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On Some New Concepts of Weakly Neutrosophic Crisp Separation Axioms

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Abstract: In this paper, we shall study some new concepts of weakly neutrosophic crisp separation axioms, which are called "neutrosophic crisp α-separation and neutrosophic crisp semi-α-separation axioms" such as neutrosophic crisp α-T_i and neutrosophic crisp semi-α-$T_i, \forall\, i = 0, 1, \dots, 4$. Moreover, we shall study the relationship between usual neutrosophic crisp separation axioms and these kinds of weakly neutrosophic crisp separation axioms.

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Keywords: NC^{α}-T_i, $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_i, \forall\, i = 0, 1, \dots, 4$, NC^{α}-regular, $\mathrm{NC}^{\mathrm{S}\alpha}$-regular, NC^{α}-normal and $\mathrm{NC}^{\mathrm{S}\alpha}$-normal spaces.

1. Introduction

A. A. Salama et al. [1] give a concept of neutrosophic crisp topological space (briefly NCTS). A. A. Salama [2] provided some classes of neutrosophic crisp nearly open sets. A. H. M. Al-Obaidi et al. [3,4] give concepts of weakly neutrosophic crisp functions. Md. Hanif PAGE et al. [5] examined the view of neutrosophic generalized homeomorphism. Q. H. Imran et al. [6-8] established neutrosophic semi-α-open sets, new types of weakly neutrosophic crisp continuity and new concepts of neutrosophic crisp open sets. R. Dhavaseelan et al. [9] examined the view of neutrosophic α^m-continuity. R. K. Al-Hamido et al. [10] tendered the interpretation of neutrosophic crisp semi-α-closed sets. A. B. Al-Nafee et al. [11] demonstrated the principle of separation axioms in neutrosophic crisp topological spaces. R. K. Al-Hamido et al. [12] provided neutrosophic crisp semi separation axioms. The objective of this paper is to study some new concepts of weakly neutrosophic crisp separation axioms, which are called "neutrosophic crisp α-separation and neutrosophic crisp semi-α-separation axioms" such as neutrosophic crisp α-T_i and neutrosophic crisp semi-α-$T_i, \forall\, i = 0, 1, \dots, 4$. Moreover, we shall study the relationship between usual neutrosophic crisp separation axioms and these kinds of weakly neutrosophic crisp separation axioms.

2. Preliminaries

Throughout this paper, $(\mathcal{S}, \zeta)$ and $(\mathcal{J}, \eta)$ (or simply $\mathcal{S}$ and $\mathcal{J}$) always mean NCTSs. The complement of a neutrosophic crisp open set (briefly NC-OS) is called a neutrosophic crisp closed set (briefly NC-CS) in $(\mathcal{S},\zeta)$. For a NCS $\mathfrak{B}$ in a NCTS $(\mathcal{S},\zeta)$, $NCcl(\mathfrak{B})$, $NCint(\mathfrak{B})$ and $\mathfrak{B}^c$ denote the NC-closure of $\mathfrak{B}$, the NC-interior of $\mathfrak{B}$ and the NC-complement of $\mathfrak{B}$ respectively.

Definition 2.1 [1]:

For any nonempty under-consideration set $\mathcal{S}$, a neutrosophic crisp set (in short NCS) $\mathfrak{B}$ is an object holding the establish $\mathfrak{B} = \langle \mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3 \rangle$ where $\mathfrak{B}_1, \mathfrak{B}_2$ and $\mathfrak{B}_3$ are mutually disjoint sets included in $\mathcal{S}$.

Definition 2.2:

A NC-subset $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is said to be:

(i) neutrosophic crisp α-open set (in short NC^{α}-OS) [2] if $\mathfrak{B} \sqsubseteq NCint(NCcl(NCint(\mathfrak{B})))$. The family of all NC^{α}-OSs of $\mathcal{S}$ is denoted by $NC^{\alpha}O(\mathcal{S})$. The complement of NC^{α}-OS is called a neutrosophic crisp α-closed set (in short NC^{α}-CS). The family of all NC^{α}-CSs of $\mathcal{S}$ is denoted by $NC^{\alpha}C(\mathcal{S})$.

(ii) neutrosophic crisp semi-α-open set (in short $\mathrm{NC}^{\mathrm{S}\alpha}$-OS) [10] if there exists a NC^{α}-OS $\mathfrak{D}$ in $\mathcal{S}$ such that $\mathfrak{D} \sqsubseteq \mathfrak{B} \sqsubseteq NCcl(\mathfrak{D})$ or equivalently if $\mathfrak{B} \sqsubseteq NCcl(NCint(NCcl(NCint(\mathfrak{B}))))$. The family of all $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs of $\mathcal{S}$ is denoted by $NC^{\mathrm{S}\alpha}O(\mathcal{S})$. The complement of $\mathrm{NC}^{\mathrm{S}\alpha}$-OS is called a neutrosophic crisp semi-α-closed set (in short $\mathrm{NC}^{\mathrm{S}\alpha}$-CS). The family of all $\mathrm{NC}^{\mathrm{S}\alpha}$-CSs of $\mathcal{S}$ is denoted by $NC^{\mathrm{S}\alpha}C(\mathcal{S})$.

Example 2.3:

Let $\mathcal{S} = \{k_1, k_2, k_3, k_4\}$. Then $\zeta = \{\emptyset_{\mathcal{N}}, \langle\{k_1\},\emptyset,\emptyset\rangle, \langle\{k_2\},\emptyset,\emptyset\rangle, \langle\{k_1,k_2\},\emptyset,\emptyset\rangle, \langle\{k_1,k_2,k_3\},\emptyset,\emptyset\rangle, \mathcal{S}_{\mathcal{N}}\}$ is a NCTS. The family of all NC^{α}-OSs of $\mathcal{S}$ is : $NC^{\alpha}O(\mathcal{S}) = \zeta \sqcup \langle\{k_1,k_2,k_4\},\emptyset,\emptyset\rangle$.

