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A neutrosophic folding on a neutrosophic fundamental group

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Abstract. In this paper, we create a new type of fundamental group called the neutrosophic fundamental group. We obtain some kinds of conditional foldings that are confined to the elements of the neutrosophic fundamental groups. Also, we deduce the limit foldings of a neutrosophic fundamental group. We present the variant and invariant of the neutrosophic fundamental group under the folding of the neutrosophic manifold into itself. We show that the neutrosophic fundamental group at the ending limits of neutrosophic foldings on the n-dimensional neutrosophic manifold into itself is the neutrosophic identity group.

Keywords: Manifold; Neutrosophic folding; Neutrosophic fundamental group.

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1. **Introduction**

In daily natural life, there are many uncertainties. However, standard mathematical logic is inadequate to account for these uncertainties to describe these uncertainties mathematically and to employ them in practice. The theory of the fuzzy set has occupied just about all areas of mathematics waz introduced by Zadeh [1]. The concept of "intuitionistic fuzzy set" was first introduced by Krassimir Atanassov [2]. A neutrosophic controller has been applied to many industrial applications, a neutrosophic controller uses scaling functions of physical variables to cope with uncertainly in process dynamics or the control environment [3]. Robertson proposed the folding of a manifold [4]. Many kinds of foldings and retractions were discussed in [5–9]. The fundamental group of quotient spaces was studied in [10]. Different groups are very significant in algebraic structures since they perform the role of a fundamental in almost all algebraic structures theories. Groups are as well important in plentiful other areas such as combinatorics, biology, physics, chemistry, etc., in order to study the symmetries and other performance among their components. For a continuous map $F : (W,\ w) \to (V,\ v)$ and $\widetilde{F} : \pi_1(W,\ w) \to \pi_1(V,\ v)$ is an induced map gained by using fundamental group functor [11].

One of the standard problems in the transformation of the fundamental groups has been to study the properties of a manifold and to illustrate them whenever possible.

section Preliminaries

Some essential concepts related to single-valued neutrosophic sets and the fundamental groups are shown in this section.

Definition 1.1. Let W be a topological space. Then the homotopy classes of loops at a given point $\mathrm{w}_\circ$ with an operation $[\alpha][\beta] = [\alpha \cdot \beta]$ is called the fundamental group and denoted by $\pi_1(\mathrm{W}, w_\circ)$ [12].

Definition 1.2. Given spaces V and W with chosen points $\mathrm{v}_\circ \in V$ and $w_\circ \in W$, then the wedge sum $V \vee W$ is the quotient of the disjoint union $V \cup W$obtained by identifying $\mathrm{v}_\circ$ and $w_\circ$ to a single point [13].

Definition 1.3. Let $\check{\mathrm{W}}$ be a space of objects, a neutrosophic set $\check{\mathrm{B}}$ in $\check{\mathrm{W}}$ is branded by three functions called truth membership function $\lambda_{\check{\mathrm{B}}}(\mathrm{w})$, indeterminacy membership function $\xi_{\check{\mathrm{B}}}(\mathrm{w})$, and falsity membership function $\sigma_{\check{\mathrm{B}}}(\mathrm{y})$, for which $\lambda_{\check{\mathrm{B}}},\ \xi_{\check{\mathrm{B}}},\ \sigma_{\check{\mathrm{B}}} : \check{\mathrm{W}} \rightarrow\ ]\ ^{-}0,\ 1^{+}[$ and $^{-}0 \leq \lambda_{\check{\mathrm{B}}}(\mathrm{w}) + \xi_{\check{\mathrm{B}}}(\mathrm{w}) + \sigma_{\check{\mathrm{B}}}(\mathrm{w}) \leq 3^{+}$. But in real-life application in scientific and engineering problems it is hard to utilize neutrosophic set on a value of $]\ ^{-}0,\ 1^{+}[$ [3, 14, 15]. We also note that in [16] for (SVNS) all values are taken as the subsets of $[0,\ 1]$. We'll utilize the symbol for convenience's sake $\langle\ \lambda_{\check{\mathrm{B}}},\ \xi_{\check{\mathrm{B}}}, \sigma_{\check{\mathrm{B}}}\ \rangle$ for the neutrosophic set $\check{\mathrm{B}} = \left\{\langle \mathrm{w}, \lambda_{\check{\mathrm{B}}}(\mathrm{w}),\ \xi_{\check{\mathrm{B}}}(\mathrm{w}),\ \sigma_{\check{\mathrm{B}}}(\mathrm{w})\ \rangle : \mathrm{w}\epsilon\check{\mathrm{W}}\right\}$, $\mathcal{F}_{\check{\mathrm{B}}} = \langle\lambda_{\check{\mathrm{B}}},\ \xi_{\check{\mathrm{B}}},\ \sigma_{\check{\mathrm{B}}}\rangle$ and $\mathcal{F}_{\check{\mathrm{B}}}(\mathrm{w}) = \langle\lambda_{\check{\mathrm{B}}}(\mathrm{w}),\ \xi_{\check{\mathrm{B}}}(\mathrm{w}),\ \sigma_{\check{\mathrm{B}}}(\mathrm{w})\rangle$, for which $\lambda_{\check{\mathrm{B}}}(\mathrm{w}),\ \xi_{\check{\mathrm{B}}}(\mathrm{w}),\ \sigma_{\check{\mathrm{B}}}(\mathrm{w}) \in [0,\ 1]$ for all $\mathrm{w} \in \check{\mathrm{W}}$. For a neutrosophic point and simlicity we use $(w,\ \mathcal{F}(\mathrm{w}))$ for $(w,\ \mathcal{F}_{\check{\mathrm{B}}}(\mathrm{w}))$.

