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New approach to bisemiring theory via the bipolar valued neutrosophic normal sets

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Abstract. In this paper, we introduce the notion of bipolar-valued neutrosophic subbisemiring (BVNSBS), level sets of BVNSBS, and bipolar valued neutrosophic normal subbisemiring (BVNNSBS) of a bisemiring. The concept of BVNSBS is a new generalization of subbisemiring over bisemirings. We discussed the theory of (ξ, τ)-BVNSBS and (ξ, τ)-BVNNSBS over bisemirings and presented several illustrative examples to demonstrate the sufficiency and validity of the proposed theorems, lemmas, and propositions.

Keywords: Fuzzy set; Bipolar valued neutrosophic subbisemiring; Bipolar valued neutrosophic bisemiring; Homomorphism; Normal.

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1. Introduction

Classical mathematics may not always be the solution for practical situations in economics, medical sciences, engineering, social sciences, and environmental sciences, which involves various uncertainties, imprecise and incomplete information. The limitation of classical mathematics that is unable to deal with uncertainties and fuzziness motivated the introduction of mathematical theory such as probability theory, fuzzy set theory [1], rough set theory [2], vague set theory [3], interval mathematics [4], and soft set theory [5]. However, these theories were insufficient and have limitations in dealing with uncertainties. Probability theory can only deal with stochastically stable problems, which may not apply to many problems in the field of economic, environmental, and social sciences. Interval mathematics takes calculation errors into account by constructing an interval estimate for the solution that is useful in many areas, but it is not appropriately adaptable for problems that arise from unreliable, inadequate, and change of information. On the other hand, the fuzzy set theory introduced by Zadeh [1] is most appropriate for dealing with uncertainties and vagueness. Membership of an element in a fuzzy set is a single value between the interval, but in real-life problems, the degree of non-membership may not always be equal to 1 minus the degree of membership as there may be some degree of hesitation. Works on fuzzy set theory are progressing rapidly and have resulted in the conception of many hybrid fuzzy models. In 1983, Atanasov [6] proposed intuitionistic fuzzy sets as a generalization of the notion of fuzzy set, which incorporated the degree of hesitation. Later, Zhang [7] introduced bipolar fuzzy sets in which the membership function is mapped to intervals, thereby allowing it to deal with complex problems in both positive and negative aspects. Later, Zhang [8] proposed that bipolar fuzzy logic should combine both fuzziness and polarity by introducing the (Yin) (Yang) bipolar fuzzy sets. Lee [9] introduced the operation in bipolar-valued fuzzy sets, whereas Lee [10] discovered that bipolar-valued fuzzy sets can represent the degree of satisfaction to counter property but fail to express uncertainties in assigning membership degree. These concepts have been widely applied to handle incomplete information arising from practical situations. However, these were still unable to address uncertainties such as indeterminate and inconsistent information.

In 1999, Smarandache [11] proposed the neutrosophic theory that deals with "the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra". The idea of neutrosophic logic is a logic that states that each proposition is estimated to have a degree of trust, degree of indeterminacy, and degree of falsity. Smarandache [12] further generalized the theory of intuitionistic fuzzy sets to the neutrosophic model, and introduced the truth, indeterminacy, and falsity components that represent the membership, indeterminacy, and non-membership values of a neutrosophic set, respectively. In contrast to intuitionistic fuzzy sets, neutrosophic sets used indeterminacy as a completely independent measure of the membership and non-membership information, and thus it can effectively describe uncertain and inconsistent information and overcome the limitation of the existing approaches in handling uncertain information.

The original neutrosophic theory was introduced from a philosophical standpoint. Hence, it may be difficult to be applied in practical problems. Subsequently, Wang et al. [13] generalized the neutrosophic set from a technical point of view and specified the set-theoretic operators on an instance of a neutrosophic set, called the single-valued neutrosophic set, which takes values from the subset of [0, 1], thereby enabling it to be used feasibly for real-world problems. Over the years, subsequent developments and extensions of the neutrosophic set were proposed. Deli et al. [14] proposed bipolar neutrosophic sets as an extension of bipolar fuzzy sets [7]. Ye [15] introduced the concept of simplified neutrosophic sets. Peng et al. [16] introduced multi-valued neutrosophic sets that allow the truth, indeterminacy, and falsity membership degrees to have a set of crisp values between zero and one, respectively. Das et al. [17] introduced the notion of neutrosophic fuzzy sets by combining fuzzy sets with neutrosophic fuzzy sets to overcome the difficulties in handling the non-standard interval of neutrosophic components.

On the other hand, the fuzzy set theory had been applied and contributed to the generalization of many fundamental concepts in algebra. Extensive research has been done on the fuzzy algebraic structure of semirings introduced by Vandiver [18], which is a generalization of a ring by relaxing the conditions on the additive structure requiring just a monoid rather than a group and have been proven useful for dealing with problems in various areas. The application of semirings had been studied extensively by Golan [19] and Glazek [20].

Ahsan, Saifullah and Farid Khan [21] initiated the study of fuzzy semirings, while Feng, Jun and Zhao [22], and Yousafzai et al. [23] studied semigroups and semirings using fuzzy set and soft sets, respectively. Furthermore, Mockor [24] introduced the notion of a semiring-valued fuzzy set for special commutative partially pre-ordered semiring and introduced F-transform and inverse F-transform for these fuzzy-type structures. Other than that, palanikumar et al. [25–30] studied the algebraic structure of various semirings that constitute a natural generalization of semirings.

Recently, many studies applied bipolar fuzzy information in various algebraic structures, for instance, semigroups [31–33] and BCK/BCI-algebras [34–37]. Zararsz et al. [38] discussed the notion of bipolar fuzzy metric spaces with application. Selvachandran and Salleh [39] introduced vague soft hyperrings and vague soft hyperideals. Jun, Kim and Lee [40] introduced bipolar fuzzy translation in BCK/BCI-algebra and investigated its properties, whereas Jun and Park [41] introduced bipolar fuzzy regularity, bipolar fuzzy regular subalgebra, bipolar fuzzy filter, and bipolar fuzzy closed quasi filter in BCH-algebras. Apart from that, Sen, Ghosh and Ghosh [42] extended the study of semirings and proposed the concept of bisemiring in 2004. Later, Hussain [43] defined the congruence relation between bisemiring and bisemiring homomorphisms, followed by the factor bisemiring. Hussain et al. [44] further generalized bisemiring to a new algebraic structure called -semiring and congruence relations on homomorphisms and n-semirings.

To the best of our knowledge, studies on bisemiring theory using bipolar valued neutrosophic sets have not been studied extensively, and further generalization for bisemiring is still needed for various practical problems. In this paper, we introduce the notion of bipolar valued neutrosophic subbisemiring (BVNSBS), level sets of BVNSBS, and bipolar valued neutrosophic normal subbisemiring (BVNNSBS) of a bisemiring. The concept of BVNSBS is a new generalization of subbisemiring over bisemirings. We discussed the theory for (ξ,τ)-BVNSBS and (ξ,τ)-BVNNSBS over bisemiring theory and presented several illustrative examples. The rest of the paper is organized as follows: Section 2 outlines the preliminary definitions and results, Section 3 introduces the notion of BVNSBS, Section 4 discusses the (ξ,τ)-BVNSBS and Section 5 discusses the (ξ,τ)-BVNNSBS.

2. Preliminaries

Definition 2.1. [9] Let U be the universe set. A bipolar valued fuzzy set ϑ in U is an object having the form $\vartheta = \{(u, \vartheta^+(u), \vartheta^-(u))|u \in U\}$, where $\vartheta^- : U \to [-1, 0]$ and $\vartheta^+ : U \to [0, 1]$ are mappings. The positive membership degree $\vartheta^+(u)$ denoted the satisfaction degree of an element u to the property corresponding to a bipolar valued fuzzy set $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$, and the negative membership degree $\vartheta^-(u)$ denotes the satisfaction degree of u to some implicit counter-property of $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$. If $\vartheta^+(u) \neq 0$ and $\vartheta^-(u) = 0$, it is the situation that u is regarded as having only positive satisfaction for $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$. If $\vartheta^+(u) = 0$ and $\vartheta^-(u) \neq 0$, it is the situation that u does not satisfy the property of $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$ but somewhat satisfies the counter property of $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$. It is possible for an element u to be $\vartheta^+(u) \neq 0$ and $\vartheta^-(u) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain. For the sake of simplicity, we shall use the symbol $\vartheta = \langle U; \vartheta^-, \vartheta^+\rangle$ for the bipolar valued fuzzy set $\vartheta = \{\langle u, \vartheta^+(u), \vartheta^-(u)\rangle|u \in U\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar valued fuzzy sets.

Definition 2.2. [11] A neutrosophic set K in a universe set U is an object having the structure $K = \left\{\left\langle m, \vartheta_K^T(m), \vartheta_K^I(m), \vartheta_K^F(m)\right\rangle | m \in U\right\}$, where $\vartheta_K^T(m), \vartheta_K^I(m), \vartheta_K^F(m) : U \to [0, 1]$ represents the truth-membership function , the indeterminacy membership function and the falsity-membership function respectively. There is no restriction on the sum of $\vartheta_K^T, \vartheta_K^I, \vartheta_K^F$ and so $0 \leq \vartheta_K^T + \vartheta_K^I + \vartheta_K^F \leq 3$.

Definition 2.3. [11] Let $K = \left\{\left\langle m, \vartheta_K^T(m), \vartheta_K^I(m), \vartheta_K^F(m)\right\rangle | m \in U\right\}$ and $L = \left\{\left\langle m, \vartheta_L^T(m), \vartheta_L^I(m), \vartheta_L^F(m)\right\rangle | m \in U\right\}$ be any two neutrosophic sets of a set U. Then
$K \cap L = \left\{ \left\langle m, \min\{\vartheta_K^T(m), \vartheta_L^T(m)\}, \min\{\vartheta_K^I(m), \vartheta_L^I(m)\}, \max\{\vartheta_K^F(m), \vartheta_L^F(m)\}\right\rangle \Big| m \in U\right\}$,
$K \cup L = \left\{ \left(\langle m, \max\{\vartheta_K^T(m), \vartheta_L^T(m)\}, \max\{\vartheta_K^I(m), \vartheta_L^I(m)\}, \min\{\vartheta_K^F(m), \vartheta_L^F(m)\}\right\rangle \Big| m \in U\right\}$.

Definition 2.4. [11] For any neutrosophic set $K = \left\{\left\langle m, \vartheta_K^T(m), \vartheta_K^I(m), \vartheta_K^F(m)\right\rangle | m \in U\right\}$ of a set U, we defined a (ξ, τ)-cut of as the crisp subset $\{\vartheta_K^T(m) \geq \xi, \vartheta_K^I(m) \geq \xi, \vartheta_K^F(m) \leq \tau | m \in U\}$ of U.

Definition 2.5. [11] Let K and L be any two neutrosophic set of U. Then $K \times L = \{\vartheta_{K\times L}^T(m,n), \vartheta_{K\times L}^I(m,n), \vartheta_{K\times L}^F(m,n)|\forall m,n \in U\}$, where $\vartheta_{K\times L}^T(m,n) = \min\{\vartheta_K^T(m), \vartheta_L^T(n)\}, \vartheta_{K\times L}^I(m,n) = \frac{\vartheta_K^I(m)+\vartheta_L^I(n)}{2}, \vartheta_{K\times L}^F(m,n) = \max\{\vartheta_K^F(m), \vartheta_L^F(n)\}$.

Definition 2.6. [44] A fuzzy subset K of a bisemiring $(S, \uplus_1, \uplus_2, \uplus_3)$ is said to be a fuzzy subbisemiring of S if $\vartheta_K(m \uplus_1 n) \geq \min\{\vartheta_K(m), \vartheta_K(n)\}, \vartheta_K(m \uplus_2 n) \geq \min\{\vartheta_K(m), \vartheta_K(n)\}, \vartheta_K(m \uplus_3 n) \geq \min\{\vartheta_K(m), \vartheta_K(n)\}$, for all $m, n \in S$.

Definition 2.7. [44] Let $(S_1, +, \cdot, \times)$ and $(S_2, \boxplus, \circ, \otimes)$ be any two bisemirings. A function $\phi : S_1 \to S_2$ is said to be a homomorphism if $\phi(m + n) = \phi(m) \boxplus \phi(n), \phi(m \cdot n) = \phi(m) \circ \phi(n), \phi(m \times n) = \phi(m) \otimes \phi(n)$, for all $m, n \in S_1$.

3. Bipolar Valued Neutrosophic Subbisemiring (BVNSBS)

In what follows, let $\mathcal{S}$ denote a bisemiring unless otherwise noted. In this section, we communication the concept of bipolar valued neutrosophic subbisemiring, strongest neutrosophic relation on $\mathcal{S}$. Furthermore, we introduce the arbitrary intersection bipolar valued neutrosophic subbisemiring and list some properties.

