Neutrosophic ℵ-interior ideals in semigroups

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Abstract: We define the concepts of neutrosophic ℵ-interior ideal and neutrosophic ℵ-characteristic interior ideal structures of a semigroup. We infer different types of semigroups using neutrosophic ℵ-interior ideal structures. We also show that the intersection of neutrosophic ℵ-interior ideals and the union of neutrosophic ℵ-interior ideals is also a neutrosophic ℵ-interior ideal.

Keywords: Semi group, neutrosophic ℵ-ideals, neutrosophic ℵ-interior ideals, neutrosophic ℵ-product.

1. Introduction

Nowadays, the theory of uncertainty plays a vital role to manage different issues relating to modelling engineering problems, networking, real-life problem relating to decision making and so on. In 1965, Zadeh[24] introduced the idea of fuzzy sets for modelling vague concepts in the globe. In 1986, Atanassov [1] generalized fuzzy set and named as Intuitionistic fuzzy set. Also, from his viewpoint, there are two degrees of freedom in the real world, one a degree of membership to a vague subset and the other is a degree of non-membership to that given subset.

Smarandache generalized fuzzy set and intuitionistic fuzzy set, and named as neutrosophic set (see [4, 7, 8, 14, 19, 22-23]). These sets are characterized by a truth membership function, an indeterminacy membership function and a falsity membership function. These sets are applied to many branches of mathematics to overcome the complexities arising from uncertain data. A Neutrosophic set can distinguish between absolute membership and relative membership. Smarandache used this in non-standard analysis such as the result of sports games (winning/defeating/tie), decision making and control theory, etc. This area has been studied by several authors (see [3, 11, 12, 16-18]).

For more details on neutrosophic set theory, the readers visit the website http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In [2], Abdel Basset et al. designed a framework to manage scheduling problems using neutrosophic theory. As the concept of time-cost tradeoffs and deterministic project scheduling disagree with the real situation, some data were changed during the implementation process. Here fuzzy scheduling and time-cost tradeoffs models assumed only truth-membership functions dealing
with uncertainties of the project and their activities duration which were unable to treat indeterminacy and inconsistency.

In [6], Abdel Basset et al. evaluated the performance of smart disaster response systems under uncertainty. In [5], Abdel Basset et al. introduced different hybrid neutrosophic multi-criteria decision-making framework for professional selection that employed a collection of neutrosophic analytical network process and order preference by similarity to the ideal solution under bipolar neutrosophic numbers.

In [21], Prakasam Muralikrishna1 et al. presented the characterization of MBJ – Neutrosophic $\beta$ Ideal of $\beta$ – algebra. They analyzed homomorphic image, pre–image, cartesian product and related results, and these concepts were explored to other substructures of a $\beta$ – algebra. In [9], Chalapathi et al. constructed certain Neutrosophic Boolean rings, introduced Neutrosophic complement elements and mainly obtained some properties satisfied by the Neutrosophic complement elements of Neutrosophic Boolean rings.

In [14], M. Khan et al. presented the notion of neutrosophic $\aleph$-subsemigroup in semigroup and explored several properties. In [11], Gulistan et al. have studied the idea of complex neutrosophic subsemigroups and introduced the concept of the characteristic function of complex neutrosophic sets, direct product of complex neutrosophic sets.

In [10], B. Elavarasan et al. introduced the notion of neutrosophic $\aleph$-ideal in semigroup and explored its properties. Also, the conditions for neutrosophic $\aleph$-structure to be neutrosophic $\aleph$-ideal are given, and discussed the idea of characteristic neutrosophic $\aleph$-structure in semigroups and obtained several properties. In [20], we have introduced and discussed several properties of neutrosophic $\aleph$-product and the intersection of neutrosophic $\aleph$-ideals were identical for regular semigroups. In this paper, we define and discuss the concepts of neutrosophic $\aleph$-interior ideal and neutrosophic $\aleph$-characteristic interior ideal structures of a semigroup.

Throughout this paper, $X$ denotes a semigroup. Now, we present the important definitions of semigroup that we need in sequel.

Recall that for any $X_1, X_2 \subseteq X$, $X_1X_2 = \{ab | a \in X_1 \text{ and } b \in X_2\}$, multiplication of $X_1$ and $X_2$.

Let $X$ be a semigroup and $\emptyset \neq X_1 \subseteq X$. Then

(i) $X_1$ is known as subsemigroup if $X_1^2 \subseteq X_1$.

(ii) A subsemigroup $X_1$ is known as left (resp., right) ideal if $X_1X \subseteq X_1$(resp., $XX_1 \subseteq X_1$).

(iii) $X_1$ is known as ideal if $X_1$ is both a left and a right ideal.

(iv) $X$ is known as left (resp., right) regular if for each $r \in X$, there exists $i \in X$ such that $r = ir^2$(resp., $r = r^2i$) [13].

