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Neutrosophic n-Valued Refined Sets and Topologies

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Abstract. In n-Valued refined logic truth value T can be split into many types of truths: $T_1, T_2, ..., T_p$ and I into many types of indeterminacies: $I_1, I_2, ..., I_r$ and F into many types of falsities: $F_1, F_2, ..., F_s$, where p, r and s are integers greater than 1, and $p + r + s = n$. Importance of n-valued refined logic and sets appeared in different applications specially in medical diagnosis. In this paper we post a condition on neutrosophic n-valued refined sets to make them functional to be applied in different mathematical branches. We define and study n-valued refined topological spaces. We defined neutrosophic n-valued refined α-open, β-open, pre-open and semi-open sets and studied their properties. We constructed different counter examples to clarify the relations between these different types of neutrosophic n-valued refined generalized open sets.

Keywords: n-valued refined topology; refined logic; refined sets; n-valued refined α-open; semi-open sets; n-valued refined generalized open sets.)

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1. Introduction

Neutrosophic sets are, first, introduced in 2005 by [26,27] as a generalization of intuitionistic fuzzy sets [13], where any element $x \in X$ we have three degrees; the degree of membership(T), indeterminacy(I), and non-membership(F). Neurosophic vague sets are introduced in 2015 by [30]. Neutrosophic vague topological spaces introduced in [21] we are many different notations are introduced and studied such as neurosophic vague continuity and compactness.

Neutrosophic topologies are defined and studied by Smarandache [27], Lupianez [19,20] and Salama [?]. Open and closed neutrosophic sets, interior, exterior, closure and boundary of neutrosophic sets can be found in [29].

Neutrosophic sets applied to generalize many notaions about soft topology and applications [18], [23], [16], generalized open and closed sets [31] , fixed point theorems [18] , graph theory

[17]and rough topology and applications [22]. Neutrosophy has many applications especially in decision making, for more details about new trends of neutrosophic applications one can consult [1]- [7].
Generalized topology and continuity introduced in 2002 in [**?**] which is a generalization of topological spaces and has different properties than general topology, see for example [8], [11] and [12]. Neutrosophic generalized sets and topologies are introduced and studies by Murad M. Arar in 2020 see [9] and [10]. In n-valued refined logic truth value T can be split into many types of truths: $T_1, T_2, ..., T_p$ and I into many types of indeterminacies: $I_1, I_2, ..., I_r$ and F into many types of falsities: $F_1, F_2, ..., F_s$, where p, r and s are integers greater than 1, and $p + r + s = n$ see [28]. Importance of n-valued refined logic and sets appeared in different applications specially in medial diagnosis see [25] and [14], where a strong assumption is assumed to make them functional; that is $p = r = s$.

Definition 1.1. [26]: We say that the set A is *neutrosophic* on X if
$A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle; x \in X\}$; $\mu, \sigma, \nu : X \to]^{-}0, 1^{+}[$ and $^{-}0 \leq \mu(x)+\sigma(x)+\nu(x) \leq 3^{+}$.

The class of all neutrosophic sets on the universe X will be denoted by $\mathcal{N}(X)$. The basic neutrosophic operations (inclusion, union, and intersection) where first introduced by [24].

Definition 1.2 (*Neutrosophic sets operations*)**.** Let $A, A_\alpha, B \in \mathcal{N}(X)$ such that $\alpha \in \Delta$. Then we define the neutrsophic:

(1) (*Inclusion*): $A \sqsubseteq B$ If $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \geq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
(2) (*Equality*): $A = B \Leftrightarrow A \sqsubseteq B$ and $B \sqsubseteq A$.
(3) (*Intersection*) $\underset{\alpha\in\Delta}{\sqcap} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\wedge} \mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\vee} \sigma_{A_\alpha}(x), \underset{\alpha\in\Delta}{\vee} \nu_{A_\alpha}(x)\rangle; x \in X\}$.
(4) (*Union*) $\underset{\alpha\in\Delta}{\sqcup} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\vee} \mu_{A_\alpha}(x), \underset{\alpha\in\Delta}{\wedge} \sigma_{A_\alpha}(x), \underset{\alpha\in\Delta}{\wedge} \nu_{A_\alpha}(x)\rangle; x \in X\}$.
(5) (*Complement*) $A^c = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle; x \in X\}$
(6) (*Universal set*) $1_X = \{\langle x, 1, 0, 0\rangle; x \in X\}$; called the *neutrosophic universal set.*
(7) (*Empty set*) $0_X = \{\langle x, 0, 1, 1\rangle; x \in X\}$; called the *neutrosophic empty set.*

Proposition 1.3. *[24] For $A, A_\alpha \in \mathcal{N}(X)$ for every $\alpha \in \Delta$ we have:*

(1) $A \sqcap (\underset{\alpha\in\Delta}{\sqcup} A_\alpha) = \underset{\alpha\in\Delta}{\sqcup} (A \sqcap A_\alpha)$.
(2) $A \sqcup (\underset{\alpha\in\Delta}{\sqcap} A_\alpha) = \underset{\alpha\in\Delta}{\sqcap} (A \sqcup A_\alpha)$.

Definition 1.4. [24] [*Neutrosophic Topology*] $\tau \subseteq \mathcal{N}(X)$ is called a *neutrosophic topology* for X if

(1) $0_X, 1_X \in \tau$.
(2) If $A_\alpha \in \tau$ for every $\alpha \in \Delta$, then $\underset{\alpha\in\Delta}{\sqcup} A_\alpha \in \tau$,
(3) For every $A, B \in \tau$, we have $A \sqcap B \in \tau$.

