A))$ for every NCS $A$ in $Y$.

**Proof:**

(1) $\Rightarrow$ (2). Let $p(a_{11}, a_{12}, a_{13})$ be a NCP $p(a_{11}, a_{12}, a_{13})$ in $X$ and let $A$ be a NCN of $f(p(a_{11}, a_{12}, a_{13}))$. Then there exists a NCPOS $B$ in $Y$ such that $f(A) = f^{-1}(A)$. Since $f$ is neutrosophic crisp pre-continuous, we know that $f^{-1}(B)$ is a NCPOS in $X$ and

$$p(a_{11}, a_{12}, a_{13}) \subseteq f^{-1}(f(p(a_{11}, a_{12}, a_{13}))) \subseteq f^{-1}(B) \subseteq f^{-1}(A).$$

Thus (2) is valid.

(2) $\Rightarrow$ (3). Let $p(a_{11}, a_{12}, a_{13})$ be a NCP in $X$ and let $A$ be a NCN of $f(p(a_{11}, a_{12}, a_{13}))$. The condition (2) implies that there exists a NCPOS $B$ in $X$ such that $p(a_{11}, a_{12}, a_{13}) \subseteq f^{-1}(A)$ so that $p(a_{11}, a_{12}, a_{13}) \subseteq B$ and $f(B) \subseteq f(f^{-1}(A)) \subseteq A$. Hence (3) is true.

(3) $\Rightarrow$ (1). Let $B$ be a NCOS in $Y$ and let $p(a_{11}, a_{12}, a_{13}) \subseteq f^{-1}(B)$. Then $f(p(a_{11}, a_{12}, a_{13})) \subseteq B$, so and $B$ is a NCN of $f(p(a_{11}, a_{12}, a_{13}))$ since $B$ is a NCOS. It follows from (3) that there exists a NCPOS $B$ in $X$ such that $p(a_{11}, a_{12}, a_{13}) \subseteq A$ and $f(A) \subseteq B$ so that

$$p(a_{11}, a_{12}, a_{13}) \subseteq A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B).$$

Applying Theorem 20 induces that $f^{-1}(B)$ is a NCPOS in $X$. Therefore, $f$ is neutrosophic crisp pre-continuous.
Theorem 34 Let $f$ be a mapping from NCTS $(X, \Gamma_1)$ to NCTS $(Y, \Gamma_2)$ that satisfies

$$NC\text{cl}(NC\text{int}(f^{-1}(NC\text{cl}(B)))) \subseteq f^{-1}(NC\text{cl}(B))$$

for every NCS $B$ in $Y$. Then $f$ is neutrosophic crisp $\alpha$-continuous.

Proof. Let $B$ be a NCOS in $Y$. Then $B$ is a NCCS in $Y$, which implies from hypothesis that

$$NC\text{cl}(NC\text{int}(f^{-1}(NC\text{cl}(B)))) \subseteq f^{-1}(NC\text{cl}(B)) = f^{-1}(B).$$

Its follows

$$NC\text{int}(NC\text{cl}(NC\text{int}(f^{-1}(B))))$$

$$= NC\text{cl}(NC\text{int}(NC\text{int}(f^{-1}(B))))$$

$$= NC\text{cl}(NC\text{int}(f^{-1}(B)))$$

$$= NC\text{cl}(NC\text{int}(f^{-1}(B)))$$

$$\subseteq f^{-1}(B)$$

$$= f^{-1}(B)$$

so that $f^{-1}(B) \subseteq NC\text{int}(NC\text{int}(f^{-1}(B)))$. This shows that $f^{-1}(B)$ is a NCαOS in $X$. Hence, $f$ is neutrosophic crisp $\alpha$-continuous.

Theorem 35 Let $f$ be a mapping from a NCTS $(X, \Gamma_1)$ to a NCTS $(Y, \Gamma_2)$. Then the following assertions are equivalent.

1. $f$ is neutrosophic crisp $\alpha$-continuous.

2. For each NCP $p(a_1, a_2, a_3) \in X$ and every (NCN) $A$ of $f(p(a_1, a_2, a_3))$, there exists a NCαOS $B$ in $X$ such that $p(a_1, a_2, a_3) \in B \subseteq f^{-1}(A)$.

3. For each NCP $p(a_1, a_2, a_3) \in X$ and every (NCN) $A$ of $f(p(a_1, a_2, a_3))$, there exists a NCαOS $B$ in $X$ such that $p(a_1, a_2, a_3) \in B$ and $f(B) \subseteq A$.

Proof. (1) $\Rightarrow$ (2). Let $p(a_1, a_2, a_3)$ be a NCP in $X$ and let $A$ be a NCN of $f(p(a_1, a_2, a_3))$. Then there exists a NCOS $B$ in $Y$ such that $f(p(a_1, a_2, a_3)) \in C \subseteq A$. Since $f$ is neutrosophic crisp $\alpha$-continuous, we know that $f^{-1}(B)$ is a NCαOS in $X$ and

$$p(a_1, a_2, a_3) \in f^{-1}(f(p(a_1, a_2, a_3))) \subseteq f^{-1}(C) = B \subseteq f^{-1}(A).$$

Thus (2) is valid.

(2) $\Rightarrow$ (3). Let $p(a_1, a_2, a_3)$ be a NCP in $X$ and let $A$ be a NCN of $f(p(a_1, a_2, a_3))$. The condition (2) implies that there exists a NCαOS $B$ in $X$ such that $p(a_1, a_2, a_3) \in B \subseteq f^{-1}(A)$, by (2). Thus, we have $p(a_1, a_2, a_3) \in B$ and $f(B) \subseteq f(f^{-1}(A)) \subseteq A$. Hence (3) is true.

