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,

On Neutrosophic Homeomorphisms via Neutrosophic Functions

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Abstract. By using neutrosophic m-alpha closed sets in neutrosophic topological spaces, we introduce the space known as neutrosophic t-alpha space in this paper. We also introduce the mappings referred as neutrosophic m-alpha continuous functions, homeomorphisms, and connectedness, and we research the characterizations and their properties.

Keywords: neutrosophic t-alpha space, neutrosophic m-alpha closed sets, neutrosophic m-alpha continuous functions, neutrosophic m-alpha homeomorphisms and neutrosophic m-alpha connectedness.

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1. Introduction

L. A. Zadeh [14] first put forward the idea of fuzzy sets in 1965, C. L. Chang [5] created fuzzy topological spaces in 1968, built around the idea of fuzzy sets, In 1986 [2] K. Atanassov derived the intuitionistic fuzzy sets, In 1997 [6] D.Coker have introduced the intuitionistic fuzzy topological spaces, F. Smarandache [9] proposed A unified field approach in neutrosophic logic in 1999 and analyzed some of its characteristics, F. Smarandache [10] started researching neutrosophy and neutrosophic logic in 2002. A. A. Salama and S. A. Alblowi [8] examined the neutrosophic set and neutrosophic topological spaces in 2012 and mentioned some of their findings. Broumi Said and Florentin Smarandache [4] proposed the intuitionistic neutrosophic soft set concept and derived some results in 2013. Smarandache, Florentin, Said Broumi, Mamoni Dhar, and Pinaki Majumdar [11] brought new intuitionistic fuzzy soft set results and derived some results in 2014. Wadel Faris Al-omeri and Florentin Smarandache [13] suggested a new neutrosophic sets using neutrosophic topological spaces in Wadel Faris Al-omeri and

Florentin Smarandache's article.

In 2017 [1] I.Arokiarani, R.Dhavaseelan, S. Jafari and M.Parimala have derived some new notions and functions in neutrosophic topological spaces, In 2021 [3] P. Basker, Broumi Said have introduced $N\psi_{\alpha}^{\#0}$ and $N\psi_{\alpha}^{\#1}$-spaces in neutrosophic topological spaces, In 2021 [7] D. Nagarajan, S. Broumi, F. Smarandache, and J. Kavikumar derived the analysis of neutrosophic multiple regression and have given some properties, In 2021 [12] A. Vadivel1, C. John Sundar derived the neutrosophic δ-open maps and neutrosophic δ-closed maps

The abbreviations NS and NTS refer to the neutrosophic set and neutrosophic toplogical spaces, respectively, throughout this study.

2. Preliminaries

We should review and analyze definitions before we begin our study.

Definition 2.1. A NS, A in a NTS is referred to as a neutrosophic set, $N\alpha$-open set ($N\alpha OS$), if A is a subset of $Nint(Ncl(Nint(A)))$. The complement of $N\alpha OS$ is called $N\alpha CS$.

Definition 2.2. (a) Assume N is an NTS and $n \in N$. N_1 is a subset of N is called as $N\alpha$-nbhd of n $\exists$ $N\alpha$-open set N_2 such that $n \in N_2 \subset N_1$.
The collection of all $N\alpha$-nbhd of $n \in N$ is called $N\alpha$-neighbourhood system at n and shall be denoted by $NBH_{N\alpha}(n)$.

(b) Let N be a NTS and N_1 be a subset of N, A subset N_2 of N is supposed to be $N\alpha$-nbhd of N_1 $\exists$ $N\alpha$-open set M such that $N_1 \in M \subseteq N_2$.

(c) Let N_1 be a subset of N. A point $n_1 \in N_1$ is supposed to be $N\alpha$-interior point of N_1, if N_1 is an $NBH_{N\alpha}(n_1)$. The entirety of everything $N\alpha$-interior points of N_1 is referred to as an $N\alpha$-interior of N_1 and is denoted by $NBH_{N\alpha}(n_1)$.

(d) $N\alpha$-interior of N_1 is the union of all $N\alpha OS \subset N_1$ and it is denoted by $INT_N\alpha(N_1)$. $INT_N\alpha(N_1) = \bigcup\{M : M\ is\ N\alpha OS, \ \ M \subseteq N_1\}$.

(e) $N\alpha$-closure of N_1 is the intersection of all $N\alpha CS \supset N_1$ and it is denoted by $CL_N\alpha(N_1)$. $CL_N\alpha(N_1) = \bigcap\{M : M\ is\ a\ N\alpha\text{-}closed\ set\ and\ N_1 \subseteq M\}$.

