Neutrosophic ℵ −interior ideals in semigroups

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Abstract: We define the concepts of neutrosophic ℵ -interior ideal and neutrosophic ℵ −characteristic interior ideal structures of a semigroup. We infer different types of semigroups using neutrosophic ℵ-interior ideal structures. We also show that the intersection of neutrosophic ℵ-interior ideals and the union of neutrosophic ℵ-interior ideals is also a neutrosophic ℵ-interior ideal.

Keywords: Semi group, neutrosophic ℵ − ideals, neutrosophic ℵ -interior ideals, neutrosophic ℵ −product.

1. Introduction

Nowadays, the theory of uncertainty plays a vital role to manage different issues relating to modelling engineering problems, networking, real-life problem relating to decision making and so on. In 1965, Zadeh[24] introduced the idea of fuzzy sets for modelling vague concepts in the globe. In 1986, Atanassov [1] generalized fuzzy set and named as Intuitionistic fuzzy set. Also, from his viewpoint, there are two degrees of freedom in the real world, one a degree of membership to a vague subset and the other is a degree of non-membership to that given subset.

Smarandache generalized fuzzy set and intuitionistic fuzzy set, and named as neutrosophic set (see [4, 7, 8, 14, 19, 22-23]). These sets are characterized by a truth membership function, an indeterminacy membership function and a falsity membership function. These sets are applied to many branches of mathematics to overcome the complexities arising from uncertain data. A Neutrosophic set can distinguish between absolute membership and relative membership. Smarandache used this in non-standard analysis such as the result of sports games (winning/defeating/tie), decision making and control theory, etc. This area has been studied by several authors (see [3, 11, 12, 16-18]).

For more details on neutrosophic set theory, the readers visit the website http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In [2], Abdel Basset et al. designed a framework to manage scheduling problems using neutrosophic theory. As the concept of time-cost tradeoffs and deterministic project scheduling disagree with the real situation, some data were changed during the implementation process. Here fuzzy scheduling and time-cost tradeoffs models assumed only truth-membership functions dealing with uncertainties of the project and their activities duration which were unable to treat indeterminacy and inconsistency.

In [6], Abdel Basset et al. evaluated the performance of smart disaster response systems under uncertainty. In [5], Abdel Basset et al. introduced different hybrid neutrosophic multi-criteria decision-making framework for professional selection that employed a collection of neutrosophic analytical network process and order preference by similarity to the ideal solution under bipolar neutrosophic numbers.

In [21], Prakasam Muralikrishna1 et al. presented the characterization of MBJ – Neutrosophic β – Ideal of β – algebra. They analyzed homomorphic image, pre–image, cartesian product and related results, and these concepts were explored to other substructures of a β – algebra. In [9], Chalapathi et al. constructed certain Neutrosophic Boolean rings, introduced Neutrosophic complement elements and mainly obtained some properties satisfied by the Neutrosophic complement elements of Neutrosophic Boolean rings.

In [14], M. Khan et al. presented the notion of neutrosophic ℵ-subsemigroup in semigroup and explored several properties. In [11], Gulistan et al. have studied the idea of complex neutrosophic subsemigroups and introduced the concept of the characteristic function of complex neutrosophic sets, direct product of complex neutrosophic sets.

In [10], B. Elavarasan et al. introduced the notion of neutrosophic ℵ-ideal in semigroup and explored its properties. Also, the conditions for neutrosophic ℵ-structure to be neutrosophic ℵ-ideal are given, and discussed the idea of characteristic neutrosophic ℵ-structure in semigroups and obtained several properties. In [20], we have introduced and discussed several properties of neutrosophic ℵ-bi-ideal in the semigroup. We have proved that neutrosophic ℵ-product and the intersection of neutrosophic ℵ-ideals were identical for regular semigroups. In this paper, we define and discuss the concepts of neutrosophic ℵ-interior ideal and neutrosophic ℵ-characteristic interior ideal structures of a semigroup.

Throughout this paper, X denotes a semigroup. Now, we present the important definitions of semigroup that we need in sequel.

Recall that for any $X_1, X_2 \subseteq X$, $X_1X_2 = \{ab | a \in X_1 \text{ and } b \in X_2\}$, multiplication of X_1 and X_2.

Let X be a semigroup and $\emptyset \neq X_1 \subseteq X$. Then

(i) X_1 is known as subsemigroup if ${X_1}^2 \subseteq X_1$.

(ii) A subsemigroup X_1 is known as left (resp., right) ideal if $X_1X \subseteq X_1$(resp., $XX_1 \subseteq X_1$).

(iii) X_1 is known as ideal if X_1 is both a left and a right ideal.

(iv) X is known as left (resp., right) regular if for each $r \in X$, there exists $i \in X$ such that $r = ir^2$(resp., $r = r^2 i$) [13].

(v) X is known as regular if for each $b_1 \in X$, there exists $i \in X$ such that $b_1 = b_1 i b_1$

(vi) X is known as intra-regular if for each $x_1 \in X$, there exist $i, j \in X$ such that $x_1 = i x_1^2 j$ [15].

2. Definitions of neutrosophic ℵ - structures

We present definitions of neutrosophic ℵ −structures namely neutrosophic ℵ −subsemigroup, neutrosophic ℵ −ideal, neutrosophic ℵ −interior ideal of a semigroup X

The set of all the functions from X to $[-1,0]$ is denoted by $\Im(X,[-1,0])$. We call that an element of $\Im(X,[-1,0])$ is $\aleph$ −function on X. A $\aleph$ −structure means an ordered pair (X,g) of X and an $\aleph$ −function g on X.

