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Neutrosophic Fuzzy Soft BCK-submodules

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Abstract: The target of this study is to apply the notion of neutrosophic soft sets to the theory of BCK-modules by introducing the notion of neutrosophic fuzzy soft BCK-submodules and deriving their basic properties. Also, (α,β,γ)-soft top of neutrosophic fuzzy soft sets in BCK-modules is presented. The concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules is defined and some results are investigated. Finally, an application of neutrosophic fuzzy soft sets in decision making is investigated and an example demonstrating the successfully application of this method is provided.

Keywords: BCK-algebras; BCK-modules; soft sets, fuzzy soft sets, neutrosophic sets; neutrosophic soft sets; neutrosophic fuzzy soft BCK-submodules.

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1. Introduction

A soft set theory as a new mathematical tool for dealing with uncertainties was proposed by Molodtsov in 1999 [21]. He pointed out several directions for the applications of soft sets. In 2002, Maji et al. [19] described the application of soft set theory to a decision-making problem. They [18] also studied several operations on the theory of soft sets. Few years later, Chen et al. [11] presented a new definition of soft set parametrization reduction and compared this definition to the related concept of attributes reduction in rough set theory. At present, works on the soft set theory are progressing rapidly. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with vagueness is the theory of fuzzy sets developed by Zadeh [34]. Since then it has become a vigorous area of research in different domains such as engineering, medical science, social science, physics, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making and automata theory.

Neutrosophic set theory was introduced by F. Smarandache in 1998 [28]. It is considered as a generalization of the fuzzy set. For the first time V.Kandasamy and F. Smarandache [14] introduced the concept of algebraic structures which has caused a pattern shift in the study of algebraic structures. Maji [17] had combined the neutrosophic sets with soft sets and introduced a new mathematical model neutrosophic soft set. The neutrosophic sets aims to model vagueness and ambiguity in complex system. In recent years, it is applied by many researchers in various fields such as group of decision making [3, 22], Project scheduling [1,2] and image processing [26, 33] etc.

In 1994, the notion of BCK-modules was introduced by H. Abujable, M. Aslam and A. Thaheem as an action of BCK-algebras on abelian group [4]. BCK-modules theory then was developed by Z. perveen, M. Aslam and A. Thaheem [25]. Bakhshi [8] presented the concept of fuzzy BCK-submodules and investigated their properties. Recently, H. Bashir and Z. Zahid applied the theory of soft sets on BCK-modules in [16].

In this paper, the concept of neutrosophic fuzzy soft BCK-submodules of BCK-algebra will be introduced and some related properties will be established. Also, (α,β,γ)-soft top of neutrosophic fuzzy soft sets in BCK-modules will be presented. We will define the concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules and investigate some results. Finally, an application of neutrosophic fuzzy soft sets in decision making is going to be investigated and an example demonstrating the successfully application of this method will be given.

This paper is classified as follows. Section 2 gives a brief introduction of neutrosophic fuzzy set, neutrosophic fuzzy soft set, BCK-algebra and BCK-submodule. The notion of neutrosophic fuzzy soft BCK-submodules and some related results are introduced in section 3. The concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules and some properties are obtained in section 4. Section 5 investigates the application of neutrosophic fuzzy soft set in group decision making problems. Finally, in section 6 conclusion is given.

2. Preliminaries

In this section, some preliminaries from the soft set theory, neutrosophic soft sets, BCK-algebras and BCK-modules are induced.

Definition 2.1.[21] Let U be an initial universe and E be a set of parameters. Let $P\ (U)$ denote the power set of U and let A be a nonempty subset of E. A pair $F_A = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to P\ (U)$ is a set-valued mapping, called the approximate function of the soft set (F, A). It is easy to represent a soft set (F, A) by a set of ordered pairs as follows:

$$(F, A) = \{(x, F\ (x)) : x \in A\}$$

Neutrosophic set is a generalization of the fuzzy set especially of intuitionistic fuzzy set. The intuitionistic fuzzy set has the degree of non-membership as introduced by K. Atanassov [7]. Smarandache in 1998 [28] has introduced the degree of indeterminacy as an independent component and defined the neutrosophic set on three components: truth, indeterminacy and falsity.