(f) $\bigcap$ of all $N\alpha$-open subsets of $(N,\ \tau_N)$ containing N_1 is called the $N\alpha$-kernel of N_1 (briefly, $nk_{\#}^{N\alpha}(N_1)$). $nk_{\#}^{N\alpha}(N_1) = \cap\{M \in N\alpha(N,\ \tau_N) : N_1 \subseteq M\}$.

(g) Let $n \in N_1$. Then $N\alpha$-kernel of n is meant to refer to as $nk_{\#}^{N\alpha}(\{n\}) = \cap\{M \in N\alpha(N,\ \tau_N) : n \in M\}$. $CL_N\alpha(N_1) = \bigcap\{M : N_1 \subset M \in N\alpha(N,\ \tau_N)\}$.

3. On $t_{\#}^{N\alpha}$-space via $N\alpha OS$

Definition 3.1. L is NS in a NTS, $N\alpha^{M_\#}CS$ if $Nint(Ncl(L))$ is a subset of Q, only when L is a $\subset$ of Q and Q is $N\alpha OS$. The opponent of $N\alpha^{M_\#}CS$ is called an $N\alpha^{M_\#}OS$.

Example 3.2. Here $N = \{n_1, n_2, n_3\}$ with $\tau_N = \{0_N, 1_N, O_1, O_2\}$ where
$O_1 = \langle(\frac{7}{10}, \frac{7}{10}, \frac{5}{10}), (\frac{3}{10}, \frac{8}{10}, 1), (1, \frac{8}{10}, \frac{6}{10})\rangle$,
$O_2 = \langle(\frac{2}{10}, \frac{5}{10}, \frac{9}{10}), (\frac{3}{10}, \frac{7}{10}, 1), (\frac{7}{10}, \frac{6}{10}, 1)\rangle$,
$O_3 = \langle(\frac{3}{10}, \frac{3}{10}, \frac{5}{10}), (\frac{7}{10}, \frac{2}{10}, 0), (0, \frac{2}{10}, \frac{4}{10})\rangle$,
$O_4 = \langle(\frac{8}{10}, \frac{5}{10}, \frac{1}{10}), (\frac{7}{10}, \frac{3}{10}, 0), (\frac{3}{10}, \frac{4}{10}, 0)\rangle$,
$O_5 = \langle(\frac{4}{10}, \frac{5}{10}, 1), (\frac{2}{10}, \frac{3}{10}, 1), (\frac{5}{10}, \frac{3}{10}, 1)\rangle$. Here the sets $O_3,\ O_4\ and\ O_5$ are the $N\alpha^{M_\#}CS$.

Definition 3.3. A NTS is neutrosophic in nature which is $t_{\#}^{N\alpha}$-space if every $N\alpha^{M_\#}CS$ is CS.

Theorem 3.4. *For a TS that is neutrosophic $(N,\ \tau_N)$ The criteria listed below are equivalent.*

(a) $(N,\ \tau_N)$ *is* $t_{\#}^{N\alpha}$*-space.*

(b) *Every singleton* $\{n_1\}$ *is either* $N\alpha CS$ *(or)* $NclNopen$.

Proof. $(a) \Rightarrow (b)$ Let $n_1 \in N$. Suppose $\{n_1\}$ is not an $N\alpha CS$ of $(N,\ \tau_N)$. Then $N - \{n_1\}$ is not an $N\alpha OS$. Thus $N - \{n_1\}$ is an $N\alpha CS$ of $(N,\ \tau_N)$. Since $(N,\ \tau_N)$ is a $t_{\#}^{N\alpha}$-space, $N - \{n_1\}$ is a $N\alpha CS$ of $(N,\ \tau_N)$, i.e., $\{n_1\}$ is $N\alpha OS$ of $(N,\ \tau_N)$.

$(b) \Rightarrow (a)$ Let N_1 be an $N\alpha^{M_\#}CS$ of $(N,\ \tau_N)$. Let $n_1 \in Nint(Ncl(N_1))$ by (b), $\{n_1\}$ is either $N\alpha CS$ (or) $NclNopen$.

$Case(i)$: Let $\{n_1\}$ be an $N\alpha CS$. If we take the presumption that $n_1 \notin N_1$, we would now have $n_1 \in Nint(Ncl(N_1)) - N_1$ which isn't possible. Hence $n_1 \in N_1$.

$Case(ii)$: Let $\{n_1\}$ be a $NclNopen$. Since $n_1 \in Nint(Ncl(N_1))$, then $\{n_1\} \bigcap N_1 \neq \phi_N$. This demonstrates that $n_1 \in N_1$. As a result, in both circumstances, we have $Nint(Ncl(A)) \subseteq N_1$.

Trivially $N_1 \subseteq Nint(Ncl(N_1))$. Therefore $N_1 = Nint(Ncl(N_1))$ (or) equivalently N_1 is $NclNopen$. Hence $(N,\ \tau_N)$ is a $t_{\#}^{N\alpha}$-space.