Definition 2.1.[14] A neutrosophic $\aleph-$ structure of $\boldsymbol{X}$ is defined to be the structure:

$$X_M := \frac{X}{(T_M,\ I_M,\ F_M)} = \left\{\frac{r}{T_M(r), I_M(r),\ F_M(r)} \mid r \in X\right\},$$

where T_M, I_M and F_M are the negative truth, negative indeterminacy and negative falsity membership function on X ($\aleph-$ functions).

It is evident that $-3 \le T_M(r) + I_M(r) + F_M(r) \le 0$ for all $r \in X$.

Definition 2.2.[14] A neutrosophic $\aleph-$ structure X_M of X is called a neutrosophic $\aleph$ −subsemigroup of X if the following assertion is valid:

$$(\forall\, g_i, h_j \in X)\begin{pmatrix} T_M(g_ih_j) \le T_M(g_i) \vee T_M(h_j) \\ I_M(g_ih_j) \ge I_M(g_i) \wedge I_M(h_j) \\ F_M(g_ih_j) \le F_M(g_i) \vee F_M(h_j) \end{pmatrix}.$$

.Let X_M be a neutrosophic $\aleph$ −structure and $\gamma, \delta, \varepsilon \in [-1, 0]$ with $-3 \le \gamma + \delta + \varepsilon \le 0$. Consider the sets:

$$T_M^{\gamma} = \{r_i \in X | T_M(r_i) \le \gamma\}$$
$$I_M^{\delta} = \{r_i \in X | I_M(r_i) \ge \delta\}$$
$$F_M^{\varepsilon} = \{r_i \in X | F_M(r_i) \le \varepsilon\}.$$

The set $X_M(\gamma,\delta,\varepsilon) := \{r_i \in X\, | T_M(r_i) \le \gamma,\ I_M(r_i) \ge \delta,\ F_M(r_i) \le \epsilon\}$ is known as $(\gamma,\delta,\varepsilon)$-level set of X_M. It is easy to observe that $X_M(\gamma,\delta,\varepsilon) = T_M^{\gamma} \cap I_M^{\delta} \cap F_M^{\varepsilon}$.

Definition 2.3.[10] A neutrosophic $\aleph$ −structure X_M of X is called a neutrosophic $\aleph$ −left (resp., right) ideal of X if

$$(\forall\, g_i, h_j \in X)\begin{pmatrix} T_M(g_ih_j) \le T_M(h_j)\ (resp., T_M(g_ih_j) \le T_M(g_i)) \\ I_M(g_ih_j) \ge I_M(h_j)\ (resp., I_M(g_ih_j) \ge I_M(g_i)) \\ F_M(g_ih_j) \le F_M(h_j)\ (resp., F_M(g_ih_j) \le F_M(g_i)) \end{pmatrix}.$$

X_M is neutrosophic $\aleph$ −ideal of $\mathbf{X}$ if X_M is neutrosophic $\aleph$ −left and $\aleph$ −right ideal of X.

Definition 2.4. A neutrosophic $\aleph$ −subsemigroup X_M of X is known as neutrosophic $\aleph$ −interior ideal if

$$(\forall\, x, a, y \in X)\begin{pmatrix} T_M(xay) \le T_M(a) \\ I_M(xay) \ge I_M(a) \\ F_M(xay) \le F_M(a) \end{pmatrix}.$$

It is easy to observe that every neutrosophic $\aleph$ −ideal is neutrosophic $\aleph$ −interior ideal, but neutrosophic $\aleph$ −interior ideal need not be a neutrosophic $\aleph-$ ideal, as shown by an example.

Example 2.5. Let X be the set of all non-negative integers except 1. Then X is a semigroup with usual multiplication.

Let $X_M = \left\{\frac{0}{(-0.9,-0.1,-0.7)}, \frac{2}{(-0.4\ -0.6,-0.5)}, \frac{5}{(-0.3,-0.8,-0.3)}, \frac{10}{(-0.3,-0.8,-0.1)}, \frac{otherwise}{(-0.7,-0.4,-0.6)}\right\}$. Then X_M is neutrosophic $\aleph$ −interior ideal, but not neutrosophic $\aleph-$ ideal with $T_N(2.5) = -0.3 \nleq T_N(2)$.

Definition 2.6.[14] For any $E \subseteq X$, the characteristic neutrosophic $\aleph$ −structure is defined as

$$\chi_E(X_M) = \frac{X}{(\chi_E(T)_M,\ \chi_E(I)_M,\ \chi_E(F)_M)}$$

where

$$\chi_E(T)_M: X\to [-1,0],\ r \to \begin{cases} -1\ if\ r \in E \\ 0\ \ otherwise, \end{cases}$$

$$\chi_E(I)_M: X\to [-1,0],\ r \to \begin{cases} 0\ if\ r \in E \\ -1\ otherwise, \end{cases}$$

$$\chi_E(F)_M: X\to [-1,0],\ r \to \begin{cases} -1\ \ if\ r \in E \\ 0\ \ otherwise. \end{cases}$$

Definition 2.7.[14] Let $X_N := \frac{X}{(T_N,\ I_N,\ F_N)}$ and $X_M := \frac{X}{(T_M,\ I_M,\ F_M)}$ be neutrosophic $\aleph-$structures of X. Then

(i) X_N is called a neutrosophic $\aleph-$ substructure of X_M, denote by $X_M \subseteq X_N$, if $T_M(r) \geq T_N(r)$, $I_M(r) \leq I_N(r)$, $F_M(r) \geq F_N(r)$ for all $r \in X$.

(ii) If $X_N \subseteq X_M$ and $X_M \subseteq X_N$, then we say that $X_N = X_M$.