2. Main Results

Intending to our study we will create the following definitions.

Definition 2.1. A neutrosophic path in a topological space $\tilde{\mathrm{W}}$ from $(\mathrm{w}_0, \mathcal{F}(\mathrm{w}_0))$ to $(\mathrm{w}_1,\ \mathcal{F}(\mathrm{w}_1))$ is a neutrosophic continuous map $\check{\eta} : \widetilde{[0,\ 1]} \longrightarrow \tilde{\mathrm{W}}$ in which $\check{\eta}(0,\ \mathcal{F}(0)) = (\mathrm{w}_0,\ \mathcal{F}(\mathrm{w}_0))$ and $\check{\eta}(1,\ \mathcal{F}(1)) = (\mathrm{w}_1,\ \mathcal{F}(\mathrm{w}_1))$.

Definition 2.2. A space $\tilde{\mathrm{W}}$ is called neutrosophic arcwise connected if for any two points $(\mathrm{w}_0,\ \mathcal{F}(\mathrm{w}_0))$ and $(\mathrm{w}_1,\ \mathcal{F}(\mathrm{w}_1))$ in $\tilde{\mathrm{W}}$, there exists a neutrosophic path with begin $(\mathrm{w}_0,\ \mathcal{F}(\mathrm{w}_0))$ and end $(\mathrm{w}_1,\ \mathcal{F}(\mathrm{w}_1))$.

Definition 2.3. Two neutrosophic continuous maps $\check{\eta}$, $\check{\xi}$: $\tilde{\mathrm{V}} \longrightarrow \tilde{\mathrm{W}}$ are neutrosophy homotopic written ($\check{\eta} \cong \check{\xi}$) if there exists a neutrosophic continuous map $\check{\phi} : \tilde{\mathrm{V}} \times \widetilde{[0,\ 1]} \rightarrow \tilde{\mathrm{W}}$, for $(\mathrm{v},\ \mathcal{F}(\mathrm{v})) \in \tilde{\mathrm{V}}$,

$\check{\phi}((\mathrm{v},\ \digamma(\mathrm{v}))\,,\ (0,\ \digamma(0))) =\ \check{\eta}(\mathrm{v},\digamma(\mathrm{v})),$
$\check{\phi}((\mathrm{v},\ \digamma(\mathrm{v}))\,,\ (1,\ \digamma(1))) =\ \check{\xi}(\mathrm{v},\ \digamma(\mathrm{v})).$

Definition 2.4. A neutrosophic path $\check{\mathrm{W}}$ is called a neutrosophic loop if $\check{\eta}(0,\ \digamma(0)) = \check{\eta}(1\,,\ \digamma(1))$.

Definition 2.5. Let $\check{\eta}$ is a path from $(\mathrm{w}_0,\ \digamma(\mathrm{w}_0))$ to $(\mathrm{w}_1,\ \digamma(\mathrm{w}_1))$ and let $\check{\xi}$ is a path from $(\mathrm{w}_1,\ \digamma(\mathrm{w}_1))$to $(\mathrm{w}_2,\ \digamma(\mathrm{w}_2))$, then

$$\check{\eta}\cdot\check{\xi}(t,\ \digamma(t)) = \begin{cases} \check{\eta}(2t,\ \digamma(2t)) & (0,\ \digamma(0)) \leq (t,\ \digamma(t)) \leq (\frac{1}{2},\ \digamma(\frac{1}{2})) \\ \check{\xi}(2t-1),\ \digamma(2t-1)) & (\frac{1}{2},\ \digamma(\frac{1}{2})) \leq (t,\ \digamma(t)) \leq (1,\ \digamma(1)\,) \end{cases}$$

Definition 2.6. The neutrosophic fundamental group in neutrosophic space $\check{\mathrm{W}}$ at the neutrosophic base point $\check{\mathrm{b}}$ is the set of neutrosophic homotopy classes of neutrosophic loops with the product operation $[\ \check{\eta}][\check{\xi}] = [\ \check{\eta}\cdot\check{\xi}]$ and denoted as $\overset{\mathrm{ne}}{\pi}(\check{\mathrm{W}},\ \check{\mathrm{b}})$.

Definition 2.7. The neutrosophic group $\tilde{G}$ is a group with neutrosophic elements, i.e. the neutrosophic elements $\tilde{\mathrm{g}}$ can be represented as $\tilde{\mathrm{g}}=(g,\ \Upsilon(g))$**.**

Definition 2.8. Let $\tilde{N}_1$ and $\tilde{N}_2$ be two neutrosophic manifolds of dimension n_1 and n_2 respectively. A neutrosophic map $\Im : \tilde{N}_1 \to \tilde{N}_2$ is called a neutrosophic topological folding iff $\Im\circ\tilde{\delta} : \widetilde{[0,\ 1]} \to \tilde{N}_2$ is an induced piecewise neutrosophic geodesic that does not preserve length as $\tilde{\delta}$, whenever $\tilde{\delta} : \widetilde{[0,\ 1]} \to \tilde{N}_1$ is a piecewise geodesic neutrosophic path **.** For simplicity, we denote the neutrosophic topological folding by neutrosophic folding.