Definition 3.1. A bipolar valued neutrosophic subset K of $\mathcal{S}$ is said to be BVNSBS of $\mathcal{S}$ if it satisfies the following conditions:

$$\left\{\begin{matrix} \begin{pmatrix} \vartheta_K^{T+}(m \uplus_1 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}, \\ \vartheta_K^{T-}(m \uplus_1 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{T+}(m \uplus_2 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}, \\ \vartheta_K^{T-}(m \uplus_2 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{T+}(m \uplus_3 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}, \\ \vartheta_K^{T-}(m \uplus_3 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \end{pmatrix} \end{matrix}\right\} \quad \left\{\begin{matrix} \begin{pmatrix} \vartheta_K^{I+}(m \uplus_1 n) \geq \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}, \\ \vartheta_K^{I-}(m \uplus_1 n) \leq \frac{\vartheta_K^{I-}(m)-\vartheta_K^{I-}(n)}{2} \end{pmatrix} \\ OR \\ \begin{pmatrix} \vartheta_K^{I+}(m \uplus_2 n) \geq \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}, \\ \vartheta_K^{I-}(m \uplus_2 n) \leq \frac{\vartheta_K^{I-}(m)-\vartheta_K^{I-}(n)}{2} \end{pmatrix} \\ OR \\ \begin{pmatrix} \vartheta_K^{I+}(m \uplus_3 n) \geq \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}, \\ \vartheta_K^{I-}(m \uplus_3 n) \leq \frac{\vartheta_K^{I-}(m)-\vartheta_K^{I-}(n)}{2} \end{pmatrix} \end{matrix}\right\}$$

$$\left\{\begin{matrix} \begin{pmatrix} \vartheta_K^{F+}(m \uplus_1 n) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}, \\ \vartheta_K^{F-}(m \uplus_1 n) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{F+}(m \uplus_2 n) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}, \\ \vartheta_K^{F-}(m \uplus_2 n) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{F+}(m \uplus_3 n) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}, \\ \vartheta_K^{F-}(m \uplus_3 n) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} \end{pmatrix} \end{matrix}\right\}$$

for all $m, n \in \mathcal{S}$.

Example 3.2. Let $\mathcal{S} = \{l_1, l_2, l_3, l_4\}$ be the bisemiring with the following Cayley table:

$\uplus_1$	l_1	l_2	l_3	l_4
l_1	l_1	l_1	l_1	l_1
l_2	l_1	l_2	l_1	l_2
l_3	l_1	l_1	l_3	l_3
l_4	l_1	l_2	l_3	l_4

$\uplus_2$	l_1	l_2	l_3	l_4
l_1	l_1	l_2	l_3	l_4
l_2	l_2	l_2	l_4	l_4
l_3	l_3	l_4	l_3	l_4
l_4	l_4	l_4	l_4	l_4

$\uplus_3$	l_1	l_2	l_3	l_4
l_1	l_1	l_1	l_1	l_1
l_2	l_1	l_2	l_3	l_4
l_3	l_4	l_4	l_4	l_4
l_4	l_4	l_4	l_4	l_4

$(\vartheta^+_K(l), \vartheta^-_K(l))$	$l = l_1$	$l = l_2$	$l = l_3$	$l = l_4$
$\left(\vartheta^{T+}_K(l), \vartheta^{T-}_K(l)\right)$	$(0.55, -0.7)$	$(0.35, -0.6)$	$(0.15, -0.3)$	$(0.25, -0.4)$
$\left(\vartheta^{I+}_K(l), \vartheta^{I-}_K(l)\right)$	$(0.65, -0.8)$	$(0.5, -0.5)$	$(0.3, -0.1)$	$(0.4, -0.2)$
$\left(\vartheta^{F+}_K(l), \vartheta^{F-}_K(l)\right)$	$(0.25, -0.15)$	$(0.35, -0.25)$	$(0.65, -0.65)$	$(0.55, -0.45)$

Clearly, K is an BVNSBS of $\mathcal{S}$.

Theorem 3.3. *The intersection of a family of $BVNSBS^s$ of $\mathcal{S}$ is a $BVNSBS$ of $\mathcal{S}$.*

Proof. Let $\{O_i | i \in I\}$ be a family of $BVNSBS^s$ of $\mathcal{S}$ and $K = \bigcap\limits_{i\in I} O_i$.
Let m and n in $\mathcal{S}$. Now,

$$\begin{aligned}
\vartheta^{T+}_K(m \uplus_1 n) &= \inf_{i\in I} \vartheta^{T+}_{O_i}(m \uplus_1 n) \\
&\geq \inf_{i\in I} \min\{\vartheta^{T+}_{O_i}(m), \vartheta^{T+}_{O_i}(n)\} \\
&= \min\left\{\inf_{i\in I} \vartheta^{T+}_{O_i}(m), \inf_{i\in I} \vartheta^{T+}_{O_i}(n)\right\} \\
&= \min\{\vartheta^{T+}_K(m), \vartheta^{T+}_K(n)\} \\
\vartheta^{T-}_K(m \uplus_1 n) &= \sup_{i\in I} \vartheta^{T-}_{O_i}(m \uplus_1 n) \\
&\leq \sup_{i\in I} \max\{\vartheta^{T-}_{O_i}(m), \vartheta^{T-}_{O_i}(n)\} \\
&= \max\left\{\sup_{i\in I} \vartheta^{T-}_{O_i}(m), \sup_{i\in I} \vartheta^{T-}_{O_i}(n)\right\} \\
&= \max\{\vartheta^{T-}_K(m), \vartheta^{T-}_K(n)\}.
\end{aligned}$$

Now,

$$\begin{aligned}
\vartheta^{I+}_K(m \uplus_1 n) &= \inf_{i\in I} \vartheta^{I+}_{O_i}(m \uplus_1 n) \\
&\geq \inf_{i\in I} \frac{\vartheta^{I+}_{O_i}(m) + \vartheta^{I+}_{O_i}(n)}{2} \\
&= \frac{\inf_{i\in I} \vartheta^{I+}_{O_i}(m) + \inf_{i\in I} \vartheta^{I+}_{O_i}(n)}{2} \\
&= \frac{\vartheta^{I+}_K(m) + \vartheta^{I+}_K(n)}{2}
\end{aligned}$$

$$\begin{aligned}
\vartheta_K^{I-}(m \uplus_1 n) &= \sup_{i\in I} \vartheta_{O_i}^{I-}(m \uplus_1 n) \\
&\leq \sup_{i\in I} \frac{\vartheta_{O_i}^{I-}(m) + \vartheta_{O_i}^{I-}(n)}{2} \\
&= \frac{\sup_{i\in I} \vartheta_{O_i}^{I-}(m) + \sup_{i\in I} \vartheta_{O_i}^{I-}(n)}{2} \\
&= \frac{\vartheta_K^{I-}(m) + \vartheta_K^{I-}(n)}{2}.
\end{aligned}$$

Now,

$$\begin{aligned}
\vartheta_K^{F+}(m \uplus_1 n) &= \sup_{i\in I} \vartheta_{O_i}^{F+}(m \uplus_1 n) \\
&\leq \sup_{i\in I} \max\{\vartheta_{O_i}^{F+}(m), \vartheta_{O_i}^{F+}(n)\} \\
&= \max\left\{\sup_{i\in I} \vartheta_{O_i}^{F+}(m), \sup_{i\in I} \vartheta_{O_i}^{F+}(n)\right\} \\
&= \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}
\end{aligned}$$

$$\begin{aligned}
\vartheta_K^{F-}(m \uplus_1 n) &= \inf_{i\in I} \vartheta_{O_i}^{F-}(m \uplus_1 n) \\
&\geq \inf_{i\in I} \min\{\vartheta_{O_i}^{F-}(m), \vartheta_{O_i}^{F-}(n)\} \\
&= \min\left\{\inf_{i\in I} \vartheta_{O_i}^{F-}(m), \inf_{i\in I} \vartheta_{O_i}^{F-}(n)\right\} \\
&= \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\}.
\end{aligned}$$

Similarly, we can prove that other two operations. Hence K is an BVNSBS of $\mathcal{S}$.

Theorem 3.4. *If K and L are any two $BVNSBS^s$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, then $K \times L$ is a BVNSBS of $\mathcal{S}_1 \times \mathcal{S}_2$.*

Proof. Let K and L be two $BVNSBS^s$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Let $m_1, m_2 \in \mathcal{S}_1$ and $n_1, n_2 \in \mathcal{S}_2$. Then (m_1, n_1) and (m_2, n_2) are in $\mathcal{S}_1 \times \mathcal{S}_2$. Now,

$$\begin{aligned}
\vartheta_{K\times L}^{T+}[(m_1, n_1) \uplus_1 (m_2, n_2)] &= \vartheta_{K\times L}^{T+}(m_1 \uplus_1 m_2, n_1 \uplus_1 n_2) \\
&= \min\{\vartheta_K^{T+}(m_1 \uplus_1 m_2), \vartheta_L^{T+}(n_1 \uplus_1 n_2)\} \\
&\geq \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(m_2)\}, \min\{\vartheta_L^{T+}(n_1), \vartheta_L^{T+}(n_2)\}\} \\
&= \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_L^{T+}(n_1)\}, \min\{\vartheta_K^{T+}(m_2), \vartheta_L^{T+}(n_2)\}\} \\
&= \min\{\vartheta_{K\times L}^{T+}(m_1, n_1), \vartheta_{K\times L}^{T+}(m_2, n_2)\}.
\end{aligned}$$

Similarly, $\vartheta^{T-}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)] \leq \max\{\vartheta^{T-}_{K\times L}(m_1,n_1),\vartheta^{T-}_{K\times L}(m_2,n_2)\}$.
Now,

$$\begin{aligned}\vartheta^{I+}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)] &= \vartheta^{I+}_{K\times L}(m_1\uplus_1 m_2, n_1\uplus_1 n_2)\\ &= \frac{\vartheta^{I+}_{K}(m_1\uplus_1 m_2)+\vartheta^{I+}_{L}(n_1\uplus_1 n_2)}{2}\\ &\geq \frac{1}{2}\left[\frac{\vartheta^{I+}_{K}(m_1)+\vartheta^{I+}_{K}(m_2)}{2}+\frac{\vartheta^{I+}_{L}(n_1)+\vartheta^{I+}_{L}(n_2)}{2}\right]\\ &= \frac{1}{2}\left[\frac{\vartheta^{I+}_{K}(m_1)+\vartheta^{I+}_{L}(n_1)}{2}+\frac{\vartheta^{I+}_{K}(m_2)+\vartheta^{I+}_{L}(n_2)}{2}\right]\\ &= \frac{1}{2}\Big[\vartheta^{I+}_{K\times L}(m_1,n_1)+\vartheta^{I+}_{K\times L}(m_2,n_2)\Big].\end{aligned}$$

Similarly, $\vartheta^{I-}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)] \leq \frac{1}{2}\Big[\vartheta^{I-}_{K\times L}(m_1,n_1)+\vartheta^{I-}_{K\times L}(m_2,n_2)\Big]$.
Now,

$$\begin{aligned}\vartheta^{F+}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)] &= \vartheta^{F+}_{K\times L}(m_1\uplus_1 m_2, n_1\uplus_1 n_2)\\ &= \max\{\vartheta^{F+}_{K}(m_1\uplus_1 m_2),\vartheta^{F+}_{L}(n_1\uplus_1 n_2)\}\\ &\leq \max\{\max\{\vartheta^{F+}_{K}(m_1),\vartheta^{F+}_{K}(m_2)\},\max\{\vartheta^{F+}_{L}(n_1),\vartheta^{F+}_{L}(n_2)\}\}\\ &= \max\{\max\{\vartheta^{F+}_{K}(m_1),\vartheta^{F+}_{L}(n_1)\},\max\{\vartheta^{F+}_{K}(m_2),\vartheta^{F+}_{L}(n_2)\}\}\\ &= \max\{\vartheta^{F+}_{K\times L}(m_1,n_1),\vartheta^{F+}_{K\times L}(m_2,n_2)\}.\end{aligned}$$

Similarly, $\vartheta^{F-}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)] \geq \min\{\vartheta^{F-}_{K\times L}(m_1,n_1),\vartheta^{F-}_{K\times L}(m_2,n_2)\}$.
Similarly, we can prove other two operations. Hence, $K\times L$ is an BVNSBS of $\mathcal{S}$.