(iii) The neutrosophic $\aleph-$ product of X_N and X_M is defined to be a neutrosophic $\aleph-$structure of X,

$$X_N \odot X_M := \frac{X}{(T_{N\circ M},\ I_{N\circ M},\ F_{N\circ M})} = \left\{\frac{h}{T_{N\circ M}(h),\ I_{N\circ M}(h),\ F_{N\circ M}(h)} \mid h\ \in X\right\},$$

where

$$(T_N \circ T_M)(h) = T_{N\circ M}(h) = \begin{cases} \bigwedge_{h=rs}\{T_N(r) \vee T_M(s)\} & if\ \exists\ r,s\ \in X\ such\ that\ h = rs \\ 0 & otherwise, \end{cases}$$

$$(I_N \circ I_M)(h) = I_{N\circ M}(h) = \begin{cases} \bigvee_{h=rs}\{I_N(r) \wedge I_M(s)\} & if\ \exists\ u,v\ \in X\ such\ that\ h = rs \\ -1 & otherwise, \end{cases}$$

$$(F_N \circ F_M)(h) = F_{N\circ M}(h) = \begin{cases} \bigwedge_{h=rs}\{F_N(r) \vee F_M(s)\} & if\ \exists\ u,v\ \in X\ such\ that\ h = rs \\ 0 & otherwise. \end{cases}$$

For $i \in X$, the element $\frac{i}{(T_{N\circ M}(i),\ I_{N\circ M}(i),\ F_{N\circ M}(i))}$ is simply denoted by $(X_N \odot X_M)(i) = (T_{N\circ M}(i),\ I_{N\circ M}(i),\ F_{N\circ M}(i))$.

(iii) The union of X_N and X_M, a neutrosophic $\aleph-$structure over X is defined as

$$X_N \cup X_M = X_{N\cup M} = (X; T_{N\cup M},\ I_{N\cup M},\ F_{N\cup M}),$$

where

$$(T_N \cup T_M)(h_i) = T_{N\cup M}(h_i) = T_N(h_i) \wedge T_M(h_i),$$
$$(I_N \cup I_M)(h_i) = I_{N\cup M}(h_i) = I_N(h_i) \vee I_M(h_i),$$
$$(F_N \cup F_M)(h_i) = F_{N\cup M}(h_i) = F_N(h_i) \wedge F_M(h_i)\ \ \forall h_i \in X.$$

(iv) The intersection of X_N and X_M, a neutrosophic $\aleph-$structure over X is defined as

$$X_N \cap X_M = X_{N\cap M} = (X; T_{N\cap M},\ I_{N\cap M},\ F_{N\cap M}),$$

where

$$(T_N \cap T_M)(h_i) = T_{N\cap M}(h_i) = T_N(h_i) \vee T_M(h_i),$$
$$(I_N \cap I_M)(h_i) = I_{N\cap M}(h_i) = I_N(h_i) \wedge I_M(h_i),$$
$$(F_N \cap F_M)(h_i) = F_{N\cap M}(h_i) = F_N(h_i) \vee F_M(h_i)\ \ \forall\ \ h_i \in X.$$

3. **Neutrosophic $\aleph-$interior ideals**

We study different properties of neutrosophic $\aleph-$interior ideals of X. It is evident that neutrosophic $\aleph-$ ideal is a neutrosophic $\aleph-$interior ideal of X, but not the converse. Further, for a regular and for an intra-regular semigroup, every neutrosophic $\aleph-$interior ideal is neutrosophic $\aleph-$ideal.

All throughout this part, we consider X_M and X_N are neutrosophic $\aleph-$structures of X.

Theorem 3.1. For any $L \subseteq X$, the equivalent assertions are:

(i) L is an interior ideal,

(ii) The characteristic neutrosophic $\aleph-$structure $\chi_L(X_N)$ is a neutrosophic $\aleph-$interior ideal.

Proof: Suppose L is an interior ideal and let $x, a, y \in X$.

If $a \in L$, then $xay \in L$, so $\chi_L(T)_N(xay) = -1 = \chi_L(T)_N(a)$, $\chi_L(I)_N(xay) = 0 = \chi_L(I)_N(a)$ and $\chi_L(F)_N(xay) = -1 = \chi_L(F)_N(a)$.

If $a \notin L$, then $\chi_L(T)_N(xay) \leq 0 = \chi_L(T)_N(a)$, $\chi_L(I)_N(xay) \geq -1 = \chi_L(I)_N(a)$ and $\chi_L(F)_N(xay) \leq 0 = \chi_L(F)_N(a)$.

Therefore $\chi_L(X_N)$ is a neutrosophic $\aleph-$interior ideal.

Conversely, assume that $\chi_L(X_N)$ is a neutrosophic $\aleph-$ interior ideal. Let $u \in L$ and $x, y \in X$. Then

$$\chi_L(T)_N(xuy) \leq \chi_L(T)_N(u) = -1,$$
$$\chi_L(I)_N(xuy) \geq \chi_L(I)_N(u) = 0,$$
$$\chi_L(F)_N(xuy) \leq \chi_L(F)_N(u) = -1 .$$

So $xuy \in L$. □

Theorem 3.2. If X_M and X_N are neutrosophic $\aleph-$ interior ideals, then $X_{M\cap N}$ is neutrosophic $\aleph-$ interior ideal.

Proof: Let X_M and X_N be neutrosophic $\aleph-$ interior ideals. For any $r, s, t \in X$, we have

$$T_{M\cap N}(rst) = T_M(rst) \vee T_N(rst) \leq T_M(s) \vee T_N(s) = T_{M\cap N}(s),$$
$$I_{M\cap N}(rst) = I_M(rst) \wedge I_N(rst) \geq I_M(s) \wedge I_N(s) = I_{M\cap N}(s),$$
$$F_{M\cap N}(rst) = F_M(rst) \vee F_N(rst) \leq F_M(s) \vee F_N(s) = F_{M\cap N}(s).$$

Therefore $X_{M\cap N}$ is neutrosophic $\aleph-$ interior ideal. □

Corollary 3.3. The arbitrary intersection of neutrosophic $\aleph-$ interior ideals is a neutrosophic $\aleph-$ interior ideal.