(v) $X$ is known as regular if for each $b_1 \in X$, there exists $i \in X$ such that $b_1 = b_1i$ $b_1$

(vi) $X$ is known as intra-regular if for each $x_1 \in X$, there exist $i,j \in X$ such that $x_1 = ix_1^2j$ [15].

2. Definitions of neutrosophic $\aleph$ - structures

We present definitions of neutrosophic $\aleph$ – structures namely neutrosophic $\aleph$ – subsemigroup, neutrosophic $\aleph$ – ideal, neutrosophic $\aleph$ – interior ideal of a semigroup $X$
The set of all the functions from $X$ to $[-1, 0]$ is denoted by $\mathcal{Z}(X, [-1, 0])$. We call that an element of $\mathcal{Z}(X, [-1, 0])$ is $\mathcal{K}$-function on $X$. A $\mathcal{K}$-structure means an ordered pair $(X, g)$ of $X$ and an $\mathcal{K}$-function $g$ on $X$.

**Definition 2.1**[14] A neutrosophic $\mathcal{K}$-structure of $X$ is defined to be the structure:

$$X_M := \left\{ x \in (T_M, I_M, F_M) : \exists r \in X \right\},$$

where $T_M$, $I_M$ and $F_M$ are the negative truth, negative indeterminacy and negative falsity membership functions on $X (\mathcal{K}$-functions).

It is evident that $-3 \leq T_M(r) + I_M(r) + F_M(r) \leq 0$ for all $r \in X$.

**Definition 2.2**[14] A neutrosophic $\mathcal{K}$-structure $X_M$ of $X$ is called a neutrosophic $\mathcal{K}$-subsemigroup of $X$ if the following assertion is valid:

$$(\forall g, h \in X) \left( T_M(g, h) \leq T_M(g) \lor T_M(h) \right)$$

Let $X_M$ be a neutrosophic $\mathcal{K}$-structure and $\gamma, \delta, \varepsilon \in [-1, 0]$ with $-3 \leq \gamma + \delta + \varepsilon \leq 0$. Consider the sets:

$T_M^\gamma = \{ r \in X : T_M(r) \leq \gamma \}$

$I_M^\delta = \{ r \in X : I_M(r) \geq \delta \}$

$F_M^\varepsilon = \{ r \in X : F_M(r) \leq \varepsilon \}$

The set $X_M(\gamma, \delta, \varepsilon) := \{ r \in X : T_M(r) \leq \gamma, I_M(r) \geq \delta, F_M(r) \leq \varepsilon \}$ is known as $(\gamma, \delta, \varepsilon)$-level set of $X_M$. It is easy to observe that $X_M(\gamma, \delta, \varepsilon) = T_M^\gamma \cap I_M^\delta \cap F_M^\varepsilon$.

**Definition 2.3**[10] A neutrosophic $\mathcal{K}$-structure $X_M$ of $X$ is called a neutrosophic $\mathcal{K}$-left (resp., right) ideal of $X$ if

$$(\forall g, h \in X) \left( T_M(g, h) \leq T_M(h) \quad (\text{resp.,} \quad T_M(g, h) \leq T_M(g)) \right)$$

$X_M$ is neutrosophic $\mathcal{K}$-ideal of $X$ if $X_M$ is neutrosophic $\mathcal{K}$-left and $\mathcal{K}$-right ideal of $X$.

**Definition 2.4** A neutrosophic $\mathcal{K}$-subsemigroup $X_M$ of $X$ is known as neutrosophic $\mathcal{K}$-interior ideal if

$$(\forall x, a, y \in X) \left( T_M(x, y) \leq T_M(a) \quad \text{resp.,} \quad T_M(x, y) \leq T_M(a) \right)$$

It is easy to observe that every neutrosophic $\mathcal{K}$-ideal is neutrosophic $\mathcal{K}$-interior ideal, but neutrosophic $\mathcal{K}$-interior ideal need not be a neutrosophic $\mathcal{K}$-ideal, as shown by an example.

**Example 2.5** Let $X$ be the set of all non-negative integers except 1. Then $X$ is a semigroup with usual multiplication.

Let $X_M = \{ 0, 2, 5, 10 \}$ then $X_M$ is neutrosophic $\mathcal{K}$-interior ideal, but not neutrosophic $\mathcal{K}$-ideal with $T_M(2.5) = -0.3 \leq T_M(2)$.