The family of all $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs of $\mathcal{S}$ is : $NC^{\mathrm{S}\alpha}O(\mathcal{S}) = NC^{\alpha}O(\mathcal{S}) \sqcup \{\langle\{k_1,k_3\},\emptyset,\emptyset\rangle, \langle\{k_1,k_4\},\emptyset,\emptyset\rangle, \langle\{k_2,k_3\},\emptyset,\emptyset\rangle, \langle\{k_2,k_4\},\emptyset,\emptyset\rangle, \langle\{k_1,k_3,k_4\},\emptyset,\emptyset\rangle, \langle\{k_2,k_3,k_4\},\emptyset,\emptyset\rangle\}$.

Remark 2.4 [10,14]:

In a NCTS $(\mathcal{S},\zeta)$, then the following statements hold, and the opposite of each statement is not true:

(i) Every NC-OS (resp. NC-CS) is a NC^{α}-OS (resp. NC^{α}-CS) and $\mathrm{NC}^{\mathrm{S}\alpha}$-OS (resp. $\mathrm{NC}^{\mathrm{S}\alpha}$-CS).

(ii) Every NC^{α}-OS (resp. NC^{α}-CS) is a $\mathrm{NC}^{\mathrm{S}\alpha}$-OS (resp. $\mathrm{NC}^{\mathrm{S}\alpha}$-CS).

Definition 2.5:

(i) The NC^{α}-interior of a NCS $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is the union of all NC^{α}-OSs contained in $\mathfrak{B}$ and is denoted by $NC^{\alpha}int(\mathfrak{B})$[3].

(ii) The $\mathrm{NC}^{\mathrm{S}\alpha}$-interior of a NCS $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is the union of all $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs contained in $\mathfrak{B}$ and is denoted by $NC^{\mathrm{S}\alpha}int(\mathfrak{B})$[10].

Definition 2.6:

(i) The NC^{α}-closure of a NCS $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is the intersection of all NC^{α}-CSs containing $\mathfrak{B}$ and is denoted by $NC^{\alpha}cl(\mathfrak{B})$[3].

(ii) The $\mathrm{NC}^{\mathrm{S}\alpha}$-closure of a NCS $\mathfrak{B}$ of a NCTS $(\mathcal{S},\zeta)$ is the intersection of all $\mathrm{NC}^{\mathrm{S}\alpha}$-CSs containing $\mathfrak{B}$ and is denoted by $NC^{\mathrm{S}\alpha}cl(\mathfrak{B})$[10].

Theorem 2.7:

Let $(\mathcal{S},\zeta)$ and $(\mathcal{J},\eta)$ be two NCTSs. If $\mathfrak{B} \in NC^{\alpha}O(\mathcal{S})$(resp. $\mathfrak{B} \in NC^{\mathrm{S}\alpha}O(\mathcal{S})$), $\mathfrak{D} \in NC^{\alpha}O(\mathcal{J})$(resp. $\mathfrak{D} \in NC^{\mathrm{S}\alpha}O(\mathcal{J})$), then $\mathfrak{B} \times \mathfrak{D} \in NC^{\alpha}O(\mathcal{S} \times \mathcal{J})$ (resp. $\mathfrak{B} \times \mathfrak{D} \in NC^{\mathrm{S}\alpha}O(\mathcal{S} \times \mathcal{J})$).

Proof:

Since $\mathfrak{B} \sqsubseteq NCint(NCcl(NCint(\mathfrak{B})))$, $\mathfrak{D} \sqsubseteq NCint(NCcl(NCint(\mathfrak{D})))$.

Hence $\mathfrak{B} \times \mathfrak{D} \sqsubseteq NCint(NCcl(NCint(\mathfrak{B}))) \times NCint(NCcl(NCint(\mathfrak{D}))) = NCint(NCcl(NCint(\mathfrak{B} \times \mathfrak{D})))$.

Therefore $\mathfrak{B} \times \mathfrak{D} \sqsubseteq NCint(NCcl(NCint(\mathfrak{B} \times \mathfrak{D}))) \Rightarrow \mathfrak{B} \times \mathfrak{D} \in NC^{\alpha}O(\mathcal{S} \times \mathcal{I})$. The second case is similar. ▪

Corollary 2.8:

Let $(\mathcal{S}, \zeta)$ and $(\mathcal{I}, \eta)$ be two NCTSs. If $\mathfrak{B} \in NC^{\alpha}C(\mathcal{S})$(resp. $\mathfrak{B} \in NC^{S\alpha}C(\mathcal{S})$), $\mathfrak{D} \in NC^{\alpha}C(\mathcal{I})$(resp. $\mathfrak{D} \in NC^{S\alpha}C(\mathcal{I})$), then $\mathfrak{B} \times \mathfrak{D} \in NC^{\alpha}C(\mathcal{S} \times \mathcal{I})$ (resp. $\mathfrak{B} \times \mathfrak{D} \in NC^{S\alpha}C(\mathcal{S} \times \mathcal{I})$).

Proof:

The proof of this is similar to that of theorem (2.6). ▪

Proposition 2.9 [10]:

For any NC-subset $\mathfrak{B}$ of a NCTS $(\mathcal{S}, \zeta)$, then:

(i) $NCint(\mathfrak{B}) \sqsubseteq NC^{\alpha}int(\mathfrak{B}) \sqsubseteq NC^{S\alpha}int(\mathfrak{B}) \sqsubseteq NC^{S\alpha}cl(\mathfrak{B}) \sqsubseteq NC^{\alpha}cl(\mathfrak{B}) \sqsubseteq NCcl(\mathfrak{B})$.

(ii) $NCint(NC^{S\alpha}int(\mathfrak{B})) = NC^{S\alpha}int\big(NCint(\mathfrak{B})\big) = NCint(\mathfrak{B})$.

(iii) $NC^{\alpha}int(NC^{S\alpha}int(\mathfrak{B})) = NC^{S\alpha}int\big(NC^{\alpha}int(\mathfrak{B})\big) = NC^{\alpha}int(\mathfrak{B})$.

(iv) $NCcl(NC^{S\alpha}cl(\mathfrak{B})) = NC^{S\alpha}cl\big(NCcl(\mathfrak{B})\big) = NCcl(\mathfrak{B})$.