The ordered pair (X, τ) will be said a *neutrosophic space* over X. The elements of τ will be called *neutrosophic open sets*. For any $A \in \mathcal{N}(X)$, If $A^c \in \tau$, then we say A is *neutrosophic closed*.

2. Neutrosophic n-valued refined sets and topology

In neutrosophic n-valued refined logic (see [28]) the membership degree refined (split) into r values $\mu_1, \mu_2, ..., \mu_r$, the indetermancy refined into s values $\sigma_1, \sigma_2, ..., \sigma_s$ and the nonmebership refined into t values $\nu_1, \nu_2, ..., \nu_t$ such that $n = r + s + t$ and

$$^{-}0 \leq \sum_{i=1}^{r} \mu_i + \sum_{i=1}^{s} \sigma_i + \sum_{i=1}^{t} \nu_i \leq n^{+}$$

Some authors assumes that $r = s = t$ see for example [14]. Actually, there is no guarntee that the membership, intedermancy and nonmembership degrees refined or split into the same number of values, and we will not get a functional system of Neutrosophic n-valued refined sets if no more restrictions are assumed on r, s and t. This accurse when we define the basic set operations on the neutrosophic n-valued refined sets, especially when we try to define the neutrosophic n-valued refined complement of a given neutrosophic n-valued refined set; where r plays the role of t and vice versa. We will be back to this discussion after stating some definitions and theorems.

Definition 2.1. [26]: A is called a *neutrosophic n-valued refined set* on a universe X if $A = \{\langle x, \mu_A^1(x), \mu_A^2(x), ..., \mu_A^r(x); \sigma_A^1(x), \sigma_A^2(x), ..., \sigma_A^s(x); \nu_A^1(x), \nu_A^2(x), ..., \nu_A^t(x)\rangle; x \in X\}$; $\mu_A^i, \sigma_A^j, \nu_A^k : X \to]^{-}0, 1^{+}[$ for every $i = 1, ..., r$, $j = 1, ..., s$, $k = 1, ..., t$ such that $r + s + t = n$ and

$$^{-}0 \leq \sum_{i=1}^{r} \mu_A^i(x) + \sum_{j=1}^{s} \sigma_A^j + \sum_{k=1}^{t} \nu_A^k \leq n^{+}.$$

The class of all neutrosophic n-valued refined sets on the universe X will be denoted by $\mathcal{R}_n(X)$.

The following is the definition of the basic operations (inclusion, union, intersection and complement) on neutrosophic n-valued refined sets.

Definition 2.2. [*Neutrosophic n-valued refined sets operations*] Let $A, A_\alpha, B \in \mathcal{R}_n(X)$ such that $\alpha \in \Delta$. Then we define the neutrsophic n-valued refined:

(1) (*Inclusion*): $A \sqsubseteq_R B$ If $\mu_A^i(x) \leq \mu_B^i(x)$, $\sigma_A^j(x) \geq \sigma_B^j(x)$ and $\nu_A^k(x) \geq \nu_B^k(x)$ for every $i = 1, ..., r$, $j = 1, ..., s$, $k = 1, ..., t$.

(2) (*Equality*): $A = B \Leftrightarrow A \sqsubseteq_R B$ and $B \sqsubseteq_R A$.

(3) (*Intersection*) $\underset{\alpha \in \Delta_R}{\sqcap} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\wedge} \mu^1_{A_\alpha}(x), ..., \underset{\alpha\in\Delta}{\wedge} \mu^r_{A_\alpha}(x); \underset{\alpha\in\Delta}{\vee} \sigma^1_A(x), ..., \underset{\alpha\in\Delta}{\vee} \sigma^s_A(x); \underset{\alpha\in\Delta}{\vee} \nu^1_A(x), ..., \underset{\alpha\in\Delta}{\vee} \nu^t_A(x)\rangle; x \in X\}$.

(4) $(Union)$ $\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha(x) = \{\langle x, \underset{\alpha\in\Delta}{\vee}\mu^1_{A_\alpha}(x), ..., \underset{\alpha\in\Delta}{\vee}\mu^r_{A_\alpha}(x); \underset{\alpha\in\Delta}{\wedge}\sigma^1_A(x), ..., \underset{\alpha\in\Delta}{\wedge}\sigma^s_A(x); \underset{\alpha\in\Delta}{\wedge}\nu^1_A(x), ..., \underset{\alpha\in\Delta}{\wedge}\nu^t_A(x)\rangle; x \in X\}$.

(5) $(Complement)$ $A^c = \{\langle x, \nu^1_A(x), ..., \nu^t_A(x); 1-\sigma^1_A(x), ..., 1-\sigma^s_A(x); \mu^1_A(x), ..., \mu^r_A(x)\rangle; x \in X\}$

(6) $(Universal\ set)$ $1_X = \{\langle x, 1, ..., 1; 0, ..., 0; 0, ..., 0\rangle; x \in X\}$; called the *neutrosophic n-valued refined universal set.*

(7) (*Empty set*) $0_X = \{\langle x, 0, ..., 0; 1, ..., 1; 1, ..., 1\rangle; x \in X\}$; called the *neutrosophic n-valued refined empty set.*

Theorem 2.3. *Let $A_\alpha, A, B \in \mathcal{R}_n(X)$ such that $\alpha \in \Delta$. Then we have*

(1) *If $A \sqsubseteq_R B \sqsubseteq_R C$, then $A \sqsubseteq_R C$.*

(2) *If $A \sqsubseteq_R B$, then $B^c \sqsubseteq_R A^c$.*

(3) $(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha) \sqcap_R A = \underset{\alpha\in\Delta}{\sqcup_R}(A_\alpha \sqcap_R A)$

