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Neutrosophic Quadratic Residues and Non-Residues

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Abstract: In this paper, we present the Neutrosophic quadratic residues and nonresidues with their basic interpretation as graphs in an algebraic manner and analog to the algebraic graphs. We establish the Neutrosophic, number-theoretic, and graph-theoretic properties of the set of Neutrosophic quadratic residues and nonresidues, many of which mirror those of the classical quadratic residues and nonresidues of modulo an odd prime. These properties, especially the algebraic ones, are connected to algebraic graphs, and thus we conclude the paper by studying the structural properties of Neutrosophic quadratic residue and quadratic nonresidue graphs.

Keywords: Quadratic residues; Quadratic nonresidues; Neutrosophic quadratic residues; Neutrosophic quadratic nonresidues; Neutrosophic quadratic residue graph; Neutrosophic quadratic nonresidue graph.

1. Introduction

For any positive integer $n \geq 1$, the set $Z_n = \{0, 1, 2, \ldots, n-1\}$ is a ring under the usual addition and multiplication modulo n. Moreover, for any prime p, the ring Z_p is a field of order p and hence $Z_p^* = \{1, 2, 3, \ldots, p-1\}$ is a group under multiplication modulo p, see [1-2].

For $a \in Z_p^*$, a is a quadratic residue modulo p if and only if $a = x^2$ for some $x \in Z_p^*$. Now suppose Q_p denote the set of all quadratic residues modulo p. Then Q_p is a nonempty subset of Z_p^*, given by $Q_p = \{x^2 \in Z_p^* : x = 1, 2, \ldots, \frac{p-1}{2}\}$. It is clear that for any $a, b \in Q_p$, there exists x and y in Z_p^* such that $a^{-1}b = (x^{-1}y)^2 \in Q_p$. Therefore, Q_p is a subgroup of Z_p^* and also the index $[Z_p^* : Q_p] = 2$. This implies that $xy \in Q_p$ if and only if x and y are both in Q_p or neither of them is in Q_p. This specifies that an element in Z_p^* as a residue or nonresidue according to whether or not it is a quadratic residue modulo p. In particular, the set of all quadratic nonresidues modulo p in Z_p^* is denoted by $\overline{Q_p}$. Hence $|Q_p| = |\overline{Q_p}| = \frac{p-1}{2}$. So, Q_p is the normal subgroup and $\overline{Q_p}$ is the only nonempty subset of Z_p^* whose orders are equal. For more information about Q_p and $\overline{Q_p}$, reader refer [3].

Much of the specific power and utility of modern mathematics arises from its abstraction of important features similar to various circumstances and illustrations. But many sets and systems we encounter have a usual addition and multiplication defined on their elements. These operations often satisfy a few common properties that we want to isolate and study. Besides the obvious illustrations in different number systems and algebraic systems, we can operate polynomials, functions, matrices, etc. Studying the algebraic structure of groups, rings, and fields based on number theoretic and combinatorial properties has caught the interest of many researchers order the last decades. Recently, algebraic systems associated with neutrosophic elements and sets [4] seem to be more interesting and active area compare to those associated with classical algebraic structures. For instance, the Neutrosophic set $N(Z_p^*, I)$ is generated by the multiplicative group Z_p^* and the neutrosophic unit element I ($I^2 = I$ and I^{-1} does not exist), that is, $N(Z_p^*, I)$ or equivalently $N(Z_p^*) = < Z_p^*, I >= Z_p^* \cup Z_p^* I$, where p is prime. This is a Neutrosophic group [5] concerning Neutrosophic multiplication $(aI)(bI) = abI$ for every $aI, bI \in N(Z_p^*, I)$.

The concept of the Neutrosophic graph of Neutrosophic structures was first introduced by Vasanth Kandasamy and Smarandache [6], but this work was mostly concerned with the basic properties of Neutrosophic algebraic structures. Recently, the authors Chalapathi and Kiran studied the Neutrosophic graphs [5] of finite groups. The Neutrosophic graph of a finite group G, which is denoted by $Ne(G, I)$, is an undirected simple graph whose vertices are elements of the neutrosophic group $N(G)$ with two distinct vertices x and y which are adjacent if and only if either $xy = x$ or $xy = y$.

In 1879, author Cayley considered the Cayley graph for finite groups. After that, a lot of research has been done on various families of Cayley graphs. For instance, Quadratic residue Cayley graphs [7], Quartic residue Cayley graphs [8]. Many researchers exist in the literature on Cayley graphs quadratic residues on odd prime and prime power modules. The authors studied quadratic residues modulo 2^n Cayley graphs in [9]. In this paper, we will focus on Neutrosophic quadratic residues and their corresponding algebraic graphs, which are not Cayley graphs.