(v) $NC^{\alpha}cl(NC^{S\alpha}cl(\mathfrak{B})) = NC^{S\alpha}cl\big(NC^{\alpha}cl(\mathfrak{B})\big) = NC^{\alpha}cl(\mathfrak{B})$.

(vi) $NC^{S\alpha}cl(\mathfrak{B}) = \mathfrak{B} \sqcup NCint(NCcl(NCint(NCcl(\mathfrak{B}))))$.

(vii) $NC^{S\alpha}int(\mathfrak{B}) = \mathfrak{B} \sqcap NCcl(NCint(NCcl(NCint(\mathfrak{B}))))$.

(viii) $NCint(NCcl(\mathfrak{B})) \sqsubseteq NC^{S\alpha}int(NC^{S\alpha}cl(\mathfrak{B}))$

Definition 2.10 [1]:

Let $\rho: (\mathcal{S}, \zeta) \longrightarrow (\mathcal{I}, \eta)$ be a function, then ρ is said to be NC-continuous (in short NC-CF) iff $\forall$ $\mathfrak{B}$ NC-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC-OS in $\mathcal{S}$.

Definition 2.11 [13]:

Let $\rho: (\mathcal{S}, \zeta) \longrightarrow (\mathcal{I}, \eta)$ be a function, then ρ is said to be NC^{α}-continuous (in short NC^{α}-CF) iff $\forall$ $\mathfrak{B}$ NC-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC^{α}-OS in $\mathcal{S}$.

Definition 2.12 [10]:

Let $\rho: (\mathcal{S}, \zeta) \longrightarrow (\mathcal{I}, \eta)$ be a function, then ρ is said to be:

(i) NC^{α^*}-continuous (in short NC^{α^*}-CF) iff $\forall$ $\mathfrak{B}$ NC^{α}-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC^{α}-OS in $\mathcal{S}$.

(ii) $NC^{\alpha^{**}}$-continuous (in short $NC^{\alpha^{**}}$-CF) iff $\forall$ $\mathfrak{B}$ NC^{α}-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC-OS in $\mathcal{S}$.

Definition 2.13 [10]:

Let $\rho: (\mathcal{S}, \zeta) \longrightarrow (\mathcal{I}, \eta)$ be a function, then ρ is said to be:

(i) $NC^{S\alpha}$-continuous (in short $NC^{S\alpha}$-CF) iff $\forall$ $\mathfrak{B}$ NC-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a $NC^{S\alpha}$-OS in $\mathcal{S}$.

(ii) $NC^{S\alpha^*}$-continuous (in short $NC^{S\alpha^*}$-CF) iff $\forall$ $\mathfrak{B}$ $NC^{S\alpha}$-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a $NC^{S\alpha}$-OS in $\mathcal{S}$.

(iii) $NC^{S\alpha^{**}}$-continuous (in short $NC^{S\alpha^{**}}$-CF) iff $\forall$ $\mathfrak{B}$ $NC^{S\alpha}$-OS in $\mathcal{I}$, then $\rho^{-1}(\mathfrak{B})$ is a NC-OS in $\mathcal{S}$.

3. Some New Concepts of Weakly Neutrosophic Crisp Separation Axioms

Definition 3.1:

(i) A NCTS $(\mathcal{S}, \zeta)$ is said to be a NC^{α}-T_0-space if for each pair of distinct neutrosophic crisp points in $(\mathcal{S}, \zeta)$ there exists NC^{α}-OS of $(\mathcal{S}, \zeta)$ containing one neutrosophic crisp point but not the other.

(ii) A NCTS $(\mathcal{S}, \zeta)$ is said to be a $NC^{S\alpha}$-T_0-space if for each pair of distinct neutrosophic crisp points in $(\mathcal{S}, \zeta)$ there exists $NC^{S\alpha}$-OS of $(\mathcal{S}, \zeta)$ containing one neutrosophic crisp point but not the other.

Theorem 3.3:

A NCTS $(\mathcal{S},\zeta)$ is NC$^{\alpha}$-T_0-space (NC$^{S\alpha}$-T_0-space respectively) iff $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \neq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$ $(NC^{S\alpha}cl(<\{u\},\emptyset,\emptyset>) \neq NC^{S\alpha}cl(<\{v\},\emptyset,\emptyset>)$ receptively) for each $u \neq v$ in $\mathcal{S}$.

Proof:

$\Rightarrow$ Let $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \neq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$, $\forall u \neq v \in \mathcal{S}$. Hence $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \not\sqsubseteq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$ or $NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>) \not\sqsubseteq NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>)$. Suppose that $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \not\sqsubseteq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$ $\Rightarrow u \notin NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>) \Rightarrow u \in (NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>))^c$ but $(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>))^c$ is a NC$^{\alpha}$-OS and $v \notin (NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>))^c$. Therefore $\mathcal{S}$ is a NC$^{\alpha}$-T_0-space.

$\Leftarrow$ Let $\mathcal{S}$ be a NC$^{\alpha}$-T_0-space, $\forall u \neq v \in \mathcal{S}$. Hence there exists a NC$^{\alpha}$-OS $\mathfrak{B}$ in $\mathcal{S}$ such that $u \in \mathfrak{B}, v \notin \mathfrak{B}$ or $u \notin \mathfrak{B}, v \in \mathfrak{B}$. Then $\mathfrak{B}^c$ is a NC$^{\alpha}$-CS and $u \notin \mathfrak{B}^c, v \in \mathfrak{B}^c$. Therefore $u \notin NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$ (since $u \notin \mathfrak{B}^c$). Hence $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \not\sqsubseteq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)$. The second case is similar. ▪

Theorem 3.4:

If $(\mathcal{S},\zeta)$ is a NC$^{\alpha}$-T_0-space (NC$^{S\alpha}$-T_0-space respectively), then $NC^{\alpha}int(NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>)) \sqcap NC^{\alpha}int(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)) = \emptyset_{\mathcal{N}}$ ($NC^{S\alpha}int(NC^{S\alpha}cl(<\{u\},\emptyset,\emptyset>)) \sqcap NC^{S\alpha}int(NC^{S\alpha}cl(<\{v\},\emptyset,\emptyset>)) = \emptyset_{\mathcal{N}}$ receptively), $\forall u \neq v$ in $\mathcal{S}$.

