Neutrosophic Real Inner Product Spaces

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Abstract: The objective of this work is to define for the first time the concept of inner product in a neutrosophic vector space V(I) and a refined neutrosophic vector space V(I_1, I_2) . Also, this paper introduces many interesting properties of neutrosophic real inner products and establishes the theoretical foundations of neutrosophic functional analysis such as neutrosophic normed spaces, refined neutrosophic normed spaces, neutrosophic real inner product spaces, refined neutrosophic inner product spaces, orthogonal basis, neutrosophic and refined neutrosophic Cauchy- Schwartz inequality.

Keywords: Neutrosophic Inner product, refined neutrosophic inner product, neutrosophic normed space, neutrosophic orthogonal basis, neutrosophic functional analysis

1. Introduction

Neutrosophy is a new branch of philosophy founded by Smarandache to study the indeterminacy in all activities and branches of science. Neutrosophic set and its generalizations such as refined neutrosophic set and n-refined neutrosophic set became useful tools in the study of topology [7,8,32], neutrosophic algebraic structures such as modules, groups, matrices, and different kinds of rings [1,3,5,6,9,11,14,15,16,22,23,24,30,33,34], and equations [31]. On the other hand, neutrosophic sets were used widely in applied mathematics and engineering such as optimization, artificial intelligence, health care, decision making , and industry [25,26,27,28,29,35,36,37,40,41]. On the other hand, neutrosophy is very effective and applicative in the study of many applied fields such as probability, statistics, analysis and Diophantine equations [39,42].

The concept of inner product was defined in classical vector spaces as a linear function takes its values in a field such as R or C, which plays an important role in the study of norms, metrics, and functional analysis [12,13].

Agboola et.al presented the concept of weak and strong neutrosophic vector space in [10], and refined neutrosophic vector space in [17,18] from an algebraic view as new generalizations of classical vector spaces. Recently, Smarandache et.al defined n-refined neutrosophic vector space in [20], with many interesting substructures [2]. Sankari et.al have proved that the weak neutrosophic vector space is isomorphic to the direct product of V with itself in [21]. This means that inner products defined over a weak neutrosophic vector space V(I) can be studied easily by taking its isomorphic image to the classical case.

In this work, we aim to extend classical real inner product to the neutrosophic and refined neutrosophic case. We define the concept of inner product over a strong neutrosophic vector space V(I) and over a strong refined neutrosophic vector space V(I_1, I_2) and we study its basic properties, which is considered as a first step in the theory of neutrosophic functional analysis.

The main result of this work is to prove that Cauchy-Schwartz inequality is still true in neutrosophic and refined neutrosophic spaces.

One of the most difficulties is that neutrosophic and refined neutrosophic real numbers has no order relation, so that we define an order relation, so we can study inequalities between such numbers.

Motivation

We regard that there is not a strict definition of inner products in neutrosophic systems based on neutrosophic spaces, thus our motivation is to close this important research gap by defining the basic theoretical concepts of functional analysis based on neutrosophic numbers and spaces.

2. Preliminaries

Definition 2.1 : [10]

Let $(V, +, .)$ be a vector space over the field K, $(V(I), +, .)$ is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Elements of V(I) have the form $x + yI$; $x, y \in V$, i.e V(I) can be written as $V(I) = V + VI$.

Definition 2.2 : [10]

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I), then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition 2.3 : [10]

Let U(I) , W(I) be two strong neutrosophic subspaces of V(I), then we say that V(I) is a direct sum of U(I) and W(I) if and only if for each element x $\in V(I)$, then x can be written uniquely as x = y + z such y$\in U(I)$ and z $\in W(I)$

Definition 2.4 :[10]

Let $v_1, v_2 .. v_s \in V(I) and\ x \in V(I)$, we say that x is a linear combination of { $v_i; i = 1..s$} if x= $a_1 v_1 + \cdots + a_s v_s$ such $a_i \in K(I)$.

The set { $v_i; i = 1..s$} is called linearly independent if $a_1 v_1 + \cdots + a_s v_s = 0$ implies $a_i = 0$ for all i.

Theorem 2.5 : [10]

If { v_1 , ... v_s } is a basis of V(I) and $f: V(I) \to W(I)$ is a neutrosophic vector space homomorphism, then { f(v_1) , ... $f(v_s)$} is a basis of W(I).

Definition 2.6: [13]

Let V be a vector space over the field R, consider the following function $g: V \times V \to R$, then g is called real inner product if and only if:

(a) $g(a, a) \geq 0, g(a, a) = 0\ if\ and\ only\ if\ a = 0$.

$(b) g(a, b) = g(b, a)$.

(c) $g(ma + nb, c) = mg(a, c) + ng(b, c)$. For all $a, b, c \in V, m, n \in R$.

V is called a real inner product space.

Definition 2.7: [12]

Let V be a real inner product space, the norm of any element $x \in V$ is defined as follows:
$\|x\| = \sqrt{g(x,x)}$.

Definition 2.8: [13]

Let V be any vector space over the field R, the function $\| \ \|: V \to R$ is called a norm if and only if:

(a) $\|x\| \geq 0, \|m.x\| = |m|.\|x\|$.

$(b) \|x+y\| \leq \|x\| + \|y\| \ for\ all\ x, y \in V and\ m \in R$.

(c) $\|x\| = 0\ if\ and\ only\ if\ x = 0$.

Theorem 2.9: [12] (Cauchy- Schwartz inequality)

Let V be any real inner product space. Then $|g(x,y)| \leq \|x\|\|y\| for\ all\ x, y \in V$.

3. Neutrosophic inner product spaces

First of all, we shall define a partial order relation (≤) on the neutrosophic field of real numbers $R(I)$.

Definition 3.1: [39]

Let $R(I) = \{a + bI;\ a, b \in R\}$ be the real neutrosophic field, we say that $a + bI \leq c + dI$ if and only if $a \leq c\ and\ a + b \leq c + d$.

Theorem 3.2: [39]

The relation defined in Definition 3.1 is a partial order relation.

Remark 3.3:

According to Theorem 3.2, we are able to define positive neutrosophic real numbers as follows:

$a + bI \geq 0 = 0 + 0.I$ implies that $a \geq 0, a + b \geq 0$.

Absolute value on R(I) can be defined as follows:

$|a + bI| = |a| + I[|a + b| - |a|]$, we can see that $|a + bI| \geq 0$.