2. Neutrosophic Quadratic and Non Quadratic Residues

In this section, for convenience and also for later use, we define some definitions and notations concerning integers modulo an odd prime p , and Neutrosophic quadratic and nonquadratic residue modulo p.

First, we recall some results about neutrosophic groups from [5].

Theorem 2.1:

1. $Z_p^* I = \{aI : a \in Z_p^*\}$
2. $N(Z_p^*) = Z_p^* \cup Z_p^* I$, where $Z_p^* \cap Z_p^* I = \emptyset$

Theorem 2.2: Let Z_p^* be a finite group with respect to multiplication modulo n. Then

1. $|Z_p^*| = p - 1$ and $|Z_p^* I| = p - 1$
2. $|N(Z_p^*)| = 2(p - 1)$

Let $aI \in N(Q_p)$. Then aI is a neutrosophic quadratic residue modulo p if and only if $aI = (xI)^2$ for some $xI \in Z_p^* I$. Now suppose $N(Q_p)$ denote the set of all neutrosophic quadratic residues modulo p . Then $Q_p I$ is a nonempty subset of $N(Z_p^*)$ given by $Q_p I = \{(xI)^2 \in N(Z_p^*) : x = 1,2, \dots, \frac{p-1}{2}\}$.

Further, if for any $aI, bI \in Q_p I$, then $aI = (xI)^2$ and $bI = (yI)^2$ for some $xI, yI \in Z_p^* I$, so $(aI)(bI) = (xyI)^2 \in Q_p I$

Hence $Q_p I$ is a neutrosophic subgroup of $N(Z_p^*) = Z_p^* \cup Z_p^* I$ with neutrosophic index, by the Theorem 2.1.

$$[N(Z_p^*) : Q_p I] = \frac{|N(Z_p^*)|}{|Q_p I|} = \frac{2(p-1)}{\left(\frac{p-1}{2}\right)} = 4.$$

Similarly, the set of all neutrosophic quadratic non-residues modulo p in $Z_p^* I$ is denoted by $\overline{Q_p I}$ with $|Q_p I| = |\overline{Q_p I}| = \frac{p-1}{2}$.

Example 2.3: The following shortlist shows that the Neutrosophic quadratic and nonquadratic residues modulo $3,\ 5, 7$, respectively.

$N(Q_3, I) = \{1, I\}$, $N(\overline{Q_3}, I) = \{2, 2I\}$,

$N(Q_5, I) = \{1, 4, I, 4I\}$, $N(\overline{Q_5}, I) = \{2, 3, 2I, 3I\}$,

$N(Q_7, I) = \{1, 3, 4, 5, 9, I, 3I, 4I, 5I, 9I\}$, $N(\overline{Q_7}, I) = \{2, 6, 7, 8, 10, 2I, 6I, 7I, 8I, 10I\}$.

From the above example, we observe the following:

$N(Q_p, I) = Q_p \cup Q_p I$ and $N(\overline{Q_p}, I) = \overline{Q_p} \cup \overline{Q_p I}$. In particular,

$|N(Q_p, I)| = |Q_p| + |Q_p I| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1$ and

$|N(\overline{Q_p}, I)| = |\overline{Q_p}| + |\overline{Q_p I}| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1$.

Theorem 2.4: Given $p > 2$, $N(W_p^*, I) = W_p^* \cup W_p^* I$, is the neutrosophic prime subgroup of $N(Z_p^*, I)$, where $W_p^* = \{1, p-1\}$.

Proof: It is clear from the well-known result that W_p^* is a subgroup of the group Z_p^*, because $(p-1)^2 \equiv 1 \ (mod\ p)$.

Theorem 2.5: Fundamental Theorem of Neutrosophic Quadratic Residues Modulo p

For each $p > 2$, we have the neutrosophic quotient group $\frac{N(Z_p^*, I)}{N(W_p^*, I)}$ is isomorphic to the neutrosophic group $N(Q_p, I)$.