Example 2.9. Let $\tilde{\mathfrak{S}}^n$ be a neutrosophic sphere of dimension n. Then, $\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1,\ \breve{\mathrm{a}}) \approx \tilde{Z}$**,** $\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^n,\ \breve{\mathrm{a}}) = \check{0}$ (neutrosophic identity group) for $n \geq 2$. Also, $\overset{\mathrm{ne}}{\pi}(\tilde{R}^n,\ \breve{\mathrm{a}}) = \check{0}$ for $n \geq 1$.

Lemma 2.10. *Two types of neutrosophic foldings* $\Im_{\mathrm{j}}$:$\tilde{\mathfrak{S}}^1_1 \to \tilde{\mathfrak{S}}^1_2$, $(\mathrm{j} = 1,\ 2)$ *without singularities induce neutrosophic foldings*
$\widehat{\Im}_{\mathrm{j}}$: $\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1_1\,) \to \overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1_2)$ *such that* $\widehat{\Im_{\mathrm{j}}}\left(\overset{\mathrm{ne}}{\pi}(\ \tilde{\mathfrak{S}}^1_1\,)\right) = \overset{\mathrm{ne}}{\pi}\left(\Im_{\mathrm{j}}\left(\tilde{\mathfrak{S}}^1_1\right)\right)$.

Proof. Let $\Im_1\ :\tilde{\mathfrak{S}}^1_1\ \to\ \tilde{\mathfrak{S}}^1_2$ be a neutrosophic folding such that $\Im_1\left(e^{i\theta},\digamma\left(e^{i\theta}\right)\right) = (re^{i\theta},\ \digamma(re^{i\theta}))$, $r > 0$, $\theta \in [0,\ 2\pi)$ then we obtain an induced neutrosophic folding $\widehat{\Im}_1$:$\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1_1\,) \to \overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1_2)$ such that $\widehat{\Im}_1\,[\alpha,\ \digamma(\alpha)] = [r\alpha,\ \digamma(r\alpha)]$, where $\alpha = e^{i(2m\pi\theta)}$, $m \in Z$, and so $\widehat{\Im}_1\left(\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1_1\,)\right) = \overset{\mathrm{ne}}{\pi}\left(\Im_1\left(\tilde{\mathfrak{S}}^1_1\right)\right)$. Also, let
$\Im_2\ :\tilde{\mathfrak{S}}^1_1\ \to\ \tilde{\mathfrak{S}}^1_2$ be a neutrosophic folding such that
$\Im_2\left(e^{i\theta},\ \digamma\left(e^{i\theta}\right)\right) = (re^{i\varphi},\ \digamma\left(re^{i\varphi}\right))$, $\theta,\ \varphi \in [0,\ 2\pi)\ 0 \leq \theta < 2\pi,\ \theta-\ \varphi \in [0,\ 2\pi)$, then we get an induced neutrosophic folding $\widehat{\Im}_2$:$\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1_1\,) \to \overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1_2)$ such that $\widehat{\Im}\left([\alpha,\digamma(\alpha)]\right) = [\beta,\digamma(\beta)]$, where $\alpha = e^{i(2m\pi\theta)},\ \beta = e^{i(2m\pi\varphi)},\ m \in Z$, and so $\widehat{\Im}\left(\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^1_1\,)\right) = \overset{\mathrm{ne}}{\pi}\left(\Im_2\left(\tilde{\mathfrak{S}}^1_1\right)\right)$. □

Now we use some interesting transformations on a neutrosophic manifold to describe the structure of a neutrosophic fundamental group. However, in the following theorem, we will describe many types of neutrosophic foldings on the neutrosophic sphere of dimension 1 and neutrosophic torus.

Theorem 2.11. *There are different types of neutrosophic foldings* $\Im : \tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ *which induced neutrosophic foldings* $\widehat{\Im}: \overset{ne}{\pi}(\tilde{\mathfrak{S}}_1^1) \to \overset{ne}{\pi}(\tilde{\mathfrak{S}}_2^1)$ *such that* $\widehat{\Im}(\overset{ne}{\pi}(\tilde{\mathfrak{S}}_1^1))$ *is either isomorphic to* $\check{0}$, $\tilde{Z}$ *or* $\tilde{G}$, *where* $\tilde{G} = \{(n,\ 0,\ 0,\ 0) : n \in Z\}$.

Proof. (i) If $\Im$: $\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ is a neutrosophic folding by a cut as in Fig.(1.a), then clearly $\widehat{\Im}_{\mathrm{j}}\left(\overset{ne}{\pi}(\ \tilde{\mathfrak{S}}_1^1)\right) = \overset{ne}{\pi}\left(\Im\left(\tilde{\mathfrak{S}}_1^1\right)\right) = \tilde{0}$.

(ii) If $\Im$: $\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ is a neutrosophic folding without singularity of on $\tilde{\mathfrak{S}}_1^1$, then $\Im(\tilde{\mathrm{S}}^1)$ is a neutrosophic manifold which is homeomorphic to $\tilde{\mathfrak{S}}_1^1$ as in Fig.(1.b) , and so $\widehat{\Im}_{\mathrm{j}}\left(\overset{ne}{\pi}(\tilde{\mathfrak{S}}_1^1)\right) = \overset{ne}{\pi}\left(\Im\left(\ \tilde{\mathfrak{S}}_1^1\right)\right) \approx \tilde{Z}$.