□

Definition 3.5. A function $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is called

(a) an $N\alpha^{M\#}$-continuous if $D^{-1}(Y)$ is $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$ for every closed set Y of $(N^{II},\ \tau_N^{ii})$.

(b) an $N\alpha^{M\#}$-irresolute if $D^{-1}(Y)$ is $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$ for every $N\alpha^{M\#}CS$ Y of $(N^{II},\ \tau_N^{ii})$.

Example 3.6. Let $N = \{n_1, n_2, n_3\}$ with $\tau_N = \{0_N, 1_N, \eta_1^{\#}, \eta_2^{\#}, \eta_3^{\#}, \eta_4^{\#}\}$ and $\delta_N = \{0_N, 1_N, \eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*\}$ where
$\eta_1^{\#} = \langle(\frac{4}{10}, \frac{4}{10}, \frac{6}{10}), (\frac{5}{10}, \frac{4}{10}, \frac{6}{10}), (\frac{5}{10}, \frac{8}{10}, \frac{7}{10})\rangle$,
$\eta_2^{\#} = \langle(\frac{5}{10}, \frac{7}{10}, \frac{7}{10}), (\frac{6}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{4}{10})\rangle$,
$\eta_3^{\#} = \langle(\frac{5}{10}, \frac{7}{10}, \frac{7}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{4}{10})\rangle$,
$\eta_4^{\#} = \langle(\frac{4}{10}, \frac{4}{10}, \frac{3}{10}), (\frac{6}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{5}{10}, \frac{7}{10}, \frac{7}{10})\rangle$,
$\eta_1^* = \langle(\frac{5}{10}, \frac{7}{10}, \frac{6}{10}), (\frac{4}{10}, \frac{5}{10}, \frac{4}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{6}{10})\rangle$,
$\eta_2^* = \langle(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{4}{10}, \frac{4}{10}, \frac{3}{10}), (\frac{5}{10}, \frac{7}{10}, \frac{6}{10})\rangle$,
$\eta_3^* = \langle(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{4}{10}, \frac{5}{10}, \frac{4}{10}), (\frac{5}{10}, \frac{7}{10}, \frac{6}{10})\rangle$,
$\eta_4^* = \langle(\frac{5}{10}, \frac{7}{10}, \frac{6}{10}), (\frac{4}{10}, \frac{4}{10}, \frac{3}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{6}{10})\rangle$. Thus, $(N,\ \tau_N)$ and $(N,\ \delta_N)$ are Nutrosophic Topologies. Define $\Lambda : (N,\ \tau_N) \longrightarrow (N,\ \delta_N)$ as $\Lambda(n_1) = n_1$, $\Lambda(n_2) = n_3$, $\Lambda(n_3) = n_2$. Then Λ is $N\alpha^{M\#}$-continuous, since $\Lambda^{-1}(L_{\#})$ is $N\alpha^{M\#}CS$ in $(N,\ \tau_N)$ for every closed set $L_{\#}$ of $(N,\ \delta_N)$ where $L_{\#} = \langle(\frac{3}{10}, \frac{5}{10}, \frac{5}{10}), (\frac{4}{10}, \frac{4}{10}, \frac{4}{10}), (\frac{5}{10}, \frac{5}{10}, \frac{6}{10})\rangle$.

Proposition 3.7. *If $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-continuous function and $(N^I,\ \tau_N^i)$ be a $t_{\#}^{N\alpha}$-space, D is continuous.*

Proof. Assume Y to be closed in $(N^{II},\ \tau_N^{ii})$. As such D is an $N\alpha^{M\#}$-continuous function, $D^{-1}(Y)$ is an $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$. Since $(N^I,\ \tau_N^i)$ is a $t_{\#}^{N\alpha}$-space, $D^{-1}(Y)$ is closed set in $(N^I,\ \tau_N^i)$. Hence D is continuous.

□

Remark 3.8. Let $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be a mapping and $(N^I,\ \tau_N^i)$ be a $t_{\#}^{N\alpha}$-space, then D is continuous if one of the following conditions is satisfied.

(a) f is $N\alpha^{M\#}$-continuous.

(b) f is $N\alpha^{M\#}$-irresolute.

Theorem 3.9. *A map $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is an $N\alpha^{M\#}$-continuous function $\Longleftrightarrow$ every open set's inverted image in $(N^{II},\ \tau_N^{ii})$ are the $N\alpha^{M\#}OS$ in the $(N^I,\ \tau_N^i)$.*

Proof. $Necessity$: Assume $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-continuous function and Z be a collection that is open in $(N^{II},\ \tau_N^{ii})$, $N^{II} - Z$ is closed $(N^{II},\ \tau_N^{ii})$. As such D is an $N\alpha^{M\#}$-continuous function, $f^{-1}(N^{II} - Z) = N^I - D^{-1}(Z)$ is an $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$ and hence $D^{-1}(Z)$ is an $N\alpha^{M\#}OS$ in $(N^I,\ \tau_N^i)$.