(4) $(\underset{\alpha\in\Delta}{\sqcap_R} A_\alpha) \sqcup_R A = \underset{\alpha\in\Delta}{\sqcap_R}(A_\alpha \sqcup_R A)$

[Demorgan's Laws]

(5) $(A \sqcup_R B)^c = A^c \sqcap_R B^c$

(6) $(A \sqcap_R B)^c = A^c \sqcup_R B^c$

Proof. (1) and (2) are Straight forward! (3) and (4) can be proved using the following two propositions:

$-\ (\underset{\alpha\in\Delta}{\vee} a_\alpha) \wedge b = \underset{\alpha\in\Delta}{\vee}(a_\alpha \wedge b)$

$-\ (\underset{\alpha\in\Delta}{\wedge} a_\alpha) \vee b = \underset{\alpha\in\Delta}{\wedge}(a_\alpha \vee b)$

Now, we prove (3) and (4) can be proved by duality:

$(A \sqcup_R B)^c = (\{\langle x, \mu^1_A(x) \vee \mu^1_B(x), ..., \mu^r_A(x) \vee \mu^r_B(x); \sigma^1_A(x) \wedge \sigma^1_B(x), ..., \sigma^s_A(x) \wedge \sigma^s_B(x); \nu^1_A(x) \wedge \nu^1_B(x), ..., \nu^t_A(x) \wedge \nu^t_A(x)\rangle; x \in X\})^c$

$= \{\langle x, \nu^1_A(x) \wedge \nu^1_B(x), ..., \nu^t_A(x) \wedge \nu^t_A(x); 1 - (\sigma^1_A(x) \wedge \sigma^1_B(x)), ..., 1 - (\sigma^s_A(x) \wedge \sigma^s_B(x)); \mu^1_A(x) \vee \mu^1_B(x), ..., \mu^r_A(x) \vee \mu^r_B(x)\rangle; x \in X\}$

$= \{\langle x, \nu^1_A(x) \wedge \nu^1_B(x), ..., \nu^t_A(x) \wedge \nu^t_A(x); (1 - \sigma^1_A(x)) \vee (1 - \sigma^1_B(x)), ..., (1 - \sigma^s_A(x)) \vee (1 - \sigma^s_B(x)); \mu^1_A(x) \vee \mu^1_B(x), ..., \mu^r_A(x) \vee \mu^r_B(x)\rangle; x \in X\}$

$= \{\langle x, \nu^1_A(x), ..., \nu^t_A(x); 1 - \sigma^1_A(x), ..., 1 - \sigma^s_A(x); \mu^1_A(x), ..., \mu^r_A(x)\rangle; x \in X\} \sqcap_R$
$\{\langle x, \nu^1_B(x), ..., \nu^t_A(x); 1 - \sigma^1_B(x), ..., 1 - \sigma^s_B(x); \mu^1_B(x), ..., \mu^r_B(x)\rangle; x \in X\} = A^c \sqcap_R B^c$ □

So, as the above theorem shows, the system defined in Definition 2.2 is rich to a certain extent, but it still needs to be stronger to deal with some situations: for example $A \sqcap_R A^c$ is not well-defined if $r \neq t$. The concept $True$ (membership) and $False$ (nonmembership) are related, it is reasonable to discuss them in any world simultaneously, so we can assume $r = t$, and this is what F. Smarandache did in [28] when he discussed the relative (absolute) truth and falsity simultaneously. The condition $r = s = t$ mentioned in [14] is very strong and will not add any value to us, actually it implies that n is divisible by 3, since $n = r + s + t$, so it does not include some worlds, for example a world of seven and five-valued logic which discussed in [28]. On the other hand if we, only, assume $r = t$, then n can be any value since we have not assumed any condition on s and worlds of any n-valued logic will be included.

Definition 2.4. : Let A be a *neutrosophic n-valued refined set* on a universe X. If $r = s$, then we call A a homogeneous neutrosophic n-valued refined set. n will be called the dimension of A, and r, s will be called the sub-dimensions of A. The class of all *homogeneous neutrosophic n-valued refined sets* on the universe X with sub-dimensions r, s will be denoted by $\mathcal{R}_{(n,r,s)}(X)$.

The following is obvious:

Proposition 2.5. *Let $A, B \in \mathcal{R}_{(n,r,s)}(X)$. Then*

(1) $A \sqcap_R B \in \mathcal{R}_{(n,r,s)}(X)$.
(2) $A \sqcup_R B \in \mathcal{R}_{(n,r,s)}(X)$.
(3) $A^c \in \mathcal{R}_{(n,r,s)}(X)$.

Example 2.6. Let $X = \{a, b\}$, and let $A, B \in \mathcal{R}_{(5,2,1)}(X)$ such that
$A = \{\langle a, 0.2, 0.1; 0.7; 0.1, 0.4\rangle, \langle b, 0.5, 0.3; 0.2; 0.9, 0.5\rangle\}$ and
$B = \{\langle a, 0.4, 0.01; 0.3; 0.4, 0.3\rangle, \langle b, 0.4, 0.2; 0.1; 0.7, 0.7\rangle\}$. Then we have:
$A \sqcap_R B = \{\langle a, 0.2, 0.01; 0.7; 0.4, 0.4\rangle, \langle b, 0.4, 0.2; 0.2; 0.9, 0.7\rangle\} \in \mathcal{R}_{(5,2,1)}$
$A \sqcup_R B = \{\langle a, 0.4, 0.1; 0.3; 0.1, 0.3\rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5\rangle\} \in \mathcal{R}_{(5,2,1)}$
$A^c = \{\langle a, 0.1, 0.4; 0.3; 0.2, 0.1\rangle, \langle b, 0.9, 0.5; 0.8; 0.5, 0.3\rangle\} \in \mathcal{R}_{(5,2,1)}$