Corollary 3.5. *If $K_1, K_2, ..., K_n$ are the family of $BVNSBS^s$ of $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n$ respectively, then $K_1\times K_2\times ...\times K_n$ is an BVNSBS of $\mathcal{S}_1\times\mathcal{S}_2\times ...\times\mathcal{S}_n$.*

Definition 3.6. Let K be a bipolar valued neutrosophic subset in $\mathcal{S}$, the strongest neutrosophic relation on $\mathcal{S}$, that is a bipolar valued neutrosophic relation on K is O such that

$$\left\{\begin{array}{l}\begin{pmatrix}\vartheta^{T+}_{O}(m,n)=\min\{\vartheta^{T+}_{K}(m),\vartheta^{T+}_{K}(n)\},\\ \vartheta^{T-}_{O}(m,n)=\max\{\vartheta^{T-}_{K}(m),\vartheta^{T-}_{K}(n)\}\end{pmatrix},\begin{pmatrix}\vartheta^{I+}_{O}(m,n)=\frac{\vartheta^{I+}_{K}(m)+\vartheta^{I+}_{K}(n)}{2},\\ \vartheta^{I-}_{O}(m,n)=\frac{\vartheta^{I-}_{K}(m)+\vartheta^{I-}_{K}(n)}{2}\end{pmatrix},\\ \begin{pmatrix}\vartheta^{F+}_{O}(m,n)=\max\{\vartheta^{F+}_{K}(m),\vartheta^{F+}_{K}(n)\},\\ \vartheta^{F-}_{O}(m,n)=\min\{\vartheta^{F-}_{K}(m),\vartheta^{F-}_{K}(n)\}\end{pmatrix}\end{array}\right\}.$$

Theorem 3.7. *Let K be the BVNSBS of $\mathcal{S}$ and O be the strongest bipolar valued neutrosophic relation of $\mathcal{S}$. Then K is an BVNSBS of $\mathcal{S}$ if and only if O is an BVNSBS of $\mathcal{S}\times\mathcal{S}$.*

Proof. Let K be the BVNSBS of $\mathcal{S}$ and O be the strongest bipolar valued neutrosophic relation of $\mathcal{S}$. Then for any $m = (m_1, m_2)$ and $n = (n_1, n_2)$ are in $\mathcal{S} \times \mathcal{S}$. Now,

$$\begin{aligned}
\vartheta_O^{T+}(m \uplus_1 n) &= \vartheta_O^{T+}[((m_1, m_2) \uplus_1 (n_1, n_2)] \\
&= \vartheta_O^{T+}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2) \\
&= \min\{\vartheta_K^{T+}(m_1 \uplus_1 n_1), \vartheta_K^{T+}(m_2 \uplus_1 n_2)\} \\
&\geq \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(n_1)\}, \min\{\vartheta_K^{T+}(m_2), \vartheta_K^{T+}(n_2)\}\} \\
&= \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(m_2)\}, \min\{\vartheta_K^{T+}(n_1), \vartheta_K^{T+}(n_2)\}\} \\
&= \min\{\vartheta_O^{T+}(m_1, m_2), \vartheta_O^{T+}(n_1, n_2)\} \\
&= \min\{\vartheta_O^{T+}(m), \vartheta_O^{T+}(n)\}.
\end{aligned}$$

Similarly, $\vartheta_O^{T-}(m \uplus_2 n) \leq \max\{\vartheta_O^{T-}(m), \vartheta_O^{T-}(n)\}$.
Now,

$$\begin{aligned}
\vartheta_O^{I+}(m \uplus_1 n) &= \vartheta_O^{I+}[((m_1, m_2) \uplus_1 (n_1, n_2)] \\
&= \vartheta_O^{I+}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2) \\
&= \frac{\vartheta_K^{I+}(m_1 \uplus_1 n_1) + \vartheta_K^{I+}(m_2 \uplus_1 n_2)}{2} \\
&\geq \frac{1}{2}\left[\frac{\vartheta_K^{I+}(m_1) + \vartheta_K^{I+}(n_1)}{2} + \frac{\vartheta_K^{I+}(m_2) + \vartheta_K^{I+}(n_2)}{2}\right] \\
&= \frac{1}{2}\left[\frac{\vartheta_K^{I+}(m_1) + \vartheta_K^{I+}(m_2)}{2} + \frac{\vartheta_K^{I+}(n_1) + \vartheta_K^{I+}(n_2)}{2}\right] \\
&= \frac{\vartheta_O^{I+}(m_1, m_2) + \vartheta_O^{I+}(n_1, n_2)}{2} \\
&= \frac{\vartheta_O^{I+}(m) + \vartheta_O^{I+}(n)}{2}.
\end{aligned}$$

Similarly, $\vartheta_O^{I-}(m \uplus_1 n) \leq \frac{\vartheta_O^{I-}(m) + \vartheta_O^{I-}(n)}{2}$.
Similarly, $\vartheta_O^{F+}(m \uplus_1 n) \leq \max\{\vartheta_O^{F+}(m), \vartheta_O^{F+}(n)\}$ and $\vartheta_O^{F-}(m \uplus_1 n) \geq \min\{\vartheta_O^{F-}(m), \vartheta_O^{F-}(n)\}$.
Similarly to prove other two operations. Hence O is an BVNSBS of $\mathcal{S} \times \mathcal{S}$.

Conversely assume that O is an BVNSBS of $\mathcal{S} \times \mathcal{S}$, then for any $m = (m_1, m_2)$ and $n = (n_1, n_2)$ are in $\mathcal{S} \times \mathcal{S}$. Now,

$$\begin{aligned}
\min\{\vartheta_K^{T+}(m_1 \uplus_1 n_1), \vartheta_K^{T+}(m_2 \uplus_1 n_2)\} &= \vartheta_O^{T+}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2) \\
&= \vartheta_O^{T+}[(m_1, m_2) \uplus_1 (n_1, n_2)] \\
&= \vartheta_O^{T+}(m \uplus_1 n)
\end{aligned}$$

$$\begin{aligned}
&\geq \min\{\vartheta_O^{T+}(m), \vartheta_O^{T+}(n)\}\\
&= \min\{\vartheta_O^{T+}(m_1, m_2), \vartheta_O^{T+}(n_1, n_2)\}\\
&= \min\{\min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(m_2)\}, \min\{\vartheta_K^{T+}(n_1), \vartheta_K^{T+}(n_2)\}\}.
\end{aligned}$$

If $\vartheta_K^{T+}(m_1 \uplus_1 n_1) \leq \vartheta_K^{T+}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{T+}(m_1) \leq \vartheta_K^{T+}(m_2)$ and $\vartheta_K^{T+}(n_1) \leq \vartheta_K^{T+}(n_2)$.
We get $\vartheta_K^{T+}(m_1 \uplus_1 n_1) \geq \min\{\vartheta_K^{T+}(m_1), \vartheta_K^{T+}(n_1)\}$.

$$\begin{aligned}
\max\{\vartheta_K^{T-}(m_1 \uplus_1 n_1), \vartheta_K^{T-}(m_2 \uplus_1 n_2)\} &= \vartheta_O^{T-}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2)\\
&= \vartheta_O^{T-}[(m_1, m_2) \uplus_1 (n_1, n_2)]\\
&= \vartheta_O^{T-}(m \uplus_1 n)\\
&\leq \max\{\vartheta_O^{T-}(m), \vartheta_O^{T-}(n)\}\\
&= \max\{\vartheta_O^{T-}(m_1, m_2), \vartheta_O^{T-}(n_1, n_2)\}\\
&= \max\{\max\{\vartheta_K^{T-}(m_1), \vartheta_K^{T-}(m_2)\}, \max\{\vartheta_K^{T-}(n_1), \vartheta_K^{T-}(n_2)\}\}.
\end{aligned}$$

If $\vartheta_K^{T-}(m_1 \uplus_1 n_1) \geq \vartheta_K^{T-}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{T-}(m_1) \geq \vartheta_K^{T-}(m_2)$ and $\vartheta_K^{T-}(n_1) \geq \vartheta_K^{T-}(n_2)$.
We get $\vartheta_K^{T-}(m_1 \uplus_1 n_1) \leq \max\{\vartheta_K^{T-}(m_1), \vartheta_K^{T-}(n_1)\}$ for all $m_1, n_1 \in \mathcal{S}$. Now,

$$\begin{aligned}
\frac{1}{2}\Big[\vartheta_K^{I+}(m_1 \uplus_1 n_1) + \vartheta_K^{I+}(m_2 \uplus_1 n_2)\Big] &= \vartheta_O^{I+}(m_1 \uplus_1 n_1, m_2 \uplus_1 n_2)\\
&= \vartheta_O^{I+}[(m_1, m_2) \uplus_1 (n_1, n_2)]\\
&= \vartheta_O^{I+}(m \uplus_1 n)\\
&\geq \frac{\vartheta_O^{I+}(m) + \vartheta_O^{I+}(n)}{2}\\
&= \frac{\vartheta_O^{I+}(m_1, m_2) + \vartheta_O^{I+}(n_1, n_2)}{2}\\
&= \frac{1}{2}\left[\frac{\vartheta_K^{I+}(m_1) + \vartheta_K^{I+}(m_2)}{2} + \frac{\vartheta_K^{I+}(n_1) + \vartheta_K^{I+}(n_2)}{2}\right].
\end{aligned}$$

If $\vartheta_K^{I+}(m_1 \uplus_1 n_1) \leq \vartheta_K^{I+}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{I+}(m_1) \leq \vartheta_K^{I+}(m_2)$ and $\vartheta_K^{I+}(n_1) \leq \vartheta_K^{I+}(n_2)$.
We get, $\vartheta_K^{I+}(m_1 \uplus_1 n_1) \geq \frac{\vartheta_K^{I+}(m_1)+\vartheta_K^{I+}(n_1)}{2}$.
Similarly, $\frac{1}{2}\Big[\vartheta_K^{I-}(m_1 \uplus_1 n_1) + \vartheta_K^{I-}(m_2 \uplus_1 n_2)\Big] \leq \frac{1}{2}\left[\frac{\vartheta_K^{I-}(m_1)+\vartheta_K^{I-}(m_2)}{2} + \frac{\vartheta_K^{I-}(n_1)+\vartheta_K^{I-}(n_2)}{2}\right]$.
If $\vartheta_K^{I-}(m_1 \uplus_1 n_1) \geq \vartheta_K^{I-}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{I-}(m_1) \geq \vartheta_K^{I-}(m_2)$ and $\vartheta_K^{I-}(n_1) \geq \vartheta_K^{I-}(n_2)$.
We get, $\vartheta_K^{I-}(m_1 \uplus_1 n_1) \leq \frac{\vartheta_K^{I-}(m_1)+\vartheta_K^{I-}(n_1)}{2}$.
Similarly, $\max\{\vartheta_K^{F+}(m_1 \uplus_1 n_1), \vartheta_K^{F+}(m_2 \uplus_1 n_2)\} \leq \max\{\max\{\vartheta_K^{F+}(m_1), \vartheta_K^{F+}(m_2)\},$
$\max\{\vartheta_K^{F+}(n_1), \vartheta_K^{F+}(n_2)\}\}$.
If $\vartheta_K^{F+}(m_1 \uplus_1 n_1) \geq \vartheta_K^{F+}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{F+}(m_1) \geq \vartheta_K^{F+}(m_2)$ and $\vartheta_K^{F+}(n_1) \geq \vartheta_K^{F+}(n_2)$.
We get, $\vartheta_K^{F+}(m_1 \uplus_1 n_1) \leq \max\{\vartheta_K^{F+}(m_1), \vartheta_K^{F+}(n_1)\}$.
Similarly, $\min\{\vartheta_K^{F-}(m_1 \uplus_1 n_1), \vartheta_K^{F-}(m_2 \uplus_1 n_2)\} \geq \min\{\min\{\vartheta_K^{F-}(m_1), \vartheta_K^{F-}(m_2)\},$
$\min\{\vartheta_K^{F-}(n_1), \vartheta_K^{F-}(n_2)\}\}$.

If $\vartheta_K^{F-}(m_1 \uplus_1 n_1) \leq \vartheta_K^{F-}(m_2 \uplus_1 n_2)$, then $\vartheta_K^{F-}(m_1) \leq \vartheta_K^{F-}(m_2)$ and $\vartheta_K^{F-}(n_1) \leq \vartheta_K^{F-}(n_2)$.
We get, $\vartheta_K^{F-}(m_1 \uplus_1 n_1) \geq \min\{\vartheta_K^{F-}(m_1), \vartheta_K^{F-}(n_1)\}$.
Similarly to prove other two operations. Hence K is an BVNSBS of $\mathcal{S}$.