$Sufficiency$: Assume that $D^{-1}(Y)$ is an $N\alpha^{M\#}OS$ in $(N^I,\ \tau_N^i)$ for each open set N^{II} in $(N^{II},\ \tau_N^{ii})$. Assume Y is a closed set in $(N^{II},\ \tau_N^{ii})$, $N^{II} - Y$ is a set that is open in $(N^{II},\ \tau_N^{ii})$. By assumption, $D^{-1}(N^{II} - Y) = N^I - D^{-1}(Y)$ is an $N\alpha^{M\#}OS$ in $(N^I,\ \tau_N^i)$, which implies that $D^{-1}(Y)$ is an $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$. Hence D is an $N\alpha^{M\#}$-continuous.

□

Proposition 3.10. *Let $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be any topological space that is neutrosophic $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space. If $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ and $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ are $N\alpha^{M\#}$-continuous functions, then their composition $D_2 \circ D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{III},\ \tau_N^{iii})$ is an $N\alpha^{M\#}$-continuous.*

Proof. Assume Y is a closed set in $(N^{III},\ \tau_N^{iii})$. As such $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ is an $N\alpha^{M\#}$-continuous function, $D_2^{-1}(Y)$ is an $N\alpha^{M\#}CS$ in $(N^{II},\ \tau_N^{ii})$. Since $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space, $D_2^{-1}(Y)$ is a closed set in $(N^{II},\ \tau_N^{ii})$. Since $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is an $N\alpha^{M\#}$-continuous function, $D_1^{-1}(D_2^{-1}(Y)) = (D_2 \circ D_1)^{-1}(Y)$ is an $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$. Hence $D_2 \circ D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{III},\ \tau_N^{iii})$ is an $N\alpha^{M\#}$-continuous function.

□

Definition 3.11. A map $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is said to be

(p) $N\alpha^{M\#}$-closed map if $D(Y)$ is $N\alpha^{M\#}$-closed in $(N^{II},\ \tau_N^{ii})$ for every NCS Y of $(N^I,\ \tau_N^i)$.

(q) $N\alpha^{M\#}$-open map if $D(Y)$ is $N\alpha^{M\#}$-open in $(N^{II},\ \tau_N^{ii})$ for every NOS Y of $(N^{I},\ \tau_N^{i})$.

Theorem 3.12. *Let $D_1 : (N^{I},\ \tau_N^{i}) \longrightarrow (N^{II},\ \tau_N^{ii})$ and $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ be two mappings and $(N^{II},\ \tau_N^{ii})$ be a $t_{\#}^{N\alpha}$-space, then*

(a) $D_2 \circ D_1$ is $N\alpha^{M\#}$-continuous, if D_1 and D_2 are $N\alpha^{M\#}$-continuous.

(b) $D_2 \circ D_1$ is $N\alpha^{M\#}$-closed, if D_1 and D_2 are $N\alpha^{M\#}$-closed.

Proof. (a) Let Y be a NCS of $(N^{III},\ \tau_N^{iii})$, then $D_2^{-1}(Y)$ is $N\alpha^{M\#}$-closed set in $(N^{II},\ \tau_N^{ii})$. Since $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space, then $D_2^{-1}(Y)$ is a NCS in $(N^{II},\ \tau_N^{ii})$. But D_1 is $N\alpha^{M\#}$-continuous, then $(D_2 \circ D_1)^{-1}(Y) = D_1^{-1}(D_2^{-1}(Y))$ is $N\alpha^{M\#}$-closed in $(N^{I},\ \tau_N^{i})$ this implies that $(D_2 \circ D_1)$ is $N\alpha^{M\#}$-continuous mappings.

(b) The proof is similar.

□

Remark 3.13. Let $D : (N^{I},\ \tau_N^{i}) \longrightarrow (N^{II},\ \tau_N^{ii})$ be a mapping from a $t_{\#}^{N\alpha}$-space $(N^{I},\ \tau_N^{i})$ into a space $(N^{II},\ \tau_N^{ii})$, then

(p) D_1 is continuous mapping if, D_1 is $N\alpha^{M\#}$-continuous.

(q) D_1 is closed mapping if, D_1 is $N\alpha^{M\#}$-closed.