Definition 2.2.[28] A neutrosophic set A on the universe of discourse U is defined as $A = \{(x, T_A(x), I_A(x), F_A(x)), x \in U\}$ where $T_A : U \to]-0, 1+[$ is a truth membership function, $I_A : U \to]-0, 1+[$ is an indeterminate membership function, and $F_A : U \to]-0, 1+[$ is a false membership function and $0- \leq T_A(x) + I_A(x) + F_A(x) \leq 3+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or nonstandard subsets of $]-0, 1+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]-0, 1+[$. Hence, we consider the neutrosophic set which takes the value from the subset of $[0, 1]$.

Definition 2.3.[17] Let U be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let $P\ (U)$ denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the neutrosophic soft set (NSS) over U, where F is a mapping given by $F : A \to P\ (U)$.

Definition 2.4.[17] Let (F, A) and (G, B) be two neutrosophic soft sets over the common universe U. (F, A) is said to be neutrosophic soft subset of (G, B) if $A \subset B$, and $T_{F(e)}(x) \leq T_{G(e)}(x), I_{F(e)}(x) \leq I_{G(e)}(x), F_{F(e)}(x) \geq F_{G(e)}(x), \forall e \in A, x \in U$. We denote it by $(F, A) \subseteq (G, B)$.

Definition 2.5. [17] The complement of a neutrosophic soft set (F, A) denoted by $(F, A)^c$ and is defined as $(F, A)^c = (F^c, \sim A)$, where $F^c: \sim A \to P\ (U)$ is a mapping given by

$$F^c(e) = (T_{F^c(e)} = F_{F(e)}, I_{F^c(e)} = I_{F(e)}, F_{F^c(e)} = T_{F(e)}) \quad \text{for all } e \in \sim A$$

Definition 2.6.[12,13] An algebra $(X,*,0)$ of type $(2,0)$ is called BCK-algebra if it is satisfying the following axioms:

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCK-2) $(x * (x * y)) * y = 0$,
(BCK-3) $x * x = 0$,
(BCK-4) $0 * x = 0$,
(BCK-5) $x * y = 0$ and $y * x = 0$ $imply$ $x = y$, for all $x, y, z \in X$.

A partial ordering " $\leq$ " is defined on X by $x \leq y \Leftrightarrow x * y = 0$. A BCK-algebra X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$, for all $x \in X$, commutative if it satisfies the identity $x \wedge y = y \wedge x$, where $x \wedge y = y * (y * x)$, for all $x, y \in X$ and implicative if $x * (y * x) = x$, for all $x, y \in X$.

Definition 2.7.[4] Let X be a BCK-algebra. Then by a left X-module (abbreviated X-module), we mean an abelian group M with an operation $X \times M \rightarrow M$ with $(x,m) \mapsto xm$ satisfies the following axioms for all $x, y \in X$ and $m, n \in M$:

(i) $(x \wedge y)m = x(ym)$,
(ii) $x(m + n) = xm + xn$,
(iii) $0m = 0$.

A subgroup N of a BCK-module M is called submodule of M if N is also a BCK-module.

Definition 2.8.[8] Let X be a BCK-algebra. A subset N of a BCK-module M is a BCK-submodule of M if and only if $n_1 - n_2 \in N$ and $xn \in N$ for all $n, n_1, n_2 \in N$ and $x \in X$.

Moreover, if X is bounded and M satisfies $1m = m$, for all $m \in M$, then M is said to be unitary. A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy set in a BCK-algebra X. For any fuzzy set μ in X and any $t \in [0,1]$, we define set $U(\mu; t) = \mu_t = \{x \in X | \mu(x) \geq t\}$, which is called an upper t-level cut of μ.

Definition 2.9.[8] A fuzzy subset μ of M is said to be a fuzzy BCK-submodule if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold:

(FBCKM1) $\mu(m_1 + m_2) \geq \min\{\mu(m_1), \mu(m_2)\}$,
(FBCKM2) $\mu(-m) = \mu(m)$,
(FBCKM3) $\mu(xm) \geq \mu(m)$.

Definition 2.10.[16] A soft set (F, A) over a BCK-module M is said to be a soft BCK-submodule over M if for all $\varepsilon \in A$, $F(\varepsilon)$ is a BCK-submodule of M.

3. Neutrosophic fuzzy soft BCK-submodules

In this section, we introduce the notion of neutrosophic fuzzy soft BCK-submodules and some related results.