Definition 2.7 (*Neutrosophic n-valued Refined Topology*)**.** $\tau \subset \mathcal{R}_{(n,r,s)}(X)$ is called a *neutrosophic n-valued refined topology* on X if

(1) $0_X, 1_X \in \tau$.
(2) For every $A, B \in \tau$, we have $A \sqcap_R B \in \tau$.
(3) If $A_\alpha \in \tau$ for every $\alpha \in \Delta$, then $\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha \in \tau$,

Elements of τ are called *neutrosophic n-valued refined open sets.* $A \in \mathcal{R}_{(n,r,s)}(X)$ is said *neutrosophic n-valued refined closed set* if $A^c \in \tau$.
The class of all neutrosophic n-valued refined topologies on X with sub-dimensions r, s will be denoted by $TOP_{(n,r,s)}(X)$.

Definition 2.8. Let $\tau \subseteq \mathcal{R}_{(n,r,s)}(X)$ be a neutrosophic n-valued refined topology on X and let $A \in \mathcal{R}_{(n,r,s)}(X)$. Then:

(1) The neutrosophic n-valued refined interior of A is defined to be
$Int_R(A) = \sqcup_R\{O \in \tau; O \sqsubseteq_R A\}$.

(2) The neutrosophic n-valued refined closure of A is defined to be
$Cl_R(A) = \sqcap_R\{C \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \quad \text{and} \quad A \sqsubseteq_R C\}$

Example 2.9. Let $X = \{a, b\}$, and let $\tau = \{0_X, 1_X, A, B, C, D\} \subset \mathcal{R}_{(5,2,1)}(X)$ where
$A = \{\langle a, 0.2, 0.1; 0.7; 0.1, 0.4\rangle, \langle b, 0.5, 0.3; 0.2; 0.9, 0.5\rangle\}$,
$B = \{\langle a, 0.4, 0.01; 0.3; 0.4, 0.3\rangle, \langle b, 0.4, 0.2; 0.1; 0.7, 0.7\rangle\}$,
$C = \{\langle a, 0.2, 0.01; 0.7; 0.4, 0.4\rangle, \langle b, 0.4, 0.2; 0.2; 0.9, 0.7\rangle\}$
$D = \{\langle a, 0.4, 0.1; 0.3; 0.1, 0.3\rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5\rangle\}$
Then τ is a Neutrosophic 5-valued refined topology on X. All closed set are: $0_X, 1_X, A^c, B^c, C^c, D^c$ where
$A^c = \{\langle a, 0.1, 0.4; 0.3; 0.2, 0.1\rangle, \langle b, 0.9, 0.5; 0.8; 0.5, 0.3\rangle\}$
$B^c = \{\langle a, 0.4, 0.3; 0.7; 0.4, 0.01\rangle, \langle b, 0.7, 0.7; 0.9; 0.4, 0.2\rangle\}$,
$C^c = \{\langle a, 0.4, 0.4; 0.3; 0.2, 0.01\rangle, \langle b, 0.9, 0.7; 0.8; 0.4, 0.2\rangle\}$
$D^c = \{\langle a, 0.1, 0.3; 0.7; 0.4, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}$
Let $K = \{\langle a, 0.43, 0.09; 0.2; 0.1, 0.2\rangle, \langle b, 0.5, 0.25; 0.1; 0.5, 0.6\rangle\}$. Then the open sets in τ contained in K are only $0_X, B, C$, so that $Int_R(K) = 0_X \sqcup_R B \sqcup_R C = B$. Now; we consider the set $K^c = \{\langle a, 0.1, 0.2; 0.8; 0.43, 0.09\rangle, \langle b, 0.5, 0.6; 0.9; 0.5, 0.25\rangle\}$ and compute $Cl_R(K^c)$; the only closed sets containing K^c are 1_X,B^c and C^c, so that $Cl_R(K^c) = 1_X \sqcap_R B^c \sqcap_R C^c = B^c$. Which means $Cl_R(K^c) = B^c$ and so $(Cl_R(K^c))^c = B = Int_R(K)$; that is $Int_R(K) = (Cl_R(K^c))^c$ and this leads us to the following theorem:

Theorem 2.10. *Let (X, τ) be an n-valued refined topological space with sub-dimensions r, s and let $A \in \mathcal{R}_{(n,r,s)}(X)$. Then we have:*

(1) $Int_R(A) = (Cl_R(A^c))^c$
(2) $Cl_R(K) = (Int_R(K^c))^c$

Proof. Since $\vee$ and $\wedge$ has duality, we will, only, proof part (1).
Let $A = \{\langle x, \mu_A^1(x), ..., \mu_A^r(x); \sigma_A^1(x), ..., \sigma_A^s(x); \nu_A^1(x), ..., \nu_A^r(x)\rangle; x \in X\}$. Then
$A^c = \{\langle x, \nu_A^1(x), ..., \nu_A^r(x); 1 - \sigma_A^1(x), ..., 1 - \sigma_A^s(x); \mu_A^1(x), ..., \mu_A^r(x)\rangle; x \in X\}$, so
$Cl_R(A^c) = \sqcap_R\{C \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \quad \text{and} \quad A^c \sqsubseteq_R C\}$. We apply Demorgan's Laws in Theorem 2.3 to get: $(Cl_R(A^c))^c = \sqcup_R\{C^c \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \quad \text{and} \quad C^c \sqsubseteq_R A\} = \sqcup_R\{O \in \mathcal{R}_{(n,r,s)}(X); O \in \tau \quad \text{and} \quad O \sqsubseteq_R A\} = Int_R(A)$.
□

Theorem 2.11. *Let (X, τ) be an n-valued refined topological space with sub-dimensions r, s and let $A, B \in \mathcal{R}_{(n,r,s)}(X)$. Then we have:*

(1) $Int_R(A) \sqsubseteq_R A$.
(2) *If A is a neutrosophic n-valued refined open set, then* $Int_R(A) = A$.