Theorem 3.4. If X_M and X_N are neutrosophic $\aleph-$ interior ideals, then $X_{M\cup N}$ is neutrosophic $\aleph-$ interior ideal.

Proof: Let X_M and X_N be neutrosophic $\aleph-$ interior ideals. For any $r, s, t \in X$, we have

$$T_{M\cup N}(rst) = T_M(rst) \wedge T_N(rst) \leq T_M(s) \wedge T_N(s) = T_{M\cup N}(s),$$
$$I_{M\cup N}(rst) = I_M(rst) \vee I_N(rst) \geq I_M(s) \vee I_N(s) = I_{M\cup N}(s),$$
$$F_{M\cup N}(rst) = F_M(rst) \wedge F_N(rst) \leq F_M(s) \wedge F_N(s) = F_{M\cup N}(s).$$

Therefore $X_{M\cup N}$ is neutrosophic $\aleph-$ interior ideal. □

Corollary 3.5. The arbitrary union of neutrosophic $\aleph-$ interior ideals is neutrosophic $\aleph-$ interior ideal.

Theorem 3.6. Let X be a regular semigroup. If X_M is neutrosophic $\aleph-$ interior ideal, then X_M is neutrosophic $\aleph-$ ideal.

Proof: Assume that X_M is an interior ideal, and let $u, v \in X$. As X is regular and $u \in X$, there exists $r \in X$ such that $u = uru$. Now, $T_M(uv) = T_M(uruv) \leq T_M(u)$, $I_M(uv) = I_M(uruv) \geq I_M(u)$ and $F_M(uv) = F_M(uruv) \leq F_M(u)$. Therefore X_M is neutrosophic $\aleph-$ right ideal.

Similarly, we can show that X_M is neutrosophic $\aleph-$ left ideal and hence X_M is neutrosophic $\aleph-$ ideal. □

Theorem 3.7. Let X be an intra-regular semigroup. If X_M is neutrosophic $\aleph-$ interior ideal, then X_M is neutrosophic $\aleph-$ ideal.

Proof: Suppose that X_M is neutrosophic $\aleph-$ interior ideal, and let $u, v \in X$. As X is intra regular and $u \in X$, there exist $s, t \in S$ such that $u = su^2t$. Now,

$$T_M(uv) = T_M(su^2tv) \leq T_M(u),$$
$$I_M(uv) = I_M(su^2tv) \geq I_M(u)$$
$$F_M(uv) = F_M(su^2tv) \leq F_M(u).$$

Therefore X_M is neutrosophic $\aleph-$ right ideal. similarly, we can show that X_M is neutrosophic $\aleph-$ left ideal and hence X_M is neutrosophic $\aleph-$ ideal. □

Definition 3.8. A semigroup X is left simple (resp., right simple) if it does not contain any proper left ideal (resp., right ideal) of X. A semigroup X is simple if it does not contain any proper ideal of X.

Definition 3.9. A semigroup X is said to be neutrosophic $\aleph-$simple if every neutrosophic $\aleph-$ ideal is a constant function

i.e., for every neutrosophic $\aleph-$ideal X_M of X, we have $T_M(i) = T_M(j)$, $I_M(i) = I_M(j)$ and $F_M(i) = F_M(j)$ for all $i, j \in X$.

Notation 3.10. If X is a semigroup and $s \in X$, we define a subset, denoted by I_s as follows:

$$I_s := \{i \in X \mid T_N(i) \leq T_N(s),\ I_N(i) \geq I_N(s) \ \ and \ \ F_N(i) \leq F_N(s)\}.$$

Proposition 3.11. If X_N is neutrosophic $\aleph-$ right (resp., $\aleph-$ left, $\aleph-$ ideal) ideal, then I_s is right (resp., left, ideal) ideal for every $s \in X$.

Proof: Let $s \in X$. Then it is clear that $\varphi \neq I_s \subseteq X$. Let $u \in I_s$ and $x \in X$. Then $ux \in I_s$. Indeed; Since X_N is neutrosophic $\aleph-$ right ideal and $u, x \in X$, we get $T_N(ux) \leq T_N(u)$, $I_N(ux) \geq I_N(u)$ and $F_N(ux) \leq F_N(t)$. Since $u \in I_s$, we get $T_N(u) \leq T_N(s)$, $I_N(u) \geq I_N(s)$ and $F_N(u) \leq F_N(s)$ which imply $ux \in I_s$. Therefore I_s is a right ideal for every $s \in X$. □

Theorem 3.12.[4] For any $L \subseteq X$, the equivalent assertions are:

(i) L is left (resp., right) ideal,

(ii) Characteristic neutrosophic $\aleph-$structure $\chi_L(X_N)$ is neutrosophic $\aleph-$left (resp., right) ideal.

Theorem 3.13. Let X be a semigroup. Then X is simple if and only if X is neutrosophic $\aleph-$simple.

Proof: Suppose X is simple. Let X_M be a neutrosophic $\aleph-$ ideal and $u, v \in X$. Then by Proposition 3.11, I_u is an ideal of X. As X is simple, we have $I_u = X$. Since $v \in I_u$, we have $T_M(v) \leq T_M(u)$, $I_M(v) \geq I_M(u)$ and $F_M(v) \leq F_M(u)$.

Similarly, we can prove that $T_M(u) \leq T_M(v)$, $I_M(u) \geq I_M(v)$ and $F_M(u) \leq F_M(v)$. So $T_M(u) = T_M(v)$, $I_M(u) = I_M(v)$ and $F_M(u) = F_M(v)$. Hence X is neutrosophic $\aleph-$ simple.