Proof: For any $p > 2$, we have $(p-1)^2 \equiv 1 \ (mod\ p)$ and $((p-1)I)^2 \equiv I \ (mod\ p)$. Therefore, $N(W_p^*, I) = \{1, p-1, I, (p-1)I\}$ is a neutrosophic subgroup of $N(Z_p^*, I)$. So, there exists a Neutrosophic quotient group $\frac{N(Z_p^*, I)}{N(W_p^*, I)}$. Now we claim that $\frac{N(Z_p^*, I)}{N(W_p^*, I)} \cong N(Q_p, I)$. For this, we define a map $\Psi: N(Z_p^*, I) \to N(Q_p, I)$ by the relation

$$\Psi(x) = \begin{cases} x^2, & if\ x \in Z_p^* \\ (xI)^2, & if\ xI \in Z_p^* I \end{cases}$$

Clearly, Ψ is a well-defined group and Neutrosophic group homomorphism, because $(ab)^2 = a^2 b^2, \ \forall\ a, b \in Z_p^*$ and $((aI)(bI))^2 = (aI)^2 (bI)^2, \ \forall\ aI, bI \in Z_p^* I$.

Now to find a kernel of Ψ. If $x \in Z_p^*$, then by the definition of kernel of group (classical) homomorphism,

$$K = \{x \in Z_p^* : x^2 = 1\}$$

$$= \{1, -1\}$$

$$= \{1,\ p-1\}.$$

Similarly, if $xI \in Z_p^* I$, then by the definition of a kernel of a Neutrosophic group homomorphism,

$$K' = \{xI \in Z_p^* I : (xI)^2 = I\}$$

$$= \{I, -I\}$$

$$= \{I, (p-1)I\}.$$

Hence, $Ker\ \Psi = K \cup K'$

$$= \{1,\ p-1,\ I, (p-1)I\ \}$$

$$= N(W_p^*,\ I).$$

Finally, to find image of Ψ.

$$Im\,(\Psi) = \{\Psi(x) \in N\left(Z_p^*,\ I\right) : x \in N\left(Z_p^*,\ I\right)\}$$

$$= \{x^2 \in Z_p^* : x \in Z_p^*\} \cup \{(xI)^2 \in Z_p^* I : xI \in Z_p^* I\}$$

$$= Q_p \cup Q_p I$$

$$= N(Q_p,\ I).$$

By the fundamental theorem of a Neutrosophic group homomorphism, $\frac{N(Z_p^*, I)}{Ker\ \Psi} \cong Im(\Psi)$. This shows that $\frac{N(Z_p^*,\ I)}{N(W_p^*,\ I)} \cong N(Q_p,\ I)$.

Remark 2.6: $x \in N(Z_p^*,\ I)$ is a Neutrosophic quadratic residue if and only if $x \in Im\ (\Psi)$, otherwise, it is called neutrosophic quadratic residue modulo p.

Example 2.7: For the prime $p = 5$, we have $Z_5^* = \{1, 2, 3, 4\}$, $N(Z_5^*, I) = \{1, 2, 3, 4, I, 2I, 3I, 4I\}$, $W_5^* = \{1, 4\}$, $N(W_5^*, I) = \{1, 4, I, 4I\}$, $\frac{N(Z_5^*, I)}{N(W_5^*, I)} = \{N(W_5^*, I), 2N(W_5^*, I), 3N(W_5^*, I), 4N(W_5^*, I), I\,N(W_5^*, I), 2I\,N(W_5^*, I), 3I\,N(W_5^*, I), 4I\,N(W_5^*, I)\}$.

Theorem 2.8: The neutrosophic product of two neutrosophic quadratic residues is again a neutrosophic a quadratic residue modulo p. Similarly, the neutrosophic product of two Neutrosophic quadratic nonresidues is a Neutrosophic quadratic residue modulo p.

Proof: Since $N(Q_p, I)$ is a Neutrosophic normal subgroup of the Neutrosophic group $N(Z_p^*, I)$ whose index is 4. So there exists a Neutrosophic quotient group $\frac{N(Z_p^*, I)}{N(Q_p, I)}$ such that $\left|\frac{N(Z_p^*, I)}{N(Q_p, I)}\right| = 4$, that is $\frac{N(Z_p^*, I)}{N(Q_p, I)} = \{x\,N\,(Q_p, I) : x \in N(Z_p^*, I)\}$.

Let $x \in Z_p^*$ such that $x \in Q_p$. Then $(x\,Q_p)^2 = Q_p{}^2 = Q_p$, since $hH = Hh = H$. Let $a \in Z_p^*$ such that $a \notin Q_p$. Then $(a\,Q_p)^2 \neq Q_p$.

Let $x \in Z_p^* I$ such that $x \in Q_p\,I$. Then $(x\,Q_p I)^2 = (Q_p I)^2 = Q_p{}^2\,I^2 = Q_p I$. Let $a \in Z_p^* I$ such that $a \notin Q_p I$. Then $(a\,Q_p I)^2 \neq Q_p I$.