Example 3.4:

$x = 2 - I$ is a neutrosophic positive real number, since $2 \geq 0\ and\ (2-1) = 1 \geq 0$.

$2 + I \geq 2$, that is because $2 \geq 2\ and\ (2+1) = 3 \geq (2+0) = 2$.

Definition 3.5:

Let V be a vector space over R, V(I) be its corresponding strong neutrosophic vector space over R(I).

Let $f: V(I) \times V(I) \to R(I)$ be a map, we call it a neutrosophic real inner product if it has the following properties:

(a) $f(x,y) = f(y,x)$.

$(b) f(x,x) \geq 0, if\ f(x,x) = 0\ if\ and\ only\ if\ x = 0$.

(c) $f(ax+by,z)=af(x,z)+bf(y,z)$ for all $x,y,z\in V(I), a,b\in R(I)$.

V(I) is called a neutrosophic real inner product space.

Definition 3.6:

Let V(I) be any neutrosophic real inner product space, consider any two elements $x,y\in V(I)$. We say that $x\perp y$ if and only if $f(x,y)=0$.

Now, we suggest a kind of real neutrosophic inner products which can be derived from any classical inner product on the space V.

Theorem 3.7:

Let V be any inner product space over R, consider $g: V\times V\to R$ as its inner product. Then the corresponding neutrosophic strong vector space $V(I)$ has a neutrosophic real inner product.

Proof:

We define $f: V(I)\times V(I)\to R(I); f(a+bI,c+dI)=g(a,c)+I[g(a+b,c+d)-g(a,c)]$ $for\ all\ a+bI, c+dI\in V(I)$. We prove that f is a neutrosophic inner product.

Let $x=a+bI, y=c+dI, z=m+nI\in V(I), hence\ a,b,c,d,m,n\in V$, let $t=k+sI, l=r+pI\in R(I)$, we have

$f(x,y)=f(a+bI,c+dI)=g(a,c)+I[g(a+b,c+d)-g(a,c)]$=

$g(c,a)+I[g(c+d,a+b)-g(c,a)]$=$f(y,x)$.

$f(x,x)=g(a,a)+I[g(a+b,a+b)-g(a,a)]\geq 0$, that is because $g(a,a)\geq 0\ and\ [g(a+b,a+b)-g(a,a)+g(a,a)]=g(a+b,a+b)\geq 0$.

$f(x,x)=0$ implies $g(a,a)+I[g(a+b,a+b)-g(a,a)]=0$, hence $g(a,a)=0\ and\ g(a+b,a+b)=0$, thus $a=0\ and\ a+b=0$, so that $a=b=0\ and\ x=0$.

Now, we shall compute $tx+ly$.

$tx=ka+I[kb+sa+sb]=ka+I[(k+s)(a+b)-ka], ly=rc+I[rd+pc+pd]=rc+I[(r+p)(c+d)-rc]$, hence

$tx+ly=[ka+rc]+I[(k+s)(a+b)-ka+(r+p)(c+d)-rc]$,

$f(tx+ly,z)=g((ka+rc,m))+I\big[g\big((k+s)(a+b)+(r+p)(c+d),m+n\big)-g\big((ka+rc),m\big)\big]=$

$kg(a,m)+rg(c,m)+I[(k+s)g(a+b,m+n)+(r+p)g(c+d,m+n)-\ kg(a,m)-rg(c,m)]$.

On the other hand we have:

$t.f(x,z) = (k + sI).[g(a,m) + I[g(a+b,m+n) - g(a,m)] = k.g(a,m) + I[k.g(a+b,m+n) - k.g(a,m) + s.g(a,m) + s.g(a+b,m+n) - s.g(a,m)] = k.g(a,m) + I[k.g(a+b,m+n) - k.g(a,m) + s.g(a+b,m+n)]$.

$lf(y,z) = (r + pI).[g(c,m) + I[g(c+d,m+n) - g(c,m)] = r.g(c,m) + I[r.g(c+d,m+n) - r.g(c,m) + p.g(c,m) + p.g(c+d,m+n) - p.g(c,m)] = r.g(c,m) + I[r.g(c+d,m+n) - r.g(c,m) + p.g(c+d,m+n)]$.

Now, we can find that

$f(tx + ly, z) = tf(x,z) + lf(y,z), thus\ f\ is\ a\ neutrosophic\ inner\ product.$

Definition 3.8:

(a) The neutrosophic real inner product introduced in Theorem 3.7 is called the canonical neutrosophic real inner product generated by g.

(b) Let V be any vector space over R, with a classical real inner product g, V(I) be its corresponding neutrosophic strong vector space, let f be the canonical inner product generated by g, the canonical norm of $x = a + bI$ is defined as follows:

$\|x = a + bI\| = \sqrt{f(x,x)}$.

Theorem 3.9:

Let V be any vector space over R, with a classical real inner product g, V(I) be its corresponding neutrosophic strong vector space, let f be the canonical inner product generated by g, we have

(a) $\|x\| = \|a\| + I[\|a+b\| - \|a\|]$ for all $x = a + bI \in V(I)$.

$(b) For\ x = a + bI, y = c + dI,\ x \perp y$ if and only if $a \perp$c, and $a + b \perp c + d$.

(c) $\|a + bI\| = 1$ if and only if $\|a\| = \|a+b\| = 1$.

(d) If $\|a + bI\| = 1$, then $g(a,b) \leq 0$.

Proof:

(a) We use definition 3.8 to compute $\|a + bI\|^2 = f(a+bI, a+bI) = g(a,a) + I[g(a+b,a+b) - g(a,a)]=$

$\|a\|^2 + I[\|a+b\|^2 - \|a\|^2]$.

Now, we prove that $\sqrt{f(x,x)} = \|a\| + I[\|a+b\| - \|a\|]$. By easy computing, we find

$[\|a\| + I[\|a+b\| - \|a\|]]^2 = \|a\|^2 + I[\|a+b\|^2 - \|a\|^2] = f(x,x)$, thus

$\|x\| = \|a\| + I[\|a+b\| - \|a\|]$.

(b) $x \perp y$ if and only if $f(x,y) = 0$, hence $g(a,c) + I[g(a+b,c+d) - g(a,c)] = 0$, this implies that $g(a,c) = 0, g(a+b,c+d) = 0$, thus $a \perp c$, and $a+b \perp c+d$.