Definition 3.1. A neutrosophic fuzzy soft set (F, A) over a BCK-module M in a BCK-algebra X is said to be a neutrosophic fuzzy soft BCK-submodule over M if for all $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$ the following axioms hold:

(NFSS1) $T_{F(\varepsilon)}(m_1 + m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 + m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 + m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$,

(NFSS2) $T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$, $F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$,

(NFSS3) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m)$, $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$.

Example 3.2. Let $X = \{0, a, b, c, d, 1\}$ be a set with a binary operation $*$ defined in Table 1, then $(X,*,0)$ forms a bounded commutative, non-implicative BCK-algebra (see [20]). $(M,+)$ forms a commutative group defined in Table 2 where $M = \{0, a, c, d\}$ be a subset of X. Consequently, M forms an X-module (see [15]).

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	a	0	b	a	0
c	c	c	c	0	0	0
d	d	c	c	a	0	0
1	1	d	c	b	a	0

Table 1

$+$	0	a	c	d
0	0	a	c	d
a	a	0	d	c
c	c	d	0	a
d	d	c	a	0

Table 2

$\wedge$	0	a	c	d
0	0	0	0	0
a	0	a	0	a
b	0	a	0	a
c	0	0	c	c
d	0	a	c	d
1	0	a	c	d

Table 3

Let $A = \{0, a\}$. Define a neutrosophic fuzzy soft set (F, A) over M as shown in Table 4.

(F, A)	0	a	c	d
$T_{F(0)}$	0.9	0.7	0.8	0.7
$I_{F(0)}$	0.8	0.5	0.6	0.5
$F_{F(0)}$	0.1	0.1	0.1	0.1
$T_{F(a)}$	0.5	0.2	0.3	0.2
$I_{F(a)}$	0.3	0.1	0.3	0.1
$F_{F(a)}$	0.1	0.5	0.4	0.5

Table 4

Consequently, a routine calculations shows that (F, A) forms a neutrosophic fuzzy soft BCK-submodule over M. Note that, Table 3 explains the action of X on M under the operation $xm = x \wedge m$ for all $x \in X$ and $m \in M$.

For the sake of simplicity, we shall use the symbol $NFSS(M)$ for the set of all neutrosophic fuzzy soft BCK-submodules over M.

Theorem 3.3. Let X be a BCK-algebra then a neutrosophic fuzzy soft set $(F, A) \in NFSS(M)$ if and only if

(i) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m), I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m), F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$,

(ii) $T_{F(\varepsilon)}(m_1 - m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

for all $m, m_1, m_2 \in M, x \in X$ and $\varepsilon \in A$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M then by the Definition (3.1)

condition (i) holds.
(ii) $T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}\big(m_1 + (-m_2)\big) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,
Similarly for

$$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$$

and

$$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Conversely suppose (F, A) satisfies the conditions (i), (ii). Then we have by (i)
$T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}((-1)m) \geq T_{F(\varepsilon)}(m)$, and $T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}((-1)(-1)m) \geq T_{F(\varepsilon)}(-m)$.
Thus, $T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}(-m)$. Similarly for $I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$ *and* $F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$.
(ii)$T_{F(\varepsilon)}(m_1 + m_2) = T_{F(\varepsilon)}\big(m_1 - (-m_2)\big) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,
Similarly for

$$I_{F(\varepsilon)}(m_1 + m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$$

and

$$F_{F(\varepsilon)}(m_1 + m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Hence (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Theorem 3.4. A neutrosophic fuzzy soft set (F, A) belongs to $NFSS(M)$ in a BCK-algebra X if and only if for all $m, m_1, m_2 \in M, x, y \in X$ *and* $\varepsilon \in A$ the following statements hold:

(i) $T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m)$, $F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m)$,

(ii) $T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,
$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,
$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

Proof. Let (F, A) be a $NFSS(M)$, by Theorem (3.3) and since $0m = 0$ for all $m \in M$, we have
(i) $T_{F(\varepsilon)}(0) = T_{F(\varepsilon)}(0m) \geq T_{F(\varepsilon)}(m)$.

The same way for $I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m)$.

(ii) $T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(xm_1), T_{F(\varepsilon)}(ym_2)\} \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$.