Theorem 3.8. *Let K be bipolar valued neutrosophic subset in $\mathcal{S}$. Then $\vartheta = \{(\vartheta_K^{T+}, \vartheta_K^{T-}), (\vartheta_K^{I+}, \vartheta_K^{I-}), (\vartheta_K^{F+}, \vartheta_K^{F-})\}$ is an BVNSBS of $\mathcal{S}$ if and only if all non empty level set $\vartheta^{(t,s)}$ is a subbisemiring of $\mathcal{S}$ for $t, s \in [-1, 0] \times [0, 1]$.*

Proof. Assume that ϑ is an BVNSBS of $\mathcal{S}$. For each $t, s \in [-1, 0] \times [0, 1]$ and $a_1, a_2 \in \vartheta^{(t,s)}$. We have $\vartheta_K^{T+}(a_1) \geq t, \vartheta_K^{T+}(a_2) \geq t$ and $\vartheta_K^{I+}(a_1) \geq t, \vartheta_K^{I+}(a_2) \geq t$ and $\vartheta_K^{F+}(a_1) \leq s, \vartheta_K^{F+}(a_2) \leq s$. Now, $\vartheta_K^{T+}(a_1 \uplus_1 a_2) \geq \min\{\vartheta_K^{T+}(a_1), \vartheta_K^{T+}(a_2)\} \geq t$ and $\vartheta_K^{I+}(a_1 \uplus_1 a_2) \geq \frac{\vartheta_K^{I+}(a_1)+\vartheta_K^{I+}(a_2)}{2} \geq \frac{t+t}{2} = t$ and $\vartheta_K^{F+}(a_1 \uplus_1 a_2) \leq \max\{\vartheta_K^{F+}(a_1), \vartheta_K^{F+}(a_2)\} \leq s$. Since, $t, s \in [-1, 0] \times [0, 1]$, we have $\vartheta_K^{T-}(a_1) \leq t, \vartheta_K^{T-}(a_2) \leq t$ and $\vartheta_K^{I-}(a_1) \leq t, \vartheta_K^{I-}(a_2) \leq t$ and $\vartheta_K^{F-}(a_1) \geq s, \vartheta_K^{F-}(a_2) \geq s$. Now, $\vartheta_K^{T-}(a_1 \uplus_1 a_2) \leq \max\{\vartheta_K^{T-}(a_1), \vartheta_K^{T-}(a_2)\} \leq t$ and $\vartheta_K^{I-}(a_1 \uplus_1 a_2) \leq \frac{\vartheta_K^{I-}(a_1)+\vartheta_K^{I-}(a_2)}{2} \leq \frac{t+t}{2} = t$ and $\vartheta_K^{F-}(a_1 \uplus_1 a_2) \geq \min\{\vartheta_K^{F-}(a_1), \vartheta_K^{F-}(a_2)\} \geq s$. This implies that $a_1 \uplus_1 a_2 \in \vartheta^{(t,s)}$. Similarly, to prove other two operations. Hence, $\vartheta^{(t,s)}$ is a subbisemiring of $\mathcal{S}$ for each $t, s \in [-1, 0] \times [0, 1]$.

Conversely, assume that $\vartheta^{(t,s)}$ is a subbisemiring of $\mathcal{S}$ for each $t, s \in [-1, 0] \times [0, 1]$. Suppose if there exist $a_1, a_2 \in \mathcal{S}$ such that $\vartheta_K^{T+}(a_1 \uplus_1 a_2) < \min\{\vartheta_K^{T+}(a_1), \vartheta_K^{T+}(a_2)\}$, $\vartheta_K^{I+}(a_1 \uplus_1 a_2) < \frac{\vartheta_K^{I+}(a_1)+\vartheta_K^{I+}(a_2)}{2}$ and $\vartheta_K^{F+}(a_1 \uplus_1 a_2) > \max\{\vartheta_K^{F+}(a_1), \vartheta_K^{F+}(a_2)\}$. Select $t, s \in [0, 1]$ such that $\vartheta_K^{T+}(a_1 \uplus_1 a_2) < t \leq \min\{\vartheta_K^{T+}(a_1), \vartheta_K^{T+}(a_2)\}$ and $\vartheta_K^{I+}(a_1 \uplus_1 a_2) < t \leq \frac{\vartheta_K^{I+}(a_1)+\vartheta_K^{I+}(a_2)}{2}$ and $\vartheta_K^{F+}(a_1 \uplus_1 a_2) > s \geq \max\{\vartheta_K^{F+}(a_1), \vartheta_K^{F+}(a_2)\}$. Then $a_1, a_2 \in \vartheta^{(t,s)}$, but $a_1 \uplus_1 a_2 \notin \vartheta^{(t,s)}$. Suppose if there exist $a_1, a_2 \in \mathcal{S}$ such that $\vartheta_K^{T-}(a_1 \uplus_1 a_2) > \max\{\vartheta_K^{T-}(a_1), \vartheta_K^{T-}(a_2)\}$, $\vartheta_K^{I-}(a_1 \uplus_1 a_2) > \frac{\vartheta_K^{I-}(a_1)+\vartheta_K^{I-}(a_2)}{2}$ and $\vartheta_K^{F-}(a_1 \uplus_1 a_2) < \min\{\vartheta_K^{F-}(a_1), \vartheta_K^{F-}(a_2)\}$. Select $t, s \in [-1, 0]$ such that $\vartheta_K^{T-}(a_1 \uplus_1 a_2) > t \geq \max\{\vartheta_K^{T-}(a_1), \vartheta_K^{T-}(a_2)\}$ and $\vartheta_K^{I-}(a_1 \uplus_1 a_2) > t \geq \frac{\vartheta_K^{I-}(a_1)+\vartheta_K^{I-}(a_2)}{2}$ and $\vartheta_K^{F-}(a_1 \uplus_1 a_2) < s \leq \min\{\vartheta_K^{F-}(a_1), \vartheta_K^{F-}(a_2)\}$. Then $a_1, a_2 \in \vartheta^{(t,s)}$, but $a_1 \uplus_1 a_2 \notin \vartheta^{(t,s)}$. This contradicts to that $\vartheta^{(t,s)}$ is a subbisemiring of $\mathcal{S}$. Hence $\vartheta_K^{T+}(a_1 \uplus_1 a_2) \geq \min\{\vartheta_K^{T+}(a_1), \vartheta_K^{T+}(a_2)\}$, $\vartheta_K^{T-}(a_1 \uplus_1 a_2) \leq \max\{\vartheta_K^{T-}(a_1), \vartheta_K^{T-}(a_2)\}$, $\vartheta_K^{I+}(a_1 \uplus_1 a_2) \geq \frac{\vartheta_K^{I+}(a_1)+\vartheta_K^{I+}(a_2)}{2}$, $\vartheta_K^{I-}(a_1 \uplus_1 a_2) \leq \frac{\vartheta_K^{I-}(a_1)+\vartheta_K^{I-}(a_2)}{2}$ and $\vartheta_K^{F+}(a_1 \uplus_1 a_2) \leq \max\{\vartheta_K^{F+}(a_1), \vartheta_K^{F+}(a_2)\}$, $\vartheta_K^{F-}(a_1 \uplus_1 a_2) \geq \min\{\vartheta_K^{F-}(a_1), \vartheta_K^{F-}(a_2)\}$. Similarly to prove other two operations such as $\uplus_2$ and $\uplus_3$. Hence $\tilde{\vartheta} = \{(\vartheta_K^{T+}, \vartheta_K^{T-}), (\vartheta_K^{I+}, \vartheta_K^{I-}), (\vartheta_K^{F+}, \vartheta_K^{F-})\}$ is an BVNSBS of $\mathcal{S}$.

Definition 3.9. Let K be any BVNSBS of $\mathcal{S}$ and $a \in \mathcal{S}$. Then the pseudo bipolar valued neutrosophic coset $(aA)^z$ is defined by

$$\left\{ \begin{matrix} \begin{pmatrix} ((a\vartheta_K^{T+})^z)(m) = z(a)\vartheta_K^{T+}(m), \\ ((a\vartheta_K^{T-})^z)(m) = z(a)\vartheta_K^{T-}(m) \end{pmatrix}, \begin{pmatrix} ((a\vartheta_K^{I+})^z)(m) = z(a)\vartheta_K^{I+}(m), \\ ((a\vartheta_K^{I-})^z)(m) = z(a)\vartheta_K^{I-}(m) \end{pmatrix}, \\ \begin{pmatrix} ((a\vartheta_K^{F+})^z)(m) = z(a)\vartheta_K^{F+}(m), \\ ((a\vartheta_K^{F-})^z)(m) = z(a)\vartheta_K^{F-}(m) \end{pmatrix} \end{matrix} \right\}.$$

for every $m \in \mathcal{S}$ and for some $z \in P$, where P is a any non-empty set.

Theorem 3.10. *Let K be any BVNSBS of $\mathcal{S}$, then the pseudo bipolar valued neutrosophic coset $(aA)^z$ is an BVNSBS of $\mathcal{S}$, for every $a \in \mathcal{S}$.*

Proof. Now, $((a\vartheta_K^{T+})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{T+}(m \uplus_1 n) \geq z(a)\ \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\} = \min\{z(a)\ \vartheta_K^{T+}(m), z(a)\ \vartheta_K^{T+}(n)\} = \min\{((a\vartheta_K^{T+})^z)(m), ((a\vartheta_K^{T+})^z)(n)\}$. Thus, $((a\vartheta_K^{T+})^z)(m \uplus_1 n) \geq \min\{((a\vartheta_K^{T+})^z)(m), ((a\vartheta_K^{T+})^z)(n)\}$. Now, $((a\vartheta_K^{I+})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{I+}(m \uplus_1 n) \geq z(a)\left[\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}\right] = \frac{z(a)\ \vartheta_K^{I+}(m)+z(a)\ \vartheta_K^{I+}(n)}{2} = \frac{((a\vartheta_K^{I+})^z)(m)+((a\vartheta_K^{I+})^z)(n)}{2}$. Thus, $((a\vartheta_K^{I+})^z)(m \uplus_1 n) \geq \frac{((a\vartheta_K^{I+})^z)(m)+((a\vartheta_K^{I+})^z)(n)}{2}$. Now, $((a\vartheta_K^{F+})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{F+}(m \uplus_1 n) \leq z(a)\ \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\} = \max\{z(a)\ \vartheta_K^{F+}(m), z(a)\ \vartheta_K^{F+}(n)\} = \max\{((a\vartheta_K^{F+})^z)(m), ((a\vartheta_K^{F+})^z)(n)\}$. Thus, $((a\vartheta_K^{F+})^z)(m \uplus_1 n) \leq \max\{((a\vartheta_K^{F+})^z)(m), ((a\vartheta_K^{F+})^z)(n)\}$. Also, $((a\vartheta_K^{T-})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{T-}(m \uplus_1 n) \leq z(a)\ \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} = \max\{z(a)\ \vartheta_K^{T-}(m), z(a)\ \vartheta_K^{T-}(n)\} = \max\{((a\vartheta_K^{T-})^z)(m), ((a\vartheta_K^{T-})^z)(n)\}$.
Thus, $((a\vartheta_K^{T-})^z)(m \uplus_1 n) \leq \max\{((a\vartheta_K^{T-})^z)(m), ((a\vartheta_K^{T-})^z)(n)\}$. Now, $((a\vartheta_K^{I-})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{I-}(m \uplus_1 n) \leq z(a)\left[\frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2}\right] = \frac{z(a)\ \vartheta_K^{I-}(m)+z(a)\ \vartheta_K^{I-}(n)}{2} = \frac{((a\vartheta_K^{I-})^z)(m)+((a\vartheta_K^{I-})^z)(n)}{2}$. Thus, $((a\vartheta_K^{I-})^z)(m \uplus_1 n) \leq \frac{((a\vartheta_K^{I-})^z)(m)+((a\vartheta_K^{I-})^z)(n)}{2}$. Now, $((a\vartheta_K^{F-})^z)(m \uplus_1 n) = z(a)\ \vartheta_K^{F-}(m \uplus_1 n) \geq z(a)\ \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} = \min\{z(a)\ \vartheta_K^{F-}(m), z(a)\ \vartheta_K^{F-}(n)\} = \min\{((a\vartheta_K^{F-})^z)(m), ((a\vartheta_K^{F-})^z)(n)\}$. Thus, $((a\vartheta_K^{F-})^z)(m \uplus_1 n) \geq \min\{((a\vartheta_K^{F-})^z)(m), ((a\vartheta_K^{F-})^z)(n)\}$. Similarly to prove other two operations such as $\uplus_2$ and $\uplus_3$. Hence $(aA)^z$ is an BVNSBS of $\mathcal{S}$.

Definition 3.11. Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any function and K be any BVNSBS in $\mathcal{S}_1$, O be any BVNSBS in $\Lambda(\mathcal{S}_1) = \mathcal{S}_2$. If $\vartheta_K = \left\{(\vartheta_K^{T+}, \vartheta_K^{T-}), (\vartheta_K^{I+}, \vartheta_K^{I-}), (\vartheta_K^{F+}, \vartheta_K^{F-})\right\}$ is a bipolar valued neutrosophic set in $\mathcal{S}_1$, then ϑ_O is a bipolar valued neutrosophic set in $\mathcal{S}_2$, defined by

$$\vartheta_O^{T+}(n) = \begin{cases} \sup \vartheta_K^{T+}(m) & if\ \ m \in \Lambda^{-1}(n) \\ 0 & otherwise \end{cases} ; \vartheta_O^{T-}(n) = \begin{cases} \inf \vartheta_K^{T-}(m) & if\ \ m \in \Lambda^{-1}(n) \\ -1 & otherwise \end{cases}$$

$$\vartheta_O^{I+}(n) = \begin{cases} \sup \vartheta_K^{I+}(m) & if\ \ m \in \Lambda^{-1}(n) \\ 0 & otherwise \end{cases} ; \vartheta_O^{I-}(n) = \begin{cases} \inf \vartheta_K^{I-}(m) & if\ \ m \in \Lambda^{-1}(n) \\ -1 & otherwise \end{cases}$$

$$\vartheta_O^{F+}(n) = \begin{cases} \inf \vartheta_K^{F+}(m) & if\ \ m \in \Lambda^{-1}(n) \\ 1 & otherwise \end{cases} ; \vartheta_O^{F-}(n) = \begin{cases} \sup \vartheta_K^{F-}(m) & if\ \ m \in \Lambda^{-1}(n) \\ 0 & otherwise \end{cases} .$$

for all $m \in \mathcal{S}_1$ and $n \in \mathcal{S}_2$ is called the image of ϑ_K under Λ.
If $\vartheta_O = \left\{(\vartheta_O^{T+}, \vartheta_O^{T-}), (\vartheta_O^{I+}, \vartheta_O^{I-}), (\vartheta_O^{F+}, \vartheta_O^{F-})\right\}$ is a bipolar valued neutrosophic set in $\mathcal{S}_2$, then neutrosophic set $\vartheta_K = \Lambda \circ \vartheta_O$ in $\mathcal{S}_1$ [ie, the bipolar valued neutrosophic set defined by $\vartheta_K(m) = \vartheta_O(\Lambda(m))$] is called the preimage of ϑ_O under Λ.