**Definition 2.6**[14] For any $E \subseteq X$, the characteristic neutrosophic $\mathcal{K}$-structure is defined as

$$X_E(X_M) = \frac{X}{(X_E(T_M) \cap X_E(I_M) \cap X_E(F_M))}$$
where

\[
X_E(T)_M: X \to [-1, 0], \ r \to \begin{cases} -1 & \text{if } r \in E \\ 0 & \text{otherwise}, \end{cases}
\]

\[
X_E(I)_M: X \to [-1, 0], \ r \to \begin{cases} 0 & \text{if } r \in E \\ -1 & \text{otherwise}, \end{cases}
\]

\[
X_E(F)_M: X \to [-1, 0], \ r \to \begin{cases} -1 & \text{if } r \in E \\ 0 & \text{otherwise}. \end{cases}
\]

**Definition 2.7.[14]** Let \(X_N := \frac{x}{(T_N, I_N, F_N)}\) and \(X_M := \frac{x}{(T_M, I_M, F_M)}\) be neutrosophic \(\kappa\)-structures of \(X\). Then

(i) \(X_N\) is called a neutrosophic \(\kappa\)-substructure of \(X_M\), denote by \(X_M \subseteq X_N\), if \(T_M(r) \geq T_N(r), I_M(r) \leq I_N(r), F_M(r) \geq F_N(r)\) for all \(r \in X\).

(ii) If \(X_N \subseteq X_M\) and \(X_M \subseteq X_N\), then we say that \(X_N = X_M\).

(iii) The neutrosophic \(\kappa\)-product of \(X_N\) and \(X_M\) is defined to be a neutrosophic \(\kappa\)-structure of \(X\),

\[
X_N \otimes X_M := \frac{x}{(T_N \circ T_M, I_N \circ I_M, F_N \circ F_M)} = \left\{ \frac{h}{T_{N \circ M}(h), I_{N \circ M}(h), F_{N \circ M}(h)} \mid h \in X \right\},
\]

where

\[
(T_N \circ T_M)(h) = T_{N \circ M}(h) = \begin{cases} \bigwedge_{r \in rs} (T_N(r) \lor T_M(s)) & \text{if } \exists r, s \in X \text{ such that } h = rs \\ 0 & \text{otherwise}, \end{cases}
\]

\[
(I_N \circ I_M)(h) = I_{N \circ M}(h) = \begin{cases} \bigvee_{r \in rs} (I_N(r) \land I_M(s)) & \text{if } \exists u, v \in X \text{ such that } h = rs \\ -1 & \text{otherwise}, \end{cases}
\]

\[
(F_N \circ F_M)(h) = F_{N \circ M}(h) = \begin{cases} \bigwedge_{r \in rs} (F_N(r) \lor F_M(s)) & \text{if } \exists u, v \in X \text{ such that } h = rs \\ 0 & \text{otherwise}. \end{cases}
\]

For \(i \in X\), the element \(\frac{i}{(T_{N \circ M}(i), I_{N \circ M}(i), F_{N \circ M}(i))}\) is simply denoted by \((X_N \otimes X_M)(i) = (T_{N \circ M}(i), I_{N \circ M}(i), F_{N \circ M}(i))\).

(iii) The union of \(X_N\) and \(X_M\), a neutrosophic \(\kappa\)-structure over \(X\) is defined as

\[
X_N \cup X_M = X_{N \cup M} = (X; T_{N \cup M}, I_{N \cup M}, F_{N \cup M}),
\]

where

\[
(T_{N \cup M}(h)) = T_{N \cup M}(h) = T_N(h) \lor T_M(h),
\]

\[
(I_{N \cup M}(h)) = I_{N \cup M}(h) = I_N(h) \land I_M(h),
\]

\[
(F_{N \cup M}(h)) = F_{N \cup M}(h) = F_N(h) \lor F_M(h) \forall h \in X.
\]

(iv) The intersection of \(X_N\) and \(X_M\), a neutrosophic \(\kappa\)-structure over \(X\) is defined as

\[
X_N \cap X_M = X_{N \cap M} = (X; T_{N \cap M}, I_{N \cap M}, F_{N \cap M}),
\]

where

\[
(T_{N \cap M}(h)) = T_{N \cap M}(h) = T_N(h) \land T_M(h),
\]

\[
(I_{N \cap M}(h)) = I_{N \cap M}(h) = I_N(h) \lor I_M(h),
\]

\[
(F_{N \cap M}(h)) = F_{N \cap M}(h) = F_N(h) \land F_M(h) \forall h \in X.
\]

3. Neutrosophic \(\kappa\)-interior ideals

We study different properties of neutrosophic \(\kappa\)-interior ideals of \(X\). It is evident that neutrosophic \(\kappa\)-ideal is a neutrosophic \(\kappa\)-interior ideal of \(X\), but not the converse. Further, for a regular and for an intra-regular semigroup, every neutrosophic \(\kappa\)-interior ideal is neutrosophic \(\kappa\)-ideal.
All throughout this part, we consider $X_m$ and $X_n$ are neutrosophic $\aleph$—structures of $X$.

**Theorem 3.1.** For any $L \subseteq X$, the equivalent assertions are:

(i) $L$ is an interior ideal,

(ii) The characteristic neutrosophic $\aleph$—structure $\chi_L(X_N)$ is a neutrosophic $\aleph$—interior ideal.