Similarly for

$$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}.$$

and

$$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Conversely suppose (F, A) satisfies (i), (ii), then

$$T_{F(\varepsilon)}(xm) = T_{F(\varepsilon)}(x(m - 0)) = T_{F(\varepsilon)}(xm - x0) \geq \min\{T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(0)\} = T_{F(\varepsilon)}(m).$$

Similarly for $I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$.
Also,

$$T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(1m_1 - 1m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}.$$

Similarly for

$$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$$

and

$$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$$

Hence by Theorem (3.3), (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Definition 3.5. Let (F, A) be a neutrosophic fuzzy soft set over M. Then (α, β, γ)-soft top of (F, A) is a soft set given by $(H, C_{(\alpha,\beta,\gamma)}(A)) = ((T)_\alpha, (I)_\beta, (F)^\gamma)$ where

$$H(a) = \{ m \in M : T_{F(\varepsilon)}(m) \geq \alpha, I_{F(\varepsilon)}(m) \geq \beta, F_{F(\varepsilon)}(m) \leq \gamma\}$$

for all $\varepsilon \in A, a \in C_{(\alpha,\beta,\gamma)}(A)$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$.

Proposition 3.6. A soft set over BCK-module M is a neutrosophic fuzzy soft BCK-submodule over M if and only if the (α, β, γ)-soft top is either empty or soft BCK-submodule over M for all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$.

Proof. Let (F, A) be a $NFSS(M)$, $(H, C_{(\alpha,\beta,\gamma)}(A))$ is non-empty (α, β, γ)-soft top of (F, A) and X is a BCK-algebra. Let $m, n \in H(a)$ then by Definition (3.5) we have

$$T_{F(\varepsilon)}(m) \geq \alpha, T_{F(\varepsilon)}(n) \geq \alpha \Rightarrow \min\{ T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(n)\} \geq \alpha,$$
$$I_{F(\varepsilon)}(m) \geq \beta, I_{F(\varepsilon)}(n) \geq \beta \Rightarrow \min\{I_{F(\varepsilon)}(m), I_{F(\varepsilon)}(n)\} \geq \beta,$$
$$F_{F(\varepsilon)}(m) \leq \gamma, F_{F(\varepsilon)}(n) \leq \gamma \Rightarrow \max\{ F_{F(\varepsilon)}(m), F_{F(\varepsilon)}(n)\} \leq \gamma.$$

By Theorem (3.3), we have

$$T_{F(\varepsilon)}(m - n) \geq \min\{ T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(n)\} \geq \alpha,$$
$$I_{F(\varepsilon)}(m - n) \geq \min\{I_{F(\varepsilon)}(m), I_{F(\varepsilon)}(n)\} \geq \beta,$$
$$F_{F(\varepsilon)}(m - n) \leq \max\{ F_{F(\varepsilon)}(m), F_{F(\varepsilon)}(n)\} \leq \gamma.$$

Hence $m - n \in H(a)$.
Now let $m \in H(a), x \in X$. Then

$$T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m) \geq \alpha,$$
$$I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) \geq \beta,$$
$$F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m) \leq \gamma.$$

Hence $xm \in H(a)$. Therfore $H(a)$ is a BCK-submodule of M and $(H, C_{(\alpha,\beta,\gamma)}(A))$ is a soft BCK-submodule over M.
Conversely, let $(H, C_{(\alpha,\beta,\gamma)}(A))$ is a soft BCK-submodule over M for all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma \leq 3$. Let$\alpha = \min\{ T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(n)\}, \beta = \min\{ I_{F(\varepsilon)}(m), I_{F(\varepsilon)}(n)\}$ and $\gamma = \max\{ F_{F(\varepsilon)}(m), F_{F(\varepsilon)}(n)\}$ for $m, n \in M$. Then $m, n \in H(a)$. Since $H(a)$is a BCK-submodule of M, therefore $m - n \in H(a)$ which mean

$$T_{F(\varepsilon)}(m - n) \geq \alpha = \min\{ T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(n)\},$$
$$I_{F(\varepsilon)}(m - n) \geq \beta = \min\{I_{F(\varepsilon)}(m), I_{F(\varepsilon)}(n)\},$$
$$F_{F(\varepsilon)}(m - n) \leq \gamma = \max\{ F_{F(\varepsilon)}(m), F_{F(\varepsilon)}(n)\}.$$