(c) $\|a+bI\| = 1$ if and only if $\|a\| + I[\|a+b\| - \|a\|] = 1$, hence $\|a\| = 1, \|a+b\| - \|a\| = 0$, thus $\|a+b\| = \|a\| = 1$.

(d) By section (c), we find that $\|a+b\| = \|a\| = 1$, this means $g(a+b,a+b) = g(a,a) = 1$, hence $g(a,a) + 2g(a,b) + g(b,b) = g(a,a), thus\ \|b\|^2 = g(b,b) = -2g(a,b) \geq 0$, *thus* $g(a,b) \leq 0$.

Example 3.10:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I) = R^2(I) = \{(a,b) + (c,d)I; a,b,c,d \in R\}$ is defined as follows:

$f\big[\big((a,b) + (c,d)I, (m,n) + (t,s)I\big)\big] = g[(a,b),(m,n)] + I[g\big((a+c,b+d),(m+t,n+s)\big) - g\big((a,b),(m,n)\big)]$= $(a.m + b.n) + I[(a+c).(m+t) + (b+d).(n+s) - a.m - b.n]$, where $a,b,c,d,m,n,s,t \in R$.

(b) Let $x = (1,2) + (1,0)I, y = (-2,1) + (2,-1)I$, we have

$f(x,y) = (1)(-2) + (2)(1) + I[(3)(0) + (2)(0) - (1)(-2) - (2)(1)] = 0, hence\ x \perp y$.

$\|x\| = \|(1,2)\| + I[\|(2,2)\| - \|(1,2)\|] = \sqrt{5} + I[\sqrt{8} - \sqrt{5}]$.

(c) Let $x = (1,0) + (-1,1)I$, we have $\|x\| = \|(1,0)\| + I[\|(0,1)\| - \|(1,0)\|] = 1 + I[1-1] = 1$.

We can see that $\|(1,0)\| = \|(1,0) + (-1,1)\| = 1 = \|(1,0)\|$.

Theorem 3.11: (Neutrosophic Cauchy-Schwartz inequality)

Let $x = a + bI, y = c + dI$ any two elements in a strong neutrosophic canonical inner product vector space. Then

$|f(x,y)| \leq \|x\|\|y\|$.

Proof:

We have $|f(x,y)| = |g(a,c)| + I[|g(a+b,c+d)| - |g(a,c)|]$.

$\|x\|\|y\| = \|a\|\|c\| + I[\|a\|\|c+d\| - \|a\|\|c\| + \|a+b\|\|c\| - \|a\|\|c\| + \|a+b\|\|c+d\| - \|c\|\|a+b\| - \|a\|\|c+d\| + \|a\|\|c\|] = \|a\|\|c\| + I[\|a+b\|\|c+d\| - \|a\|\|c\|]$.

By classical Cauchy – Schwartz inequality, we find $|g(a,c)| \leq \|a\|\|c\|$, and $|g(a+b,c+d)| \leq \|a+b\|\|c+d\|$, thus

$|g(a,c)| + I[|g(a+b,c+d)| - |g(a,c)|] \leq \|a\|\|c\| + I[\|a+b\|\|c+d\| - \|a\|\|c\|]$, so that

$|f(x,y)| \leq \|x\|\|y\|$.

Example 3.12:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I) = R^2(I) = \{(a,b) + (c,d)I; a,b,c,d \in R\}$ is defined as follows:

$f\big[\big((a,b) + (c,d)I, (m,n) + (t,s)I\big)\big] = g[(a,b),(m,n)] + I[g\big((a+c,b+d),(m+t,n+s)\big) - g\big((a,b),(m,n)\big)]$= $(a.m + b.n) + I[(a+c).(m+t) + (b+d).(n+s) - a.m - b.n]$, where $a,b,c,d,m,n,s,t \in R$.

(b) Let $x = (1,1) + (2,-1)I, y = (1,0) + (0,1)I$, we have

$f(x,y) = 1 + I[3-1] = 1 + 2I, |f(x,y)| = 1 + 2I, \|x\| = \sqrt{2} + I\big[3 - \sqrt{2}\big], \|y\| = 1 + I[\sqrt{2} - 1]$.

$\|x\|\|y\| = \sqrt{2} + I\big[2 - \sqrt{2} + 3 - \sqrt{2} + 3\sqrt{2} - 3 - 2 + \sqrt{2}\big] = \sqrt{2} + 2\sqrt{2}I,$

$|f(x,y)| = 1 + 2I \leq \sqrt{2} + 2\sqrt{2}I$. That is because $1 \leq \sqrt{2}, 1 + 2 = 3 \leq \sqrt{2} + 2\sqrt{2}$. (see definition 3.13).

Theorem 3.13:

Let V(I) be a neutrosophic strong real inner product vector space, let $x = a + bI$ be any element in V(I). We have

(a) $\|x\| \geq 0, \|m.x\| = |m|.\|x\|$.

$(b) \|x+y\| \leq \|x\| + \|y\| \; for \; all \; x,y \in V(I) and \; m \in R(I)$.

(c) $\|x\| = 0 \; if \; and \; only \; if \; x = 0$.

Proof:

(a) Since$\|x\| = \|a\| + I[\|a+b\| - \|a\|]$, and $\|a\| \geq 0, (\|a+b\| - \|a\|) + \|a\| = \|a+b\| \geq 0$, we get that $\|x\| \geq 0$.

Let $m = c + dI \in R(I); c,d \in R$, we have $m.x = c.a + I[(c+d)(a+b) - c.a]$, hence

$\|m.x\| = \|\mathrm{c.a}\| + I[\|(c+d)(a+b) - c.a + c.a\| - \|c.a\|]$=

$|c|\|a\| + I[|(c+d)|\|a+b\| - |c|\|a\|]$=

$[|c| + I[|c+d| - |c|]][\|a\| + I[\|a+b\| - \|a\|]] = |m|.\|x\|$.

(b) Let $x = a + bI, y = c + dI \in V(I); a,b,c,d \in V$, $\|x+y\| = \|(a+c) + (b+d)I\|$=

$\|a+c\| + I[\|a+c+b+d\| - \|a+c\|]$, by regarding classical properties of classical norms, we get

$\|a+c\| \leq \|a\| + \|c\|, \|a+c+b+d\| \leq \|a+b\| + \|c+d\|$, thus

$\|a+c\|+I[\|a+c+b+d\|-\|a+c\|] \leq \|a\|+\|c\|+I[\|a+b\|+\|c+d\|-\|a\|-\|c\|]=\|x\|+\|y\|$.