Theorem 3.14. *Let $D : (N^{I},\ \tau_N^{i}) \longrightarrow (N^{II},\ \tau_N^{ii})$ is surjective closed and $N\alpha^{M\#}$-irresolute, then $(N^{II},\ \tau_N^{ii})$ $t_{\#}^{N\alpha}$-space if $(N^{I},\ \tau_N^{i})$ is also $t_{\#}^{N\alpha}$-space.*

Proof. Let Y be an $N\alpha^{M\#}$-closed subset of $(N^{II},\ \tau_N^{ii})$. Then $D_1^{-1}(Y)$ is $N\alpha^{M\#}$-closed set in $(N^{I},\ \tau_N^{i})$. Since, $(N^{I},\ \tau_N^{i})$ is a $t_{\#}^{N\alpha}$-space, then $D_1^{-1}(Y)$ is closed set in $(N^{I},\ \tau_N^{i})$. Hence, Y is closed set in $(N^{II},\ \tau_N^{ii})$ and so, $(N^{II},\ \tau_N^{ii})$ is $t_{\#}^{N\alpha}$-space.

□

Proposition 3.15. *If $D_1 : (N^{I},\ \tau_N^{i}) \longrightarrow (N^{II},\ \tau_N^{ii})$ is $N\alpha^{M\#}$-closed, $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ is an $N\alpha^{M\#}$-closed, and $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space, then their composition*

$D_2 \circ D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{III},\ \tau_N^{iii})$ *is* $N\alpha^{M\#}$*-closed.*

Proof. Let N_1 be a NCS of $(N^I,\ \tau_N^i)$. Then by assumption $D_1(N_1)$ is $N\alpha^{M\#}$-closed in $(N^{II},\ \tau_N^{ii})$. Since $(N^{II},\ \tau_N^{ii})$ is a $t_{\#}^{N\alpha}$-space, $D_1(N_1)$ is NCS in $(N^{II},\ \tau_N^{ii})$ and again by assumption $D_2(D_1(N_1))$ is $N\alpha^{M\#}$-closed in $(N^{III},\ \tau_N^{iii})$. i.e., $(D_2 \circ D_1)(N_1)$ is $N\alpha^{M\#}$-closed in $(N^{III},\ \tau_N^{iii})$ and so $D_2 \circ D_1$ is $N\alpha^{M\#}$-closed.

□

Proposition 3.16. *For any bijection* $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ *the following statements are equivalent:*

(p) $D^{-1} : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^I,\ \tau_N^i)$ *is* $N\alpha^{M\#}$*-continuous.*

(q) D *is* $N\alpha^{M\#}$*-open map.*

(r) D *is* $N\alpha^{M\#}$*-closed map.*

Proof. $(p) \Longrightarrow (q)$ Let U be a NOS of $(N^I,\ \tau_N^i)$. By assumption, $(D^{-1})^{-1}(U) = D(U)$ is $N\alpha^{M\#}$-open in $(N^{II},\ \tau_N^{ii})$ and so D is $N\alpha^{M\#}$-open.

$(q) \Longrightarrow (r)$ Let F be a NCS of $(N^I,\ \tau_N^i)$. Then F^c is NOS in $(N^I,\ \tau_N^i)$. By assumption, $D(F^c)$ is $N\alpha^{M\#}$-open in $(N^{II},\ \tau_N^{ii})$. That is $D(F^c) = (D(F))^c$ is $N\alpha^{M\#}$-open in $(N^{II},\ \tau_N^{ii})$ and therefore $D(F)$ is $N\alpha^{M\#}$-closed in $(N^{II},\ \tau_N^{ii})$. Hence D is $N\alpha^{M\#}$-closed.

$(r) \Longrightarrow (p)$ Let F be a NCS of $(N^I,\ \tau_N^i)$. By assumption, $D(F)$ is $N\alpha^{M\#}$-closed in $(N^{II},\ \tau_N^{ii})$. But $D(F) = (D^{-1})^{-1}(F)$ and therefore D^{-1} is $N\alpha^{M\#}$-continuous.

□

4. On $N\alpha^{M\#}$-homeomorphisms

Definition 4.1. A function $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is supposed to be an $N\alpha^{M\#}$-homeomorphism $[(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}]$ if both D and D^{-1} are $N\alpha^{M\#}$-irresolute.

We are using the entire family of all $N\alpha^{M\#}$-homeomorphisms of a NTS $(N^I,\ \tau_N^i)$ onto itself by $N\alpha^{M\#}$-$H(N,\ \tau_N)$.