Because $N(Z_p^*, I) = Z_p^* \cup Z_p^* I$ and $N\,(Q_p, I) = Q_p \cup Q_p I$, we know that the neutrosophic quotient group defined as $\frac{N(Z_p^*, I)}{N(Q_p, I)} = \{\,Q_p, aQ_p\,, IQ_p, a\,IQ_p\,\}$.

(1) If $x, y \in Q_p I$, then

$$\begin{aligned} xy\,Q_p I &= (x\,Q_p I)(y\,Q_p I) \\ &= (Q_p I)(Q_p I) \\ &= (Q_p I)^2 \\ &= Q_p{}^2 I^2 \end{aligned}$$

$= Q_p I$, since ${Q_p}^2 = Q_p$,

and thus $xy \in Q_p I$.

(2) If $x, y \notin Q_p I$, then $x,\ y \in \overline{Q_p}$. So there exists $\bar{a}, \bar{b} \in \overline{Q_p}$ such that $x = \bar{a}\, I$ and $y = \bar{b}\, I$. Then

$$xy\, Q_p I = (\bar{a}\, I)(\bar{b}\, I)\, Q_p$$

$$= (\bar{a}\, \bar{b}) I\, Q_p$$

$$= I\left((\bar{a}\, \bar{b})\, Q_p\right)$$

$$= I\, Q_p, \text{ since } \bar{a}, \bar{b} \in \overline{Q_p} \Rightarrow \bar{a}\bar{b} \in Q_p \text{ and } \bar{a}\bar{b}Q_p = Q_p.$$

Hence $xy \in Q_p I$.

(3) If $x \in Q_p I$ and $y \notin Q_p I$, then

$$xy\, Q_p I = (x\, Q_p I)(y\, Q_p I)$$

$$= (Q_p I)(y\, Q_p I), \text{ since } x \in Q_p I \Leftrightarrow x\, Q_p I = Q_p I$$

$$= y(Q_p I)^2$$

$$= y\, {Q_p}^2 I^2$$

$$= y\, Q_p I$$

$$\neq Q_p I, \text{ since } y \notin Q_p I \text{ iff } y\, Q_p I \neq Q_p I.$$

Hence $xy \notin Q_p I$. This proves the theorem.

Now, let us start with simple undirected graphs of neutrosophic quadratic residue and Neutrosophic quadratic Nonresidue graphs of the Neutrosophic graph $N(Z_p^*, I)$ whose vertices are members in the Neutrosophic graph $N(Z_p^*, I)$ where p is an odd prime.

3. Neutrosophic Quadratic Residue Graphs

Structurally, many real-world concepts, aspects, and situations can be described by using and applying diagrams of a set of vertices with edges joining pairs of these vertices. So, a mathematical abstraction of this type of diagram gives rise to the concept of a graph. A graph G and is denoted by

$G = (V, E)$, where $V = V(G)$ and $E = E(G)$ vertex and edge sets of G, respectively. A graph G is said to be connected if there is at least one path between every two vertices in G and disconnected if G has at least one pair of vertices between which there is no path. Every graph G consists of one or more connected graphs as subgraphs, and each such connected subgraph of G is called a component of G , and each component of G is denoted by $Comp(G)$. It is clear that every connected graph contains only one component and every disconnected graph of more than one vertex contains two or more components. Now a graph G is said to be complete if every vertex in G is connected to another vertex in G.

A complete graph of order n is denoted by K_n and it has exactly $\frac{n(n-1)}{2} = n_{C_2}$ edges, and it is called the size of K_n. If u is a vertex of G, then the number of edges incident on a vertex u is called the degree of u and it is denoted by $\deg(u)$. In particular, if $\deg(u) = k$ for every vertex u in G, then G is called a k −regular graph. A graph G is said to be bipartite if its vertex set V can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that each edge of G connects a vertex of V_1 to a vertex of V_2, and the pair (V_1, V_2) is called bipartite of G. Similarly, G is called a complete bipartite graph if each vertex of V_1 is adjacent to each vertex of V_2. Now, consider two graphs $G = (V, E)$ and $G' = (V', E')$, then G and G' are isomorphic to each other and it is denoted by $G \cong G'$ if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved, see [10].

Definition 3.1: An undirected simple graph $G(Z_p^*, Q_p, I)$ is called a Neutrosophic quadratic residue graph of the Neutrosophic group $N(Z_p^*, I)$ whose vertex set is $N(Z_p^*, I)$ and two distinct vertices x and y are adjacent in $G(Z_p^*, Q_p, I)$ if and only if $xy \in N(Q_p, I)$.

Before studying the properties of neutrosophic quadratic residue graphs, we give two examples to illustrate their usefulness.