Proof:

Let $(\mathcal{S},\zeta)$ be a NC$^{\alpha}$-T_0-space. Then there exists a NC$^{\alpha}$-OS $\mathfrak{B}$ such that $u \in \mathfrak{B}, v \notin \mathfrak{B}$ or $u \notin \mathfrak{B}, v \in \mathfrak{B}$. If $u \in \mathfrak{B}, v \notin \mathfrak{B} \Rightarrow u \notin \mathfrak{B}^c, v \in \mathfrak{B}^c$. Thus $NC^{\alpha}int(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)) \sqsubseteq NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>) \sqsubseteq \mathfrak{B}^c = NC^{\alpha}cl(\mathfrak{B}^c)$ (since $\mathfrak{B}^c$ is a NC$^{\alpha}$ -CS). Hence $NC^{\alpha}int(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)) \sqsubseteq \mathfrak{B}^c \Rightarrow NC^{\alpha}int(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)) \sqcap \mathfrak{B} = \emptyset_{\mathcal{N}}$. Therefore, $u \in \mathfrak{B} \sqsubseteq (NC^{\alpha}int(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)))^c$. Hence $NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>) \sqsubseteq (NC^{\alpha}int(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)))^c \Rightarrow NC^{\alpha}int(NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>))$

$\sqsubseteq \left(NC^{\alpha}int(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>))\right)^c \Rightarrow NC^{\alpha}int(NC^{\alpha}cl(<\{u\},\emptyset,\emptyset>)) \sqcap NC^{\alpha}int(NC^{\alpha}cl(<\{v\},\emptyset,\emptyset>)) = \emptyset_{\mathcal{N}}$. The second case is similar. ▪

Remark 3.5:

(i) Every NC-T_0-space is a NC$^{\alpha}$-T_0-space and NC$^{S\alpha}$-T_0-space.

(ii) Every NC$^{\alpha}$-T_0-space is a NC$^{S\alpha}$-T_0-space.

Remark 3.6:

(i) NC$^{\alpha}$-T_0 (NC$^{S\alpha}$-T_0 respectively) property is a NC$^{\alpha^*}$ (NC$^{S\alpha^*}$ respectively) topological property.

(ii) NC$^{\alpha}$-T_0 (NC$^{S\alpha}$-T_0 respectively) property is a NC$^{\alpha^{**}}$ (NC$^{S\alpha^{**}}$ respectively) topological property.

(iii) NC$^{\alpha}$-T_0 is a NC$^{\alpha}$-hereditary property.

Proposition 3.7:

(i) Let $(\mathcal{S},\zeta)$ and $(\mathcal{I},\eta)$ be NC$^{\alpha}$-T_0-spaces if and only if $\mathcal{S} \times \mathcal{I}$ is a NC$^{\alpha}$-T_0-space.

(ii) If $(\mathcal{S},\zeta)$ and $(\mathcal{I},\eta)$ are NC$^{S\alpha}$-T_0-spaces, then $\mathcal{S} \times \mathcal{I}$ is a NC$^{S\alpha}$-T_0-space.

Proof:

(i) $\Rightarrow$ Let $\mathcal{S}$ and $\mathcal{I}$ be NC$^{\alpha}$-T_0-spaces. Let $(u_1,v_1) \neq (u_2,v_2)$ in $\mathcal{S} \times \mathcal{I}$. Then $u_1 \neq u_2$ in $\mathcal{S} \Rightarrow$ there exists $\mathfrak{B}_1 \in NC^{\alpha}O(\mathcal{S})$ such that $u_1 \in \mathfrak{B}_1, u_2 \notin \mathfrak{B}_1$ or $u_1 \notin \mathfrak{B}_1, u_2 \in \mathfrak{B}_1$.

Also $v_1 \neq v_2$ in $\mathcal{I} \Rightarrow$ there exists $\mathfrak{B}_2 \in NC^{\alpha}O(\mathcal{I})$ such that $v_1 \in \mathfrak{B}_2, v_2 \notin \mathfrak{B}_2$ or $v_1 \notin \mathfrak{B}_2, v_2 \in \mathfrak{B}_2$.

Then $(u_1,v_1) \in \mathfrak{B}_1 \times \mathfrak{B}_2, (u_2,v_2) \notin \mathfrak{B}_1 \times \mathfrak{B}_2$ or $(u_1,v_1) \notin \mathfrak{B}_1 \times \mathfrak{B}_2, (u_2,v_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$.

But $\mathfrak{B}_1 \times \mathfrak{B}_2 \in NC^{\alpha}O(\mathcal{S} \times \mathcal{I})$ (since by theorem (2.6)). Hence $\mathcal{S} \times \mathcal{I}$ is a NC$^{\alpha}$-T_0-space.

$\Leftarrow$ Let $\mathcal{S}\times\mathcal{J}$ be a NC^{α}-T_0-space, to prove that $\mathcal{S}$ and $\mathcal{J}$ are NC^{α}-T_0-spaces. Since $\mathcal{S}\times\mathcal{J}$ is a NC^{α}-T_0-space, then $\mathcal{S}\times<\{\mathcal{v}_0\},\emptyset,\emptyset>$ and $<\{\mathcal{u}_0\},\emptyset,\emptyset>\times\mathcal{J}$ are NC^{α}-T_0-spaces (since NC^{α}-T_0 property is a NC^{α}-hereditary). Hence $\mathcal{S}$ and $\mathcal{J}$ are NC^{α}-T_0-spaces. The proof (ii) is evident for others. ▪

Definition 3.8:

(i) A NCTS $(\mathcal{S},\zeta)$ is said to be a NC^{α}-T_1-space if for each pair of distinct NC- points $\mathcal{u}$ and $\mathcal{v}$ of $\mathcal{S}$, there exist two NC^{α}-OSs $\mathfrak{B}$ and $\mathfrak{D}$ containing $\mathcal{u}$ and $\mathcal{v}$ respectively, such that $\mathcal{u}\in\mathfrak{B}$, $\mathcal{v}\in\mathfrak{D}$.