Now let $\alpha = T_{F(\varepsilon)}(m)$, $\beta = I_{F(\varepsilon)}(m)$ and $\gamma = F_{F(\varepsilon)}(m)$ then $m \in H(a)$. Since $H(a)$ is a BCK-submodule of M then $xm \in H(a)$ for all $x \in X$ i.e.
$T_{F(\varepsilon)}(xm) \geq \alpha = T_{F(\varepsilon)}(m)$, $I_{F(\varepsilon)}(xm) \geq \beta = I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(xm) \leq \gamma = F_{F(\varepsilon)}(m)$. By Theorem (3.3) we have, (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Definition 3.7. Let (F, A) be a neutrosophic fuzzy soft set over M, then $(\tilde{F}, A_0^1)$ is called soft support of (F, A) if it satisfies

$$\tilde{F}(\delta) = \{ m \in M : T_{F(\varepsilon)}(m) > 0, I_{F(\varepsilon)}(m) > 0, F_{F(\varepsilon)}(m) < 1\}$$

for all $\delta \in A_0^1$ and $m \in M$.

Theorem 3.8. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M, then $(\tilde{F}, A_0^1)$ is a soft BCK-submodule over M.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M in a BCK-algebra X and let $m_1, m_2 \in \tilde{F}(a)$, $a \in A_0^1$ then
$T_{F(\varepsilon)}(m_1 - m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\} > 0$, $I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\} > 0$, and $F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\} < 1$. So, $m_1 - m_2 \in \tilde{F}(a)$.
Now let $m \in \tilde{F}(a), x \in X$, then we have $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m) > 0$, $I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) > 0$, and $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m) < 1$. So, $xm \in \tilde{F}(a)$. Hence $\tilde{F}(a)$ is a BCK-submodule of M. Therefore $(\tilde{F}, A_0^1)$ a soft BCK-submodule over M.

Proposition 3.9 If a neutrosophic fuzzy soft set over M is a neutrosophic fuzzy soft BCK-submodule over M, then the complement of a neutrosophic fuzzy soft set is also neutrosophic fuzzy soft BCK-submodule over M.

Proof. The proof follow from the Theorem (3.3) and Definition (2.5).

Corollary 3.10 Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M if and only if $(F, A)^c$ is a neutrosophic fuzzy soft BCK-submodule over M.

4. Cartesian Product of Neutrosophic Fuzzy Soft BCK-submodules

In this section, we defined the concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules and obtained some properties on it.

Definition 4.1. Let (F, A) and (G, B) be two neutrosophic fuzzy soft BCK-submodules over M. Then the Cartesian product $(F, A) \times (G, B) = (H, C)$ where $C = A \times B$ and $H(\varepsilon, \delta) = F(\varepsilon) \times G(\delta)$ for all $(\varepsilon, \delta) \in A \times B$ defined as $H(\varepsilon, \delta) = (T_{F\times G}(m, n), I_{F\times G}(m, n), F_{F\times G}(m, n))$ where

$$T_{H(\varepsilon,\delta)}(m, n) = T_{F\times G}(m, n) = \min\{T_{F(\varepsilon)}(m), T_{G(\delta)}(n)\},$$
$$I_{H(\varepsilon,\delta)}(m, n) = I_{F\times G}(m, n) = \min\{I_{F(\varepsilon)}(m), I_{G(\delta)}(n)\},$$
$$F_{H(\varepsilon,\delta)}(m, n) = F_{F\times G}(m, n) = \max\{F_{F(\varepsilon)}(m), F_{G(\delta)}(n)\}.$$

For all $m, n \in M$ and $T_H, I_H, F_H: M \times M \to [0,1]$.

Theorem 4.2. Let (F, A) and (G, B) be two neutrosophic fuzzy soft BCK-submodules over M. Then $(F, A) \times (G, B)$ is a neutrosophic fuzzy soft BCK-submodule over $M \times M$.