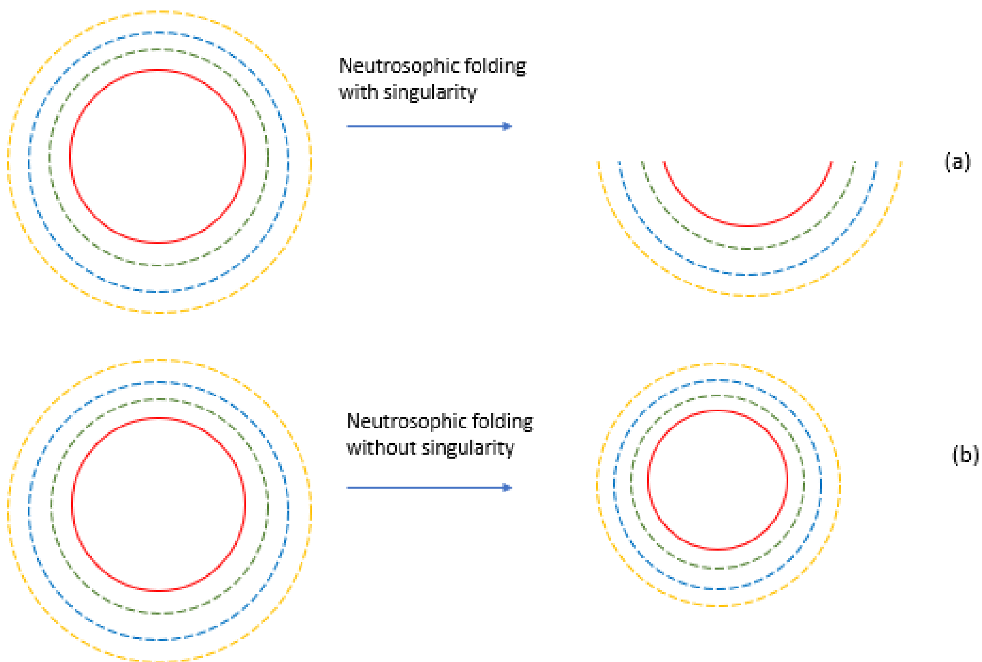

FIGURE 1

(iii) If $\Im$: $\tilde{\mathfrak{S}}_1^1 \to \tilde{\mathfrak{S}}_2^1$ is a neutrosophic folding such that $\Im$(geometry) = geometry and neutrosophic folding by a cut to all other neutrosophy as in Fig.(2) Then there is an induced neutrosophic folding $\Im : \tilde{\pi}_1(\tilde{\mathfrak{S}}_1^1) \to \overset{ne}{\pi}(\tilde{\mathfrak{S}}_2^1)$ for which $\widehat{\Im}_{\mathrm{j}}\left(\overset{ne}{\pi}(\tilde{\mathfrak{S}}_1^1)\right) = \overset{ne}{\pi}\left(\Im\left(\tilde{\mathfrak{S}}_1^1\right)\right) \approx \tilde{G}$, where $\tilde{G} = \{(n,\ 0,\ 0,\ 0) : n \in Z\}$.

□

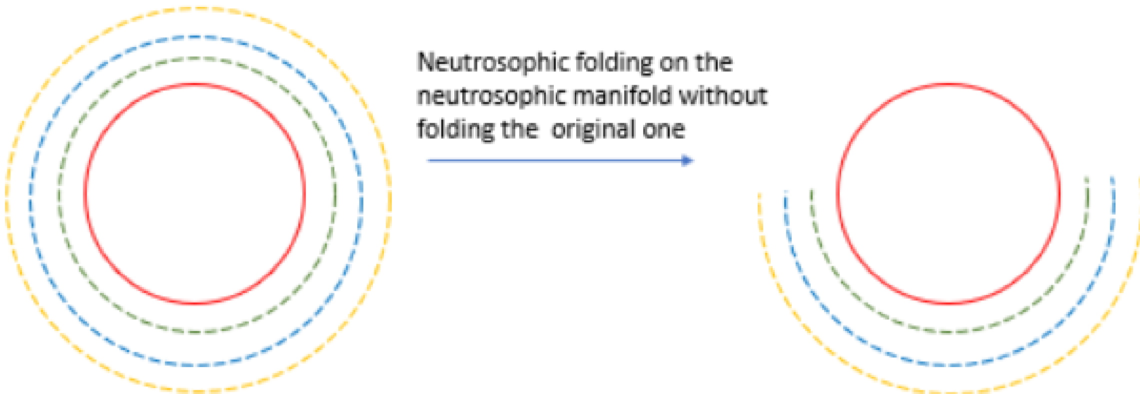

FIGURE 2

Theorem 2.12. *There are different types of neutrosophic foldings* $\Im : \tilde{\mathrm{T}}_1^1 \to \tilde{\mathrm{T}}_2^1$ *which induce neutrosophic foldings* $\widehat{\Im}_{\mathrm{j}}:\tilde{\pi}_1(\tilde{\mathrm{T}}_1^1)\to\tilde{\pi}_1(\tilde{\mathrm{T}}_2^1)$ *such that* $\widehat{\Im}_{\mathrm{j}}\left(\tilde{\pi}_1(\tilde{\mathrm{T}}_1^1)\right) \approx G_1 \times G_2$ *where* G_1 *and* G_2 *are either* $\tilde{0}$, $\tilde{Z}$ *or* $\{(n,\ 0,\ 0,\ 0) : n \in Z\ \}$.