(ii) A NCTS $(\mathcal{S},\zeta)$ is said to be a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space if for each pair of distinct NC-points $\mathcal{u}$ and $\mathcal{v}$ of $\mathcal{S}$, there exist two $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs $\mathfrak{B}$ and $\mathfrak{D}$ containing $\mathcal{u}$ and $\mathcal{v}$ respectively, such that $\mathcal{u}\in\mathfrak{B}$, $\mathcal{v}\in\mathfrak{D}$.

Proposition 3.9:

A NCTS $(\mathcal{S},\zeta)$ is NC^{α}-T_1-space ($\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space respectively) if and only if $<\{\mathcal{u}\},\emptyset,\emptyset>$ is a NC^{α}-CS ($\mathrm{NC}^{\mathrm{S}\alpha}$-CS respectively), $\forall\mathcal{u}\in\mathcal{S}$.

Proof:

$\Rightarrow$ Let $\mathcal{S}$ be a NC^{α}-T_1-space. Let $\mathcal{w}\in\mathcal{S}$, to prove that $<\{\mathcal{w}\},\emptyset,\emptyset>$ is a NC^{α}-CS. Let $\mathcal{u}\in(<\{\mathcal{w}\},\emptyset,\emptyset>)^c\Rightarrow\mathcal{u}\neq\mathcal{w}$ in $\mathcal{S}$. Hence there exists a NC^{α}-OS $\mathfrak{B}$ such that $\mathcal{u}\in\mathfrak{B},\mathcal{w}\notin\mathfrak{B}$ or $\mathcal{u}\notin\mathfrak{B},\mathcal{w}\in\mathfrak{B}$. If $\mathcal{u}\in\mathfrak{B},\mathcal{w}\notin\mathfrak{B}\Rightarrow\mathcal{u}\in\mathfrak{B}\sqsubseteq(<\{\mathcal{w}\},\emptyset,\emptyset>)^c\Rightarrow(<\{\mathcal{w}\},\emptyset,\emptyset>)^c$ is a NC^{α}-OS $\Rightarrow<\{\mathcal{w}\},\emptyset,\emptyset>$ is a NC^{α}-CS.

$\Leftarrow$ Let $<\{\mathcal{w}\},\emptyset,\emptyset>$ be a NC^{α}-CS, $\forall\mathcal{w}\in\mathcal{S}$, to prove that $\mathcal{S}$ is a NC^{α}-T_1-space. Let $\mathcal{u}\neq\mathcal{v}$ in $\mathcal{S}$. Hence $<\{\mathcal{u}\},\emptyset,\emptyset>,<\{\mathcal{v}\},\emptyset,\emptyset>$ are NC^{α}-CSs $\Rightarrow(<\{\mathcal{u}\},\emptyset,\emptyset>)^c,(<\{\mathcal{v}\},\emptyset,\emptyset>)^c$ are NC^{α}-OSs and $\mathcal{v}\in(<\{\mathcal{u}\},\emptyset,\emptyset>)^c,\mathcal{u}\notin(<\{\mathcal{u}\},\emptyset,\emptyset>)^c,\mathcal{u}\in(<\{\mathcal{v}\},\emptyset,\emptyset>)^c,\mathcal{v}\notin(<\{\mathcal{v}\},\emptyset,\emptyset>)^c$. Therefore $\mathcal{S}$ is a NC^{α}-T_1-space. The second case is similar. ▪

Remark 3.10:

(i) Every NC-T_1-space is a NC^{α}-T_1-space and $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space.

(ii) Every NC^{α}-T_1-space is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space.

(iii) Every NC^{α}-T_1-space is a NC^{α}-T_0-space.

(iv) Every $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_0$-space.

Remark 3.11:

(i) NC^{α}-T_1 ($\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$ respectively) property is a NC^{α^*} ($\mathrm{NC}^{\mathrm{S}\alpha^*}$ respectively) topological property.

(ii) NC^{α}-T_1 ($\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$ respectively) property is a $\mathrm{NC}^{\alpha^{**}}$ ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$ respectively) topological property.

(iii) NC^{α}-T_1 property is a NC^{α}-hereditary property.

Proposition 3.12:

(i) Let $\mathcal{S}$ and $\mathcal{J}$ be NC^{α}-T_1-spaces if and only if $\mathcal{S}\times\mathcal{J}$ is a NC^{α}-T_1-space.

(ii) If $\mathcal{S}$ and $\mathcal{J}$ are $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-spaces, then $\mathcal{S}\times\mathcal{J}$ is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space.

Proof:

The proof of this is similar to that of proposition (3.7). ▪

Definition 3.13:

(i) A NCTS $(\mathcal{S},\zeta)$ is said to be a NC^{α}-T_2-space if for each pair of distinct NC-points $\mathcal{u}$ and $\mathcal{v}$ in $\mathcal{S}$, there exist two NC^{α}-OSs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that $\mathcal{u}\in\mathfrak{D}_1$, $\mathcal{v}\in\mathfrak{D}_2$ and $\mathfrak{D}_1\sqcap\mathfrak{D}_2=\emptyset_{\mathcal{N}}$.

(ii) A NCTS $(\mathcal{S},\zeta)$ is said to be a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_2$-space if for each pair of distinct NC-points $\mathcal{u}$ and $\mathcal{v}$ in $\mathcal{S}$, there exist two $\mathrm{NC}^{\mathrm{S}\alpha}$-OSs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that $\mathcal{u}\in\mathfrak{D}_1$, $\mathcal{v}\in\mathfrak{D}_2$ and $\mathfrak{D}_1\sqcap\mathfrak{D}_2=\emptyset_{\mathcal{N}}$.

Proposition 3.14:

If $(\mathcal{S},\zeta)$ is a NC^{α}-T_2-space ($\text{NC}^{S\alpha}$-T_2-space respectively), then $\mathfrak{B} = \{(u,v): u = v, u,v \in \mathcal{S}\}$ is a NC^{α}-CS ($\text{NC}^{S\alpha}$-CS respectively).