**Proof:** Suppose $L$ is an interior ideal and let $x, a, y \in X$.

If $a \in L$, then $xay \in L$, so $\chi_L(T)_N(xay) = -1 = \chi_L(T)_N(a)$, $\chi_L(I)_N(xay) = 0 = \chi_L(I)_N(a)$ and $\chi_L(F)_N(xay) = -1 = \chi_L(F)_N(a)$.

If $a \notin L$, then $\chi_L(T)_N(xay) \leq 0 = \chi_L(T)_N(a)$, $\chi_L(I)_N(xay) \geq -1 = \chi_L(I)_N(a)$ and $\chi_L(F)_N(xay) \leq 0 = \chi_L(F)_N(a)$.

Therefore $\chi_L(X_N)$ is a neutrosophic $\aleph$—interior ideal.

Conversely, assume that $\chi_L(X_N)$ is a neutrosophic $\aleph$—interior ideal. Let $u \in L$ and $x, y \in X$. Then

$$\chi_L(T)_N(xay) \leq \chi_L(T)_N(u) = -1,$$

$$\chi_L(I)_N(xay) \geq \chi_L(I)_N(u) = 0,$$

$$\chi_L(F)_N(xay) \leq \chi_L(F)_N(u) = -1.$$

So $xuy \in L$. \hfill \square

**Theorem 3.2.** If $X_M$ and $X_N$ are neutrosophic $\aleph$—interior ideals, then $X_{MN}$ is neutrosophic $\aleph$—interior ideal.

**Proof:** Let $X_M$ and $X_N$ be neutrosophic $\aleph$—interior ideals. For any $r, s, t \in X$, we have

$$T_{MN}(rst) = T_M(rst) \cup T_N(rst) \subseteq T_M(s) \cup T_N(s) = T_{MN}(s),$$

$$I_{MN}(rst) = I_M(rst) \cap I_N(rst) \supseteq I_M(s) \cap I_N(s) = I_{MN}(s),$$

$$F_{MN}(rst) = F_M(rst) \cap F_N(rst) \subseteq F_M(s) \cap F_N(s) = F_{MN}(s).$$

Therefore $X_{MN}$ is neutrosophic $\aleph$—interior ideal. \hfill \square

**Corollary 3.3.** The arbitrary intersection of neutrosophic $\aleph$—interior ideals is a neutrosophic $\aleph$—interior ideal.

**Theorem 3.4.** If $X_M$ and $X_N$ are neutrosophic $\aleph$—interior ideals, then $X_{MN}$ is neutrosophic $\aleph$—interior ideal.

**Proof:** Let $X_M$ and $X_N$ be neutrosophic $\aleph$—interior ideals. For any $r, s, t \in X$, we have

$$T_{MN}(rst) = T_M(rst) \cap T_N(rst) \subseteq T_M(s) \cap T_N(s) = T_{MN}(s),$$

$$I_{MN}(rst) = I_M(rst) \cup I_N(rst) \supseteq I_M(s) \cup I_N(s) = I_{MN}(s),$$

$$F_{MN}(rst) = F_M(rst) \cup F_N(rst) \subseteq F_M(s) \cup F_N(s) = F_{MN}(s).$$

Therefore $X_{MN}$ is neutrosophic $\aleph$—interior ideal. \hfill \square

**Corollary 3.5.** The arbitrary union of neutrosophic $\aleph$—interior ideals is neutrosophic $\aleph$—interior ideal.

**Theorem 3.6.** Let $X$ be a regular semigroup. If $X_M$ is neutrosophic $\aleph$—interior ideal, then $X_M$ is neutrosophic $\aleph$—ideal.
Proof: Assume that \( X_M \) is an interior ideal, and let \( u, v \in X \). As \( X \) is regular and \( u \in X \), there exists \( r \in X \) such that \( u = uru \). Now, \( T_M(uv) = T_M(uru) \leq T_M(u) \), \( I_M(uv) = I_M(uru) \geq I_M(u) \) and \( F_M(uv) = F_M(uru) \leq F_M(u) \). Therefore \( X_M \) is neutrosophic \( \kappa - \) right ideal.

Similarly, we can show that \( X_M \) is neutrosophic \( \kappa - \) left ideal and hence \( X_M \) is neutrosophic \( \kappa - \) ideal. \( \square \)

Theorem 3.7. Let \( X \) be an intra-regular semigroup. If \( X_M \) is neutrosophic \( \kappa - \) interior ideal, then \( X_M \) is neutrosophic \( \kappa - \) ideal.