Example 4.2. Let $M^{N^1} = \{\alpha, \beta\}$, $M^{N^2} = \{\gamma, \delta\}$, $O_1^{\#} = \langle(\frac{2}{10}, \frac{6}{10}, \frac{3}{10}), (\frac{3}{10}, \frac{6}{10}, \frac{4}{10}), (\frac{3}{10}, \frac{7}{10}, \frac{4}{10})\rangle$, $O_2^{\#} = \langle(\frac{4}{10}, \frac{6}{10}, \frac{5}{10}), (\frac{5}{10}, \frac{6}{10}, \frac{6}{10}), (\frac{5}{10}, \frac{7}{10}, \frac{6}{10})\rangle$. Then $\tau_{E1} = \{0_E, 1_E, O_1^{\#}\}$ and $\tau_{E2} = \{0_E, 1_E, O_2^{\#}\}$ are neutrosophic topologies on M^{N^1} and M^{N^2} respectively. Define a bijective mapping $F_{Nf_{\#}} = (M^{N^1},\ \tau_{E1}) \longrightarrow (M^{N^2},\ \tau_{E2})$ by $F_{Nf_{\#}}(\alpha) = \gamma$ and $F_{Nf_{\#}}(\beta) = \delta$. Then $F_{Nf_{\#}}$ is a $N\alpha^{M\#}$-irresolute $F_{Nf_{\#}}^{-1}$ is also a $N\alpha^{M\#}$-irresolute. Therefore the bijection function $F_{Nf_{\#}}$ is a $(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}$.

Proposition 4.3. *Let* $D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ *and* $D_2 : (N^{II},\ \tau_N^{ii}) \longrightarrow (N^{III},\ \tau_N^{iii})$ *are* $(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}$*, then their composition*
$D_2 \circ D_1 : (N^I,\ \tau_N^i) \longrightarrow (N^{III},\ \tau_N^{iii})$ *is also* $(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}$.

Proof. Let J be an $N\alpha^{M\#}OS$ in $(N^{III},\ \tau_N^{iii})$. Since D_2 is $N\alpha^{M\#}$-irresolute, $D_2^{-1}(J)$ is $N\alpha^{M\#}OS$ in $(N^{II},\ \tau_N^{ii})$. Since D_1 is $N\alpha^{M\#}$-irresolute, $D_1^{-1}(D_2^{-1}(Y)) = (D_2 \circ D_1)^{-1}(Y)$ is $N\alpha^{M\#}OS$ in $(N^I,\ \tau_N^i)$. Therefore $D_2 \circ D_1$ is $N\alpha^{M\#}$-irresolute.

Also for an $N\alpha^{M\#}OS$, G in $(N^I,\ \tau_N^i)$, we have $(D_2 \circ D_1)(G) = D_2(D_1(G)) = D_2(W)$, where $W = D_1(G)$. By hypothesis, $D_1(G)$ is $N\alpha^{M\#}OS$ in $(N^{II},\ \tau_N^{ii})$ and so again by hypothesis, $D_2(D_1(G))$ is an $N\alpha^{M\#}OS$ in $(N^{III},\ \tau_N^{iii})$. That is $(D_2 \circ D_1)(G)$ is an $N\alpha^{M\#}OS$ in $(N^{III},\ \tau_N^{iii})$ and therefore $(D_2 \circ D_1)^{-1}$ is $N\alpha^{M\#}$-irresolute. Also $D_2 \circ D_1$ is a bijection. Hence $D_2 \circ D_1$ is $(hmpm(N,\ \tau_N))_{N\alpha^{M\#}}$.

□

Theorem 4.4. *The set* $N\alpha^{M\#}$-$H(N,\ \tau_N)$ *is a subset of the map composition.*

Proof. Establish a binary operation $* : N\alpha^{M\#}\text{-}H(N,\ \tau_N) \times N\alpha^{M\#}\text{-}H(N,\ \tau_N) \longrightarrow N\alpha^{M\#}$-$H(N,\ \tau_N)$ by $D_1 * D_2 = D_2 \circ D_1$ for all $D_1, D_2 \in N\alpha^{M\#}\text{-}H(N,\ \tau_N)$ and *circ* is the standard map composition operation. $D_2 \circ D_1 \in N\alpha^{M\#}\text{-}H(N,\ \tau_N)$.

We notice that maps are made up of associative elements, and the identity map is no exception $I : (N,\ \tau_N) \longrightarrow (N,\ \tau_N)$ belonging to $N\alpha^{M\#}$-$H(N,\ \tau_N)$ identity element as a distinguishing feature. If $D_1 \in N\alpha^{M\#}\text{-}H(N,\ \tau_N)$, then $D_1^{-1} \in N\alpha^{M\#}\text{-}H(N,\ \tau_N)$ such that $D_1 \circ D_1^{-1} = D_1^{-1} \circ D_1 = I$. As a result, there is an inverse for each element of $N\alpha^{M\#}$-$H(N,\ \tau_N)$.

Consequently $N\alpha^{M\#}$-$H(N,\ \tau_N),\ \circ)$ is a network of under the operation on map composition.