Proof:

Let $\mathcal{S}$ be a NC^{α}-T_2-space, to prove that $\mathfrak{B}$ is a NC^{α}-CS. Let $(u,v) \in \mathfrak{B}^c = \mathcal{S} \times \mathcal{S} - \mathfrak{B}$. Hence $u \neq v$ in $\mathcal{S}$ $\Longrightarrow$ there exist $\mathfrak{D}_1, \mathfrak{D}_2 \in NC^{\alpha}O(\mathcal{S})$ such that $u \in \mathfrak{D}_1$, $v \in \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$ (since $\mathcal{S}$ is a NC^{α}-T_2-space). Hence $\mathfrak{D}_1 \times \mathfrak{D}_2 \in NC^{\alpha}O(\mathcal{S} \times \mathcal{S})$ by theorem (2.7) $(u,v) \in \mathfrak{D}_1 \times \mathfrak{D}_2 \sqsubseteq \mathfrak{B}^c$, hence $\mathfrak{B}^c$ is a NC^{α}-OS. Therefore $\mathfrak{B}$ is a NC^{α}-CS. The second case is similar. ▪

Remark 3.15:

(i) Every NC-T_2-space is a NC^{α}-T_2-space and $\text{NC}^{S\alpha}$-T_2-space.

(ii) Every NC^{α}-T_2-space is a $\text{NC}^{S\alpha}$-T_2-space.

(iii) Every NC^{α}-T_2-space is a NC^{α}-T_1-space.

(iv) Every $\text{NC}^{S\alpha}$-T_2-space is a $\text{NC}^{S\alpha}$-T_1-space.

Remark 3.16:

(i) NC^{α}-T_2 ($\text{NC}^{S\alpha}$-T_2 respectively) property is a NC^{α^*} ($\text{NC}^{S\alpha^*}$ respectively) topological property.

(ii) NC^{α}-T_2 ($\text{NC}^{S\alpha}$-T_2respectively) property is a $\text{NC}^{\alpha^{**}}$ ($\text{NC}^{S\alpha^{**}}$ respectively) topological property.

(iii) NC^{α}-T_2 property is a NC^{α}-hereditary property.

Proposition 3.17:

(i) Let $\mathcal{S}$ and $\mathcal{I}$ be NC^{α}-T_2-spaces if and only if $\mathcal{S} \times \mathcal{I}$ is a NC^{α}-T_2-space.

(ii) If $\mathcal{S}$ and $\mathcal{I}$ are $\text{NC}^{S\alpha}$-T_2-spaces, then $\mathcal{S} \times \mathcal{I}$ is a $\text{NC}^{S\alpha}$-T_2-space.

Proof:

The proof of this is similar to that of proposition (3.12). ▪

Proposition 3.18:

(i) If $\rho, \mu: \mathcal{S} \longrightarrow \mathcal{I}$ are NC^{α^*}-CF and $\mathcal{I}$ is a NC^{α}-T_2-space, then the NC-set $\mathfrak{B} = \{u: u \in \mathcal{S}, \rho(u) = \mu(u)\}$ is a NC^{α}-CS.

(ii) If $\rho, \mu: \mathcal{S} \longrightarrow \mathcal{I}$ are NC^{α}-CF and $\mathcal{I}$ is a NC-T_2-space, then the NC-set $\mathfrak{B} = \{u: u \in \mathcal{S}, \rho(u) = \mu(u)\}$ is a NC^{α}-CS.

Proof:

(i) If $u \notin \mathfrak{B} \Longrightarrow u \in \mathfrak{B}^c \Longrightarrow \rho(u) \neq \mu(u)$ in $\mathcal{I}$. Hence there exist two NC^{α}-OSs $\mathfrak{D}_1$ and $\mathfrak{D}_2$ in $\mathcal{I}$ such that $\rho(u) \in \mathfrak{D}_1$, $\mu(u) \in \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$ (since $\mathcal{I}$ is a NC^{α}-T_2-space). But $\rho^{-1}(\mathfrak{D}_1), \mu^{-1}(\mathfrak{D}_2) \in NC^{\alpha}O(\mathcal{S})$ (since ρ, μ are NC^{α^*}-CF). Therefore, $u \in \rho^{-1}(\mathfrak{D}_1)$ and $u \in \mu^{-1}(\mathfrak{D}_2)$. Hence $u \in \rho^{-1}(\mathfrak{D}_1) \sqcap \mu^{-1}(\mathfrak{D}_2)$. Let $\mathfrak{U} = \rho^{-1}(\mathfrak{D}_1) \sqcap \mu^{-1}(\mathfrak{D}_2) \in NC^{\alpha}O(\mathcal{S})$. To prove $\mathfrak{U} \sqsubseteq \mathfrak{B}^c$, i.e., $\mathfrak{U} \sqcap \mathfrak{B} = \emptyset_{\mathcal{N}}$.
Suppose that $\mathfrak{U} \sqcap \mathfrak{B} \neq \emptyset_{\mathcal{N}} \Longrightarrow \exists v \in \mathfrak{U} \sqcap \mathfrak{B} \Longrightarrow v \in \mathfrak{U}$ and $v \in \mathfrak{B}$, i.e., $v \in \rho^{-1}(\mathfrak{D}_1)$ and $v \in \mu^{-1}(\mathfrak{D}_2)$ and $v \in \mathfrak{B}$. Hence $\rho(v) \in \mathfrak{D}_1$, $\mu(v) \in \mathfrak{D}_2$ and $v \in \mathfrak{B}$. Therefore $\rho(v) = \mu(v)$ (since $v \in \mathfrak{B}$). Hence $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 \neq \emptyset_{\mathcal{N}}$ which is a contradiction. Therefore $\mathfrak{U} \sqsubseteq \mathfrak{B}^c \Longrightarrow \mathfrak{B}^c \in NC^{\alpha}O(\mathcal{S}) \Longrightarrow \mathfrak{B}$ is a NC^{α}-CS. The proof (ii) is evident for others. ▪

4. Some New Concepts of Weakly Neutrosophic Crisp Regularity

Definition 4.1:

Let $(\mathcal{S},\zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) NC^{α}-regular ($\text{NC}^{S\alpha}$-regular respectively) if every $u \in \mathcal{S}$ and every Q NC-CS such that $u \notin Q$, there exist two NC^{α}-OSs ($NC^{S\alpha}$-OSs respectively) $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}$, $\mathcal{Q} \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