Theorem 3.12. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. The homomorphic image of BVNSBS of $\mathcal{S}_1$ is an BVNSBS of $\mathcal{S}_2$.*

Proof. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Then $\Lambda(m \vee_1 n) = \Lambda(m) \sqcup_1 \Lambda(n), \Lambda(m \vee_2 n) = \Lambda(m) \sqcup_2 \Lambda(n)$ and $\Lambda(m \vee_3 n) = \Lambda(m) \sqcup_3 \Lambda(n)$ for all $m, n \in \mathcal{S}_1$. Let $O = \Lambda(K)$, K is any BVNSBS of $\mathcal{S}_1$. Let $\Lambda(m), \Lambda(n) \in \mathcal{S}_2$. Let $m \in \Lambda^{-1}(\Lambda(m))$ and $n \in \Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{T+}(m) = \sup\limits_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{T+}(z)$, $\vartheta_K^{T+}(n) = \sup\limits_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{T+}(z)$ and $\vartheta_K^{T-}(m) = \inf\limits_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{T-}(z)$, $\vartheta_K^{T-}(n) = \inf\limits_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{T-}(z)$. Now,

$$\begin{aligned}
\vartheta_O^{T+}(\Lambda(m) \sqcup_1 \Lambda(n)) &= \sup_{z' \in \Lambda^{-1}(\Lambda(m) \sqcup_1 \Lambda(n))} \vartheta_K^{T+}(z') \\
&= \sup_{z' \in \Lambda^{-1}(\Lambda(m \vee_1 n))} \vartheta_K^{T+}(z') \\
&= \vartheta_K^{T+}(m \vee_1 n) \\
&\geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\} \\
&= \min\{\vartheta_O^{T+}\Lambda(m), \vartheta_O^{T+}\Lambda(n)\}.
\end{aligned}$$

Thus, $\vartheta_O^{T+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_O^{T+}\Lambda(m), \vartheta_O^{T+}\Lambda(n)\}$.

$$\begin{aligned}
\vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)) &= \inf_{z' \in \Lambda^{-1}(\Lambda(m) \sqcup_1 \Lambda(n))} \vartheta_K^{T-}(z') \\
&= \inf_{z' \in \Lambda^{-1}(\Lambda(m \vee_1 n))} \vartheta_K^{T-}(z') \\
&= \vartheta_K^{T-}(m \vee_1 n) \\
&\leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \\
&= \max\{\vartheta_O^{T-}\Lambda(m), \vartheta_O^{T-}\Lambda(n)\}.
\end{aligned}$$

Thus, $\vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_O^{T-}\Lambda(m), \vartheta_O^{T-}\Lambda(n)\}$.
Let $m \in \Lambda^{-1}(\Lambda(m))$ and $n \in \Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{I+}(m) = \sup\limits_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{I+}(z)$, $\vartheta_K^{I+}(n) = \sup\limits_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{I+}(z)$, $\vartheta_K^{I-}(m) = \inf\limits_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{I-}(z)$, $\vartheta_K^{I-}(n) = \inf\limits_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{I-}(z)$.
Now,

$$\begin{aligned}
\vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)) &= \sup_{z' \in \Lambda^{-1}(\Lambda(m) \sqcup_1 \Lambda(n))} \vartheta_K^{I+}(z') \\
&= \sup_{z' \in \Lambda^{-1}(\Lambda(m \vee_1 n))} \vartheta_K^{I+}(z') \\
&= \vartheta_K^{I+}(m \vee_1 n) \\
&\geq \frac{\vartheta_K^{I+}(m) + \vartheta_K^{I+}(n)}{2} \\
&= \frac{\vartheta_O^{I+}\Lambda(m) + \vartheta_O^{I+}\Lambda(n)}{2}.
\end{aligned}$$

Thus, $\vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \frac{\vartheta_O^{I+}\Lambda(m)+\vartheta_O^{I+}\Lambda(n)}{2}$.
Similarly, $\vartheta_O^{I-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \frac{\vartheta_O^{I-}\Lambda(m)+\vartheta_O^{I-}\Lambda(n)}{2}$.
Let $\Lambda(m), \Lambda(n) \in \mathcal{S}_2$. Let $m \in \Lambda^{-1}(\Lambda(m))$ and $n \in \Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{F+}(m) = \inf_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{F+}(z)$, $\vartheta_K^{F+}(n) = \inf_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{F+}(z)$, $\vartheta_K^{F-}(m) = \sup_{z \in \Lambda^{-1}(\Lambda(m))} \vartheta_K^{F-}(z)$ and $\vartheta_K^{F-}(n) = \sup_{z \in \Lambda^{-1}(\Lambda(n))} \vartheta_K^{F-}(z)$. Now,

$$\begin{aligned}
\vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) &= \inf_{z' \in \Lambda^{-1}(\Lambda(m) \sqcup_1 \Lambda(n))} \vartheta_K^{F+}(z') \\
&= \inf_{z' \in \Lambda^{-1}(\Lambda(m \vee_1 n)} \vartheta_K^{F+}(z') \\
&= \vartheta_K^{F+}(m \vee_1 n) \\
&\leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\} \\
&= \max\{\vartheta_O^{F+}\Lambda(m), \vartheta_O^{F+}\Lambda(n)\}.
\end{aligned}$$

Thus, $\vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_O^{F+}\Lambda(m), \vartheta_O^{F+}\Lambda(n)\}$.
Similarly, $\vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_O^{F-}\Lambda(m), \vartheta_O^{F-}\Lambda(n)\}$.
Similarly, to prove other two operations. Hence O is an BVNSBS of $\mathcal{S}_2$.

Theorem 3.13. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. The homomorphic preimage of BVNSBS of $\mathcal{S}_2$ is an BVNSBS of $\mathcal{S}_1$.*

Proof. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Then $\Lambda(m \vee_1 n) = \Lambda(m) \sqcup_1 \Lambda(n), \Lambda(m \vee_2 n) = \Lambda(m) \sqcup_2 \Lambda(n)$ and $\Lambda(m \vee_3 n) = \Lambda(m) \sqcup_3 \Lambda(n)$ for all $m, n \in \mathcal{S}_1$. Let $O = \Lambda(K)$, where O is any BVNSBS of $\mathcal{S}_2$. Let $m, n \in \mathcal{S}_1$. Now, $\vartheta_K^{T+}(m \vee_1 n) = \vartheta_O^{T+}(\Lambda(m \vee_1 n)) = \vartheta_O^{T+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_O^{T+}\Lambda(m), \vartheta_O^{T+}\Lambda(n)\} = \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}$. Thus, $\vartheta_K^{T+}(m \vee_1 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\}$. Now, $\vartheta_K^{I+}(m \vee_1 n) = \vartheta_O^{I+}(\Lambda(m \vee_1 n)) = \vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \frac{\vartheta_O^{I+}\Lambda(m)+\vartheta_O^{I+}\Lambda(n)}{2} = \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}$. Thus, $\vartheta_K^{I+}(m \vee_1 n) \geq \frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2}$. Now, $\vartheta_K^{F+}(m \vee_1 n) = \vartheta_O^{F+}(\Lambda(m \vee_1 n)) = \vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_O^{F+}\Lambda(m), \vartheta_O^{F+}\Lambda(n)\} = \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}$. Thus, $\vartheta_K^{F+}(m \vee_1 n) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\}$. Also, $\vartheta_K^{T-}(m \vee_1 n) = \vartheta_O^{T-}(\Lambda(m \vee_1 n)) = \vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_O^{T-}\Lambda(m), \vartheta_O^{T-}\Lambda(n)\} = \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\}$. Thus, $\vartheta_K^{T-}(m \vee_1 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\}$. We have, $\vartheta_K^{I-}(m \vee_1 n) = \vartheta_O^{I-}(\Lambda(m \vee_1 n)) = \vartheta_O^{I-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \frac{\vartheta_O^{I-}\Lambda(m)+\vartheta_O^{I-}\Lambda(n)}{2} = \frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2}$. Thus, $\vartheta_K^{I-}(m \vee_1 n) \leq \frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2}$. Now, $\vartheta_K^{F-}(m \vee_1 n) = \vartheta_O^{F-}(\Lambda(m \vee_1 n)) = \vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_O^{F-}\Lambda(m), \vartheta_O^{F-}\Lambda(n)\} = \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\}$. Thus, $\vartheta_K^{F-}(m \vee_1 n) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\}$. Similarly to prove other two operations, hence K is an BVNSBS of $\mathcal{S}_1$.

Theorem 3.14. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. If $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ is a homomorphism, then $\Lambda(K_{(t,s)})$ is a level subbisemiring of BVNSBS O of $\mathcal{S}_2$.*

Proof. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Then $\Lambda(m \vee_1 n) = \Lambda(m) \sqcup_1 \Lambda(n), \Lambda(m \vee_2 n) = \Lambda(m) \sqcup_2 \Lambda(n)$ and $\Lambda(m \vee_3 n) = \Lambda(m) \sqcup_3 \Lambda(n)$ for all $m, n \in \mathcal{S}_1$. Let $O = \Lambda(K)$, K is an BVNSBS of $\mathcal{S}_1$. By Theorem 3.12, O is an BVNSBS of $\mathcal{S}_2$. Let $K_{(t,s)}$ be any level subbisemiring of K. Suppose that $m, n \in K_{(t,s)}$. Then $\Lambda(m \vee_1 n), \Lambda(m \vee_2 n)$ and $\Lambda(m \vee_3 n) \in K_{(t,s)}$. Now, $\vartheta_O^{T+}(\Lambda(m)) = \vartheta_K^{T+}(m) \geq t, \vartheta_O^{T+}(\Lambda(n)) = \vartheta_K^{T+}(n) \geq t$. Thus, $\vartheta_O^{T+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \vartheta_K^{T+}(m \vee_1 n) \geq t$. Now, $\vartheta_O^{I+}(\Lambda(m)) = \vartheta_K^{I+}(m) \geq t, \vartheta_O^{I+}(\Lambda(n)) = \vartheta_K^{I+}(n) \geq t$. Thus, $\vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \vartheta_K^{I+}(m \vee_1 n) \geq t$. Now, $\vartheta_O^{F+}(\Lambda(m)) = \vartheta_K^{F+}(m) \leq s, \vartheta_O^{F+}(\Lambda(n)) = \vartheta_K^{F+}(n) \leq s$. Thus, $\vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \vartheta_K^{F+}(m \vee_1 n) \leq s$, for all $\Lambda(m), \Lambda(n) \in \mathcal{S}_2$. Also, $\vartheta_O^{T-}(\Lambda(m)) = \vartheta_K^{T-}(m) \leq t, \vartheta_O^{T-}(\Lambda(n)) = \vartheta_K^{T-}(n) \leq t$. Thus, $\vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \vartheta_K^{T-}(m \vee_1 n) \leq t$. Now, $\vartheta_O^{I-}(\Lambda(m)) = \vartheta_K^{I-}(m) \leq t, \vartheta_O^{I-}(\Lambda(n)) = \vartheta_K^{I-}(n) \leq t$. Thus, $\vartheta_O^{I-}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \vartheta_K^{I-}(m \vee_1 n) \leq t$. Now, $\vartheta_O^{F-}(\Lambda(m)) = \vartheta_K^{F-}(m) \geq s, \vartheta_O^{F-}(\Lambda(n)) = \vartheta_K^{F-}(n) \geq s$. Thus, $\vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \vartheta_K^{F-}(m \vee_1 n) \geq s$, for all $\Lambda(m), \Lambda(n) \in \mathcal{S}_2$. Similarly to prove other operations, hence $\Lambda(K_{(t,s)})$ is a level subbisemiring of BVNSBS O of $\mathcal{S}_2$.