(3) $Int_R(Int_R(A)) = Int_R(A)$.

(4) *If* $A \sqsubseteq_R B$, *then* $Int_R(A) \sqsubseteq_R Int_R(B)$.

(5) $Int_R(A \sqcap_R B) = Int_R(A) \sqcap_R Int_R(B)$

(6) $Int_R(A \sqcup_R B) \sqsupseteq_R Int_R(A) \sqcup_R Int_R(B)$

(7) $Int_R(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha) \sqsupseteq_R \underset{\alpha\in\Delta}{\sqcup_R} Int_R(A_\alpha)$

(8) $A \sqsubseteq_R Cl_R(A)$.

(9) *If* A *is a neutrosophic* n*-valued refined closed set, then* $Cl_R(A) = A$.

(10) $Cl_R(Cl_R(A)) = Cl_R(A)$.

(11) *If* $A \sqsubseteq_R B$, *then* $Int_R(A) \sqsubseteq_R Int_R(B)$.

(12) $Cl_R(A \sqcup_R B) = Cl_R(A) \sqcup_R Cl_R(B)$

(13) $Cl_R(A \sqcap_R B) \sqsubseteq_R Cl_R(A) \sqcap_R Cl_R(B)$

(14) $Cl_R(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha) \sqsupseteq_R \underset{\alpha\in\Delta}{\sqcup_R} Cl_R(A_\alpha)$

Proof. (1) Let $O \in \tau$ such that $O \sqsubseteq_R A$. Then for every $x \in X$ we have $\mu^i_O(x) \leq \mu^i_A(x)$ for every $i = 1,..,r$, $\sigma^i_O(x) \geq \sigma^i_A(x)$ for every $i = 1,..,s$ and $\nu^i_O(x) \geq \nu^i_A(x)$ for every $i = 1,..,r$, which implies that $\bigvee\limits_{O\in\tau, O\sqsubseteq_R A} \mu^i_O(x) \leq \mu^i_A(x)$ for every $i = 1,..,r$, $\bigwedge\limits_{O\in\tau, O\sqsubseteq_R A} \sigma^i_O(x) \geq \sigma^i_O(x)$ for every $i = 1,..,s$ and $\bigwedge\limits_{O\in\tau, O\sqsubseteq A} \nu^i_O(x) \geq \nu^i_A(x)$ for every $i = 1,..,r$; that is $Int_R(A) \sqsubseteq A$.

(2) Since A is open, then, from the definition of $Int_R(A)$, we have $A \sqsubseteq_R Int_R(A)$, and from part (1) we have the converse, and we done.

(3) Since $Int_R(A)$ is a neutrosophic n-valued refined open set, we have (from part (2)) $Int_R(Int_R(A)) = Int_R(A)$.

(4) Let O be a *neutrosophic* n*-valued refined open* set such that $O \sqsubseteq_R A$. Then since $A \sqsubseteq_R B$, we have $O \sqsubseteq_R B$, that is $Int_R(A) \sqsubseteq_R Int_R(B)$

(5) From part (4) we have $Int_R(A \sqcap_R B) \sqsubseteq_R Int_R(A) \sqcap_R Int_R(B)$. On the other hand, $Int_R(A) \sqcap_R Int_R(B)$ is a neutrosophic n-valued refined open set contained in A and B, so that $Int_R(A) \sqcap_R Int_R(B) \sqsubseteq_R Int_R(A \sqcap_R B)$, and we done.

(6) Since $Int_R(A) \sqsubseteq_R A$ and $Int_R(B) \sqsubseteq_R B$, we have $Int_R(A) \sqcup_R Int_R(B)$ is a neutrosophic n-valued refined open set contained in $A \sqcup_R B$, which implies that $Int_R(A) \sqcup_R Int_R(B) \sqsubseteq_R Int_R(A \sqcup_R B)$.

(7) Since $A_\alpha \sqsubseteq_R \underset{\alpha\in\Delta}{\sqcup_R} A_\alpha$ for every $\alpha \in \Delta$, $Int_R(A_\alpha) \sqsubseteq_R Int_R(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha)$ for every $\alpha \in \Delta$, that is $\underset{\alpha\in\Delta}{\sqcup_R} Int_R(A_\alpha) \sqsubseteq_R Int_R(\underset{\alpha\in\Delta}{\sqcup_R} A_\alpha)$.

The remaining 5 parts can be proved by duality. □

Equality in parts (7) and (13) of Theorem 2.11 does not hold.