Proof. The proof of this theorem is similar to the proof of theorem 2.11. □

Theorem 2.13. *Let* $\tilde{\mathcal{M}}$ *be the neutrosophic annulus and let* $\tilde{E}^2$ *denote the closed neutrosophic unit ball in* $\tilde{R}^2$. *Then there is a sequence of neutrosophic foldings* $\Im_{\mathrm{m}} : \tilde{\mathcal{M}} \longrightarrow \tilde{E}^2, m = 1, 2, .., k$ *for which* $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty}(\Im_{\mathrm{k}}(\tilde{\mathcal{M}})))=\check{0}$.

Proof. We can define a sequence of neutrosophic foldings as follows:

$$\Im_1 : \tilde{\mathcal{M}}_1 \longrightarrow \tilde{\mathcal{M}}_2, \quad \tilde{\mathcal{M}}_1 \subseteq \tilde{\mathcal{M}}_2 \subseteq \tilde{E}$$
$$\Im_2 : \tilde{\mathcal{M}}_2 \longrightarrow \tilde{\mathcal{M}}_3, \quad \tilde{\mathcal{M}}_2 \subseteq \tilde{\mathcal{M}}_3 \subseteq \tilde{E}$$
$$\vdots \qquad\qquad \vdots$$
$$\Im_k : \tilde{\mathcal{M}}_{k-1} \subseteq \tilde{\mathcal{M}}_k \quad \tilde{\mathcal{M}}_{k-1} \subseteq \tilde{\mathcal{M}}_k \subseteq \tilde{E}^2$$

and so $\lim_{\mathrm{k}\to\infty}(\Im_{\mathrm{k}}(\tilde{\mathcal{M}}))=\tilde{0}$ as in Fig.(3). Hence, $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty}(\Im_{\mathrm{k}}(\tilde{\mathcal{M}}))) = \overset{\mathrm{ne}}{\pi}(\tilde{E}^2)$, thus $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty}(\Im_{\mathrm{k}}(\tilde{\mathcal{M}})))=\check{0}$. □

Lemma 2.14. *The neutrosophic fundamental group of the limit neutrosophic foldings of a neutrosophic manifold which is homeomorphic to* $\tilde{\mathfrak{S}}^n, \mathrm{n} \geq 2$ *is the neutrosophic identity group.*

Proof. The proof follows explicitly from the concept of a neutrosophic folding. □

Theorem 2.15. *Let* $\tilde{\mathrm{D}}_{\mathrm{n}}$ *be the disjoint union of neutrosophic n discs on the neutrosophic sphere* $\tilde{\mathfrak{S}}^2$. *Then there is a sequence of neutrosophic foldings*

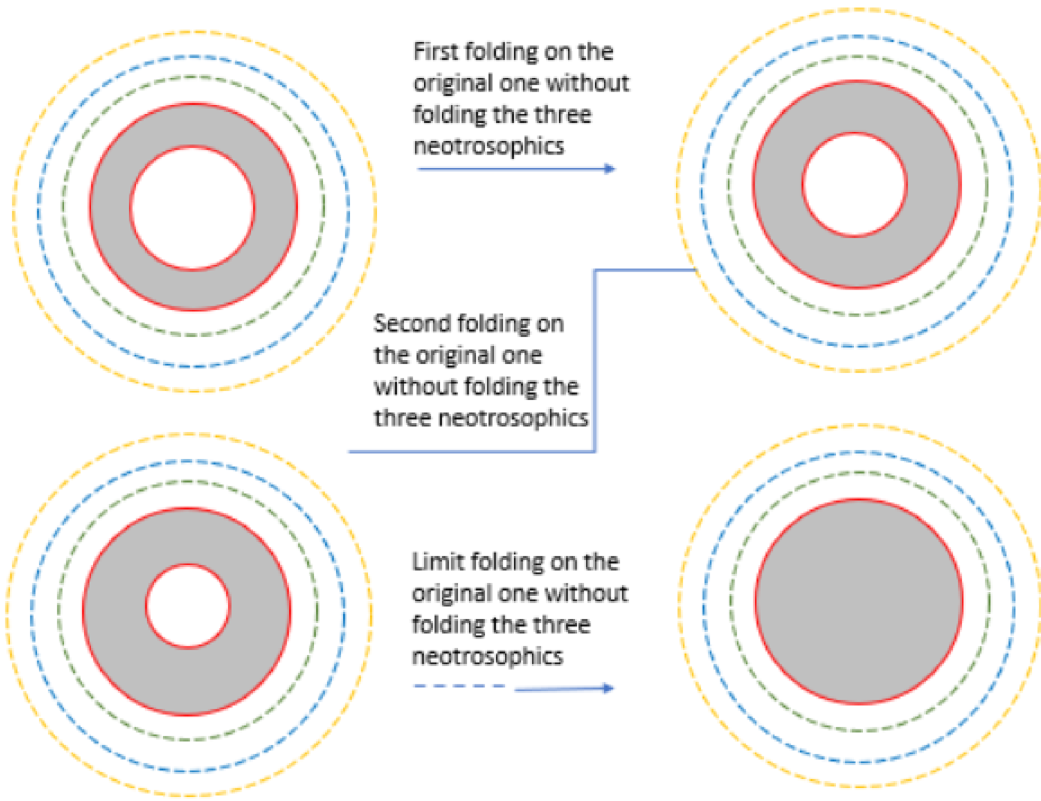

FIGURE 3

$\Im_m$: $\left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{m-1} \to \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_m : m = 1, 2, \ldots k$ *in which* $\overset{\mathrm{ne}}{\pi}(\lim\limits_{\mathrm{k}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}})) = \check{0}$.