(c) The proof is trivial and similar to the classical case.

According to the previous theorem, we can define any neutrosophic norm on a strong neutrosophic vector space V(I) as a function $\| \quad \|: V(I) \rightarrow R(I)$, where conditions (a), (b), and (c) are true. V(I) is called a strong neutrosophic normed space in this case.

Example 3.14:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I)=R^2(I)=\{(a,b)+(c,d)I; a,b,c,d \in R\}$ is defined as follows:

$f[((a,b)+(c,d)I,(m,n)+(t,s)I)]=g[(a,b),(m,n)]+I[g((a+c,b+d),(m+t,n+s))-g((a,b),(m,n))]= (a.m+b.n)+I[(a+c).(m+t)+(b+d).(n+s)-a.m-b.n]$, where $a,b,c,d,m,n,s,t \in R$.

(b) Let $x=(1,1)+(1,0)I, y=(1,-1)+(0,1)I, m=2+3I$, we have

$x+y=(2,0)+I(1,1), \|x+y\|=\|(2,0)\|+I[\|(3,1)\|-\|(2,0)\|]=2+I[\sqrt{10}-2]$,

$\|x\|=\sqrt{2}+I[\sqrt{5}-\sqrt{2}], \|y\|=\sqrt{2}+I[1-\sqrt{2}]$, it is easy to check that

$\|x+y\| \leq \|x\|+\|y\|$.

(c) $\|m.x\|=\|(2,2)+I[(3,3)+(2,0)+(3,0)\|=\|(2,2)+I(8,3)\|=\sqrt{8}+I[\|(10,5)\|-\sqrt{8}]=\sqrt{8}+I[5\sqrt{5}-\sqrt{8}]$,

$|m|=|2|+I[|3+2|-|2|]=2+3I, \|x\|=\sqrt{2}+I[\sqrt{5}-\sqrt{2}]$, it is easy to see that

$\|m.x\|=|m|.\|x\|$.

It is clear that $R^2(I)$ is a neutrosophic normed space.

Definition 3.15:

Let W be a subspace of V(I), we define the canonical orthogonal complement to be the set

$W^{\perp}=\{x \in V(I); f(x,y)=0 \; for \; all \; y \in W\}$.

Definition 3.16:

Let S be any basis of V(I), we say that S is a canonical orthogonal basis if and only if

$f(x,y)=0 \; for \; all \; x,y \in S$.

Definition 3.17:

Let S be any canonical orthogonal basis of V(I), we say that S is standard if and only if

$\|x\| = 1 \, for \, all \, x \in S$.

Theorem 3.18:

Let W be a subspace of V(I), and $W^{\perp} = \{x \in V(I); f(x,y) = 0 \, for \, all \, y \in W\}$be the canonical orthogonal complement, then $W^{\perp}$ is a strong neutrosophic subspace of V(I).

Proof:

Let x, y be any two elements in $W^{\perp}$, z be any element in W, $m = a + bI$ be any element in R(I), we have

$f(x-y,z) = f(x,z) - f(y,z) = 0 - 0 = 0, thus \, x - y \in W^{\perp}$. On the other hand

$f(m.x,z) = m.f(x,z) = m.0 = 0, thus \, m.x \in W^{\perp}$, hence $W^{\perp}$ is a strong neutrosophic subspace of V(I).

Theorem 3.19:

$W^{\perp^{\perp}} = W$.

Proof:

The proof is similar to the classical case.

Example 3.20:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I) = R^2(I) = \{(a,b) + (c,d)I; a,b,c,d \in R\}$ is defined as follows:

$f[((a,b) + (c,d)I, (m,n) + (t,s)I)] = g[(a,b),(m,n)] + I[g((a+c, b+d),(m+t, n+s)) -$

$g((a,b),(m,n))]$= $(a.m + b.n) + I[(a+c).(m+t) + (b+d).(n+s) - a.m - b.n]$, where

$a,b,c,d,m,n,s,t \in R$.

(b) $W = \{v = (x,0) + (0,y)I; x,y \in R\}$ is a strong neutrosophic subspace of V(I).

$W^{\perp} = \{w = (t,z) + (k,s)I; t,z,k,s \in R\}; f(v,w) = 0$, this implies

$xt = 0, x(t+k) + y(z+s) - xt = 0, thus \, t = 0$, hence $xk + y(z+s) = 0$ for all $x, y \in R$, thus

$k = z + s = 0$, so that $s = -z$ and $W^{\perp} = \{w = (0,z) + (0,-z)I; z \in R\}$.

4. Refined neutrosophic inner product spaces

First of all, we shall define an order relation ($\leq$) on the refined neutrosophic field of real numbers R(I_1, I_2).

Definition 4.1:

Let $R(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in R\}$ be the real refined neutrosophic field, we say that

$(a, bI_1, cI_2) \leq (x, yI_1, zI_2)$ if and only if $a \leq x$ *and* $a + c \leq x + z, a + b + c \leq x + y + z.$

Theorem 4.2:

The relation defined in Definition 4.1 is an order relation.

Proof:

Let $s = (a, bI_1, cI_2), k = (x, yI_1, zI_2), l = (m, nI_1, tI_2) \in R(I_1, I_2)$, we have

$s \leq s$ that is because $a \leq a$ *and* $a + b \leq a + b, a + b + c \leq a + b + c.$

Now, suppose that $s \leq k$ *and* $k \leq s$, then $a \leq x, a + c \leq x + z, x \leq a, x + z \leq a + c, a + b + c \leq x + y + z, x + y + z \leq a + b + c$, hence

$a = x, a + c = x + z,\ a + b + c = x + y + z$, which means that $c = z, b = y$ and $s = k.$

Assume that $s \leq k$ *and* $k \leq l$, hence $a \leq x, a + c \leq x + z, a + b + c \leq x + y + z, x \leq m, x + z \leq m + t, x + y + z \leq m + n + t$, this implies that

$a \leq m, a + c \leq m + t, a + b + c \leq m + n + t$, hence $s \leq l$. Thus $\leq$ is an order relation on $R(I_1, I_2)$.