(ii) NC^{α^*}-regular ($NC^{S\alpha^*}$-regular respectively) if every $u \in \mathcal{S}$ and every $\mathcal{Q}$ NC^{α}-CS ($NC^{S\alpha}$-CS respectively) such that $u \notin \mathcal{Q}$, there exist two NC^{α}-OSs ($NC^{S\alpha}$-OSs respectively) $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}$, $\mathcal{Q} \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

(iii) $NC^{\alpha^{**}}$-regular ($NC^{S\alpha^{**}}$-regular respectively) if every $u \in \mathcal{S}$ and every $\mathcal{Q}$ NC^{α}-CS ($NC^{S\alpha}$-CS respectively) such that $u \notin \mathcal{Q}$, there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}$, $\mathcal{Q} \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

Remark 4.2:

The following diagram shows the relation between the different types of weakly NC-regular and weakly NC^{α}-regular ($NC^{S\alpha}$-regular respectively) spaces:

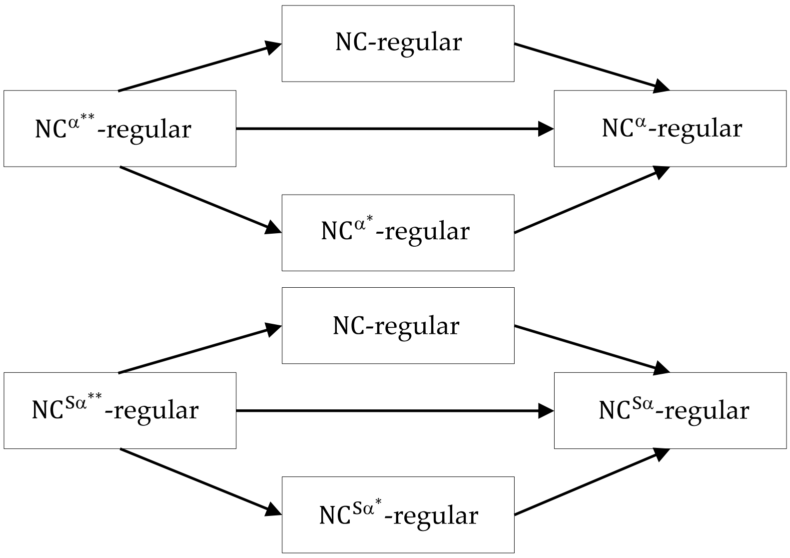


Fig. 4.1

Theorem 4.3:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a NC^{α}-regular if and only if for each $\mathfrak{B}$ NC-OS containing u, there exists $\mathfrak{D}$ NC^{α}-OS containing u such that $u \in \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(ii) $\mathcal{S}$ is a NC^{α^*}-regular if and only if for each $\mathfrak{B}$ NC^{α}-OS contains u, there exists $\mathfrak{D}$ NC^{α}-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(iii) $\mathcal{S}$ is a $NC^{\alpha^{**}}$-regular if and only if for each $\mathfrak{B}$ NC^{α}-OS contains u, there exists $\mathfrak{D}$ NC-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

Proof:

(i) $\Rightarrow$ Let $\mathcal{S}$ be a NC^{α}-regular space and let $\mathfrak{B}$ be a NC-OS containing u. Hence $\mathfrak{B}^c$ is a NC-CS and $u \notin \mathfrak{B}^c$. Then there exist $\mathfrak{D}_1, \mathfrak{D}_2$ NC^{α}-OSs in $\mathcal{S}$ such that $u \in \mathfrak{D}_1$, $\mathfrak{B}^c \sqsubseteq \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$ (since $\mathcal{S}$ is a NC^{α}-regular space). Hence $u \in \mathfrak{D}_1 \sqsubseteq {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B}$ (since $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}} \Longrightarrow \mathfrak{D}_1 \sqsubseteq {\mathfrak{D}_2}^c$). Therefore $u \in \mathfrak{D}_1 \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq NC^{\alpha}cl({\mathfrak{D}_2}^c) \sqsubseteq NC^{\alpha}cl(\mathfrak{B})$. Therefore $u \in \mathfrak{D}_1 \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B}$. The implies that $u \in \mathfrak{D}_1 \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq \mathfrak{B}$, where $\mathfrak{D}_1$ is a NC^{α}-OS.

$\Leftarrow$ Let $\mathcal{Q}$ be a NC-CS such that $u \notin \mathcal{Q} \Longrightarrow \mathcal{Q}^c$ is a NC-OS contains u. Hence there exists $\mathfrak{D}$ NC^α-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NC^\alpha cl(\mathfrak{D}) \sqsubseteq \mathcal{Q}^c$. We get $\mathcal{Q} \sqsubseteq (NC^\alpha cl(\mathfrak{D}))^c$, so it is $(NC^\alpha cl(\mathfrak{D}))^c$ is a NC^α-OS and contains $\mathcal{Q}$. Now, to prove $\mathfrak{D} \sqcap (NC^\alpha cl(\mathfrak{D}))^c = \emptyset_{\mathcal{N}}$. Since $\mathfrak{D} \sqsubseteq NC^\alpha cl(\mathfrak{D})$, but $NC^\alpha cl(\mathfrak{D}) \sqcap (NC^\alpha cl(\mathfrak{D}))^c = \emptyset_{\mathcal{N}} \Longrightarrow \mathfrak{D} \sqcap (NC^\alpha cl(\mathfrak{D}))^c = \emptyset_{\mathcal{N}}$. Hence $\mathcal{S}$ is a NC^α-regular space. The proofs (ii), (iii) are evident for others. ▪

Theorem 4.4:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a $NC^{S\alpha^*}$-regular if and only if for each $\mathfrak{B}$ $NC^{S\alpha}$-OS contains u, there exists $\mathfrak{D}$ $NC^{S\alpha}$-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NC^{S\alpha} cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(ii) $\mathcal{S}$ is a $NC^{S\alpha^{**}}$-regular if and only if for each $\mathfrak{B}$ $NC^{S\alpha}$-OS contains u, there exists $\mathfrak{D}$ NC-OS contains u such that $u \in \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

Proof:

The proof of this is similar to that of theorem (4.3). ▪

Theorem 4.5:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a NC^{α^*}-regular if and only if $u \notin \mathcal{Q}$ where $\mathcal{Q}$ is a NC^α-CS, there exist two NC^α-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}, \mathcal{Q} \sqsubseteq \mathfrak{D}$ and $NC^\alpha cl(\mathfrak{B}) \sqcap NC^\alpha cl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$.