Proof. Let $\tilde{\mathrm{D}}_{\mathrm{n}}$ be the disjoint union of neutrosophic n discs on the neutrosophic sphere $\tilde{\mathfrak{S}}^2$. Then, we can define a sequence of neutrosophic foldings as:

$$
\begin{array}{llll}
\Im_1\colon \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_0 & \to & \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_1 \subseteq & \tilde{\mathfrak{S}}^2 \\
\Im_2\colon \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_1 & \to & \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_2 \subseteq & \tilde{\mathfrak{S}}^2 \\
\quad\vdots & \vdots & & \vdots \\
\Im_k\colon \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{k-1} & \to & \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_k \subseteq & \tilde{\mathfrak{S}}^2,
\end{array}
$$

for which $\lim\limits_{\mathrm{k}\to\infty} \Im_{\mathrm{k}} \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{k-1} = \tilde{\mathfrak{S}}^2$ as in Fig.(4) for $n = 2$. Hence, $\overset{\mathrm{ne}}{\pi}(\lim\limits_{\mathrm{k}\to\infty} \Im_{\mathrm{k}} \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{k-1}) = \overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^2)$. Therefore, $\overset{\mathrm{ne}}{\pi}(\Im_{\mathrm{k}} \left(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}\right)_{k-1}) = \tilde{0}$. □

Theorem 2.16. *There is a kind of a neutrosophic folding on* $\tilde{\mathfrak{S}}^2$ *for which* $\overset{\mathrm{ne}}{\pi}(\lim\limits_{\mathrm{m}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2))$ *is either a free neutrosophic group of rank n or a neutrosophic identity group.*

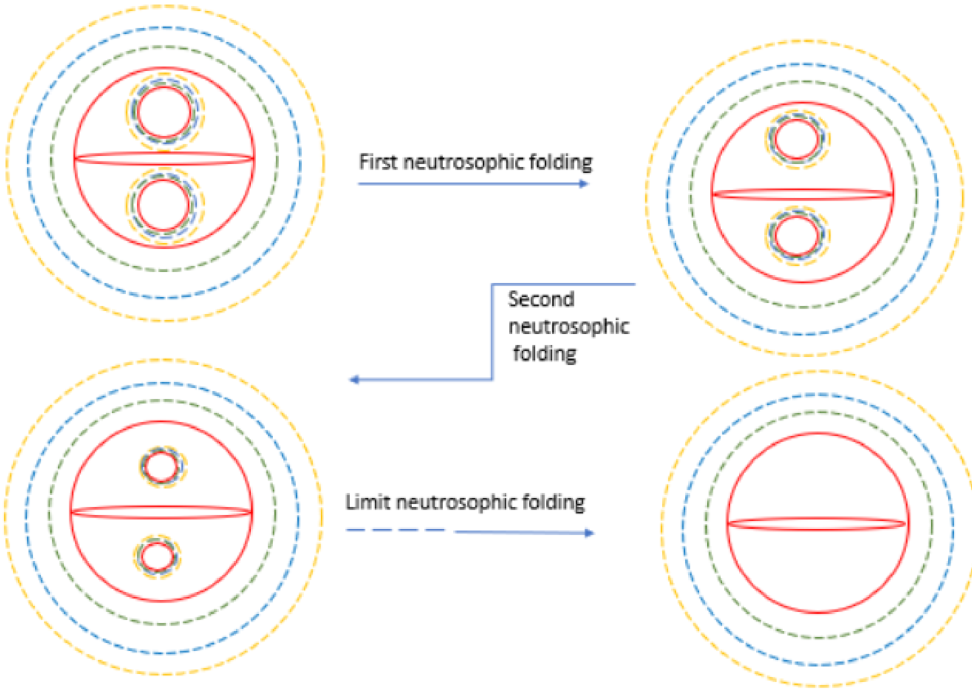

FIGURE 4

Proof. Consider the following chain of neutrosophic folding on $\tilde{\mathfrak{S}}^2$

$\Im_1 : \tilde{\mathfrak{S}}^2 \to \tilde{\mathfrak{S}}_1^2$, radius $(\tilde{\mathfrak{S}}^2) <$ radius $(\tilde{\mathfrak{S}}_1^2)$,

$\Im_2 : \tilde{\mathfrak{S}}_1^2 \to \tilde{\mathfrak{S}}_2^2$, radius $(\tilde{\mathfrak{S}}_1^2) <$ radius $(\tilde{\mathfrak{S}}_2^2)$,

$\vdots \qquad \vdots$

$\Im_{\mathrm{k}} : \tilde{\mathfrak{S}}_{k-1}^2 \to \tilde{\mathfrak{S}}_k^2$, radius $(\tilde{\mathfrak{S}}_{k-1}^2) <$ radius $(\tilde{\mathfrak{S}}_k^2)$,

then we have two cases:

Case (1) If $\lim_{\mathrm{k}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2) = \tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}$, for some n as in Fig. (5) for n = 2 then $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2)) = \overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty} (\Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}})))$. Since $\bigvee_{j=1}^{n} \tilde{\mathfrak{S}}_j$ is a neutrosophic deformation retract of $\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}}$, it follows that $\overset{\mathrm{ne}}{\pi}(\tilde{\mathfrak{S}}^2 - \tilde{\mathrm{D}}_{\mathrm{n}})$ is a free neutrosophic group of rank n. Thus, $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{m}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2))$ is a free neutrosophic group of rank n.