Example 3.2: Since $N(Z_5^*, I) = \{1,\ 2,\ 3,\ 4,\ I,\ 2I,\ 3I,\ 4I\}$ is the vertex set of the graph $G(Z_5^*,\ Q_5,\ I)$, where $N(Q_5,\ I) = \{1, 4,\ I,\ 4I\}$.

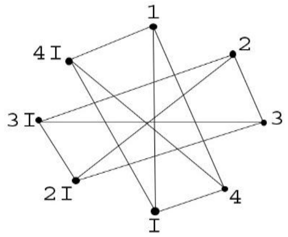


Figure 1. Neutrosophic Quadratic Residue Graph $G(Z_5^*,\ Q_5,\ I)$ of modulo 5.

Example 3.3: For $p = 7$, we have $N(Z_7^*, I) = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ I,\ 2I,\ 3I,\ 4I,\ 5I,\ 6I\}$ and $N(Q_7,\ I) = \{1,\ 2,\ 4,\ I,\ 2I,\ 4I\}$. Then the graph $G(Z_7^*,\ Q_7,\ I)$ is represented as follows.

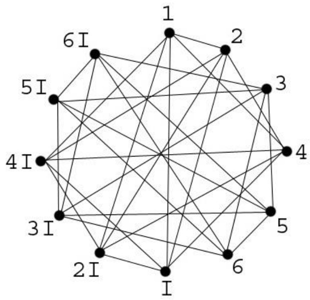


Figure 2. Neutrosophic Quadratic Residue Graph $G(Z_7^*,\ Q_7,\ I)$ of modulo 7.

In this section, the basic properties of $G(Z_p^*,\ Q_p,\ I)$ being studied. We begin with the disconnectedness of the graph $G(Z_p^*,\ Q_p,\ I)$.

Theorem 3.4: For $p > 2$, the graph $G(Z_p^*,\ Q_p,\ I)$ is disconnected. In particular, graph $G(Z_p^*,\ Q_p,\ I)$ is the disjoint union of two complete components.

Proof: Let $p > 2$ be an odd prime. Then the vertex set of neutrosophic quadratic residue graph $G(Z_p^*,\ Q_p,\ I)$ is $N(Z_p^*,\ I)$. But

$$N(Z_p^*,\ I) = N(Q_p,\ I) \cup N(\overline{Q_p},\ I)$$

$$= (Q_p \cup Q_p I) \cup (\overline{Q_p} \cup \overline{Q_p} I)$$

where $(Q_p \cup Q_p I) \cap (\overline{Q_p} \cup \overline{Q_p} I) = \emptyset$. This gives us that the vertex set $N(Z_p^*,\ I)$ is partitioned into two disjoint unions of $(Q_p \cup Q_pI)$ and $(\overline{Q_p} \cup \overline{Q_p}I)$. So, because of Theorem 2.8, we clear that $G(Z_p^*, Q_p, I)$ is disconnected. Now consider the following three cases.

Case 1: Suppose $x, y \in N(Q_p, I)$. Then obviously $xy \in N(Q_p, I)$. This implies that there exists an edge between any two vertices x and y in the graph $G(Z_p^*, Q_p, I)$. Thus, $G(Z_p^*, Q_p, I)$ has a complete subgraph, say $Comp(Z_p^*, Q_p, I)$ whose vertex set is $N(Q_p, I)$.

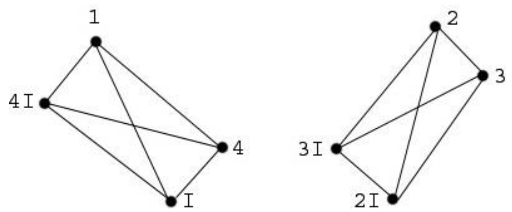
Case 2: Suppose $x, y \in N(\overline{Q_p}, I)$. Then again by Theorem 2.8, $xy \in N(\overline{Q_p}, I)$. So, in this case also there exists an edge between every two vertices x and y in the graph $G(Z_p^*, Q_p, I)$. Thus, the graph, $G(Z_p^*, Q_p, I)$ has another complete subgraph, say $Comp(Z_p^*, \overline{Q_p}, I)$ whose vertex set is $N(\overline{Q_p}, I)$.

Case 3: Suppose $x \in N(Q_p, I)$ and $y \in N(\overline{Q_p}, I)$. Then $xy \notin N(Q_p, I)$. It gives us that there is no edge between x and y when $x \in N(Q_p, I)$ and $y \in N(\overline{Q_p}, I)$.