Proof. Let X be a BCK-algebra and (F, A), (G, B) be a neutrosophic fuzzy soft BCK-submodules over M. Let $m \in M$, then by Definition (4.1) and Theorem (3.4)

$$T_{F\times G}(0,0) = \min\{T_{F(\varepsilon)}(0), T_{G(\delta)}(0)\} \geq \min\{T_{F(\varepsilon)}(m), T_{G(\delta)}(m)\} = T_{F\times G}(m, m),$$

The same for $I_{F\times G}(0,0) \geq I_{F\times G}(m, m)$ and $F_{F\times G}(0,0) \leq F_{F\times G}(m, m)$ for all $(\varepsilon, \delta) \in A \times B$.
Also, for any $(m_1, n_1), (m_2, n_2) \in M \times M$ and $x, y \in X$ we have

$$\begin{aligned} T_{F\times G}(xm_1 - ym_2, xn_1 - yn_2) &= \min\{T_{F(\varepsilon)}(xm_1 - ym_2), T_{G(\delta)}(xn_1 - yn_2)\} \\ &\geq \min\{\min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}, \min\{T_{G(\delta)}(n_1), T_{G(\delta)}(n_2)\}\} \\ &= \min\{\min\{T_{F(\varepsilon)}(m_1), T_{G(\delta)}(n_1)\}, \min\{T_{F(\varepsilon)}(m_2), T_{G(\delta)}(n_2)\}\} \\ &= \min\{T_{F\times G}(m_1, n_1), T_{F\times G}(m_2, n_2)\}. \end{aligned}$$

Similarly for

$$I_{F\times G}(xm_1 - ym_2, xn_1 - yn_2) \geq \min\{I_{F\times G}(m_1, n_1), I_{F\times G}(m_2, n_2)\}$$

and

$$F_{F\times G}(xm_1 - ym_2, xn_1 - yn_2) \leq \max\{F_{F\times G}(m_1, n_1), F_{F\times G}(m_2, n_2)\}$$

Hence $(F,A)\times(G,B)$ is a neutrosophic fuzzy soft BCK-submodule over $M\times M$. The converse of Theorem 4.2 is not true in general as seen in the following example.

Example 4.3. Let $X=\{0,a,b,c\}$ with a binary operation $*$ defined in Table 5, and then $(X,*,0)$ forms a bounded implicative BCK-algebra (see [20]). Let $M=\{0,a\}$ be a subset of X with a binary operation $+$ defined by Table 6. Then M is a commutative group. Define scalar multiplication $(X,M)\to M$ by $xm=x\wedge m$ for all $x\in X$ and $m\in M$ that is given in Table 7. Consequently, M forms an X-module (see [15]).

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 5

$+$	0	a
0	0	a
a	a	0

Table 6

$\wedge$	0	a
0	0	0
a	0	a
b	0	0
c	0	a

Table 7

Let $A=B=M$. Then $C=A\times B=\{(0,0),(0,a),(a,0),(a,a)\}$. Define a neutrosophic fuzzy soft set (H,C) on $M\times M$ as shown in Table 8.

(H,C)	$(0,0)$	$(0,a)$	$(a,0)$	(a,a)
$T_{H(0,0)}$	0.3	0.3	0.2	0.2
$I_{H(0,0)}$	0.7	0.5	0.6	0.5
$F_{H(0,0)}$	0.1	0.5	0.4	0.5
$T_{H(0,a)}$	0.1	0.1	0.1	0.1
$I_{H(0,a)}$	0.1	0.1	0.1	0.1
$F_{H(0,a)}$	0.5	0.6	0.5	0.6
$T_{H(a,0)}$	0.2	0.2	0.2	0.2
$I_{H(a,0)}$	0.1	0.1	0.1	0.1
$F_{H(a,0)}$	0.4	0.5	0.5	0.5
$T_{H(a,a)}$	0.1	0.1	0.1	0.1
$I_{H(a,a)}$	0.1	0.1	0.1	0.1
$F_{H(a,a)}$	0.5	0.6	0.5	0.6

Table 8

Then $(H,C)=(F,A)\times(G,B)$ is a neutrosophic fuzzy soft BCK-submodule over $M\times M$. But if we consider the neutrosophic fuzzy soft sets (F,A) and (G,B) defined as in Table 9 and Table 10.