Theorem 3.15. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings. If $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ is any homomorphism, then $K_{(t,s)}$ is a level subbisemiring of BVNSBS K of $\mathcal{S}_1$.*

Proof. Let $\Lambda : \mathcal{S}_1 \to \mathcal{S}_2$ be any homomorphism. Then $\Lambda(m \vee_1 n) = \Lambda(m) \sqcup_1 \Lambda(n), \Lambda(m \vee_2 n) = \Lambda(m) \sqcup_2 \Lambda(n)$ and $\Lambda(m \vee_3 n) = \Lambda(m) \sqcup_3 \Lambda(n)$ for all $m, n \in \mathcal{S}_1$. Let $O = \Lambda(K)$, O is an BVNSBS of $\mathcal{S}_2$. By Theorem 3.13, K is an BVNSBS of $\mathcal{S}_1$. Let $\Lambda(K_{(t,s)})$ be a level subbisemiring of O. Suppose that $\Lambda(m), \Lambda(n) \in \Lambda(K_{(t,s)})$. Then $\Lambda(m \vee_1 n), \Lambda(m \vee_2 n)$ and $\Lambda(m \vee_3 n) \in \Lambda(K_{(t,s)})$. Now, $\vartheta_K^{T+}(m) = \vartheta_O^{T+}(\Lambda(m)) \geq t, \vartheta_K^{T+}(n) = \vartheta_O^{T+}(\Lambda(n)) \geq t$. Thus, $\vartheta_K^{T+}(m \vee_1 n) \geq \min\{\vartheta_K^{T+}(m), \vartheta_K^{T+}(n)\} \geq t$. Now, $\vartheta_K^{I+}(m) = \vartheta_O^{I+}(\Lambda(m)) \geq t, \vartheta_K^{I+}(n) = \vartheta_O^{I+}(\Lambda(n)) \geq t$. Thus, $\vartheta_K^{I+}(m \vee_1 n) \geq \frac{\vartheta_K^{I+}(m) + \vartheta_K^{I+}(n)}{2} \geq t$. Now, $\vartheta_K^{F+}(m) = \vartheta_O^{F+}(\Lambda(m)) \leq s, \vartheta_K^{F+}(n) = \vartheta_O^{F+}(\Lambda(n)) \leq s$. Thus, $\vartheta_K^{F+}(m \vee_1 n) = \vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)) \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n)\} \leq s$, for all $m, n \in \mathcal{S}_1$. Also, $\vartheta_K^{T-}(m) = \vartheta_O^{T-}(\Lambda(m)) \leq t, \vartheta_K^{T-}(n) = \vartheta_O^{T-}(\Lambda(n)) \leq t$. Thus, $\vartheta_K^{T-}(m \vee_1 n) \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n)\} \leq t$. Now, $\vartheta_K^{I-}(m) = \vartheta_O^{I-}(\Lambda(m)) \leq t, \vartheta_K^{I-}(n) = \vartheta_O^{I-}(\Lambda(n)) \leq t$. Thus, $\vartheta_K^{I-}(m \vee_1 n) \leq \frac{\vartheta_K^{I-}(m) + \vartheta_K^{I-}(n)}{2} \leq t$. Now, $\vartheta_K^{F-}(m) = \vartheta_O^{F-}(\Lambda(m)) \geq s, \vartheta_K^{F-}(n) = \vartheta_O^{F-}(\Lambda(n)) \geq s$. Thus, $\vartheta_K^{F-}(m \vee_1 n) = \vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)) \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n)\} \geq s$, for all $m, n \in \mathcal{S}_1$. In the same way, prove the other two operations, hence $K_{(t,s)}$ is a level subbisemiring of BVNSBS K of $\mathcal{S}_1$.

4. (ξ, τ)–**Bipolar Valued Neutrosophic Subbisemiring**

In this section, we discuss (ξ, τ)–bipolar valued neutrosophic subbisemiring. In what follows that, $(\xi^+, \tau^+) \in [0, 1]$ and $(\xi^-, \tau^-) \in [-1, 0]$ be such that $0 \leq \xi^+ < \tau^+ \leq 1$ and $-1 \leq \tau^- < \xi^- \leq 0$, both $(\xi, \tau) \in [0, 1]$ are arbitrary but fixed.

Definition 4.1. Let K be any bipolar valued neutrosophic subset of $\mathcal{S}$ is called a (ξ,τ)-BVNSBS of $\mathcal{S}$ if it satisfies the following conditions:

$$\left\{\begin{array}{l}\begin{pmatrix}\max\{\vartheta_K^{T+}(m\uplus_1 n),\xi^+\}\geq\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\},\\ \min\{\vartheta_K^{T-}(m\uplus_1 n),\xi^-\}\leq\max\{\vartheta_K^{T-}(m),\vartheta_K^{T-}(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\max\{\vartheta_K^{T+}(m\uplus_2 n),\xi^+\}\geq\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\},\\ \min\{\vartheta_K^{T-}(m\uplus_2 n),\xi^-\}\leq\max\{\vartheta_K^{T-}(m),\vartheta_K^{T-}(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\max\{\vartheta_K^{T+}(m\uplus_3 n),\xi^+\}\geq\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\},\\ \min\{\vartheta_K^{T-}(m\uplus_3 n),\xi^-\}\leq\max\{\vartheta_K^{T-}(m),\vartheta_K^{T-}(n),\tau^-\}\end{pmatrix}\end{array}\right\}$$

$$\left\{\begin{array}{c}\begin{pmatrix}\max\{\vartheta_K^{I+}(m\uplus_1 n),\xi^+\}\geq\min\left\{\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2},\tau^+\right\}\\ \min\{\vartheta_K^{I-}(m\uplus_1 n),\xi^-\}\leq\max\left\{\frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2},\tau^-\right\}\end{pmatrix}\\ \text{OR}\\ \begin{pmatrix}\max\{\vartheta_K^{I+}(m\uplus_2 n),\xi^+\}\geq\min\left\{\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2},\tau^+\right\}\\ \min\{\vartheta_K^{I-}(m\uplus_2 n),\xi^-\}\leq\max\left\{\frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2},\tau^-\right\}\end{pmatrix}\\ \text{OR}\\ \begin{pmatrix}\max\{\vartheta_K^{I+}(m\uplus_3 n),\xi^+\}\geq\min\left\{\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2},\tau^+\right\}\\ \min\{\vartheta_K^{I-}(m\uplus_3 n),\xi^-\}\leq\max\left\{\frac{\vartheta_K^{I-}(m)+\vartheta_K^{I-}(n)}{2},\tau^-\right\}\end{pmatrix}\end{array}\right\}$$

$$\left\{\begin{array}{l}\begin{pmatrix}\min\{\vartheta_K^{F+}(m\uplus_1 n),\xi^+\}\leq\max\{\vartheta_K^{F+}(m),\vartheta_K^{F+}(n),\tau^+\},\\ \max\{\vartheta_K^{F-}(m\uplus_1 n),\xi^-\}\geq\min\{\vartheta_K^{F-}(m),\vartheta_K^{F-}(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\min\{\vartheta_K^{F+}(m\uplus_2 n),\xi^+\}\leq\max\{\vartheta_K^{F+}(m),\vartheta_K^{F+}(n),\tau^+\},\\ \max\{\vartheta_K^{F-}(m\uplus_2 n),\xi^-\}\geq\min\{\vartheta_K^{F-}(m),\vartheta_K^{F-}(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\min\{\vartheta_K^{F+}(m\uplus_3 n),\xi^+\}\leq\max\{\vartheta_K^{F+}(m),\vartheta_K^{F+}(n),\tau^+\},\\ \max\{\vartheta_K^{F-}(m\uplus_3 n),\xi^-\}\geq\min\{\vartheta_K^{F-}(m),\vartheta_K^{F-}(n),\tau^-\}\end{pmatrix}\end{array}\right\}$$

for all $m,n\in\mathcal{S}$.

Example 4.2. By the Example 3.2,

$\left(\vartheta_K^{+}(l),\vartheta_K^{-}(l)\right)$	$l=l_1$	$l=l_2$	$l=l_3$	$l=l_4$
$\left(\vartheta_K^{T+}(l),\vartheta_K^{T-}(l)\right)$	$(0.85,-0.95)$	$(0.8,-0.75)$	$(0.7,-0.55)$	$(0.75,-0.65)$
$\left(\vartheta_K^{I+}(l),\vartheta_K^{I-}(l)\right)$	$(0.95,-0.8)$	$(0.9,-0.7)$	$(0.8,-0.5)$	$(0.85,-0.55)$
$\left(\vartheta_K^{F+}(l),\vartheta_K^{F-}(l)\right)$	$(0.65,-0.25)$	$(0.85,-0.35)$	$(0.95,-0.45)$	$(0.90,-0.40)$

Clearly, K is a $(0.60,0.70)$-BVNSBS of $\mathcal{S}$.

Theorem 4.3. *The intersection of family of (ξ,τ)- $BVNSBS^s$ of $\mathcal{S}$ is a (ξ,τ)- $BVNSBS$ of $\mathcal{S}$.*

Proof. Let $\{O_i|i\in I\}$ be any family of (ξ,τ)- $BVNSBS^s$ of $\mathcal{S}$ and $K=\bigcap_{i\in I}O_i$.

Let m and n in $\mathcal{S}$. Now,

$$\begin{aligned}\max\{\vartheta_K^{T+}(m\uplus_1 n),\xi^+\}&=\inf_{i\in I}\ \max\{\vartheta_{O_i}^{T+}(m\uplus_1 n),\xi^+\}\\ &\geq\inf_{i\in I}\ \min\{\vartheta_{O_i}^{T+}(m),\vartheta_{O_i}^{T+}(n),\tau^+\}\\ &=\min\left\{\inf_{i\in I}\ \vartheta_{O_i}^{T+}(m),\inf_{i\in I}\ \vartheta_{O_i}^{T+}(n),\tau^+\right\}\\ &=\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\}\end{aligned}$$

$$\begin{aligned}\min\{\vartheta^{T-}_{K}(m \uplus_1 n), \xi^-\} &= \sup_{i\in I}\ \min\{\vartheta^{T-}_{O_i}(m \uplus_1 n), \xi^-\}\\ &\leq \sup_{i\in I}\ \max\{\vartheta^{T-}_{O_i}(m), \vartheta^{T-}_{O_i}(n), \tau^-\}\\ &= \max\left\{\sup_{i\in I}\ \vartheta^{T-}_{O_i}(m), \sup_{i\in I}\ \vartheta^{T-}_{O_i}(n), \tau^-\right\}\\ &= \max\{\vartheta^{T-}_{K}(m), \vartheta^{T-}_{K}(n), \tau^-\}.\end{aligned}$$

Now,

$$\begin{aligned}\max\{\vartheta^{I+}_{K}(m \uplus_1 n), \xi^+\} &= \inf_{i\in I}\ \max\{\vartheta^{I+}_{O_i}(m \uplus_1 n), \xi^+\}\\ &\geq \inf_{i\in I}\ \min\left\{\frac{\vartheta^{I+}_{O_i}(m) + \vartheta^{I+}_{O_i}(n)}{2}, \tau^+\right\}\\ &= \min\left\{\frac{\inf_{i\in I}\ \vartheta^{I+}_{O_i}(m) + \inf_{i\in I}\ \vartheta^{I+}_{O_i}(n)}{2}, \tau^+\right\}\\ &= \min\left\{\frac{\vartheta^{I+}_{K}(m) + \vartheta^{I+}_{K}(n)}{2}, \tau^+\right\}.\end{aligned}$$

$$\begin{aligned}\min\{\vartheta^{I-}_{K}(m \uplus_1 n), \xi^-\} &= \sup_{i\in I}\ \min\{\vartheta^{I-}_{O_i}(m \uplus_1 n), \xi^-\}\\ &\leq \sup_{i\in I}\ \max\left\{\frac{\vartheta^{I-}_{O_i}(m) + \vartheta^{I-}_{O_i}(n)}{2}, \tau^-\right\}\\ &= \max\left\{\frac{\sup_{i\in I}\ \vartheta^{I-}_{O_i}(m) + \sup_{i\in I}\ \vartheta^{I-}_{O_i}(n)}{2}, \tau^-\right\}\\ &= \max\left\{\frac{\vartheta^{I-}_{K}(m) + \vartheta^{I-}_{K}(n)}{2}, \tau^-\right\}.\end{aligned}$$

Now,

$$\begin{aligned}\min\{\vartheta^{F+}_{K}(m \uplus_1 n), \xi^+\} &= \sup_{i\in I}\ \min\{\vartheta^{F+}_{O_i}(m \uplus_1 n), \xi^+\}\\ &\leq \sup_{i\in I}\ \max\{\vartheta^{F+}_{O_i}(m), \vartheta^{F+}_{O_i}(n), \tau^+\}\\ &= \max\left\{\sup_{i\in I}\ \vartheta^{F+}_{O_i}(m), \sup_{i\in I}\ \vartheta^{F+}_{O_i}(n), \tau^+\right\}\\ &= \max\{\vartheta^{F+}_{K}(m), \vartheta^{F+}_{K}(n), \tau^+\}\\ \max\{\vartheta^{F-}_{K}(m \uplus_1 n), \xi^-\} &= \inf_{i\in I}\ \max\{\vartheta^{F-}_{O_i}(m \uplus_1 n), \xi^-\}\\ &\geq \inf_{i\in I}\ \min\{\vartheta^{F-}_{O_i}(m), \vartheta^{F-}_{O_i}(n), \tau^-\}\\ &= \min\left\{\inf_{i\in I}\ \vartheta^{F-}_{O_i}(m), \inf_{i\in I}\ \vartheta^{F-}_{O_i}(n), \tau^-\right\}\\ &= \min\{\vartheta^{F-}_{K}(m), \vartheta^{F-}_{K}(n), \tau^-\}.\end{aligned}$$

Similarly to prove other operations. Hence, K is a (ξ, τ)- BVNSBS of $\mathcal{S}$.