From the above three cases, we conclude that $Comp(Z_p^*, Q_p, I)$ and $Comp(Z_p^*, \overline{Q_p}, I)$ are two disjoint complete components of the graph $G(Z_p^*, Q_p, I)$ such that

$$G(Z_p^*, Q_p, I) = Comp(Z_p^*, Q_p, I) \cup Comp(Z_p^*, \overline{Q_p}, I).$$

Example 3.5: Two components of the graph $G(Z_5^*, Q_5, I)$ as shown below.



$Comp(Z_5^*, Q_5, I)$ $Comp(Z_5^*, \overline{Q_5}, I)$

Figure 3. Components of the graph $G(Z_5^*, Q_5, I)$.

For each odd prime p, the structure of $G(Z_p^*, Q_p, I)$ is easy to describe, because it contains the following properties:

1. $G(Z_p^*,\ Q_p,\ I)$ contains two disjoint connected components for each $p > 2$.

2. Each component of $G(Z_p^*,\ Q_p,\ I)$ contains even and odd cycles for $p \geq 5$.

3. Each component of $G(Z_p^*,\ Q_p,\ I)$ is not a bipartite graph for $p \geq 3$.

The next result gives useful and important properties of the components of the graph $G(Z_p^*,\ Q_p,\ I)$ when $p > 2$.

Theorem 3.6: For each prime $p > 2$, $Comp\ (Z_p^*,\ Q_p,\ I) \cong Comp\ (Z_p^*,\ \overline{Q_p},\ I)$.

Proof: For each prime $p > 2$, the Neutrosophic quadratic residue and non-residue sets of $N(Z_p^*,\ I)$ are given by $N(Q_p,\ I) = Q_p \cup Q_p I$ and $N(\overline{Q_p},\ I) = \overline{Q_p} \cup \overline{Q_p} I$.

These are the vertex sets of the components $Comp\ (Z_p^*,\ Q_p,\ I)$ and $Comp\ (Z_p^*,\ \overline{Q_p},\ I)$, respectively. Also, we have $|N(Q_p,\ I)| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1 = |N(\overline{Q_p},\ I)|$. Now to prove that $Comp\ (Z_p^*,\ Q_p,\ I)$ and $Comp\ (Z_p^*,\ \overline{Q_p},\ I)$ are isomorphic as groups. For this, we define a function $f: N(Q_p,\ I) \rightarrow N(\overline{Q_p},\ I)$ by the relation $f(u) = v$ for every $u \in N(Q_p,\ I)$ and $v \in N(\overline{Q_p},\ I)$. Because of $|N(Q_p,\ I)| = p - 1$ and $|N(\overline{Q_p},\ I)| = p - 1$, the map f is a one-to-one correspondence.

Now, suppose $\bar{e}$ be an edge with end vertices v and v' in the component $Comp\ (Z_p^*,\ \overline{Q_p},\ I)$. Then $\bar{e} = (v,\ v') \Leftrightarrow \bar{e} = (f(u), f(u'))$

$$\Leftrightarrow \bar{e} = f(u,\ u')$$

$$\Leftrightarrow \bar{e} = f(e),$$

where $e = (u,\ u')$ be an edge with end vertices u and u' in $Comp\ (Z_p^*,\ Q_p,\ I)$. This shows that there is a one-to-one correspondence between their vertices and their edges such that the incidence relationship is preserved. Hence, $Comp\ (Z_p^*,\ Q_p,\ I) \cong Comp\ (Z_p^*,\ \overline{Q_p},\ I)$.

The following example illustrates the procedure of the above theorem 3.6 clearly.

Example 3.7: Since $N(Q_5,\ I) = \{1, 4,\ I,\ 4I\}$ and $N(\overline{Q_5},\ I) = \{2, 3,\ 2I,\ 3I\}$. Using the map $f: N(Q_5,\ I) \rightarrow N(\overline{Q_5},\ I)$ as above, write the equations $f(1) = 2$, $f(4) = 3$, $f(I) = f(2I)$ and

$f(4I) = f(3I)$. These equations show that f is a one-to-one correspondence between the graph components $Comp\ (Z_5^*,\ Q_5,\ I)$ and $Comp\ (Z_5^*,\ \overline{Q_5},\ I)$, and thus which are isomorphic as graphs.

This special case of the above theorem when $p > 2$ occurs frequently and so we isolate it as a corollary.

Corollary 3.8: Each component of the neutrosophic quadratic residue graph is isomorphic to the complete graph K_{p-1}.

Proof: Due to Theorem 3.6, the only possibility of the graph $Comp\ (Z_p^*,\ Q_p,\ I)$ is $Comp\ (Z_p^*,\ Q_p,\ I) \cong Comp\ (Z_p^*,\ \overline{Q_p},\ I)$. Therefore, the order and size of each component are $p-1$ and $\binom{p-1}{2}$, respectively, and thus each component of the graph $G(Z_p^*,\ Q_p,\ I)$ is isomorphic to the complete graph K_{p-1}.