Conversely, assume that X is neutrosophic $\aleph-$ simple and I is an ideal of X. Then by Theorem 3.12, $\chi_I(X_N)$ is a neutrosophic $\aleph-$ ideal. We now claim that $X = I$. Let $w \in X$. Since X is neutrosophic $\aleph-$ simple, we have $\chi_I(X_N)$ is a constant function and $\chi_I(X_N)(w) = \chi_I(X_N)(y)$ for every $y \in X$. In particular, we have $\chi_I(T_N)(w) = \chi_I(T_N)(d) = -1$, $\chi_I(I_N)(w) = \chi_I(I_N)(d) = 0$ and $\chi_I(F_N)(w) = \chi_I(F_N)(d) = -1$ for any $d \in I$ which implies $w \in I$. Thus $X \subseteq I$ and hence $X = I$. □

Lemma 3.14. Let X be a semigroup. Then X is simple if and only for every $t \in X$, we have $X = XtX$.

Proof: Suppose X is simple and let $t \in X$. Then $X(XtX) \subseteq XtX$ and $(XtX)X \subseteq XtX$ imply that XtX is an ideal. Since X is simple, we have $XtX = X$.

Conversely, let P be an ideal and let $a \in P$. Then $X = XaX$, $XaX \subseteq XPX \subseteq P$ which implies $P = X$. Therefore X is simple. □

Theorem 3.15. Suppose X is a semigroup. Then X is simple if and only every neutrosophic $\aleph-$ interior ideal of X is a constant function.

Proof: Suppose X is simple and $s, t \in X$. Let X_N be neutrosophic $\aleph-$ interior ideal. Then by Lemma 3.14, we get $X = XsX = XtX$. As $s \in XsX$, we have $s = atb$ for $a, b \in X$. Since X_N is neutrosophic $\aleph-$ interior ideal, we have $T_N(s) = T_N(atb) \leq T_N(t)$, $I_N(s) = I_N(atb) \geq I_N(t)$ and $F_N(s) = F_N(atb) \leq F_N(t)$. Similarly, we can prove that $T_N(t) \leq T_N(s)$, $I_N(t) \geq I_N(s)$ and $F_N(t) \leq F_N(s)$. So X_N is a constant function.

Conversely, suppose X_N is neutrosophic $\aleph-$ ideal. Then X_N is neutrosophic $\aleph-$ interior ideal. By hypothesis, X_N is a constant function and so X_N is neutrosophic $\aleph-$simple. By Theorem 3.13, X is simple. □

Theorem 3.16. Let X_M be neutrosophic $\aleph-$ structure and let $\gamma, \delta, \varepsilon \in [-1, 0]$ with $-3 \leq \gamma + \delta + \varepsilon \leq 0$. If X_M is neutrosophic $\aleph-$interior ideal, then $(\gamma, \delta, \varepsilon)$-level set of X_M is neutrosophic $\aleph-$interior ideal whenever $X_M(\gamma, \delta, \varepsilon) \neq \emptyset$.

Proof: Suppose $\boldsymbol{X_M(\gamma, \delta, \varepsilon)} \neq \emptyset$ for $\boldsymbol{\gamma, \delta, \varepsilon} \in [\boldsymbol{-1, 0}]$ with $\boldsymbol{-3} \leq \boldsymbol{\gamma + \delta + \varepsilon} \leq \boldsymbol{0}$.

Let $\boldsymbol{X_M}$ be a neutrosophic $\aleph-$interior ideal and let $\boldsymbol{u, v, w} \in \boldsymbol{X_M(\gamma, \delta, \varepsilon)}$. Then $\boldsymbol{T_M(uvw)} \leq \boldsymbol{T_M(v)} \leq \boldsymbol{\alpha}$; $\boldsymbol{I_M(uvw)} \geq \boldsymbol{I_M(v)} \geq \boldsymbol{\beta}$ and $\boldsymbol{F_M(uvw)} \leq \boldsymbol{F_M(v)} \leq \boldsymbol{\gamma}$ which imply $\boldsymbol{uvw} \in \boldsymbol{X_M(\alpha, \beta, \gamma)}$. Therefore $\boldsymbol{X_M(\gamma, \delta, \varepsilon)}$ is a neutrosophic $\aleph-$interior ideal of $\boldsymbol{X}$. □

Theorem 3.17. Let X_N be neutrosophic $\aleph-$ structure with $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If T_N^{α}, I_N^{β} and F_N^{γ} are interior ideals, then X_N is neutrosophic $\aleph-$ interior ideal of X whenever it is non-empty.

Proof: Suppose that for $\boldsymbol{a, b, c} \in \boldsymbol{X}$ with $\boldsymbol{T_N(abc)} > \boldsymbol{T_N(b)}$. Then $\boldsymbol{T_N(abc)} > \boldsymbol{t_\alpha} \geq \boldsymbol{T_N(b)}$ for some $\boldsymbol{t_\alpha} \in [\boldsymbol{-1, 0})$. So $\boldsymbol{b} \in \boldsymbol{T_N^{t_\alpha}(b)}$ but $\boldsymbol{abc} \notin \boldsymbol{T_N^{t_\alpha}(b)}$, a contradiction. Thus $\boldsymbol{T_N(abc)} \leq \boldsymbol{T_N(b)}$.

Suppose that for $a, b, c \in X$ with $I_N(abc) < I_N(b)$. Then $I_N(abc) < t_\alpha \leq I_N(b)$ for some $t_\alpha \in [-1, 0)$. So $b \in I_N^{t_\alpha}(b)$ but $abc \notin I_N^{t_\alpha}(b)$, a contradiction. Thus $I_N(abc) \geq I_N(b)$.