Theorem 4.4. *If K and L are any two $(\xi,\tau)-BVNSBS^s$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, then $K \times L$ is a $(\xi,\tau)-BVNSBS$ of $\mathcal{S}_1 \times \mathcal{S}_2$.*

Proof. Let K and L be two $(\xi,\tau)-BVNSBS^s$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Let $m_1, m_2 \in \mathcal{S}_1$ and $n_1, n_2 \in \mathcal{S}_2$. Then (m_1, n_1) and (m_2, n_2) are in $\mathcal{S}_1 \times \mathcal{S}_2$. Now
$\max\left\{\vartheta^{T+}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)], \xi^+\right\}$

$$\begin{aligned}
&= \max\left\{\vartheta^{T+}_{K\times L}(m_1 \uplus_1 m_2, n_1 \uplus_1 n_2), \xi^+\right\}\\
&= \min\left\{\max\{\vartheta^{T+}_{K}(m_1 \uplus_1 m_2), \xi^+\}, \max\{\vartheta^{T+}_{L}(n_1 \uplus_1 n_2), \xi^+\}\right\}\\
&\geq \min\left\{\min\{\vartheta^{T+}_{K}(m_1), \vartheta^{T+}_{K}(m_2), \tau^+\}, \min\{\vartheta^{T+}_{L}(n_1), \vartheta^{T+}_{L}(n_2), \tau^+\}\right\}\\
&= \min\left\{\{\min\{\vartheta^{T+}_{K}(m_1), \vartheta^{T+}_{L}(n_1)\}, \min\{\vartheta^{T+}_{K}(m_2), \vartheta^{T+}_{L}(n_2)\}\}, \tau^+\right\}\\
&= \min\left\{\vartheta^{T+}_{K\times L}(m_1,n_1), \vartheta^{T+}_{K\times L}(m_2,n_2), \tau^+\right\}.
\end{aligned}$$

Also, $\min\left\{\vartheta^{T-}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)], \xi^-\right\} \leq \max\left\{\vartheta^{T-}_{K\times L}(m_1,n_1), \vartheta^{T-}_{K\times L}(m_2,n_2), \tau^-\right\}$.
Now, $\max\left\{\vartheta^{I+}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)], \xi^+\right\}$

$$\begin{aligned}
&= \max\left\{\vartheta^{I+}_{K\times L}(m_1 \uplus_1 m_2, n_1 \uplus_1 n_2), \xi^+\right\}\\
&= \min\left\{\frac{1}{2}\left[\max\left\{\vartheta^{I+}_{K}(m_1 \uplus_1 m_2), \xi^+\right\} + \max\left\{\vartheta^{I+}_{L}(n_1 \uplus_1 n_2), \xi^+\right\}\right]\right\}\\
&\geq \min\left\{\frac{1}{2}\left[\min\left\{\frac{\vartheta^{I+}_{K}(m_1) + \vartheta^{I+}_{K}(m_2)}{2}, \tau^+\right\} + \min\left\{\frac{\vartheta^{I+}_{L}(n_1) + \vartheta^{I+}_{L}(n_2)}{2}, \tau^+\right\}\right]\right\}\\
&= \min\left\{\frac{1}{2}\left[\frac{\vartheta^{I+}_{K}(m_1) + \vartheta^{I+}_{L}(n_1)}{2} + \frac{\vartheta^{I+}_{K}(m_2) + \vartheta^{I+}_{L}(n_2)}{2}\right], \tau^+\right\}\\
&= \min\left\{\frac{\vartheta^{I+}_{K\times L}(m_1,n_1) + \vartheta^{I+}_{K\times L}(m_2,n_2)}{2}, \tau^+\right\}.
\end{aligned}$$

Also, $\min\left\{\vartheta^{I-}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)], \xi^-\right\} \leq \max\left\{\frac{\vartheta^{I-}_{K\times L}(m_1,n_1) + \vartheta^{I-}_{K\times L}(m_2,n_2)}{2}, \tau^-\right\}$.
Similarly, $\min\left\{\vartheta^{F+}_{K\times L}[(m_1,n_1) \uplus_1 (m_2,n_2)], \xi^+\right\}$

$$\begin{aligned}
&= \min\left\{\vartheta^{F+}_{K\times L}(m_1 \uplus_1 m_2, n_1 \uplus_1 n_2), \xi^+\right\}\\
&= \max\left\{\min\{\vartheta^{F+}_{K}(m_1 \uplus_1 m_2), \xi^+\}, \min\{\vartheta^{F+}_{L}(n_1 \uplus_1 n_2), \xi^+\}\right\}\\
&\leq \max\left\{\max\{\vartheta^{F+}_{K}(m_1), \vartheta^{F+}_{K}(m_2), \tau^+\}, \max\{\vartheta^{F+}_{L}(n_1), \vartheta^{F+}_{L}(n_2), \tau^+\}\right\}\\
&= \max\left\{\{\max\{\vartheta^{F+}_{K}(m_1), \vartheta^{F+}_{L}(n_1)\}, \max\{\vartheta^{F+}_{K}(m_2), \vartheta^{F+}_{L}(n_2)\}\}, \tau^+\right\}\\
&= \max\left\{\vartheta^{F+}_{K\times L}(m_1,n_1), \vartheta^{F+}_{K\times L}(m_2,n_2), \tau^+\right\}.
\end{aligned}$$

Also, $\max\left\{\vartheta^{F-}_{K\times L}[(m_1,n_1)\uplus_1(m_2,n_2)],\xi^-\right\}\geq\min\left\{\vartheta^{F-}_{K\times L}(m_1,n_1),\vartheta^{F-}_{K\times L}(m_2,n_2),\tau^-\right\}$.
In the same way, prove the other two operations. Hence $K\times L$ is a (ξ,τ)- BVNSBS of $\mathcal{S}_1\times\mathcal{S}_2$.

Corollary 4.5. *If $K_1,K_2,...,K_n$ are the family of (ξ,τ)- $BVNSBS^s$ of $\mathcal{S}_1,\mathcal{S}_2,...,\mathcal{S}_n$ respectively, then $K_1\times K_2\times...\times K_n$ is a (ξ,τ)- BVNSBS of $\mathcal{S}_1\times\mathcal{S}_2\times...\times\mathcal{S}_n$.*

Definition 4.6. Let K be any (ξ,τ)- bipolar valued neutrosophic subset in $\mathcal{S}$, the strongest (ξ,τ)- bipolar valued neutrosophic relation on $\mathcal{S}$, that is a (ξ,τ)- bipolar valued neutrosophic relation on K is O such that

$$\left\{\begin{array}{l}\begin{pmatrix}\max\{\vartheta^{T+}_O(m,n),\xi^+\}=\min\{\vartheta^{T+}_K(m),\vartheta^{T+}_K(n),\tau^+\},\\ \min\{\vartheta^{T-}_O(m,n),\xi^-\}=\max\{\vartheta^{T-}_K(m),\vartheta^{T-}_K(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\max\{\vartheta^{I+}_O(m,n),\xi^+\}=\min\{\vartheta^{I+}_K(m),\vartheta^{I+}_K(n),\tau^+\},\\ \min\{\vartheta^{I-}_O(m,n),\xi^-\}=\max\{\vartheta^{I-}_K(m),\vartheta^{I-}_K(n),\tau^-\}\end{pmatrix}\\ \begin{pmatrix}\min\{\vartheta^{F+}_O(m,n),\xi^+\}=\max\{\vartheta^{F+}_K(m),\vartheta^{F+}_K(n),\tau^+\},\\ \max\{\vartheta^{F-}_O(m,n),\xi^-\}=\min\{\vartheta^{F-}_K(m),\vartheta^{F-}_K(n),\tau^-\}\end{pmatrix}\end{array}\right\}.$$

Theorem 4.7. *Let K be any $(\xi,\tau)-BVNSBS$ of $\mathcal{S}$ and O be the strongest (ξ,τ)- bipolar valued neutrosophic relation of $\mathcal{S}$. Then K is a $(\xi,\tau)-BVNSBS$ of $\mathcal{S}$ if and only if O is a $(\xi,\tau)-BVNSBS$ of $\mathcal{S}\times\mathcal{S}$.*

Theorem 4.8. *Let $(\mathcal{S}_1,\vee_1,\vee_2,\vee_3)$ and $(\mathcal{S}_2,\sqcup_1,\sqcup_2,\sqcup_3)$ be any two bisemirings. The homomorphic image of $(\xi,\tau)-BVNSBS$ of $\mathcal{S}_1$ is a $(\xi,\tau)-BVNSBS$ of $\mathcal{S}_2$.*

Proof. Let $\Lambda:\mathcal{S}_1\to\mathcal{S}_2$ be any homomorphism. Then $\Lambda(m\vee_1 n)=\Lambda(m)\sqcup_1\Lambda(n),\Lambda(m\vee_2 n)=\Lambda(m)\sqcup_2\Lambda(n)$ and $\Lambda(m\vee_3 n)=\Lambda(m)\sqcup_3\Lambda(n)$ for all $m,n\in\mathcal{S}_1$. Let $O=\Lambda(K)$, K is any (ξ,τ)-BVNSBS of $\mathcal{S}_1$. Let $\Lambda(m),\Lambda(n)\in\mathcal{S}_2$. Let $m\in\Lambda^{-1}(\Lambda(m))$ and $n\in\Lambda^{-1}(\Lambda(n))$ be such that $\vartheta^{T+}_K(m)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta^{T+}_K(z)$, $\vartheta^{T+}_K(n)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta^{T+}_K(z)$, $\vartheta^{T-}_K(m)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta^{T-}_K(z)$ and $\vartheta^{T-}_K(n)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta^{T-}_K(z)$. Now,

$$\begin{aligned}\max\left[\vartheta^{T+}_O(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^+\right]&=\max\left[\sup_{z'\in\Lambda^{-1}(\Lambda(m)\sqcup_1\Lambda(n))}\vartheta^{T+}_K(z')\ ,\xi^+\right]\\&=\max\left[\sup_{z'\in\Lambda^{-1}(\Lambda(m\vee_1 n)}\vartheta^{T+}_K(z')\ ,\xi^+\right]\\&=\max\left[\vartheta^{T+}_K(m\vee_1 n),\xi^+\right]\\&\geq\min\left\{\vartheta^{T+}_K(m),\vartheta^{T+}_K(n),\tau^+\right\}\\&=\min\left\{\vartheta^{T+}_O\Lambda(m),\vartheta^{T+}_O\Lambda(n),\tau^+\right\}.\end{aligned}$$

Similarly, $\min\left[\vartheta_O^{T-}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^-\right]\leq\max\left\{\vartheta_O^{T-}\Lambda(m),\vartheta_O^{T-}\Lambda(n),\tau^-\right\}$.
Let $\Lambda(m),\Lambda(n)\in\mathcal{S}_2$. Let $m\in\Lambda^{-1}(\Lambda(m))$ and $n\in\Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{I+}(m)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta_K^{I+}(z)$ and $\vartheta_K^{I+}(n)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta_K^{I+}(z)$, $\vartheta_K^{I-}(m)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta_K^{I-}(z)$ and $\vartheta_K^{I-}(n)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta_K^{I-}(z)$. Now,

$$\begin{aligned}\max\left[\vartheta_O^{I+}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^+\right]&=\max\left[\sup_{z'\in\Lambda^{-1}(\Lambda(m)\sqcup_1\Lambda(n))}\vartheta_K^{I+}(z')\ ,\xi^+\right]\\&=\max\left[\sup_{z'\in\Lambda^{-1}(\Lambda(m\vee_1 n)}\vartheta_K^{I+}(z')\ ,\xi^+\right]\\&=\max\left[\vartheta_K^{I+}(m\vee_1 n),\xi^+\right]\\&\geq\min\left\{\frac{\vartheta_K^{I+}(m)+\vartheta_K^{I+}(n)}{2},\tau^+\right\}\\&=\min\left\{\frac{\vartheta_O^{I+}\Lambda(m)+\vartheta_O^{I+}\Lambda(n)}{2},\tau^+\right\}\end{aligned}$$

Similarly, $\min\left[\vartheta_O^{I-}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^-\right]\leq\max\left\{\frac{\vartheta_O^{I-}\Lambda(m)+\vartheta_O^{I-}\Lambda(n)}{2},\tau^-\right\}$.
Let $m\in\Lambda^{-1}(\Lambda(m))$ and $n\in\Lambda^{-1}(\Lambda(n))$ be such that $\vartheta_K^{F+}(m)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta_K^{F+}(z)$,$\vartheta_K^{F+}(n)=\inf\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta_K^{F+}(z)$, $\vartheta_K^{F-}(m)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(m))}\vartheta_K^{F-}(z)$ and $\vartheta_K^{F-}(n)=\sup\limits_{z\in\Lambda^{-1}(\Lambda(n))}\vartheta_K^{F-}(z)$. Now,

$$\begin{aligned}\min\left[\vartheta_O^{F+}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^+\right]&=\min\left[\inf_{z'\in\Lambda^{-1}(\Lambda(m)\sqcup_1\Lambda(n))}\vartheta_K^{F+}(z')\ ,\xi^+\right]\\&=\min\left[\inf_{z'\in\Lambda^{-1}(\Lambda(m\vee_1 n)}\vartheta_K^{F+}(z')\ ,\xi^+\right]\\&=\min\left[\vartheta_K^{F+}(m\vee_1 n),\xi^+\right]\\&\leq\max\left\{\vartheta_K^{F+}(m),\vartheta_K^{F+}(n),\tau^+\right\}\\&=\max\left\{\vartheta_O^{F+}\Lambda(m),\vartheta_O^{F+}\Lambda(n),\tau^+\right\}.\end{aligned}$$

Similarly, $\max\left[\vartheta_O^{F-}(\Lambda(m)\sqcup_1\Lambda(n))\ ,\xi^-\right]\geq\min\left\{\vartheta_O^{F-}\Lambda(m),\vartheta_O^{F-}\Lambda(n),\tau^-\right\}$. In the same way, prove the other two operations. Hence O is a (ξ,τ)-BVNSBS of $\mathcal{S}_2$.