Example 3.9: $Comp\ (Z_5^*,\ Q_5,\ I) \cong K_4$ and $Comp\ (Z_7^*,\ Q_7,\ I) \cong K_6$.

The integer p is prime if and only if $p = 2$ or $p \equiv 3(mod\ 4)$ $p \equiv 1(mod\ 4)$. But, this paper p will denote odd prime integer such that either $p \equiv 1(mod\ 4)$ or $p \equiv 3(mod\ 4)$. These prime integers are weapons for verifying two components of the graph $G(Z_p^*,\ Q_p,\ I)$ are Eulerian or not. It is now the time for determining the cases in which the components of the graph $G(Z_p^*,\ Q_p,\ I)$ are Eulerian, but first, we recall the following well-known result.

Theorem 3.10 [10]: A connected graph G is Eulerian if and only if the degree of each vertex of G is even.

For $p \equiv 1\ (mod\ 4)$ or $p \equiv 3\ (mod\ 4)$, the following theorems show that $G(Z_p^*,\ Q_p,\ I)$ could not be Eulerian.

Theorem 3.11: If $p \equiv 1\ (mod\ 4)$ or $p \equiv 3\ (mod\ 4)$, then each component of $G(Z_p^*,\ Q_p,\ I)$ is not Eulerian.

Proof: Suppose on contrary that each component of $G(Z_p^*,\ Q_p,\ I)$ is Eulerian, which implies that the degree of each vertex is even. By Theorem 3.6, it is clear that

$$Comp\ (Z_p^*,\ Q_p,\ I) \cong Comp\ (Z_p^*,\ \overline{Q_p},\ I) \cong K_{p-1}.$$

So, for every vertex x in $G(Z_p^*, Q_p, I)$, we have

$$\deg(x) = (p-1) - 1 = p - 2.$$

$\deg(x) = (4q+1) - 2 = 4q - 1$, which is odd. Similarly, we can show that $\deg(x) = (4q+3) - 2 = 4q + 1$, which is also odd. Hence, we found that the degree of each vertex in the graph $G(Z_p^*, Q_p, I)$ can not be even. This contraposition shows that each component of $G(Z_p^*, Q_p, I)$ is never Eulerian when $p \equiv 1(mod\ 4)$ or $p \equiv 3(mod\ 4)$.

4. Neutrosophic Quadratic Nonresidue Graphs

In this section, we establish a complement graph of the neutrosophic quadratic residue graph $G(Z_p^*, Q_p, I)$, which is denoted by $\bar{G}(Z_p^*, \overline{Q_p}, I)$ and it is called a Neutrosophic quadratic nonresidue graph whose vertex set is the Neutrosophic group $N(Z_p^*, I)$ and edge set is $E(\bar{G}(Z_p^*, \overline{Q_p}, I)) = \{(x, y) : x, y \in N(Z_p^*, I)\ and\ xy \in N(\overline{Q_p}, I)\ \}$.

Example 4.1: Since $N(Z_3^*, I) = \{1, 2, I, 2I\}$ and $N(\overline{Q_3}, I) = \{2, 2I\}$. The Neutrosophic quadratic nonresidue graph of $N(Z_3^*, I)$ is shown below.

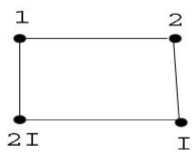


Figure 4. The graph $\bar{G}(Z_3^*, \overline{Q_3}, I)$.

Now several interesting properties of these graphs on Neutrosophic quadratic nonresidues of modulo p have been obtained.

We begin with the basic properties of $\bar{G}(Z_p^*, \overline{Q_p}, I)$.

Theorem 4.2: The Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is connected.

Proof: By the Theorem 2.8, $xy \in N(\overline{Q_p}, I)$ whenever $x \in N(Q_p, I)$ and $x \in N(\overline{Q_p}, I)$. This relates, for each $1 \le i \le \frac{p-1}{2}$, we have

$$Q_p = \left\{x_1,\ x_2,\ \ldots,\ x_{\frac{p-1}{2}}\right\},$$

$$Q_p I = \left\{x_1 I,\ x_2 I,\ \ldots,\ x_{\frac{p-1}{2}} I\right\},$$

$$\overline{Q_p} = \left\{y_1,\ y_2,\ \ldots,\ y_{\frac{p-1}{2}}\right\} \text{ and}$$

$$\overline{Q_p} I = \left\{y_1 I,\ y_2 I,\ \ldots,\ y_{\frac{p-1}{2}} I\right\}.$$