Proof: Suppose that \( X_M \) is neutrosophic \( \kappa - \) interior ideal, and let \( u, v \in X \). As \( X \) is intra regular and \( u \in X \), there exist \( s, t \in S \) such that \( u = su^2t \). Now,

\[
T_M(uv) = T_M(su^2tv) \leq T_M(u),
I_M(uv) = I_M(su^2tv) \geq I_M(u)
F_M(uv) = F_M(su^2tv) \leq F_M(u).
\]

Therefore \( X_M \) is neutrosophic \( \kappa - \) right ideal. similarly, we can show that \( X_M \) is neutrosophic \( \kappa - \) left ideal and hence \( X_M \) is neutrosophic \( \kappa - \) ideal. \( \square \)

Definition 3.8. A semigroup \( X \) is left simple (resp., right simple) if it does not contain any proper left ideal (resp., right ideal) of \( X \). A semigroup \( X \) is simple if it does not contain any proper ideal of \( X \).

Definition 3.9. A semigroup \( X \) is said to be neutrosophic \( \kappa - \)simple if every neutrosophic \( \kappa - \) ideal is a constant function

i.e., for every neutrosophic \( \kappa - \) ideal \( X_M \) of \( X \), we have \( T_M(i) = T_M(j), I_M(i) = I_M(j) \) and \( F_M(i) = F_M(j) \) for all \( i, j \in X \).

Notation 3.10. If \( X \) is a semigroup and \( s \in X \), we define a subset, denoted by \( I_s \), as follows:

\[
I_s = \{ i \in X \mid T_N(i) \leq T_N(s), I_N(i) \geq I_N(s) \ \text{and} \ F_N(i) \leq F_N(s) \}.
\]

Proposition 3.11. If \( X_N \) is neutrosophic \( \kappa - \) right (resp., \( \kappa - \) left, \( \kappa - \) ideal) ideal, then \( I_s \) is right (resp., left, ideal) ideal for every \( s \in X \).

Proof: Let \( s \in X \). Then it is clear that \( \varphi \neq I_s \subseteq X \). Let \( u \in I_s \) and \( x \in X \). Then \( ux \in I_s \). Indeed; Since \( X_N \) is neutrosophic \( \kappa - \) right ideal and \( u, x \in X \), we get \( T_N(ux) \leq T_N(u), I_N(ux) \geq I_N(u) \) and \( F_N(ux) \leq F_N(t) \). Since \( u \in I_N \), we get \( T_N(u) \leq T_N(s), I_N(u) \geq I_N(s) \) and \( F_N(u) \leq F_N(s) \) which imply \( ux \in I_s \). Therefore \( I_s \) is a right ideal for every \( s \in X \). \( \square \)

Theorem 3.12.\([4]\) For any \( L \subseteq X \), the equivalent assertions are:

(i) \( L \) is left (resp., right) ideal,
(ii) Characteristic neutrosophic \( \kappa - \) structure \( \chi_L(X_N) \) is neutrosophic \( \kappa - \) left (resp., right) ideal.

Theorem 3.13. Let \( X \) be a semigroup. Then \( X \) is simple if and only if \( X \) is neutrosophic \( \kappa - \)simple.
Proof: Suppose \( X \) is simple. Let \( X_M \) be a neutrosophic \( \mathcal{N} \)– ideal and \( u, v \in X \). Then by Proposition 3.11, \( I_u \) is an ideal of \( X \). As \( X \) is simple, we have \( I_u = X \). Since \( v \in I_u \), we have \( T_M(v) \leq T_M(u), I_M(v) \geq I_M(u) \) and \( F_M(v) \leq F_M(u) \).

Similarly, we can prove that \( T_M(u) \leq T_M(v), I_M(u) \geq I_M(v) \) and \( F_M(u) \leq F_M(v) \). So \( T_M(u) = T_M(v), I_M(u) = I_M(v) \) and \( F_M(u) = F_M(v) \). Hence \( X \) is neutrosophic \( \mathcal{N} \)– simple.

Conversely, assume that \( X \) is neutrosophic \( \mathcal{N} \)– simple and \( I \) is an ideal of \( X \). Then by Theorem 3.12, \( \chi(X_N) \) is a neutrosophic \( \mathcal{N} \)– ideal. We now claim that \( X = I \). Let \( w \in X \). Since \( X \) is neutrosophic \( \mathcal{N} \)– simple, we have \( \chi(X_N)(w) = \chi(X_N)(y) \) for every \( y \in X \). In particular, we have \( \chi(T_N)(w) = \chi(T_N)(d) = -1, \chi(I_N)(w) = \chi(I_N)(d) = 0 \) and \( \chi(F_N)(w) = \chi(F_N)(d) = -1 \) for any \( d \in I \) which implies \( w \in I \). Thus \( X \subseteq I \) and hence \( X = I \).

Lemma 3.14. Let \( X \) be a semigroup. Then \( X \) is simple if and only for every \( t \in X \), we have \( X = Xtx \).

Proof: Suppose \( X \) is simple and let \( t \in X \). Then \( X(Xtx) \subseteq Xtx \) and \( (Xtx)X \subseteq Xtx \) imply that \( Xtx \) is an ideal. Since \( X \) is simple, we have \( Xtx = X \).