Suppose that for $a, b, c \in X$ with $F_N(abc) > F_N(b)$. Then $F_N(abc) > t_\alpha \geq F_N(b)$ for some $t_\alpha \in [-1, 0)$. So $b \in F_N^{t_\alpha}(b)$ but $abc \notin F_N^{t_\alpha}(b)$, a contradiction. Thus $F_N(abc) \leq F_N(b)$.

Thus X_N is neutrosophic $\aleph -$ interior ideal. □

Theorem 3.18. Let X_M be neutrosophic $\aleph -$ structure over X. Then the equivalent assertions are:

(i) X_M is neutrosophic $\aleph -$interior ideal,

(ii) $X_N \odot X_M \odot X_N \subseteq X_M$ for any neutrosophic $\aleph -$ structure X_N.

Proof: Suppose X_M is neutrosophic $\aleph -$ interior ideal. Let $x \in X$. For any $u, v, w \in X$ such that $x = uvw$. Then $T_M(x) = T_M(uvw) \leq T_M(v) \leq T_N(u) \vee T_M(v) \vee T_N(w)$ which implies $T_M(x) \leq T_{N \circ M \circ N}(x)$. Otherwise $x \neq uvw$. Then $T_M(x) \leq 0 = T_{N \circ M \circ N}(x)$. Similarly, we can prove that $I_M(x) \geq I_{N \circ M \circ N}(x)$ and $F_M(x) \leq F_{N \circ M \circ N}(x)$. Thus $X_N \odot X_M \odot X_N \subseteq X_M$.

Conversely, assume that $X_N \odot X_M \odot X_N \subseteq X_M$ for any neutrosophic $\aleph -$structure X_N.

Let $u, v, w \in X$. If $x = uvw$, then

$$T_M(uvw) = T_M(x) \leq (\chi_X(T)_N \circ T_M \circ \chi_X(T)_N)(x) = \bigwedge_{x=rw} \{\chi_X(T)_N \circ T_M)(r) \vee \chi_X(T)_N(w)\}$$

$$= \bigwedge_{x=rc} \{ \bigwedge_{r=uv} \{\chi_X(T)_N(u) \vee (T)_M(v)\} \vee \chi_X(T)_N(w)\}$$

$$\leq \chi_X(T)_N(u) \vee (T)_M(v) \vee \chi_X(T)_N(w) = T_M(v),$$

$$I_M(uvw) = I_M(x) \leq (\chi_X(I)_N \circ I_M \circ \chi_X(I)_N)(x) = \bigvee_{x=rw} \{\chi_X(I)_N \circ I_M)(r) \wedge \chi_X(I)_N(w)\}$$

$$= \bigvee_{x=rc} \{ \bigvee_{r=uv} \{\chi_X(I)_N(u) \wedge (I)_M(v)\} \wedge \chi_X(I)_N(w)\}$$

$$\geq \chi_X(I)_N(u) \wedge (I)_M(v) \wedge \chi_X(I)_N(w) = (I)_M(v),$$

and

$$F_M(uvw) = F_M(x) \leq (\chi_X(F)_N \circ F_M \circ \chi_X(F)_N)(x) = \bigwedge_{x=rw} \{\chi_X(F)_N \circ F_M)(r) \vee \chi_X(F)_N(w)\}$$

$$= \bigwedge_{x=rc} \{ \bigwedge_{r=uv} \{\chi_X(F)_N(u) \vee (F)_M(v)\} \vee \chi_X(F)_N(w)\}$$

$$\leq \chi_X(F)_N(u) \vee (F)_M(v) \vee \chi_X(F)_N(w) = F_M(v).$$

Therefore X_M is neutrosophic $\aleph -$interior ideal. □

Notation 3.19. Let X and Z be semigroups. A mapping $g: X \to Z$ is said to be a homomorphism if $g(uv) = g(u)g(v)$ for all $u, v \in X$. Throughout this remaining section, we denote $Aut(X)$, the set of all automorphisms of X.

Definition 3.20. An interior ideal J of a semigroup X is called a characteristic interior ideal if $h(J) = J$ for all $h \in Aut(X)$.

Definition 3.21. Let $\boldsymbol{X}$ be a semigroup. A neutrosophic $\aleph-$ interior ideal $\boldsymbol{X_N}$ is called neutrosophic $\aleph-$ characteristic interior ideal if $\boldsymbol{T_N(h(u)) = T_N(u)}$, $\boldsymbol{I_N(h(u)) = I_N(u)}$ and $\boldsymbol{F_N(h(u)) = F_N(u)}$ for all $\boldsymbol{u \in X}$ and all $\boldsymbol{h \in Aut(X)}$.

Theorem 3.22. For any $\mathrm{L} \subseteq \mathrm{X}$, the equivalent assertions are:

(i) L is characteristic interior ideal,

(ii) The characteristic neutrosophic $\aleph-$structure $\chi_{\mathrm{L}}(\mathrm{X}_{\mathrm{M}})$ is neutrosophic $\aleph-$characteristic interior ideal.