Remark 4.3:

According to Theorem 4.2, we are able to define positive refined neutrosophic real numbers as follows:

$(a, bI_1, cI_2) \geq 0 = (0, 0I_1, 0I_2)$ implies that $a \geq 0, a + b + c \geq 0, a + c \geq 0.$

Absolute value on R(I) can be defined as follows:

$|(a, bI_1, cI_2)| = (|a|, (|b| - |c|)I_1, (|c| - |a|)I_2)$, we can see that $|(a, bI_1, cI_2)| \geq 0.$

Example 4.4:

$x = (5, -2I_1, -I_2)$ is a refined neutrosophic positive real number, since $5 \geq 0$ *and* $(5 - 1) = 4 \geq 0, (5 - 2 - 1) = 3 \geq 0.$

$(5, -2I_1, -I_2) \geq (2,0,0)$, that is because $5 \geq 2$ *and* $(5 - 1) = 4 \geq (2 + 0) = 2, (5 - 2 - 1) = 2 \geq 2 + 0 + 0 = 2.$

Definition 4.5:

Let V be a vector space over R, $V(I_1,I_2)$ be its corresponding strong refined neutrosophic vector space over $R(I_1,I_2)$. Let f:$V(I_1,I_2)\times V(I_1,I_2)\to R(I_1,I_2)$ be a map, we call it a refined neutrosophic real inner product if it has the following properties:

(a) $f(x,y)=f(y,x)$.

$(b) f(x,x)\ge 0, if\ f(x,x)=0, then\ x=0$.

(c) $f(ax+by,z)=af(x,z)+bf(y,z)$ for all $x,y,z\in V(I_1,I_2), a,b\in R(I_1,I_2)$.

$V(I_1,I_2)$ is called a refined neutrosophic real inner product space.

Definition 4.6:

Let $V(I_1,I_2)$ be any refined neutrosophic real inner product space, consider any two elements $x,y\in V(I_1,I_2)$. We say that $x\perp y$ if and only if $f(x,y)=0$.

Now, we suggest a kind of real refined neutrosophic inner products which can be derived from any classical inner product on the space V.

Theorem 4.7:

Let V be any inner product space over R, consider $g: V\times V\to R$ as its inner product. Then the corresponding refined neutrosophic strong vector space $V(I_1,I_2)$ has a refined neutrosophic real inner product.

Proof:

We define $f: V(I_1,I_2)\times V(I_1,I_2)\to R(I_1,I_2); f\big((a,bI_1,cI_2),(x,yI_1,xI_2)\big)=(g(a,x),I_1[g(a+b+c,x+y+z)-g(a+c,x+z)],I_2[g(a+c,x+z)-g(a,x)])\ for\ all(a,bI_1,cI_2),(x,yI_1,xI_2)\in\ V(I_1,I_2)$. We prove that f is a refined neutrosophic inner product.

Let $s=(a,bI_1,cI_2), k=(x,yI_1,zI_2), l=(m,nI_1,tI_2)\in V(I_1,I_2), hence\ a,b,c,x,y,x,t,m,n\in V$, let $i=(e,hI_1,rI_2), j=(p,qI_1,wI_2)\in R(I_1,I_2)$, we have

$f(s,k)=f\big((a,bI_1,cI_2),(x,yI_1,xI_2)\big)=(g(a,x),I_1[g(a+b+c,x+y+z)-g(a+c,x+z)],I_2[g(a+c,x+z)-g(a,x)])$=

$(g(x,a),I_1[g(x+y+z,a+b+c)-g(x+z,a+c)],I_2[g(x+z,a+c)-g(x,a)])$=$f(y,x)$.

$f(s,s)=(g(a,a),I_1[g(a+b+c,a+b+c)-g(a+c,a+c)],I_2[g(a+c,a+c)-g(a,a)])\ge 0$, that is because $g(a,a)\ge 0 and\ [g(a+c,a+c)-g(a,a)+g(a,a)]=g(a+c,a+c)\ge 0, [g(a+b+c,a+b+c)-g(a+c,a+c)+g(a+c,a+c)-g(a,a)+g(a,a)]=g(a+b+c,a+b+c)\ge 0.$

$f(s,s)=0$ implies $(g(a,a), I_1[g(a+b+c,a+b+c)-g(a+c,a+c)], I_2[g(a+c,a+c)-g(a,a)])=0$, hence $g(a,a)=0$ and $g(a+c,a+c)=0, g(a+b+c,a+b+c)=0$, thus $a=0$ and $a+c=0, a+b+c$, so that $a=b=c=0$ and $s=0$.

Now, we shall compute $is+jk$.

$is=(ea, I_1[eb+ha+hc+hb+rb], I_2[ec+rc+ra])=(ea, I_1[(e+h+r)(a+b+c)-(e+r)(a+c)], I_2[(e+r)(a+c)-ea]), jk=(px, I_1[py+qx+qz+qy+wy], I_2[pz+wz+wx])=(px, I_1[(p+q+r)(x+y+z)-(p+w)(x+z)], I_2[(p+w)(x+z)-px])$, hence

$is+jk=(ea+px, I_1[(e+h+r)(a+b+c)-(e+r)(a+c)+(p+q+r)(x+y+z)-(p+w)(x+z)], I_2[(e+r)(a+c)-ea+(p+w)(x+z)-px])$,

$f(is+jk,l)=(g(ea+px,m), I_1\left[g\big((e+h+r)(a+b+c)+(p+q+w)(x+y+z), m+n+t\big)-g\left(\big((e+r)(a+c)+(p+w)(x+z)\big), m+t\right)\right], I_2[g\big((e+r)(a+c)+(p+w)(x+z), m+t\big)-(g(ea+px,m)])=(eg(a,m)+pg(x,m), I_1[(e+h+r)g(a+b+c,m+n+t)+(p+q+w)g(x+y+z,m+n+t)-(e+r)g(a+c,m+t)-(p+w)g(x+z,m+t)], I_2[(e+r)g(a+c,m+t)+(p+w)g(x+z,m+t)-eg(a,m)-pg(x,m)])$.

On the other hand we have:

$i.f(s,l)=(e,hI_1,rI_2).(g(a,m), I_1[g(a+b+c,m+n+t)-g(a+c,m+t)], I_2[g(a+c,m+t)-g(a,m)])=(eg(a,m), I_1[(e+h+r)g(a+b+c,m+n+t)-(e+r)g(a+c,m+t)], I_2[(e+r)g(a+c,m+n)-eg(a,m)])$,

$jf(k,l)=(p,qI_1,wI_2).(g(x,m), I_1[g(x+y+z,m+n+t)-g(x+z,m+t)], I_2[g(x+z,m+t)-g(x,m)])=(pg(x,m), I_1[(p+q+w)g(x+y+z,m+n+t)-(p+w)g(x+z,m+t)], I_2[(p+w)g(x+z,m+n)-pg(x,m)])$.