Conversely, let \( P \) be an ideal and let \( a \in P \). Then \( X = Xax, Xax \subseteq XPX \subseteq P \) which implies \( P = X \). Therefore \( X \) is simple.

Theorem 3.15. Suppose \( X \) is a semigroup. Then \( X \) is simple if and only every neutrosophic \( \mathcal{N} \)– interior ideal of \( X \) is a constant function.

Proof: Suppose \( X \) is simple and \( s, t \in X \). Let \( X_N \) be neutrosophic \( \mathcal{N} \)– interior ideal. Then by Lemma 3.14, we get \( X = Xsx = Xtx \). As \( s \in Xsx \), we have \( s = abt \) for \( a, b \in X \). Since \( X_N \) is neutrosophic \( \mathcal{N} \)– interior ideal, we have \( T_N(s) = T_N(abt) \leq T_N(t), I_N(s) = I_N(abt) \geq I_N(t) \) and \( F_N(s) = F_N(abt) \leq F_N(t) \). Similarly, we can prove that \( T_N(t) \leq T_N(s), I_N(t) \geq I_N(s) \) and \( F_N(t) \leq F_N(s) \). So \( X_N \) is a constant function.

Conversely, suppose \( X_N \) is neutrosophic \( \mathcal{N} \)– ideal. Then \( X_N \) is neutrosophic \( \mathcal{N} \)– interior ideal. By hypothesis, \( X_N \) is a constant function and so \( X_N \) is neutrosophic \( \mathcal{N} \)–simple. By Theorem 3.13, \( X \) is simple.

Theorem 3.16. Let \( X_M \) be neutrosophic \( \mathcal{N} \)– structure and let \( \gamma, \delta, \epsilon \in [-1, 0] \) with \(-3 \leq \gamma + \delta + \epsilon \leq 0 \). If \( X_M \) is neutrosophic \( \mathcal{N} \)– interior ideal, then \((\gamma, \delta, \epsilon)\)-level set of \( X_M \) is neutrosophic \( \mathcal{N} \)– interior ideal whenever \( X_M(\gamma, \delta, \epsilon) \neq \emptyset \).

Proof: Suppose \( X_M(\gamma, \delta, \epsilon) \neq \emptyset \) for \( \gamma, \delta, \epsilon \in [-1, 0] \) with \(-3 \leq \gamma + \delta + \epsilon \leq 0 \).

Let \( X_M \) be a neutrosophic \( \mathcal{N} \)– interior ideal and let \( u, v, w \in X_M(\gamma, \delta, \epsilon) \). Then \( T_M(uvw) \leq T_M(v) \leq \alpha; I_M(uvw) \geq I_M(v) \geq \beta \) and \( F_M(uvw) \leq F_M(v) \leq \gamma \) which imply \( uvw \in X_M(\alpha, \beta, \gamma) \). Therefore \( X_M(\gamma, \delta, \epsilon) \) is a neutrosophic \( \mathcal{N} \)– interior ideal of \( X \).

Theorem 3.17. Let \( X_N \) be neutrosophic \( \mathcal{N} \)– structure with \( \alpha, \beta, \gamma \in [-1, 0] \) such that \(-3 \leq \alpha + \beta + \gamma \leq 0 \). If \( T^u_N, I^t_N \) and \( F^d_N \) are interior ideals, then \( X_N \) is neutrosophic \( \mathcal{N} \)– interior ideal of \( X \) whenever it is non-empty.

Proof: Suppose that for \( a, b, c \in X \) with \( T_N(abc) > T_N(b) \). Then \( T_N(abc) > t_a \geq T_N(b) \) for some \( t_a \in [-1, 0] \). So \( b \in T^b_N(b) \) but \( abc \notin T^a_N(b) \), a contradiction. Thus \( T_N(abc) \leq T_N(b) \).
Suppose that for \( a, b, c \in X \) with \( I_N(ab) \leq F_N(b) \). Then \( I_N(ab) < t_a \leq I_N(b) \) for some \( t_a \in [-1, 0) \). So \( b \in I_N^a \) but \( abc \notin I_N^a \), a contradiction. Thus \( I_N(ab) \geq I_N(b) \).

Suppose that for \( a, b, c \in X \) with \( F_N(ab) > F_N(b) \). Then \( F_N(ab) > t_a \geq F_N(b) \) for some \( t_a \in [-1, 0) \). So \( b \in I_N^a \) but \( abc \notin I_N^a \), a contradiction. Thus \( F_N(ab) \leq F_N(b) \).

Thus \( X_N \) is neutrosophic \( \mathbb{K}^- \)– interior ideal.

\[ \square \]

**Theorem 3.18.** Let \( X_M \) be neutrosophic \( \mathbb{K}^- \)– structure over \( X \). Then the equivalent assertions are:

(i) \( X_M \) is neutrosophic \( \mathbb{K}^- \)– interior ideal,

(ii) \( X_N \bigcap X_M \bigcap X_N \subseteq X_M \) for any neutrosophic \( \mathbb{K}^- \)– structure \( X_N \).