□

Proposition 4.5. *Let $J : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-homeomorphism, J causes the group to become isomorphic $N\alpha^{M\#}$-$H(N^I,\ \tau_N^i)$ onto $N\alpha^{M\#}$-$H(N^{II},\ \tau_N^{ii})$.*

Proof. Making use of the map J, We construct a map $\Psi_J : N\alpha^{M\#}$-$H(N^I,\ \tau_N^i) \longrightarrow N\alpha^{M\#}$-$H(N^{II},\ \tau_N^{ii})$ by $\Psi_J(F) = J \circ F \circ J^{-1}$ for every $F \in N\alpha^{M\#}$-$H(N^I,\ \tau_N^i)$. Then Ψ_J is a bijection. Further, for all $h_1, h_2 \in N\alpha^{M\#}$-$H(N^I,\ \tau_N^i)$, $\Psi_J(F_1 \circ F_2) = J \circ (F_1 \circ F_2) \circ J^{-1} = (J \circ F_1 \circ J^{-1}) \circ (J \circ F_2 \circ J^{-1}) = \Psi_J(F_1) \circ \Psi_J(F_2)$.

Therefore, Ψ_J It is an isomorphism caused by a homeomorphism by J.

□

5. On $N\alpha^{M\#}$-connectedness

Definition 5.1. A $NTS(N,\ \tau_N)$ is noted to be $N\alpha^{M\#}$-connected if N can't be characterized as a non-empty union of two distinct elements $N\alpha^{M\#}OS$. A subset of N is $N\alpha^{M\#}$-connected if any of this $N\alpha^{M\#}$-connected as a subspace.

Theorem 5.2. *For a $NTS(N,\ \tau_N)$, the following are better compared.*

(a) $(N,\ \tau_N)$ is $N\alpha^{M\#}$-connected.

(b) $(N,\ \tau_N)$ and ϕ_N seem to be the only subsets of $(N,\ \tau_N)$ both of which are $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed.

(c) Each $N\alpha^{M\#}$-continuous map of $(N^I,\ \tau_N^i)$ into a discrete space $(N^{II},\ \tau_N^{ii})$ the map is constant if there are at least two points.

Proof. $(a) \Longrightarrow (b)$: Suppose $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected. Let S be both a valid subset $N\alpha^{M\#}OS$ and $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$. Its complement N/S is also $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed. $N = S \cup (N/S)$, a non-empty union that is disjointed $N\alpha^{M\#}$-open sets that are incompatible (a). Therefore $S = \phi$ *or* N.

$(b) \implies (a)$: Suppose that $N = I_1 \cup I_2$ where I_1 and I_2 are disjoint non-empty $N\alpha^{M\#}$-open subsets of $(N^I,\ \tau_N^i)$. Then I_1 is both $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed. By assumption $I_1 = \phi$ *or* N. Therefore N is $N\alpha^{M\#}$-connected.

$(b) \implies (c)$: Let $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-continuous map. Then $(N^I,\ \tau_N^i)$ is covered by $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed covering $\{D^{-1}(n_{ii}) : n_{ii} \in N_{ii}\}$. By assumption $D^{-1}(n_{ii}) = \phi_N$ *or* N for each $n_{ii} \in N_{ii}$. If $D^{-1}(n_{ii}) = \phi$ for all $n_{ii} \in N_{ii}$, then D a map that isn't a map. Then $\exists$ a point $n_{ii} \in N_{ii}$ such that $D^{-1}(n_{ii}) \neq \phi_N$ and hence $D^{-1}(n_{ii}) = N$. This shows that D is a constant map.

$(c) \implies (b)$: Let S be both $N\alpha^{M\#}$-open and $N\alpha^{M\#}$-closed in N. Suppose $S \neq \phi$. Let $D : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ be an $N\alpha^{M\#}$-continuous map defined by $D(S) = n_{ii}$ and $D(S^c) = \{\omega\}$ for a few key reasons n_{ii} and ω in $(N^{II},\ \tau_N^{ii})$. By assumption D is a constant map. Therefore we have $S = N$.

□

Theorem 5.3. *Every $N\alpha^{M\#}$-Space that is linked is connected.*

Proof. Let $(N^I,\ \tau_N^i)$ be $N\alpha^{M\#}$-linked(connected). Suppose N is not connected. There is then a suitable non-empty subset. B of $(N^I,\ \tau_N^i)$ which has both an open and a closed sets in $(N^I,\ \tau_N^i)$. Since every closed set is $N\alpha^{M\#}$-closed, B is a proper non empty subset of $(N^I,\ \tau_N^i)$ as well as $N\alpha^{M\#}OS$ and $N\alpha^{M\#}CS$ in $(N^I,\ \tau_N^i)$, $(N^I,\ \tau_N^i)$ is not $N\alpha^{M\#}$-connected. This proves the theorem.