Case(2) If $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2))$ is a neutrosophic sphere of radius ∞, then $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathfrak{S}}^2) = \tilde{0}$. Hence, the proof is complete. □

Remark 2.17. Let $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_1$ be two neutrosophic manifolds of the same dimension and let $\Im : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}_1$ be any neutrosophic folding of $\tilde{\mathcal{M}}$ into $\tilde{\mathcal{M}}_1$. Then, $\overset{\mathrm{ne}}{\pi}(\lim_{\mathrm{k}\to\infty} \Im_{\mathrm{k}}(\tilde{\mathcal{M}}_{k-1}))$ and $\overset{\mathrm{ne}}{\pi}(\Im(\tilde{\mathcal{M}}))$ needed not to be equal.

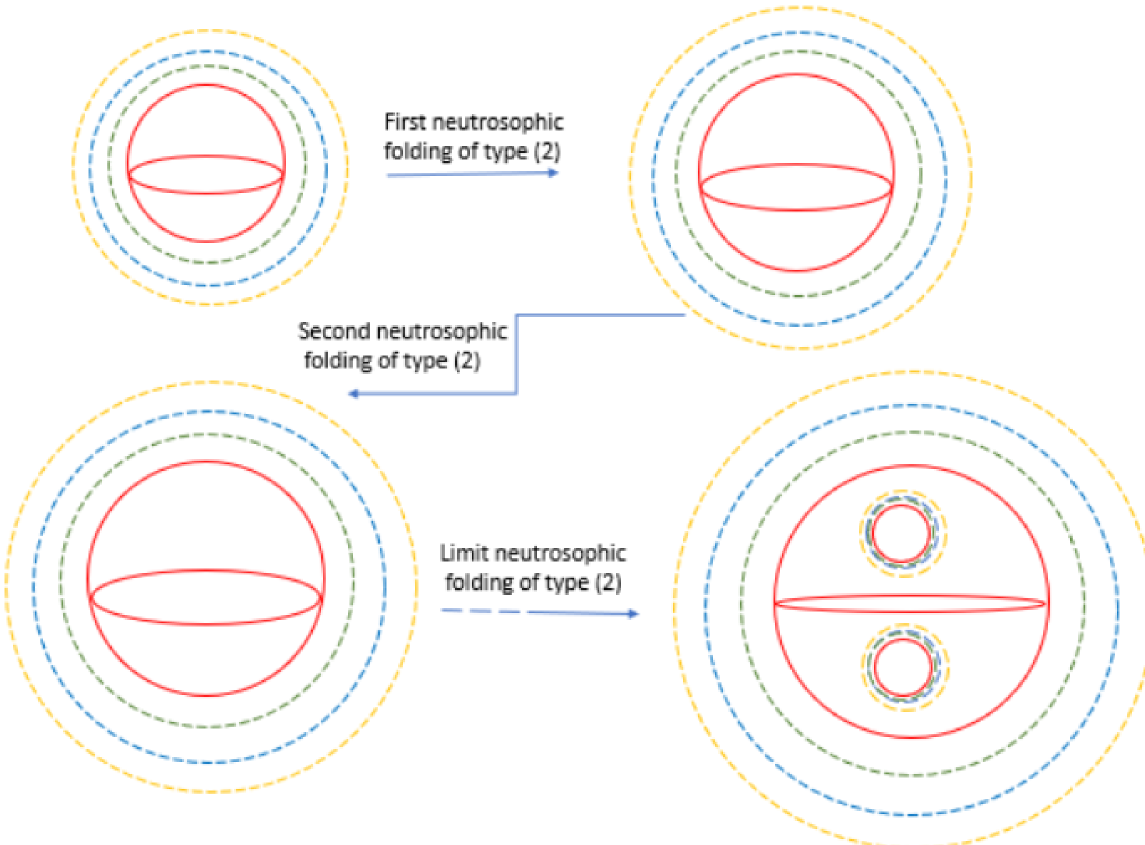

FIGURE 5

Proof. We will show this result by considering the following counter-example, Let $\tilde{\mathcal{M}} = \tilde{\mathfrak{S}}^1$, then we have a chain of neutrosophic folding as in Fig.(6), we have $\lim\limits_{k\to\infty} \Im_k(\tilde{\mathfrak{S}}^1_{k-1}) \approx \tilde{\mathrm{L}}$ (neutrosophic line), and so $\overset{ne}{\pi}(\lim\limits_{k\to\infty} \Im_k(\tilde{\mathfrak{S}}^1_{k-1})) = \overset{ne}{\pi}(\tilde{\mathrm{L}}) = \check{0}$ but $\overset{ne}{\pi}(\Im(\tilde{\mathfrak{S}}^1)) = \overset{ne}{\pi}(\tilde{\mathfrak{S}}^1) \approx \tilde{Z}$. Hence, $\overset{ne}{\pi}(\lim\limits_{k\to\infty} \Im_k(\tilde{\mathcal{M}}_{k-1})) \not\approx \overset{ne}{\pi}(\Im(\tilde{\mathcal{M}}))$. □

Theorem 2.18. *The neutrosophic fundamental group at end limits of neutrosophic foldings of an n-dimensional neutrosophic manifold $\tilde{\mathcal{M}}^n$ into itself is the neutrosophic identity group.*