Proof: Suppose L is characteristic interior ideal and let $x \in X$. Then by Theorem 3.1, $\chi_{\mathrm{L}}(\mathrm{X}_{\mathrm{M}})$ is neutrosophic $\aleph-$interior ideal. If $x \in L$, then $\chi_L(T)_M(x) = -1$, $\chi_L(I)_M(x) = 0$, and $\chi_L(F)_M(x) = -1$. Now, for any $h \in Aut(X)$, $h(x) \in h(L) = L$ which implies $\chi_L(T)_M(h(x)) = -1, \chi_L(I)_M(h(x)) = 0$, and $\chi_L(F)_M(h(x)) = -1$. If $x \notin L$, then $\chi_L(T)_M(x) = 0, \chi_L(I)_M(x) = -1$, and $\chi_L(F)_M(x) = 0$. Now, for any $h \in Aut(X)$, $h(x) \notin h(L)$ which implies $\chi_L(T)_M(h(x)) = 0, \chi_L(I)_M(h(x)) = -1$, and $\chi_L(F)_M(h(x)) = 0$. Thus $\chi_L(T)_M(h(x)) = \chi_L(T)_M(x)$, $\chi_L(I)_M(h(x)) = \chi_L(I)_M(x)$, and $\chi_L(F)_M(h(x)) = \chi_L(F)_M(x)$ for all $x \in X$ and hence $\chi_{\mathrm{L}}(\mathrm{X}_{\mathrm{M}})$ is neutrosophic $\aleph-$ characteristic interior ideal.

Conversely, assume that $\chi_{\mathrm{L}}(\mathrm{X}_{\mathrm{M}})$ is neutrosophic $\aleph-$characteristic interior ideal. Then by Theorem 3.1, L is an interior ideal . Now, let $h \in Aut(X)$ and $x \in L$. Then $\chi_L(T)_M(x) = -1$, $\chi_L(I)_M(x) = 0$ and $\chi_L(F)_M(x) = -1$. Since $\chi_{\mathrm{L}}(\mathrm{X}_{\mathrm{M}})$ is neutrosophic $\aleph-$characteristic interior ideal , we have $\chi_L(T)_M(h(x)) = \chi_L(T)_M(x)$, $\chi_L(I)_M(h(x)) = \chi_L(I)_M(x)$ and $\chi_L(F)_M(h(x)) = \chi_L(T)_M(x)$ which imply $h(x) \in L$. So $h(L) \subseteq L$ for all $h \in Aut(X)$. Again, since $h \in Aut(X)$ and $x \in L$, there exists $y \in L$ such that $h(y) = x$.

Suppose that $y \notin L$. Then $\chi_L(T)_M(y) = 0$, $\chi_L(I)_M(y) = -1$ and $\chi_L(F)_M(y) = 0$. Since $\chi_L(T)_M(h(y)) = \chi_L(T)_M(y)$, $\chi_L(I)_M(h(y)) = \chi_L(I)_M(y)$ and $\chi_L(F)_M(h(y)) = \chi_L(T)_M(y)$, we get $\chi_L(T)_M(h(y)) = 0$, $\chi_L(I)_M(h(y)) = -1$ and $\chi_L(F)_M(h(y)) = 0$ which imply $h(y) \notin L$, a contradiction. So $y \in L$ i.e., $h(y) \in L$. Thus $L \subseteq h(L)$ for all $h \in Aut(X)$ and hence L is characteristic interior ideal. □

Theorem 3.23. For a semigroup X, the equivalent statements are:

(i) X is intra-regular,

(ii) For any neutrosophic $\aleph-$interior ideal X_M, we have $X_M(w) = X_M(w^2)$ for all $w \in X$.

Proof: $(i) \Rightarrow (ii)$ Suppose X is intra-regular, and X_M is neutrosophic $\aleph-$ interior ideal and $w \in X$. Then there exist $r, s \in X$ such that $w = rw^2s$. Now $T_M(w) = T_M(rw^2s) \leq T_M(w^2) \leq T_M(w)$ and so $T_M(w) = T_M(w^2)$, $I_M(w) = I_M(rw^2s) \geq I_M(w^2) \geq I_M(w)$ and so $I_M(w) = I_M(w^2)$, and $F_M(w) = F_M(rw^2s) \leq F_M(w^2) \leq F_M(w)$ and so $F_M(w) = F_M(w^2)$. Therefore $X_M(w) = X_M(w^2)$ for all $w \in X$.

$(ii) \Rightarrow (i)$ Let (ii) holds and $s \in X$. Then $I(s^2)$ is an ideal of X. By Theorem 3.5 of [4], $\chi_{I(s^2)}(X_M)$ is neutrosophic $\aleph-$ideal. By assumption, $\chi_{I(s^2)}(X_M)(s) = \chi_{I(s^2)}(X_M)(s^2)$. Since $\chi_{I(s^2)}(T)_M(s^2) = -1 = \chi_{I(s^2)}(F)_M(s^2)$ and $\chi_{I(s^2)}(I)_M(s^2) = 0$, we get $\chi_{I(s^2)}(T)_M(s) = -1 = \chi_{I(s^2)}(F)_M(s)$ and $\chi_{I(s^2)}(I)_M(s^2) = 0$ which imply $s \in I(s^2)$. Hence X is intra-regular. □

Theorem 3.24. For a semigroup X, the equivalent statements are:

(i) X is left (resp., right) regular,

(ii) For any neutrosophic $\aleph$ −interior ideal X_M, we have $X_M(w) = X_M(w^2)$ for all $w \in X$.

Proof: $(i) \Rightarrow (ii)$ Let X be left regular. Then there exists $y \in X$ such that $w = yw^2$. Let X_M be a neutrosophic $\aleph$ −interior ideal. Then $T_M(w) = T_M(yw^2) \leq T_M(w)$ and so $T_M(w) = T_M(w^2)$, $I_M(w) = I_M(yw^2) \geq I_M(w)$ and so $I_M(w) = I_M(w^2)$, and $F_M(w) = F_M(yw^2) \leq F_M(w)$ and so $F_M(w) = F_M(w^2)$. Therefore $X_M(w) = X_M(w^2)$ for all $w \in X$.