(3) $\Rightarrow$ (1) Let $B$ be a NCOS in $Y$ and let $p(a_1, a_2, a_3) \in f^{-1}(B)$. Then $f(p(a_1, a_2, a_3)) \in \text{inf}(f^{-1}(B)) \subseteq B$ and so $B$ is a NCαOS of $f(p(a_1, a_2, a_3))$ since $B$ is a NCαOS. It follows from (3) that there exists a NCαOS $A$ in $X$ such that $p(a_1, a_2, a_3) \in A$ and $f(A) \subseteq B$ so that

$$p(a_1, a_2, a_3) \in A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B).$$

Using Theorem 20 induces that $f^{-1}(B)$ is a NCαOS in $X$, and hence $f$ is neutrosophic crisp $\alpha$-continuous.

Combining Theorems 35, 34, we have the following characterization of a neutrosophic crisp $\alpha$-continuous mapping.

Theorem 36 Let $f$ be a mapping from NCTS $(X, \Gamma_1)$ to NCTS $(Y, \Gamma_2)$. Then the following assertions are equivalent.

1. $f$ is neutrosophic crisp $\alpha$-continuous.

2. If $C$ is a NCCS in $Y$, then $f^{-1}(C)$ is a NCαOS in $X$.

3. $NC\text{cl}(NC\text{int}(f^{-1}(NC\text{cl}(B)))) \subseteq f^{-1}(NC\text{cl}(B))$ for every NCS $B$ in $Y$.

4. For each NCP $p(a_1, a_2, a_3) \in X$ and every (NCN) $A$ of $f(p(a_1, a_2, a_3))$, there exists a NCαOS $B$ in $X$ such that $p(a_1, a_2, a_3) \in B \subseteq f^{-1}(A)$.

5. For each NCP $p(a_1, a_2, a_3) \in X$ and every (NCN) $A$ of $f(p(a_1, a_2, a_3))$, there exists a NCαOS $B$ in $X$ such that $p(a_1, a_2, a_3) \in B$ and $f(B) \subseteq A$.

Some aspects of neutrosophic crisp continuity, neutrosophic crisp $\alpha$-continuity, neutrosophic crisp pre-continuity, neutrosophic crisp semi-continuity, and neutrosophic crisp $\beta$-continuity are studied in this paper and as well as in several papers, see [20]. The relation among these types of neutrosophic crisp continuity is given as follows, where $NC$ means neutrosophic crisp.

![Figure 1: Diagram 2](image-url)

**Figure 1: Diagram 2**

**Remark 37** The reverse implications are not true in the above diagram in general.

**Example 38** Let $(X, \Gamma_0)$ and $(Y, \Psi_0)$ be two NCTS. If $f : X \rightarrow Y$ is continuous in the usual sense, then in this case, $f$ is continuous in the sense of $f(A) = \{f(A_1), f(A_2), f(A_3)\}$. Here we
consider the NCTS on X and Y, respectively, as follows: $\Gamma_1 = \{ (G, \phi, G^c : G \in \Gamma_0) \}$ and $\Gamma_2 = \{ (H, \phi, H^c : H \in \Psi_0) \}$. In this case we have $\langle H, \phi, H^c \rangle \in \Gamma_2$, $H \in \Psi_0$, $f^{-1}(H, \phi, H^c) = (f^{-1}(H), f^{-1}(\phi), f^{-1}(H^c)) \in \Gamma_1$.

Example 39 Let $f$ be a mapping from a NCTS $(X, \Gamma_1)$ to a NCTS $(Y, \Gamma_2)$, and let $X \simeq Y \simeq \{ a, b, c, d \}$. Then $\Gamma_1 = \{ (a), (b), (c) \}$, $\Gamma_2 = \{ (a), (b), (c) \}$. Then $f$ is a neutrosophic crisp continuous function.

Example 40 Let $X = \{ a, b, c, d \}$, $Y = \{ u, v, w \}$ and $A_1 = \{ (a), (b), (c) \}$, $A_2 = \{ (a), (b), (c) \}$. Then $f : (X, \Gamma_1) \rightarrow (Y, \Gamma_2)$ by $f(\{ a \}) = \{ u \}$, $f(\{ b \}) = \{ v \}$, $f(\{ c \}) = \{ w \}$. Then $f$ is a neutrosophic crisp $\alpha$-continuous function.

Example 41 Let $X = \{ a, b, c, d \}$, $Y = \{ u, v, w \}$ and $A_1 = \{ (a), (b), (c) \}$. Then $f : (X, \Gamma_1) \rightarrow (Y, \Gamma_2)$ by $f(\{ a \}) = \{ u \}$, $f(\{ b \}) = \{ v \}$, $f(\{ c \}) = \{ w \}$. Then $f$ is a neutrosophic crisp $\beta$-continuous function.

Theorem 42 Let $f$ be a mapping from NCTS $(X, \Gamma_1)$ to NCTS $(Y, \Gamma_2)$. If $f$ is both neutrosophic crisp pre-continuous and neutrosophic crisp semi-continuous, then it is neutrosophic crisp $\alpha$-continuous.

5 Conclusions and Discussions

In this paper, we have introduced neutrosophic crisp $\beta$-open, Neutrosophic crisp semipre-open, Neutrosophic crisp regular-open, Neutrosophic crisp semia-open sets and studied some of their basic properties. Also we studied the relationship between the newly introduced sets and some of the Neutrosophic crisp open sets that already existed. In this paper, we also introduced Neutrosophic crisp closed sets and studied some of their basic properties. Finally, we introduced the definition of neutrosophic crisp continuous function, and studied some of its basic properties.

References


Received: December 02, 2016. Accepted: December 19, 2016