Example 2.12. Consider the neutrosophic 5-valued refined topological space (X, τ) defined in Example 2.9 and let $K = \{\langle a, 0, 1; 0; 1, 1\rangle, \langle b, 1, 1; 0; 0, 1\rangle\}$, and $L = \{\langle a, 1, 0; 1; 0, 0\rangle, \langle b, 0, 0; 1; 1, 0\rangle\}$. Then $K \sqcup_R L = \{\langle a, 1, 1; 0; 0, 0\rangle, \langle b, 1, 1; 0; 0, 0\rangle\} = 1_X$. So we have $Int_R(K \sqcup_R L) = 1_X$, and since K and L contains no neutrosophic n-valued refined open set except 0_X we have $Int_R(K) = Int_R(L) = 0_X$, which means $Int_R(K) \sqcup_R Int_R(L) = 0_X$, hence equality in parts (7) and (8) of Theorem 2.11 does not hold. For part (13) let $K = \{\langle a, 0.1, 0.4; 0.6; 0.5, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}$, $L = \{\langle a, 0.1, 0.3; 0.7; 0.3, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}$. Then
$K \sqcap_R L = \{\langle a, 0.1, 0.3; 0.7; 0.5, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}$. The only neutrosophic 5-valued Refined closed sets containing K are: $1_X, A^c$ and C^c, so that we have $Cl_R(K) = 1_X \sqcap_R A^c \sqcap_R C^c = A^c$. Again the only neutrosophic 5-valued Refined closed sets containing L are:1_X, A^c and C^c, so that we have $Cl_R(L) = 1_X \sqcap_R A^c \sqcap_R C^c = A^c$, and $Cl_R(K) \sqcap_R Cl_R(L) = A^c \sqcap_R A^c = A^c$, on the other hand the only neutrosophic 5-valued Refined closed sets containing $K \sqcap_R L$ are: $1_X, A^c, B^c$ and D^c, so that we have $Cl_R(K \sqcap_R L) = 1_X \sqcap_R A^c \sqcap_R B^c \sqcap_R D^c = D^c$. Note that D^c is a proper subset of A^c, so equality in Theorem 2.11 part (13) does not hold.

Question 2.13. *Is there a neutrosophic n-valued refined topological space (X, τ) shows that equality in part* (14) *of Theorem 2.11 does not hold.*

Definition 2.14 (*Neutrosophic n-valued refined pre-open and pre-closed sets*)**.** Let $\tau \in TOP_{(n,r,s)}(X)$ and $A \in \mathcal{R}_{(n,r,s)}(X)$. Then A is said to be:

(1) A *neutrosophic n-valued refined semi-open set*, if $A \sqsubseteq_R Cl_R(Int_R(A))$. The complement of a neutrosophic n-valued refined semi-open set is called a *neutrosophic n-valued refined semi-closed set.*

(2) A *neutrosophic n-valued refined pre-open set*, if $A \sqsubseteq_R Int_R(Cl_R(A))$. The complement of a neutrosophic n-valued refined pre-open set is called a *neutrosophic n-valued refined pre-closed set.*

(3) A *neutrosophic n-valued refined α-open set*, if $A \sqsubseteq_R Int_R(Cl_R(Int_R(A)))$. The complement of a neutrosophic n-valued refined α-open set is called a *neutrosophic n-valued refined α-closed set.*

(4) A *neutrosophic n-valued refined β-open set*, if $A \sqsubseteq_R Cl_R(Int_R(Cl_R(A)))$. The complement of a neutrosophic n-valued refined β-open set is called a *neutrosophic n-valued refined β-closed set.*

Theorem 2.15. *Let $\tau \in TOP_{(n,r,s)}(X)$ and $A \in \mathcal{R}_{(n,r,s)}(X)$. Then:*

(1) *Every* Neutrosophic n-valued refined open (closed) set*, is* neutrosophic n-valued refined α-open (closed) set.

(2) *Every* Neutrosophic n-valued refined α-open (α-closed) set*, is* neutrosophic n-valued refined pre-open (pre-closed) set *and* neutrosophic n-valued refined semi-open (semi-closed) set.

(3) *Every* Neutrosophic n-valued refined pre-open (pre-closed) or semi-open (semi-closed) set*, is a* neutrosophic n-valued refined β-open (β-closed) set.

Proof. (1) Let A be a *Neutrosophic n-valued refined open set*. Then, from Theorem 2.11 part (2) and (8), we have $Int_R(A) = A$ and $A \sqsubseteq_R Cl_R(A)$. So $Int_R(Cl_R(int_R(A))) \sqsupseteq_R Int_R(Cl_R(A)) \sqsupseteq_R Int_R(A) = A$. That is A is a *neutrosophic n-valued refined α-open set*. Now, suppose that A is a *Neutrosophic n-valued refined closed set*. Then A^c is a *Neutrosophic n-valued refined open set*, which implies A^c is a *neutrosophic n-valued refined α-open set*, and so A is a is a *neutrosophic n-valued refined α-closed set*.

(2) Obvious! we only use Theorem 2.11 part (1).

(3) Obvious! we only use Theorem 2.11 part (8) .

□

None of the above implications reverse. The following is an example of a neutrosophic 5-valued refined α-open set which is not open, and another example of a neutrosophic 5-valued refined *pre*-open (so it is β-open) set which is neither *semi*-open nor α-open.