**Proof:** Suppose \( X_M \) is neutrosophic \( \mathbb{K}^- \)– interior ideal. Let \( x \in X \). For any \( u, v, w \in X \) such that \( x = uvw \). Then \( T_M(x) = T_M(uvw) \leq T_M(v) \leq T_N(u) \cup T_M(v) \cup T_N(w) \) which implies \( T_M(x) \leq T_{N,M-N}(x) \). Otherwise \( x = uvw \). Then \( T_M(x) = 0 = T_{N,M-N}(x) \). Similarly, we can prove that \( I_M(x) \geq I_{N,M-N}(x) \) and \( F_M(x) \leq F_{N,M-N}(x) \). Thus \( \bigcap X_M \subseteq X_M \).

Conversely, assume that \( X_N \bigcap X_M \bigcap X_N \subseteq X_M \) for any neutrosophic \( \mathbb{K}^- \)–structure \( X_N \).

**Notation 3.19.** Let \( X \) and \( Z \) be semigroups. A mapping \( g:X \rightarrow Z \) is said to be a homomorphism if \( g(uv) = g(u)g(v) \) for all \( u, v \in X \). Throughout this remaining section, we denote \( \text{Aut}(X) \), the set of all automorphisms of \( X \).

**Definition 3.20.** An interior ideal \( J \) of a semigroup \( X \) is called a characteristic interior ideal if \( h(J) = J \) for all \( h \in \text{Aut}(X) \).
Definition 3.21. Let $X$ be a semigroup. A neutrosophic $\mathfrak{K}$-interior ideal $X_M$ is called neutrosophic $\mathfrak{K}$-characteristic interior ideal if $T_M(h(u)) = T_M(u)$, $I_M(h(u)) = I_M(u)$ and $F_M(h(u)) = F_M(u)$ for all $u \in X$ and all $h \in Aut(X)$.

Theorem 3.22. For any $L \subseteq X$, the equivalent assertions are:

(i) $L$ is characteristic interior ideal,

(ii) The characteristic neutrosophic $\mathfrak{K}$-structure $\chi_L(X_M)$ is neutrosophic $\mathfrak{K}$-characteristic interior ideal.

Proof: Suppose $L$ is characteristic interior ideal and let $x \in X$. Then by Theorem 3.1, $\chi_L(X_M)$ is neutrosophic $\mathfrak{K}$-ideal. If $x \in L$, then $\chi_L(T)_M(x) = -1$, $\chi_L(I)_M(x) = 0$, and $\chi_L(F)_M(x) = -1$. Now, for any $h \in Aut(X)$, $h(x) \in h(L) = L$ which implies $\chi_L(T)_M(h(x)) = -1$, $\chi_L(I)_M(h(x)) = 0$, and $\chi_L(F)_M(h(x)) = -1$. If $x \notin L$, then $\chi_L(T)_M(x) = 0$, $\chi_L(I)_M(x) = -1$, and $\chi_L(F)_M(x) = 0$. Now, for any $h \in Aut(X)$, $h(x) \notin h(L)$ which implies $\chi_L(T)_M(h(x)) = 0$, $\chi_L(I)_M(h(x)) = -1$, and $\chi_L(F)_M(h(x)) = 0$. Thus $\chi_L(T)_M(h(x)) = \chi_L(T)_M(x)$, $\chi_L(I)_M(h(x)) = \chi_L(I)_M(x)$, and $\chi_L(F)_M(h(x)) = \chi_L(F)_M(x)$ for all $x \in X$ and hence $\chi_L(X_M)$ is neutrosophic $\mathfrak{K}$-characteristic interior ideal.

Conversely, assume that $\chi_L(X_M)$ is neutrosophic $\mathfrak{K}$-characteristic interior ideal. Then by Theorem 3.1, $L$ is an interior ideal. Now, let $h \in Aut(X)$ and $x \in L$. Then $\chi_L(T)_M(x) = -1$, $\chi_L(I)_M(x) = 0$ and $\chi_L(F)_M(x) = -1$. Since $\chi_L(X_M)$ is neutrosophic $\mathfrak{K}$-characteristic interior ideal, we have $\chi_L(T)_M(h(x)) = \chi_L(T)_M(x)$, $\chi_L(I)_M(h(x)) = \chi_L(I)_M(x)$ and $\chi_L(F)_M(h(x)) = \chi_L(F)_M(x)$ which imply $h(x) \in L$. So $h(L) \subseteq L$ for all $h \in Aut(X)$. Again, since $h \in Aut(X)$ and $x \in L$, there exists $y \in L$ such that $h(y) = x$.