Theorem 4.9. *Let $(\mathcal{S}_1,\vee_1,\vee_2,\vee_3)$ and $(\mathcal{S}_2,\sqcup_1,\sqcup_2,\sqcup_3)$ be any two bisemirings. The homomorphic preimage of (ξ,τ)-BVNSBS of $\mathcal{S}_2$ is a (ξ,τ)-BVNSBS of $\mathcal{S}_1$.*

Proof. Let $\Lambda:\mathcal{S}_1\to\mathcal{S}_2$ be any homomrphism. Then $\Lambda(m\vee_1 n)=\Lambda(m)\sqcup_1\Lambda(n),\Lambda(m\vee_2 n)=\Lambda(m)\sqcup_2\Lambda(n)$ and $\Lambda(m\vee_3 n)=\Lambda(m)\sqcup_3\Lambda(n)$ for all $m,n\in\mathcal{S}_1$. Let $O=\Lambda(K)$, where O is any (ξ,τ)-BVNSBS of $\mathcal{S}_2$. Let $m,n\in\mathcal{S}_1$. Then $\max\{\vartheta_K^{T+}(m\vee_1 n),\xi^+\}=\max\{\vartheta_O^{T+}(\Lambda(m\vee_1 n)),\xi^+\}=\max\{\vartheta_O^{T+}(\Lambda(m)\sqcup_1\Lambda(n)),\xi^+\}\geq\min\{\vartheta_O^{T+}\Lambda(m),\vartheta_O^{T+}\Lambda(n),\tau^+\}=\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\}$. Thus, $\max\{\vartheta_K^{T+}(m\vee_1 n),\xi^+\}\geq\min\{\vartheta_K^{T+}(m),\vartheta_K^{T+}(n),\tau^+\}$. Also,

$\min\{\vartheta_K^{T-}(m \vee_1 n), \xi^-\} = \min\{\vartheta_O^{T-}(\Lambda(m \vee_1 n)), \xi^-\} = \min\{\vartheta_O^{T-}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^-\} \leq \max\{\vartheta_O^{T-}\Lambda(m), \vartheta_O^{T-}\Lambda(n), \tau^-\} = \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n), \tau^-\}$. Thus, $\min\{\vartheta_K^{T-}(m \vee_1 n), \xi^-\} \leq \max\{\vartheta_K^{T-}(m), \vartheta_K^{T-}(n), \tau^-\}$. Now, $\max\{\vartheta_K^{I+}(m \vee_1 n), \xi^+\} = \max\{\vartheta_O^{I+}(\Lambda(m \vee_1 n)), \xi^+\} = \max\{\vartheta_O^{I+}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^+\} \geq \min\{\vartheta_O^{I+}\Lambda(m), \vartheta_O^{I+}\Lambda(n), \tau^+\} = \min\{\vartheta_K^{I+}(m), \vartheta_K^{I+}(n), \tau^+\}$. Thus, $\max\{\vartheta_K^{I+}(m \vee_1 n), \xi^+\} \geq \min\{\vartheta_K^{I+}(m), \vartheta_K^{I+}(n), \tau^+\}$. Also, $\min\{\vartheta_K^{I-}(m \vee_1 n), \xi^-\} = \min\{\vartheta_O^{I-}(\Lambda(m \vee_1 n)), \xi^-\} = \min\{\vartheta_O^{I-}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^-\} \leq \max\{\vartheta_O^{I-}\Lambda(m), \vartheta_O^{I-}\Lambda(n), \tau^-\} = \max\{\vartheta_K^{I-}(m), \vartheta_K^{I-}(n), \tau^-\}$. Thus, $\min\{\vartheta_K^{I-}(m \vee_1 n), \xi^-\} \leq \max\{\vartheta_K^{I-}(m), \vartheta_K^{I-}(n), \tau^-\}$. Now, $\min\{\vartheta_K^{F+}(m \vee_1 n), \xi^+\} = \min\{\vartheta_O^{F+}(\Lambda(m \vee_1 n)), \xi^+\} = \min\{\vartheta_O^{F+}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^+\} \leq \max\{\vartheta_O^{F+}\Lambda(m), \vartheta_O^{F+}\Lambda(n), \tau^+\} = \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n), \tau^+\}$. Thus, $\min\{\vartheta_K^{F+}(m \vee_1 n), \xi^+\} \leq \max\{\vartheta_K^{F+}(m), \vartheta_K^{F+}(n), \tau^+\}$. Also, $\max\{\vartheta_K^{F-}(m \vee_1 n), \xi^-\} = \max\{\vartheta_O^{F-}(\Lambda(m \vee_1 n)), \xi^-\} = \max\{\vartheta_O^{F-}(\Lambda(m) \sqcup_1 \Lambda(n)), \xi^-\} \geq \min\{\vartheta_O^{F-}\Lambda(m), \vartheta_O^{F-}\Lambda(n), \tau^-\} = \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n), \tau^-\}$. Thus, $\max\{\vartheta_K^{F-}(m \vee_1 n), \xi^-\} \geq \min\{\vartheta_K^{F-}(m), \vartheta_K^{F-}(n), \tau^-\}$. In the same way, prove the other two operations, hence K is a (ξ, τ)-BVNSBS of $\mathcal{S}_1$.

5. (ξ, τ)-Bipolar Valued Neutrosophic Normal Subbisemiring

In this section, we interact the theory for (ξ, τ)-bipolar valued neutrosophic normal subbisemiring. Here $BVNNSBS$ stands for bipolar valued neutrosophic normal subbisemiring.

Definition 5.1. Let K be any bipolar valued neutrosophic subset of $\mathcal{S}$ is said to be a $BVNNSBS$ of $\mathcal{S}$ if it satisfies the following conditions:

$$\left\{\begin{matrix} \begin{pmatrix} \vartheta_K^{T+}(m \uplus_1 n) = \vartheta_K^{T+}(n \uplus_1 m), \\ \vartheta_K^{T-}(m \uplus_1 n) = \vartheta_K^{T-}(n \uplus_1 m) \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{T+}(m \uplus_2 n) = \vartheta_K^{T+}(n \uplus_2 m), \\ \vartheta_K^{T-}(m \uplus_2 n) = \vartheta_K^{T-}(n \uplus_2 m) \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{T+}(m \uplus_3 n) = \vartheta_K^{T+}(n \uplus_3 m), \\ \vartheta_K^{T-}(m \uplus_3 n) = \vartheta_K^{T-}(n \uplus_3 m) \end{pmatrix} \end{matrix}\right\} \quad \left\{\begin{matrix} \begin{pmatrix} \vartheta_K^{I+}(m \uplus_1 n) = \vartheta_K^{I+}(n \uplus_1 m), \\ \vartheta_K^{I-}(m \uplus_1 n) = \vartheta_K^{I-}(n \uplus_1 m) \end{pmatrix} \\ \text{OR} \\ \begin{pmatrix} \vartheta_K^{I+}(m \uplus_2 n) = \vartheta_K^{I+}(n \uplus_2 m), \\ \vartheta_K^{I-}(m \uplus_2 n) = \vartheta_K^{I-}(n \uplus_2 m) \end{pmatrix} \\ \text{OR} \\ \begin{pmatrix} \vartheta_K^{I+}(m \uplus_3 n) = \vartheta_K^{I+}(n \uplus_3 m), \\ \vartheta_K^{I-}(m \uplus_3 n) = \vartheta_K^{I-}(n \uplus_3 m) \end{pmatrix} \end{matrix}\right\}$$

$$\left\{\begin{matrix} \begin{pmatrix} \vartheta_K^{F+}(m \uplus_1 n) = \vartheta_K^{F+}(n \uplus_1 m), \\ \vartheta_K^{F-}(m \uplus_1 n) = \vartheta_K^{F-}(n \uplus_1 m) \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{F+}(m \uplus_2 n) = \vartheta_K^{F+}(n \uplus_2 m), \\ \vartheta_K^{F-}(m \uplus_2 n) = \vartheta_K^{F-}(n \uplus_2 m) \end{pmatrix} \\ \begin{pmatrix} \vartheta_K^{F+}(m \uplus_3 n) = \vartheta_K^{F+}(n \uplus_3 m), \\ \vartheta_K^{F-}(m \uplus_3 n) = \vartheta_K^{F-}(n \uplus_3 m) \end{pmatrix} \end{matrix}\right\}$$

for all $m, n \in \mathcal{S}$.

Theorem 5.2. *(a) The intersection of a family of $BVNNSBS^s$ of $\mathcal{S}$ is a $BVNNSBS$ of $\mathcal{S}$.*
(b) The intersection of a family of $(\xi, \tau) - BVNNSBS^s$ of $\mathcal{S}$ is a $(\xi, \tau) - BVNNSBS$ of $\mathcal{S}$.

Proof. Proof follows from Theorem 3.3 and Theorem 4.3.

Theorem 5.3. *(a) If $K_1, K_2, ..., K_n$ are the family of $BVNNSBS^s$ of $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n$ respectively, then $K_1 \times K_2 \times ... \times K_n$ is a $BVNNSBS$ of $\mathcal{S}_1 \times \mathcal{S}_2 \times ... \times \mathcal{S}_n$.*
(b) If $K_1, K_2, ..., K_n$ are the family of $(\xi, \tau) - BVNNSBS^s$ of $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_n$ respectively, then $K_1 \times K_2 \times ... \times K_n$ is a $(\xi, \tau) - BVNNSBS$ of $\mathcal{S}_1 \times \mathcal{S}_2 \times ... \times \mathcal{S}_n$.

Proof. Proof follows from Theorem 3.4 and Theorem 4.4.

Theorem 5.4. *(a) Let K be any $BVNNSBS$ of $\mathcal{S}$ and O be the strongest bipolar valued neutrosophic relation of $\mathcal{S}$. Then K is a $BVNNSBS$ of $\mathcal{S}$ if and only if O is a $BVNNSBS$ of $\mathcal{S} \times \mathcal{S}$.*
(b) Let K be any $(\xi, \tau) - BVNNSBS$ of $\mathcal{S}$ and O be the strongest (ξ, τ) bipolar valued neutrosophic relation of $\mathcal{S}$. Then K is a $(\xi, \tau) - BVNNSBS$ of $\mathcal{S}$ if and only if O is a $(\xi, \tau) - BVNNSBS$ of $\mathcal{S} \times \mathcal{S}$.

Proof. Proof follows from Theorem 3.7.

Theorem 5.5. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings.*
(a) The homomorphic image of any $BVNNSBS$ of $\mathcal{S}_1$ is a $BVNNSBS$ of $\mathcal{S}_2$.
(b) The homomorphic image of any $(\xi, \tau) - BVNNSBS$ of $\mathcal{S}_1$ is a $(\xi, \tau) - BVNNSBS$ of $\mathcal{S}_2$.

Proof. Proof follows from Theorem 3.12 and Theorem 4.8.

Theorem 5.6. *Let $(\mathcal{S}_1, \vee_1, \vee_2, \vee_3)$ and $(\mathcal{S}_2, \sqcup_1, \sqcup_2, \sqcup_3)$ be any two bisemirings.*
(a) The homomorphic preimage of any $BVNNSBS$ of $\mathcal{S}_2$ is a $BVNNSBS$ of $\mathcal{S}_1$.
(b) The homomorphic preimage of any $(\xi, \tau) - BVNNSBS$ of $\mathcal{S}_2$ is a $(\xi, \tau) - BVNNSBS$ of $\mathcal{S}_1$.

Proof. Proof follows from Theorem 3.13 and Theorem 4.9.

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