(F,A)	0	a
$T_{F(0)}$	0.3	0.2
$I_{F(0)}$	0.8	0.6
$F_{F(0)}$	0.1	0.4
$T_{F(a)}$	0.2	0.2
$I_{F(a)}$	0.1	0.1
$F_{F(a)}$	0.4	0.5

Table9

(G,B)	0	a
$T_{G(0)}$	0.3	0.4
$I_{G(0)}$	0.7	0.5
$F_{G(0)}$	0.1	0.5
$T_{G(a)}$	0.1	0.1
$I_{G(a)}$	0.1	0.1
$F_{G(a)}$	0.5	0.6

Table 10

we can observe that (G,B) is not a neutrosophic fuzzy soft BCK-submodule over M since

$$T_{G(0)}\,(0a)=T_{G(0)}\,(0\wedge a)=T_{G(0)}\,(0)=0.3\ngeq T_{G(0)}\,(a)=0.4$$

Proposition.4.4. Let (F,A) and (G,B) be two neutrosophic fuzzy soft BCK-submodules over M. Then the following equalities are satisfied for the (α,β,γ)-soft top:

$$(T_{F\times G})_\alpha = (T_{F(\varepsilon)})_\alpha \times (T_{G(\delta)})_\alpha, (I_{F\times G})_\beta = (I_{F(\varepsilon)})_\beta \times (I_{G(\delta)})_\beta \text{ and } (F_{F\times G})^\gamma = (F_{F(\varepsilon)})^\gamma \times (F_{G(\delta)})^\gamma$$

For all $(\varepsilon, \delta) \in A \times B$.

Proof: Let $(x, y) \in (T_{F\times G})_\alpha$ be arbitrary. So

$$\begin{aligned} T_{F\times G}(x,y) \geq \alpha &\Leftrightarrow \min\{T_{F(\varepsilon)}(x), T_{G(\delta)}(y)\} \geq \alpha \\ &\Leftrightarrow T_{F(\varepsilon)}(x) \geq \alpha, T_{G(\delta)}(y) \geq \alpha \\ &\Leftrightarrow (x,y) \in (T_{F(\varepsilon)})_\alpha \times (T_{G(\delta)})_\alpha. \end{aligned}$$

$(I_{F\times G})_\beta = (I_{F(\varepsilon)})_\beta \times (I_{G(\delta)})_\beta$ is proved in similar way. Now let$(x, y) \in (F_{F\times G})^\gamma$. Then

$$\begin{aligned} F_{F\times G}(x,y) \leq \gamma &\Leftrightarrow \max\{F_{F(\varepsilon)}(x), F_{G(\delta)}(y)\} \leq \gamma \\ &\Leftrightarrow F_{F(\varepsilon)}(x) \leq \gamma, F_{G(\delta)}(y) \leq \gamma \\ &\Leftrightarrow (x,y) \in (F_{F(\varepsilon)})^\gamma \times (F_{G(\delta)})^\gamma. \end{aligned}$$

Hence the equalities $(T_{F\times G})_\alpha = (T_{F(\varepsilon)})_\alpha \times (T_{G(\delta)})_\alpha, (I_{F\times G})_\beta = (I_{F(\varepsilon)})_\beta \times (I_{G(\delta)})_\beta$ and $(F_{F\times G})^\gamma = (F_{F(\varepsilon)})^\gamma \times (F_{G(\delta)})^\gamma$ are satisfied for all $(\varepsilon, \delta) \in A \times B$.

5. The Neutrosophic Fuzzy Soft Set Application in a Decision-Making Problems

In this section we have investigated the application of neutrosophic fuzzy soft set in group decision making problems. Let $U = \{u_1, u_2, \dots, u_n\}$ be a universal set consisting set of alternatives. Let $E = \{e_1, e_2, \dots, e_m\}$ be a set of criteria. We can represent a group decision making problem using the neutrosophic fuzzy soft approach in the following way.

Let (F, A) denotes the corresponding neutrosophic fuzzy soft set in which $F(e_j)$ represents the neutrosophic fuzzy set for the alternative u_i corresponding to the criteria e_j.

Definition.5.1.[17] Let $A = \langle T_A, I_A, F_A\rangle$ be a neutrosophic fuzzy number, and then the score function $S(A)$ is defined as follows

$$S(A) = (T_A + 1 - I_A + 1 - F_A)/3$$

For two neutrosophic fuzzy numbers A and B, if $S(A) > S(B)$ then $A > B$.

Algorithm

Step 1: Input the neutrosophic soft set (F, A).