Suppose that $y \notin L$. Then $\chi_L(T)_M(y) = 0$, $\chi_L(I)_M(y) = -1$ and $\chi_L(F)_M(y) = 0$. Since $\chi_L(T)_M(h(y)) = \chi_L(T)_M(y)$, $\chi_L(I)_M(h(y)) = \chi_L(I)_M(y)$ and $\chi_L(F)_M(h(y)) = \chi_L(F)_M(y)$, we get $\chi_L(T)_M(h(y)) = 0$, $\chi_L(I)_M(h(y)) = -1$ and $\chi_L(F)_M(h(y)) = 0$ which imply $h(y) \notin L$, a contradiction. So $y \in L$, i.e., $h(y) \in L$. Thus $L \subseteq h(L)$ for all $h \in Aut(X)$ and hence $L$ is characteristic interior ideal.

Theorem 3.23. For a semigroup $X$, the equivalent statements are:

(i) $X$ is intra-regular,

(ii) For any neutrosophic $\mathfrak{K}$-interior ideal $X_M$, we have $X_M(w) = X_M(w^2)$ for all $w \in X$.

Proof: (i) $\Rightarrow$ (ii) Suppose $X$ is intra-regular, and $X_M$ is neutrosophic $\mathfrak{K}$-interior ideal and $w \in X$. Then there exist $r, s \in X$ such that $w = rw^2$. Now $T_M(w) = T_M(rw^2) \leq T_M(w^2) \leq T_M(w)$ and so $T_M(w) = T_M(w^2)$, $I_M(w) = I_M(rw^2) \geq I_M(w^2) \geq I_M(w)$ and so $I_M(w) = I_M(w^2)$, and $F_M(w) = F_M(rw^2) \leq F_M(w^2) \leq F_M(w)$ and so $F_M(w) = F_M(w^2)$. Therefore $X_M(w) = X_M(w^2)$ for all $w \in X$.

(ii) $\Rightarrow$ (i) Let (ii) holds and $s \in X$. Then $I(s^2)$ is an ideal of $X$. By Theorem 3.5 of [4], $X_{I(s^2)}(X_M)$ is neutrosophic $\mathfrak{K}$-ideal. By assumption, $X_{I(s^2)}(X_M)(s) = X_{I(s^2)}(X_M)(s^2)$. Since $X_{I(s^2)}(T)_M(s^2) = -1 = X_{I(s^2)}(F)_M(s^2)$ and $X_{I(s^2)}(I)_M(s^2) = 0$, we get $X_{I(s^2)}(T)_M(s) = -1 = X_{I(s^2)}(F)_M(s)$ and $X_{I(s^2)}(I)_M(s^2) = 0$ which imply $s \in I(s^2)$. Hence $X$ is intra-regular.

Theorem 3.24. For a semigroup $X$, the equivalent statements are:

(i) $X$ is left (resp., right) regular,
(ii) For any neutrosophic $\aleph$–interior ideal $X_M$, we have $X_M(w) = X_M(w^2)$ for all $w \in X$.

**Proof:** (i) $\Rightarrow$ (ii) Let $X$ be left regular. Then there exists $y \in X$ such that $w = yw^2$. Let $X_M$ be a neutrosophic $\aleph$–interior ideal. Then $T_M(w) = T_M(yw^2) \leq T_M(w)$ and so $T_M(w) = T_M(w^2)$. $I_M(w) = I_M(yw^2) \geq I_M(w)$ and so $I_M(w) = I_M(w^2)$, and $F_M(w) = F_M(yw^2) \leq F_M(w)$ and so $F_M(w) = F_M(w^2)$. Therefore $X_M(w) = X_M(w^2)$ for all $w \in X$.

(ii) $\Rightarrow$ (i) Suppose (ii) holds and let $X_M$ be neutrosophic $\aleph$–interior ideal. Then for any $w \in X$, $\chi_L(w^2)(T)_M(w) = \chi_L(w^2)(T)_M(w^2) = -1$, $\chi_L(w^2)(I)_M(w) = \chi_L(w^2)(I)_M(w^2) = 0$ and $\chi_L(w^2)(F)_M(w) = \chi_L(w^2)(F)_M(w^2) = -1$ which imply $w \in L(w^2)$. Thus $X$ is left regular. □

**Conclusions**

In this paper, we have introduced the concepts of neutrosophic $\aleph$–interior ideals and neutrosophic $\aleph$–characteristic interior ideals in semigroups and studied their properties, and characterized regular and intra-regular semigroups using neutrosophic $\aleph$-interior ideal structures. We have also shown that $\aleph$ is a characteristic interior ideal if and only if the characteristic neutrosophic $\aleph$–structure $\chi_M(X_\aleph)$ is neutrosophic $\aleph$–characteristic interior ideal. In future, we will define neutrosophic $\aleph$–prime ideals in semigroups and study their properties.

**Reference**


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