Proof. Let $\Im_i$ be a neutrosophic folding of an n-dimensional neutrosophic manifold $\tilde{\mathcal{M}}^n$. Then we, get the following chains,

$$\begin{array}{ccccccccc}
\tilde{\mathcal{M}}^n & \xrightarrow{\Im_1^1} & \tilde{\mathcal{M}}_1^n & \xrightarrow{\Im_2^1} & \tilde{\mathcal{M}}_2^n & \rightarrow & \cdots\tilde{\mathcal{M}}_{k-1}^n & \cdots\xrightarrow{\lim_{k\to\infty}\Im_k^1} & \tilde{\mathcal{M}}^{n-1} \\
\tilde{\mathcal{M}}^{n-1} & \xrightarrow{\Im_1^2} & \tilde{\mathcal{M}}_1^{n-1} & \xrightarrow{\Im_2^2} & \tilde{\mathcal{M}}_2^{n-1} & \rightarrow & \cdots\tilde{\mathcal{M}}_{k-1}^{n-1} & \cdots\xrightarrow{\lim_{k\to\infty}\Im_k^2} & \tilde{\mathcal{M}}^{n-2} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\tilde{\mathcal{M}}^1 & \xrightarrow{\Im_1^n} & \tilde{\mathcal{M}}_1^1 & \xrightarrow{\Im_2^n} & \tilde{\mathcal{M}}_2^1 & & \cdots\tilde{\mathcal{M}}_{k-1}^1 & \cdots\xrightarrow{\lim_{k\to\infty}\Im_k^n} & \tilde{\mathcal{M}}^0.
\end{array}$$

As a result, of the last sequence, the ending limits of neutrosophic foldings is a zero-dimensional

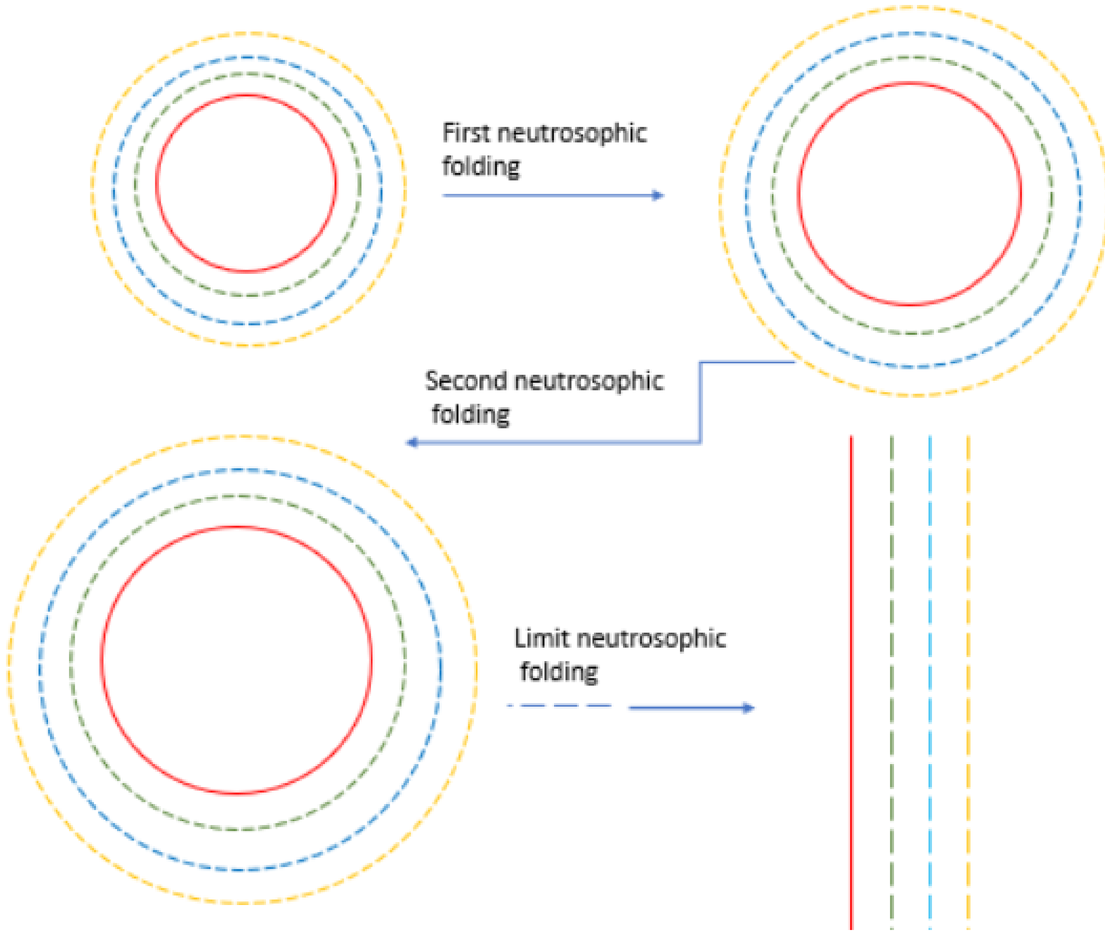

FIGURE 6

neutrosophic manifold, it is a neutrosophic point and the neutrosophic the fundamental group of a neutrosophic point is the neutrosophic identity group. □

3. conclusions

As a result, the neutrosophic fundamental group and foldings map impact on a neutrosophic fundamental group is introduced. We use the transformation to describe the elements of a neutrosophic fundamental group. Under the folding map, many kinds of the isomorphic neutrosophic group are obtained.

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