□

Theorem 5.4. *If $J : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is an $N\alpha^{M\#}$-continuous and N is $N\alpha^{M\#}$-connected, then $(N^{II},\ \tau_N^{ii})$ is linked.*

Proof. Presume that $(N^{II},\ \tau_N^{ii})$ is not linked. Let $N^{ii} = V_1 \cup V_2$ where V_1 and V_2 are disjoint non-empty OS in $(N^{II},\ \tau_N^{ii})$. As such J is $N\alpha^{M\#}$-continuous and onto, $N = J^{-1}(V_1) \cup J^{-1}(V_2)$ where $J^{-1}(V_1)$ and $J^{-1}(V_2)$ are disjoint non-empty $N\alpha^{M\#}$-open sets in $(N^I,\ \tau_N^i)$.

This is diametrically opposed to the fact that $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected. Furthermore N^{ii} is connected.

□

Theorem 5.5. *If $J : (N^I,\ \tau_N^i) \longrightarrow (N^{II},\ \tau_N^{ii})$ is an $N\alpha^{M\#}$-irresolute and $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected, then $(N^{II},\ \tau_N^{ii})$ is $N\alpha^{M\#}$-connected.*

Proof. Suppose that $(N^{II},\ \tau_N^{ii})$ is not $N\alpha^{M\#}$-connected. Let $N^{ii} = V_1 \cup V_2$ where V_1 and V_2 are disjoint non-empty $N\alpha^{M\#}$-open sets in $(N^{II},\ \tau_N^{ii})$. Since J is $N\alpha^{M\#}$-irresolute and onto, $N = j^{-1}(V_1) \cup j^{-1}(V_2)$ where $J^{-1}(V_1)$ and $J^{-1}(V_2)$ are disjoint non-empty $N\alpha^{M\#}$-open sets in $(N^I,\ \tau_N^i)$.

This contradicts the fact that $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected. Hence $(N^{II},\ \tau_N^{ii})$ is $N\alpha^{M\#}$-connected.

□

Theorem 5.6. *Suppose that $(N^I,\ \tau_N^i)$ is $t_{\#}^{N\alpha}$-space then $(N^I,\ \tau_N^i)$ is connected $\Longleftrightarrow$ $N\alpha^{M\#}$-connected.*

Proof. Suppose that $(N^I,\ \tau_N^i)$ is connected. Then $(N^I,\ \tau_N^i)$ disjoint union of two non-empty proper subsets of the set cannot be expressed in $(N^I,\ \tau_N^i)$. Suppose $(N^I,\ \tau_N^i)$ is not a $N\alpha^{M\#}$-connected space. Let V_1 and V_2 be any two $N\alpha^{M\#}$-open subsets of $(N^I,\ \tau_N^i)$ such that $N^{ii} = V_1 \cup V_2$, where $V_1 \cap V_2 = \phi_N$ and $V_1 \subset N$, $V_2 \subset N$ Since $(N^I,\ \tau_N^i)$ is $t_{\#}^{N\alpha}$-space and V_1, V_2 are $N\alpha^{M\#}$-open. V_1, V_2 are open subsets of $(N^I,\ \tau_N^i)$, which contradicts that $(N^I,\ \tau_N^i)$ is connected. Therefore $(N^I,\ \tau_N^i)$ is $N\alpha^{M\#}$-connected.

Conversely, every open set is $N\alpha^{M\#}$-open. Therefore every $N\alpha^{M\#}$-connected space is connected.

□

Theorem 5.7. *If the $N\alpha^{M\#}$-open sets Z_1 and Z_2 form a separation of $(N^I,\ \tau_N^i)$ and if $(N^{II},\ \tau_N^{ii})$ is $N\alpha^{M\#}$-connected subspace of $(N^I,\ \tau_N^i)$, then $(N^{II},\ \tau_N^{ii})$ lies entirely within Z_1 or Z_2.*

Proof. Since Z_1 and Z_2 are both $N\alpha^{M\#}$-open in $(N^I,\ \tau_N^i)$, the sets $Z_1 \cap N^{ii}$ and $Z_2 \cap N^{ii}$ are $N\alpha^{M\#}$-open in $(N^{II},\ \tau_N^{ii})$. These two sets are incompatible, thus their union is impossible is $(N^{II},\ \tau_N^{ii})$. They would represent a separation if they were both non-empty $(N^{II},\ \tau_N^{ii})$.

Therefore, one of them is empty. Hence $(N^{II},\ \tau_N^{ii})$ must lie entirely in Z_1 or in Z_2.

□

6. Conclusion

The notions of $N\alpha^{M\#}CS$ in neutrosophic topological spaces have been discussed in this research study. We have also introduced the neutrosophic $t_{\#}^{N\alpha}$-space in this paper. The mappings known as neutrosophic $N\alpha^{M\#}$-continuous functions, $N\alpha^{M\#}$-irresolute functions, homeomorphisms and connectedness have also been introduced and investigate their characterizations and distinguishing features.

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