Example 2.16. Consider $\tau = \{0_X, 1_X, A, B, C, D\}$ in Example 2.9 and let
$H = \{\langle a, 0.5, 0.1; 0.3; 0.1, 0.3\rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5\rangle\}$.
Then the neutrosophic 5-valued refined open sets contained in H are $0_X, A, B, C, D$; so we have $Int_R(H) = 0_X \sqcup_R A \sqcup_R B \sqcup_R C \sqcup_R D = D$, and since the only neutrosophic 5-valued refined close set containing D is 1_X, we have $Cl_R(Int_R(H)) = 1_X$, which implies $Int_R(Cl_R(int_R(H))) = 1_X$, hence $A \sqsubseteq_R Int_R(Cl_R(int_R(A)))$ and H is a neutrosophic 5-valued refined α-open set but not a neutrosophic 5-valued refined open set.
Consider, again, the set $K = \{\langle a, 0.1, 0.4; 0.6; 0.1, 0.3\rangle, \langle b, 0.9, 0.2; 0.4; 0.1, 0.5\rangle\}$. Since $\mu_K^1(a) < \mu_O^1(a)$ for every $O \in \tau - \{0_X\}$, we have the only Neutrosophic 5-valued refined open set contained in K is 0_X and $Int_R(K) = 0_X$, which implies $Cl_R(Int_R(K)) = 0_X$ and $Int_R(Cl_R(Int_R(K))) = 0_X$, so K is not a neutrosophic 5-valued refined semi-open nor α-open set; on the other hand, $\mu_K^1(b) > \mu_D^1(b)$ for every neutrosophic 5-valued refined closed set D in τ except for 1_X, that means $Cl_R(K) = 1_X$ and $int_R(Cl_R(A)) = 1_X$, hence $K \sqsubseteq_R Int_R(Cl_R(A))$ and K is a neutrosophic 5-valued refined pre-open set but not α-open. Since every neutrosophic 5-valued refined pre-open set is a neutrosophic 5-valued refined β-open set, K is, also, and example of a neutrosophic 5-valued refined β-open set which is not neutrosophic 5-valued refined semi-open.

Here we give an example of a a neutrosophic 5-valued refined *semi*-open (so it is β-open) set which is neither *pre*-open nor α-open.

Example 2.17. Let $X = \{a\}$, and let $\tau = \{0_X, 1_X, A, B\} \subset \mathcal{R}_{(5,2,1)}(X)$ where $A = \{\langle a, 0.2, 0.1; 0.7; 0.3, 0.4\rangle\}$, $B = \{\langle a, 0.3, 0.2; 0.5; 0.2, 0.3\rangle\}$. Since $A \sqcap_R B = A$ and $A \sqcup_R B = B$, τ is a neutrosophic 5-valued refined topology on X. The 5-valued refined closed sets in (X, τ) are: $0_X, 1_X, A^c, B^c$ where
$A^c = \{\langle a, 0.3, 0.4; 0.3; 0.2, 0.1\rangle\}$ and $B^c = \{\langle a, 0.2, 0.3; 0.5; 0.3, 0.2\rangle\}$. Consider the neutrosophic 5-valued refined set $L = \{\langle a, 0.2, 0.2; 0.5; 0.3, 0.3\rangle\}$. Then the only neutrosophic 5-valued refined open sets contained in K are $0_X, A$, so that $Int_R(L) = 0_X \sqcup_R A = A$. To find $Cl_R(Int_R(L))$ we note that the neutrosophic 5-valued refined closed sets containing $Int_R(L)$ are $1_X, A^c, B^c$, so $Cl_R(Int_R(L)) = 1_x \sqcap_R A^c \sqcap_R B^c = B^c$, and since $L \sqsubseteq_R B^c$, L is a neutrosophic 5-valued refined semi-open sets. Now, we will show that L is not α-open. First note that the neutrosophic 5-valued refined open sets contained in $Cl_R(Int_R(K)) = B^c$ are 0_X and A, so we have $Int_R(Cl_R(Int_R(L))) = A$, and since L is not contained in A, L is not a neutrosophic α-open set.
We will show L is not a neutrosophic 5-valued refined *pre*-open set. The only neutrosophic 5-valued refined closed sets containing L are $1_X, A^c$ and B^c, so $Cl_R(L) = 1_X \sqcap_R A^c \sqcap_R B^c = B^c$, and since the neutrosophic 5-valued refined open sets contained in B^c are 0_X and A, we have $Int_R(Cl_R(L)) = A$ which not containing L, that is L is not a neutrosophic 5-valued refined pre-open set. So L is, also, an example of a neutrosophic 5-valued refined *semi*-open set which is not pre-open. And since every neutrosophic 5-valued refined *semi*-open set is β-open set, K is an example of a neutrosophic 5-valued refined β-open set which is not *pre*-open.

Finally we will give an example of a a neutrosophic 5-valued refined β-open set which is neither *pre*-open nor *semi*-open.

Example 2.18. Let (X, τ) as in Example 2.17 and consider the neutrosophic 5-valued refined set $M = \{\langle a, 0.2, 0.1; 0.9; 0.3, 0.5\rangle\}$. Then the only neutrosophic 5-valued refined open sets in τ contained in K is 0_X, so $Int_R(M) = 0_X$, which implies $Cl_R(Int_R(M)) = 0_X$, and since M is not contained in 0_X, we have M is not neutrosophic 5-valued refined semi-open set; on the other hand the neutrosophic 5-valued refined closed sets containing M are $1_X, A^c$ and B^c, so that $Cl_R(M) = B^c$, and since the only neutrosophic 5-valued refined open sets contained in B^c are 0_X and A we have $Int_R(Cl_R(M)) = A$. Since $Int_R(Cl_R(M)) = A$ and A does not contain M, we have M is not a neutrosophic 5-valued refined *pre*-open set. Now, to find $Cl_R(Int_R(Cl_R(M)))$ we note that the only neutrosophic 5-valued refined closed sets in τ