These sets determine the elements

$x_1 y_1,\ x_1 y_2, \ldots,\ x_1 y_i$;

$x_2 y_1,\ x_2 y_2, \ldots,\ x_2 y_i$;

… … …

$x_i y_1,\ x_i y_2, \ldots,\ x_i y_i$;

… … …

$(x_1 I)(y_1 I),\ (x_1 I)(y_2 I), \ldots,\ (x_1 I)(y_i I)$;

$(x_2 I)(y_1 I),\ (x_2 I)(y_2 I), \ldots,\ (x_2 I)(y_i I)$;

… … …

$(x_i I)(y_1 I),\ (x_i I)(y_2 I), \ldots,\ (x_i I)(y_i I)$;

… … …

are elements in $N(\overline{Q_p},\ I)$ and which are the edges in the graph $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$. Consequently, there is a path between any two distinct vertices in $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$ and hence $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$ is connected.

Theorem 4.3: The Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$ is $(p-1)$- regular.

Proof: If x is any vertex of the Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$, then x must be an element of the Neutrosophic group $N(Z_p^*,\ I)$. So there exist Neutrosophic quadratic residues $N(Q_p,\ I)$ and nonresidues $N(\overline{Q_p},\ I)$ such that

$$N(Z_p^*,\ I) = N(Q_p,\ I) \cup N(\overline{Q_p},\ I).$$

This partition of the vertex set of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ implies that either $x \in N(Q_p, I)$ or $x \in N(\overline{Q_p}, I)$.

Now $x \in N(Q_p, I)$, and if $N(\overline{Q_p}, I) = \left\{y_1, y_2, \ldots, y_{\frac{p-1}{2}}, y_1 I, y_2 I, \ldots, y_{\frac{p-1}{2}} I\right\}$ then by Theorem 2.8 $xN(\overline{Q_p}, I) = \left\{xy_1, xy_2, \ldots, xy_{\frac{p-1}{2}}, xy_1 I, xy_2 I, \ldots, xy_{\frac{p-1}{2}} I\right\} = N(\overline{Q_p}, I)$.

It gives that the vertex x is adjacent to every element in $N(\overline{Q_p}, I)$. This means that

$$\begin{aligned}\deg(x) &= |N(\overline{Q_p}, I)| \\ &= |\overline{Q_p} \cup \overline{Q_p} I| \\ &= |\overline{Q_p}| + |\overline{Q_p} I| \\ &= \frac{p-1}{2} + \frac{p-1}{2} \\ &= p-1.\end{aligned}$$

Next $x \in N(\overline{Q_p}, I)$ and if $N(Q_p, I) = \left\{x_1, x_2, \ldots, x_{\frac{p-1}{2}}, x_1 I, x_2 I, \ldots, x_{\frac{p-1}{2}} I\right\}$. Then, again by the Theorem 2.8,

$$x\, N(Q_p, I) = N(\overline{Q_p}, I).$$

It yields that $\deg(x) = p-1$, proving that the Neutrosophic Quadratic nonresidue Graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is $(p-1)-$ regular.

Finally looking at another basic property of the Neutrosophic quadratic nonresidue graph, we state the following fundamental theorem of graph theory.

Theorem 4.4 [10]: If G is a simple undirected graph of the size $|E|$. Then $\sum_{x \in V(G)} \deg(x) = 2|E|$.

Theorem 4.5: The size of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is $(p-1)^2$.

Proof: By the Theorem 4.3 and theorem 2.5, the size of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is denoted by $|E(\bar{G})|$ and defined as

$$|E(\bar{G})| = \frac{1}{2}\sum_{x \in N(Z_p^*, I)} \deg(x)$$

$$= \frac{1}{2} \sum_{x \in N(Z_p^*,\ I)} (p-1)$$

$$= \frac{1}{2}(p-1)\left|N(Z_p^*,\ I)\right|$$

$$= \frac{1}{2}(p-1)(2p-2)$$

$$= (p-1)^2.$$

5. Conclusions

In this paper, we have studied two Neutrosophic graphical representations for determining the Neutrosophic Quadratic residues and nonresidues of the Neutrosophic group of modulo prime by using Neutrosophic algebraic theory, number theory, and classical algebraic theory. In addition to these, the Neutrosophic algebraic system can find Neutrosophic properties of Quadratic residues and nonresidues. Also, this algebraic-based application produces the complement neutrosophic graphs of each disjoint union of Neutrosophic Quadratic residue and nonresidue sets.

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