Step 2: Compute the score function $S(A)$ of a neutrosophic fuzzy number $A = \langle T_A, I_A, F_A\rangle$, based on the truth-membership degree, indeterminacy-membership degree and falsity membership degree by $S(A) = (T_A + 1 - I_A + 1 - F_A)/3$ and the induced fuzzy soft set $\Delta_{\tilde{F}} = (\tilde{F}, A)$.

Step 3: Calculate the average of $\tilde{F}(e_j)$ for each u_i and let it be denoted as a_i , this is the decision table.

Step 4: Select the optimal alternative u_i if $a_i = \max_k(a_k)$.

Step 5: If there are more than one u_i's then any one of u_i may be chosen.

Remark 5.2:

In the case of multicriteria decision making problems, sometimes every criteria e_j associated with the value $w_j \in [0, 1]$ called its weight, which used to represent the different importance of the concerned criteria. In this case there is a small change in the above algorithm. In step 3 instead of average we take weighted average

$$\frac{\sum_{j=1}^{m} \tilde{F}(e_j)w_j}{m}$$

and follows the next steps.

We adopt the following example to illustrate the idea of algorithm given above.

Example 5.3. Suppose that someone wants to invest his money in a stock exchange company. Let $U = \{u_1, u_2, u_3, u_4\}$ the set of alternative companies. Then the four alternatives are evaluated over the set of criteria $E = \{e_1, e_2, e_3, e_4\}$ where e_1=Earnings Per Share, e_2=Dividend, e_3=Book Value and e_4=Price/Earning Ratio. The Evaluation values of the four alternatives on the basis of the above four criteria using the form of neutrosophic fuzzy soft set. The problem is the selection of best company which satisfies the criteria.

Step 1: Neutrosophic fuzzy soft set (F, A) can describe in Table 11.

F	e_1	e_2	e_3	e_4
u_1	⟨0.4,0.2,0.5⟩	⟨0.5,0.3,0.3⟩	⟨0.2,0.7,0.5⟩	⟨0.4,0.6,0.5⟩
u_2	⟨0.3,0.6,0.1⟩	⟨0.2,0.6,0.1⟩	⟨0.4,0.2,0.5⟩	⟨0.2,0.7,0.5⟩
u_3	⟨0.3,0.5,0.2⟩	⟨0.4,0.5,0.2⟩	⟨0.9,0.5,0.7⟩	⟨0.3,0.7,0.6⟩
u_4	⟨0.6,0.7,0.5⟩	⟨0.8,0.4,0.6⟩	⟨0.6,0.3,0.6⟩	⟨0.8,0.3,0.2⟩

Table 11

Step 2: calculate the score of each neutrosophic fuzzy number and obtain the induced fuzzy soft set
$\Delta_{\tilde{F}} = (\tilde{F}, A)$, which is shown in Table 12.

$\tilde{F}$	e_1	e_2	e_3	e_4
u_1	0.57	0.63	0.33	0.43
u_2	0.53	0.5	0.57	0.33
u_3	0.53	0.57	0.57	0.33
u_4	0.47	0.6	0.57	0.77

Table 12

Step 3: Calculate the average of $\tilde{F}(e_j)$ and the decision table for each company u_i obtained in Table 13.

a_i	values
a_1	0.49
a_2	0.4825
a_3	0.5
a_4	0.6025

Table 13

Step 4: Rank all the alternative companies according to the average values a_i $(i = 1, 2, 3, 4)$ as:

$$u_4 \succ u_3 \succ u_1 \succ u_2$$

and thus u_4 is the most desirable alternative.

6. Conclusion

In this paper, we introduced the concept of neutrosophic fuzzy soft BCK-submodules of BCK-algebra and established some related properties. Also, (α, β, γ)-soft top of neutrosophic fuzzy soft sets in BCK-modules was presented. We defined the concept of Cartesian product of neutrosophic fuzzy soft BCK-submodules and investigated some results. Then, we presented an application method for the neutrosophic fuzzy soft set theory in decision making problem. Finally, we provided an example demonstrating the successfully application of this method. The study of neutrosophic fuzzy soft set and their properties have a considerable significance in the sense of applications as well as in understanding the fundamentals of uncertainty. In the future, we shall further develop more algorithms for neutrosophic fuzzy soft set and apply them to solve practical applications in areas such as group decision making, image processing, fusion images and so on.

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