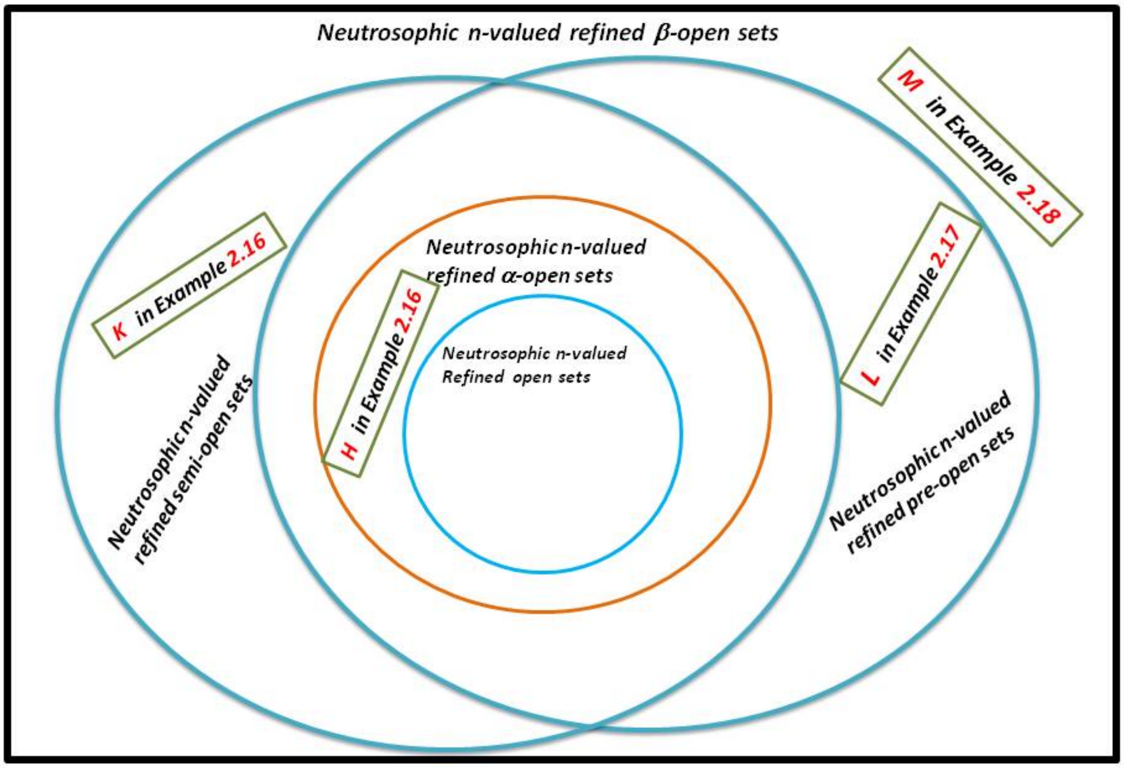

FIGURE 1. Relations between differen types of generalized neutrosophic n-valued refined open sets.

containing A are $1_X, A^c$ and B^c, so $Cl_R(Int_R(Cl_R(M))) = B^c$ which contains M, so M is a neutrosophic 5-valued refined β-open set but not *semi*-open nor *pre*-open.

The following diagram shows the relations between different types of generalized neutrosophic n-valued refined sets:

Theorem 2.19. *Let $\tau \in TOP_{(n,r,s)}(X)$ and $K \in \mathcal{R}_{(n,r,s)}(X)$. Then*

(1) *If there is a neutrosophic n-valued refined open set U such that $K \sqsubseteq_R U \sqsubseteq_R Cl_R(K)$, then K is a neutrosophic n-valued refined pre-open set.*

(2) *If there is a neutrosophic n-valued refined open set U such that $U \sqsubseteq_R K \sqsubseteq_R Cl_R(U)$, then K is a neutrosophic n-valued refined semi-open set.*

Proof. (1) $K \sqsubseteq_R U \sqsubseteq_R Int_R(Cl_R(U)) \sqsubseteq_R Int_R(Cl_R(Cl_R(K))) = Int_R(Cl_R(K))$.

(2) Since $Cl_R(Int_R(U)) = Cl_R(U)$ we have
$Cl_R(Int_R(K)) \sqsupseteq_R Cl_R(Int_R(U)) = Cl_R(U) \sqsupseteq_R K)$.

□

Theorem 2.20. *Let $\tau \in TOP_{(n,r,s)}(X)$ and $K \in \mathcal{R}_{(n,r,s)}(X)$. Then the union of any collection of neutrosophic n-valued refined α-open, β-open, pre-open or semi-open sets is a neutrosophic n-valued refined α-open, β-open, pre-open or semi-open set respectively.*

Proof. We will prove it for neutrosophic n-valued refined β-open sets, and the remaining parts can be proved in the same manner. Let A_γ be a neutrosophic n-valued refined β-open set for every $\gamma \in \Delta$. Then $A_\gamma \sqsubseteq_R Cl_R(int_R(Cl_R(A_\gamma)))$ for every $\gamma \in \Delta$. Then from parts (7) and (14) of Theorem 2.11 we have:
$Cl_R(int_R(Cl_R(\underset{\gamma\in\Delta}{\sqcup_R} A_\gamma))) \sqsupseteq_R Cl_R(int_R(\underset{\gamma\in\Delta}{\sqcup_R} Cl_R(A_\gamma))) \sqsupseteq_R Cl_R(\underset{\gamma\in\Delta}{\sqcup_R} int_R(Cl_R(A_\gamma))) \sqsupseteq_R \underset{\gamma\in\Delta}{\sqcup_R} Cl_R(int_R(Cl_R(A_\gamma))) \sqsupseteq_R \underset{\gamma\in\Delta}{\sqcup_R} A_\gamma$ □

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