Now, we can find that

$f(is+jk,l)=if(s.l)+jf(k,l), thus\ f\ is\ a\ refined\ neutrosophic\ inner\ product.$

Definition 4.8:

(a) The refined neutrosophic real inner product introduced in Theorem 3.7 is called the canonical refined neutrosophic real inner product generated by g.

(b) Let V be any vector space over R, with a classical real inner product g, $V(I_1, I_2)$ be its corresponding refined neutrosophic strong vector space, let f be the canonical refined inner product generated by g, the canonical norm of $s = (a, bI_1, cI_2)$ is defined as follows:

$\|s = (a, bI_1, cI_2)\| = \sqrt{f(s,s)}$.

Theorem 4.9:

Let V be any vector space over R, with a classical real inner product g, $V(I_1, I_2)$ be its corresponding refined neutrosophic strong vector space, let f be the canonical refined inner product generated by g, we have

(a) $\|s\| = (\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])$ for all $s = (a, bI_1, cI_2) \in V(I_1, I_2)$.

$(b) For\ s = (a, bI_1, cI_2), k = (x, yI_1, zI_2)$, $s \perp k$ if and only if $a \perp x$, and $a + c \perp x + z, a + b + c \perp x + y + z$.

(c) $\|(a, bI_1, cI_2)\| = 1$ if and only if $\|a\| = \|a+b+c\| = \|a+c\| = 1$.

Proof:

(a) We compute $\|(a, bI_1, cI_2)\|^2 = f\big((a, bI_1, cI_2), (a, bI_1, cI_2)\big) = (g(a,a), I_1[g(a+b+c, a+b+c) - g(a+c, a+c)], I_2[g(a+c, a+c) - g(a,a)])$=

$(\|a\|^2, I_1[\|a+b+c\|^2 - \|a+c\|^2], I_2[\|a+c\|^2 - \|a\|^2])$.

Now, we prove that $\sqrt{f(s,s)} = (\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])$. By easy computing, we find

$[(\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])]^2 = (\|a\|^2, I_1[\|a+b+c\|^2 - \|a+c\|^2], I_2[\|a+c\|^2 - \|a\|^2]) = f(s,s)$, thus

$\|s\| = (\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])$.

(b) $s \perp k$ if and only if $f(s,k) = 0$, hence $(g(a,x), I_1[g(a+b+c, x+y+z) - g(a+c, x+z)], I_2[g(a+c, x+z) - g(a,x)]) = 0$, this implies that

$g(a,x) = 0, g(a+c, x+z) = 0, g(a+b+c, x+y+z) = 0$, thus $a \perp x$, and $a + c \perp x + z, a + b + c \perp x + y + z$.

(c) $\|(a, bI_1, cI_2)\| = 1$ if and only if $(\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|]) = (1,0,0)$, hence $\|a\| = 1, \|a+c\| - \|a\| = 0$, thus $\|a+c\| = \|a\| = 1, \|a+b+c\| - \|a+c\|, thus\ \|a+b+c\| = \|a+c\| = 1$.

Example 4.10:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I_1, I_2) = R^2(I_1, I_2) = \{((a,b),(c,d)I_1,(m,n)I_2); a,b,c,d,m,n \in R\}$ is defined as follows:

$f[(((a,b),(c,d)I_1,(m,n)I_2),((x,y),(z,t)I_1,(k,s)I_2))] = (g[(a,b),(x,y)], I_1\big[g\big((a+c+m, b+d+n),(x+z+k, y+t+s)\big) - g\big((a+m,b+n),(x+k,y+s)\big)\big], I_2\left[g\big((a+m,b+n),(x+k,y+s)\big) - g[(a,b),(x,y)]\right])= (ax+by, I_1[(a+c+m)(x+z+k)+(b+d+n)(y+t+s)-(a+m)(x+k)-(b+n)(y+s)], I_2[(a+m)(x+k)+(b+n)(y+s)-ax-by])$, where

$a,b,c,d,m,n,x,y,z,t,k,s \in R$.

(b) Let $x = ((1,1),(2,-1)I_1,(0,0)I_2), y = ((-1,1),(1,-1)I_1,(0,0)I_2)$, we have

$f(x,y) = \big((1)(-1)+(1)(1), I_1[(3)(0)+(0)(6)-(1)(-1)-(1)(1)], I_2[(1)(-1)+(1)(1)-(1)(-1)-(1)(1)]\big) = (0,0,0), hence\ x \perp y.$

$\|x\| = (\|(1,1)\|, I_1[\|(1,1)+(2,-1)+(0,0)\| - \|(1,1)+(0,0)\|], I_2[\|(1,1)+(0,0)\| - \|(1,1)\|]) = (\sqrt{2}, I_1[3-\sqrt{2}], I_2[0])$.

Theorem 4.11: (Refined neutrosophic Cauchy-Schwartz inequality)

Let $x = (a, bI_1, cI_2), y = (m, nI_1, tI_2)$ any two elements in a refined strong neutrosophic canonical inner product vector space. Then

$|f(x,y)| \le \|x\|\|y\|$.

Proof:

We have $|f(x,y)| = (|g(a,m)|, I_1[|g(a+b+c, m+n+t)| - |g(a+c,m+t)|], I_2[|g(a+c,m+t) - |g(a,m)|])$.

$\|x\|\|y\| = (\|a\|\|m\|, I_1[\|a+b+c\|\|m+n+t\| - \|a+c\|\|m+t\|], I_2[\|a+c\|\|m+t\| - \|a\|\|m\|])$.

By classical Cauchy – Schwartz inequality, we find $|g(a,m)| \le \|a\|\|m\|$, and

$|g(a+b+c,m+n+t)| \le \|a+b+c\|\|m+n+t\|$, $|g(a+c,m+t)| \le \|a+c\|\|m+t\|$ thus

$(|g(a,m)|, I_1[|g(a+b+c,m+n+t)| - |g(a+c,m+t)|], I_2[|g(a+c,m+t)| - |g(a,m)|]) \le (\|a\|\|m\|, I_1[\|a+b+c\|\|m+n+t\| - \|a+c\|\|m+t\|], I_2[\|a+c\|\|m+t\| - \|a\|\|m\|])$, so that

$|f(x,y)| \le \|x\|\|y\|$.