$(ii) \Rightarrow (i)$ Suppose (ii) holds and let X_M be neutrosophic $\aleph$ −interior ideal. Then for any $w \in X$, $\chi_{L(w^2)}(T)_M(w) = \chi_{L(w^2)}(T)_M(w^2) = -1$, $\chi_{L(w^2)}(I)_M(w) = \chi_{L(w^2)}(I)_M(w^2) = 0$ and $\chi_{L(w^2)}(F)_M(w) = \chi_{L(w^2)}(F)_M(w^2) = -1$ which imply $w \in L(w^2)$. Thus X is left regular. □

Conclusions

In this paper, we have introduced the concepts of neutrosophic $\aleph$ − interior ideals and neutrosophic $\aleph$ − characteristic interior ideals in semigroups and studied their properties, and characterized regular and intra-regular semigroups using neutrosophic $\aleph$-interior ideal structures. We have also shown that R is a characteristic interior ideal if and only if the characteristic neutrosophic $\aleph$ −structure $\chi_R(X_N)$ is neutrosophic $\aleph$ −characteristic interior ideal. In future, we will define neutrosophic $\aleph$ −prime ideals in semigroups and study their properties.

Reference

1. Atanassov, K. T. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems* **1986**, 20, 87-96.
2. Abdel-Basset, M., Ali, M., & Atef, A. Uncertainty assessments of linear time-cost tradeoffs using neutrosophic set. *Computers & Industrial Engineering* **2020**, 141, 106286.
3. Abdel-Basset, M., Ali, M., & Atef, A. Resource levelling problem in construction projects under neutrosophic environment. *The Journal of Supercomputing* **2020**, 76(2), 964-988.
4. Abdel-Baset, M.; Chang, V.; Gamal, A.; Smarandache, F. An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field. *Computers in Industry* ***2019***, 106, 94-110.
5. Abdel-Basset, M., Gamal, A., Son, L. H., & Smarandache, F. A Bipolar Neutrosophic Multi Criteria Decision Making Framework for Professional Selection. *Applied Sciences* **2020**, 10(4), 1202.
6. Abdel-Basset, M., Mohamed, R., Elhoseny, M., & Chang, V. Evaluation framework for smart disaster response systems in uncertainty environment. *Mechanical Systems and Signal Processing* **2020**, 145, 106941.
7. Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., Gamal, A., & Smarandache, F. Solving the supply chain problem using the best-worst method based on a novel Plithogenic model. In Optimization Theory Based on Neutrosophic and Plithogenic Sets, Academic Press **2020**, 1-19.
8. Abdel-Basset, M.; Saleh, M.; Gamal, A.; Smarandache, F. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing* **2019**, 77, 438-452.
9. Chalapathi, T; Madhavi, L. Neutrosophic Boolean rings. *Neutrosophic Sets and Systems* **2020**, 33, 59-66.
10. Elavarasan, B.; Smarandache, F.; Jun, Y. B. Neutrosophic $\aleph$ −ideals in semigroups. *Neutrosophic Sets and Systems* **2019**, 28, 273-280.
11. Gulistan, M.; Khan, A.; Abdullah, A.; Yaqoob, N. Complex Neutrosophic subsemigroups and ideals. *International J. Analysis and Applications* **2018**, 16, 97-116.

12. Jun, Y. B.; Smarandache, F.; Ozturk, M. A. Commutative falling neutrosophic ideals in BCK-algebras. *Neutrosophic Sets and Systems* **2018**, 20, 44-53.
13. Kehayopulu, N. A note on strongly regular ordered semigroups. *Sci. Math.* **1998**, 1, 33-36.
14. Khan, M. S.; Anis; Smarandache, F.; Jun,Y. B. Neutrosophic $\aleph -$structures and their applications in semigroups. *Annals of Fuzzy Mathematics and Informatics* **2017**, 14(6), 583–598.
15. Mordeson, J.N.; Malik, D. S.; Kuroki. N. Regular semigroups. *Fuzzy Smigroups* **2003**, 59-100.
16. Muhiuddin, G.; Ahmad, N.; Al-Kenani; Roh, E. H.; Jun, Y. B. Implicative neutrosophic quadruple BCK-algebras and ideals. *Symmetry* **2019**, 11, 277.
17. Muhiuddin, G.; Bordbar, H.; Smarandache, F.; Jun, Y. B. Further results on (2;2)-neutrosophic subalgebras and ideals in BCK/BCI- algebras. *Neutrosophic Sets and Systems* **2018**, Vol. 20, 36-43.
18. Muhiuddin, G.; Kim, S. J.; Jun, Y. B. Implicative N-ideals of BCK-algebras based on neutrosophic N-structures. *Discrete Mathematics, Algorithms and Applications* **2019**, Vol. 11, No. 01, 1950011.
19. Muhiuddin, G.; Smarandache, F.; Jun, Y. B. Neutrosophic quadruple ideals in neutrosophic quadruple BCI-algebras. *Neutrosophic Sets and Systems* **2019**, 25, 161-173.
20. Porselvi, K., Elavarasan, B.; Smarandache, F.; Jun, Y. B. Neutrosophic $\aleph - \text{bi} -$ideals in semigroups. *Neutrosophic Sets and Systems* **2020**, 35, 422-434.
21. Prakasam Muralikrishna.; Surya Manokaran. MBJ – Neutrosophic $\boldsymbol{\beta}$ – Ideal of $\boldsymbol{\beta}$ – Algebra. *Neutrosophic Sets and Systems* **2020**, 35, 99-118.
22. Smarandache, F. A. Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. *American Research Press* **1999**, Rehoboth, NM.
23. Smarandache, F. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *Int. J. Pure Appl. Math.* **2005,** 24(3), 287-297.
24. Zadeh, L. A. Fuzzy sets. *Information and Control* **1965**, 8, 338 - 353.

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