Example 4.12:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I_1, I_2) = R^2(I_1, I_2) = \{((a,b),(c,d)I_1,(m,n)I_2); a,b,c,d,m,n \in R\}$ is defined as follows:

$f[(((a,b),(c,d)I_1,(m,n)I_2),((x,y),(z,t)I_1,(k,s)I_2))] = (g[(a,b),(x,y)], I_1\big[g\big((a+c+m,b+d+n),(x+z+k,y+t+s)\big) - g\big((a+m,b+n),(x+k,y+s)\big)\big], I_2\Big[g\big((a+m,b+n),(x+k,y+s)\big) - g[(a,b),(x,y)]\Big]) = (ax+by, I_1[(a+c+m)(x+z+k)+(b+d+n)(y+t+s)-(a+m)(x+k)-(b+n)(y+s)], I_2[(a+m)(x+k)+(b+n)(y+s)-ax-by])$, where

$a,b,c,d,m,n,x,y,z,t,k,s \in R$.

(b) Let $x = ((1,1),(2,-1)I_1,(0,2)I_2), y = ((-1,1),(3,1)I_1,(-2,4)I_2)$, we have

$f(x,y) = (0, I_1[0+8+3-15], I_2[-3+15-0]) = (0, I_1[-4], I_2[12]), |f(x,y)| = (0,-8I_1,12I_2), \|x\| = \big(\sqrt{2}, I_1\big[\sqrt{14}-\sqrt{10}\big], I_2\big[\sqrt{10}-\sqrt{2}\big]\big), \|y\| = (\sqrt{2}, I_1\big[6-\sqrt{34}\big], I_2[\sqrt{34}-\sqrt{2}])$.

$\|x\|\|y\| = \big(2, I_1\big[6\sqrt{14}-\sqrt{340}\big)\big], I_2[\sqrt{340}-2]),$

$|f(x,y)| = (0,-8I_1,12I_2) \le \big(2, I_1\big[6\sqrt{14}-\sqrt{340}\big)\big], I_2[\sqrt{340}-2])$. That is because $0 \le 2, 0+12 \le 2+\big(\sqrt{340}-2\big), 0-8+12 = 4 \le \big(\sqrt{2}+6-\sqrt{34}+\sqrt{34}-\sqrt{2}\big) = 6$.

Theorem 4.13:

Let $V(I_1, I_2)$ be a refined neutrosophic strong real inner product vector space, let $x = (a, bI_1, cI_2)$ be any element in $V(I_1, I_2)$. We have

(a) $\|x\| \ge 0, \|m.x\| = |m|.\|x\|$.

$(b)\|x+y\| \le \|x\| + \|y\|\ for\ all\ x, y \in V(I) and\ m \in R(I)$.

(c) $\|x\| = 0\ if\ and\ only\ if\ x = 0$.

Proof:

(a) Since$\|x\| = (\|a\|, I_1[\|a+b+c\| - \|a+c\|], I_2[\|a+c\| - \|a\|])$, and $\|a\| \ge 0, (\|a+c\| - \|a\|) + \|a\| = \|a+c\| \ge 0, and\ (\|a+b+c\| - \|a+c\|) + (\|a+c\| - \|a\|) + (\|a\|) = \|a+b+c\| \ge 0$, we get that $\|x\| \ge 0$.

Let $m = (n, pI_1, qI_2) \in R(I_1, I_2); n,p,q \in R$, we have $m.x = (n.a, I_1[(n+p+q)(a+b+c) - (n+q)(a+c)], I_2[(n+q)(a+c) - n.a])$, hence

$\|m.x\| = (\|n.a\|, I_1[\|(n+p+q)(a+b+c)\| - \|(n+q)(a+c)\|], I_2[\|(n+q)(a+c)\| - \|n.a\|])$= $[(|n|, (|n+p+q| - |n+q|)I_1, (|n+q| - |n|)I_2)][(\|a\|, (\|a+b+c\| - \|a+c\|)I_1, (\|a+c\| - \|a\|)I_2)]$= $|m|.\|x\|$.

(b) Let $x=(a,bI_1,cI_2)),y=(m,nI_1,tI_2)\in \mathrm{V}(I_1,I_2);a,b,c,m,n,t\in V$, $\|x+y\|=\|(a+m,I_1[b+n],I_2[c+t])\|=$ $(\|a+m\|,I_1[\|a+m+b+n+c+t\|-\|a+m+c+t\|],I_2[\|a+m+c+t\|-\|a+m\|])$, by regarding classical properties of classical norms, we get

$\|a+m\|\le\|a\|+\|m\|,\|a+m+c+t\|\le\|a+c\|+\|m+t\|,\|a+m+b+n+c+t\|\le\|a+b+c\|+\|m+n+t\|$, thus

$(\|a+m\|,I_1[\|a+m+b+n+c+t\|-\|a+m+c+t\|],I_2[\|a+m+c+t\|-\|a+m\|])\le(\|a\|+\|m\|,I_1[\|a+b+c\|+\|m+n+t\|-\|a+c\|-\|m+t\|],I_2[\|a+c\|+\|m+t\|-\|a\|-\|m\|])=\|x\|+\|y\|$.

(c) The proof is trivial and similar to the classical case.

According to the previous theorem, we can define any neutrosophic norm on a strong neutrosophic vector space $\mathrm{V}(I_1,I_2)$ as a function $\|\ \ \|:\mathrm{V}(I_1,I_2)\to \mathrm{R}(I_1,I_2)$, where conditions (a), (b), and (c) are true. $\mathrm{V}(I_1,I_2)$ is called a strong neutrosophic normed space in this case.

Example 4.14:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I_1,I_2)=R^2(I_1,I_2)=\{((a,b),(c,d)I_1,(m,n)I_2);a,b,c,d,m,n\in R\}$ is defined as follows:

$f[(((a,b),(c,d)I_1,(m,n)I_2),((x,y),(z,t)I_1,(k,s)I_2))]=(g[(a,b),(x,y)],I_1\left[g\big((a+c+m,b+d+n),(x+z+k,y+t+s)\big)-g\big((a+m,b+n),(x+k,y+s)\big)\right],I_2\left[g\big((a+m,b+n),(x+k,y+s)\big)-g[(a,b),(x,y)]\right])=$ $(ax+by,I_1[(a+c+m)(x+z+k)+(b+d+n)(y+t+s)-(a+m)(x+k)-(b+n)(y+s)],I_2[(a+m)(x+k)+(b+n)(y+s)-ax-by])$, where

$a,b,c,d,m,n,x,y,z,t,k,s\in R$.

(b) Let $x=\big((1,1),(1,0)I_1,(-1,2)I_2\big),y=\big((1,-1),(0,1)I_1,(-2,1)I_2\big),m=(2,3I_1,I_2)$, we have

$x+y=\big((2,0),I_1(1,1),I_2(-3,3)\big),\|x+y\|=(\|(2,0)\|,I_1[\|(0,4)\|-\|(-1,3)\|],I_2[\|(0,4)\|-\|(2,0)\|])=(2,I_1[4-\sqrt{10}],I_2[\sqrt{10}-2])$,

$\|x\|=\big(\sqrt{2},I_1[\sqrt{10}-3],I_2[3-\sqrt{2}]\big),\|y\|=(\sqrt{2},I_1[\sqrt{2}-1],I_2[1-\sqrt{2}])$, it is easy to check that

$\|x+y\|\le\|x\|+\|y\|$.

(c) $\|m.x\| = \|((2,2), I_1[((3,3) + (2,0) + (3,0) + (1,0) + (-3,6), I_2[(1,1) + (-2,4) + (-1,2)])\| =$

$\|((2,2), I_1(6,9), I_2(-2,7))\| = \left(\sqrt{8}, I_1[\|(6,18)\| - \|(0,9)\|], I_2[\|(0,9)\| - \|(2,2)\|]\right) = \left(\sqrt{8}, I_1\left[\sqrt{36+324} - \sqrt{81}\right], I_2\left[9 - \sqrt{8}\right]\right) = \left(\sqrt{8}, \left(6\sqrt{10} - 9\right)I_1, \left(9 - \sqrt{8}\right)I_2\right),$

$|m| = (|2|, I_1[|3+2+1| - |2+1|], I_2[|1+2| - |2|]) = (2, 3I_1, I_2), \|x\| = (\sqrt{2}, I_1\left[\sqrt{10} - 3\right], I_2\left[3 - \sqrt{2}\right]),$

it is easy to see that

$\|m.x\| = |m|.\|x\|.$

It is clear that $R^2(I_1, I_2)$ is a neutrosophic normed space.

Definition 4.15:

Let W be a subspace of $V(I_1, I_2)$, we define the canonical orthogonal complement to be the set

$W^{\perp} = \{x \in V(I_1, I_2); f(x, y) = 0 \ for\ all\ y \in W\}.$

Definition 4.16:

Let S be any basis of $V(I_1, I_2)$, we say that S is a canonical orthogonal basis if and only if

$f(x, y) = 0 \ for\ all\ x, y \in S.$

Definition 4.17:

Let S be any canonical orthogonal basis of $V(I_1, I_2)$, we say that S is standard if and only if

$\|x\| = 1 \ for\ all\ x \in S.$

Theorem 4.18:

Let W be a subspace of $V(I_1, I_2)$, and $W^{\perp} = \{x \in V(I_1, I_2); f(x, y) = (0,0,0) \ for\ all\ y \in W\}$ be the canonical orthogonal complement, then $W^{\perp}$ is a strong refined neutrosophic subspace of V(I).

Proof:

Let x, y be any two elements in $W^{\perp}$, z be any element in W, m=(a, bI_1, cI_2) be any element in $R(I_1, I_2)$, we have

$f(x - y, z) = f(x, z) - f(y, z) = (0,0,0) - (0,0,0) = (0,0,0), thus\ x - y \in W^{\perp}$. On the other hand $f(m.x, z) = m.f(x, z) = m.(0,0,0) = (0,0,0), thus\ m.x \in W^{\perp}$, hence $W^{\perp}$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.

Theorem 4.19:

$W^{\perp^{\perp}} = W.$

Proof:

The proof is similar to the classical case.

Example 4.20:

(a) Consider the Euclidean inner product on R^2. The corresponding canonical neutrosophic inner product on $V(I_1, I_2) = R^2(I_1, I_2) = \{((a,b),(c,d)I_1,(m,n)I_2); a,b,c,d,m,n \in R\}$ is defined as follows:

$f[(((a,b),(c,d)I_1,(m,n)I_2),((x,y),(z,t)I_1,(k,s)I_2))] = (g[(a,b),(x,y)], I_1\big[g\big((a+c+m, b+d+n),(x+z+k, y+t+s)\big) - g\big((a+m, b+n),(x+k, y+s)\big)\big], I_2\left[g\big((a+m,b+n),(x+k,y+s)\big) - g[(a,b),(x,y)]\right]) = (ax+by, I_1[(a+c+m)(x+z+k)+(b+d+n)(y+t+s)-(a+m)(x+k)-(b+n)(y+s)], I_2[(a+m)(x+k)+(b+n)(y+s)-ax-by])$, where

$a,b,c,d,m,n,x,y,z,t,k,s \in R$.

(b) $W = \{v = ((x,0),(0,y)I_1,(0,0)I_2); x,y \in R\}$ is a strong neutrosophic subspace of $V(I_1, I_2)$.

$W^{\perp} = \{w = \big((t,z),(k,s)I_1,(p,q)I_2\big); t,z,k,s,p,q \in R\}; f(v,w) = (0,0,0)$, this implies

$xt = 0, x(t+k+p)+y(z+s+q)-x(t+p)+y(z+q) = 0, x(t+p)+y(z+q)-xt = 0, thus\ t = 0$, hence $x(t+k+p)+y(z+s+q) = 0, x(t+p)+y(z+q) = 0$ for all $x,y \in R$, thus

$z+q = t+p = 0, and\ t+k+p = z+s+q = 0$, so that $q = -z, t = -p = 0, k = s = 0$ and $W^{\perp} = \{w = \big((0,-q),(0,0)I_1,(0,q)I_2\big); p,q \in R\}$.

4. Conclusions

In this article, we have defined the concept of real inner product over a strong neutrosophic vector space V(I) and strong refined neutrosophic space $V(I_1, I_2)$, as well as neutrosophic and refined neutrosophic normed space. Many interesting properties were studied and proved, especially neutrosophic and refined neutrosophic Cauchy- Schwartz inequality, where we have proved that it is still correct in neutrosophic spaces.

This work opens a wide door to study neutrosophic functional analysis, neutrosophic orthogonal standard basis, and neutrosophic matrices and isometrics in the future, especially inequalities, since we have determined a strong partial ordering relation between neutrosophic numbers.

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