(ii) $\mathcal{S}$ is a $NC^{\alpha^{**}}$-regular if and only if for each $\mathcal{Q}$ NC^α-CS, such that $u \notin \mathcal{Q}$, there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}, \mathcal{Q} \sqsubseteq \mathfrak{D}$ and $NCcl(\mathfrak{B}) \sqcap NCcl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$.

Proof:

(i) Let $\mathcal{S}$ be a NC^{α^*}-regular space and let $\mathcal{Q}$ be a NC^α-CS, such that $u \notin \mathcal{Q}$. Then there exist two NC^α-OSs $\mathcal{U}$ and $\mathcal{V}$ such that $u \in \mathcal{U}, \mathcal{Q} \sqsubseteq \mathcal{V}$ and $\mathcal{U} \sqcap \mathcal{V} = \emptyset_{\mathcal{N}}$. Therefore $\mathcal{U}$ is a NC^α-OS containing u in $\mathcal{S}$, where $\mathcal{S}$ is a NC^{α^*}-regular space. Then there exists $\mathfrak{B}$ NC^α-OS containing u such that $u \in \mathfrak{B} \sqsubseteq NC^\alpha cl(\mathfrak{B}) \sqsubseteq \mathcal{U}$ (since by theorem (4.3) (ii)). Hence $NC^\alpha cl(\mathfrak{B}) \sqsubseteq \mathcal{U}$. Also, $\mathcal{Q} \sqsubseteq \mathcal{V} \sqsubseteq NC^\alpha cl(\mathcal{V})$, but $NC^\alpha cl(\mathcal{V}) \sqsubseteq (NC^\alpha cl(\mathcal{U}))^c$ (since $\mathcal{U} \sqcap \mathcal{V} = \emptyset_{\mathcal{N}} \Longrightarrow \mathcal{V} \sqsubseteq \mathcal{U}^c \Longrightarrow NC^\alpha cl(\mathcal{V}) \sqsubseteq NC^\alpha cl(\mathcal{U}^c)$). Hence $\mathcal{Q} \sqsubseteq \mathcal{V} \sqsubseteq NC^\alpha cl(\mathcal{V}) \sqsubseteq NC^\alpha cl(\mathcal{U}^c) = \mathcal{U}^c$ (since $\mathcal{U}^c$ is a NC^α-CS). Suppose that $\mathcal{V} = \mathfrak{D}$, hence $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^\alpha cl(\mathfrak{D}) \sqsubseteq \mathcal{U}^c \Longrightarrow NC^\alpha cl(\mathfrak{D}) \sqsubseteq \mathcal{U}^c$. Since $\mathcal{U} \sqcap \mathcal{U}^c = \emptyset_{\mathcal{N}}$, hence $NC^\alpha cl(\mathfrak{B}) \sqcap NC^\alpha cl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$ (since $NC^\alpha cl(\mathfrak{B}) \sqsubseteq \mathcal{U}$ and $NC^\alpha cl(\mathfrak{D}) \sqsubseteq \mathcal{U}^c$). The other side is clear. The proof (ii) is evident for others. ▪

Theorem 4.6:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a $NC^{S\alpha^*}$-regular if and only if $u \notin \mathcal{Q}$, where $\mathcal{Q}$ is a $NC^{S\alpha}$-CS, there exist two $NC^{S\alpha}$-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}, \mathcal{Q} \sqsubseteq \mathfrak{D}$ and $NC^{S\alpha} cl(\mathfrak{B}) \sqcap NC^{S\alpha} cl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$.

(ii) $\mathcal{S}$ is a $NC^{S\alpha^{**}}$-regular if and only if $u \notin \mathcal{Q}$, where $\mathcal{Q}$ is a $NC^{S\alpha}$-CS, there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $u \in \mathfrak{B}, \mathcal{Q} \sqsubseteq \mathfrak{D}$ and $NCcl(\mathfrak{B}) \sqcap NCcl(\mathfrak{D}) = \emptyset_{\mathcal{N}}$.

Proof:

The proof of this is similar to that of theorem (4.5). ▪

Remark 4.7:

(i) NC^α-regular property is a $NC^{\alpha^{**}}$-topological property.

(ii) NC^{α^*}-regular property is a NC^{α^*}-topological property.

(iii) $NC^{\alpha^{**}}$-regular property is a $NC^{\alpha^{**}}$-topological property.

(iv) $NC^{S\alpha}$-regular property is a $NC^{S\alpha^{**}}$-topological property.

(v) $NC^{S\alpha^*}$-regular property is a $NC^{S\alpha^*}$-topological property.

(vi) $NC^{S\alpha^{**}}$-regular property is a $NC^{S\alpha^{**}}$-topological property.

Proposition 4.8:

(i) If $\mathcal{S} \times \mathcal{J}$ is a $NC^{\alpha^{**}}$-regular, then both $\mathcal{S}$ and $\mathcal{J}$ are $NC^{\alpha^{**}}$-regular spaces.

(ii) If $\mathcal{S} \times \mathcal{J}$ is a $NC^{S\alpha^{**}}$-regular, then both $\mathcal{S}$ and $\mathcal{J}$ are $NC^{S\alpha^{**}}$-regular spaces.

Proof:

(i) Suppose that $\mathcal{S} \times \mathcal{J}$ is a $NC^{\alpha^{**}}$-regular, to prove that $\mathcal{S}$ and $\mathcal{J}$ are $NC^{\alpha^{**}}$-regular spaces.

Let $\mathcal{U}$ and $\mathcal{V}$ be two NC^{α}-OSs in $\mathcal{S}$ and $\mathcal{J}$ containing u and v respectively. Hence $(u, v) \in \mathcal{U} \times \mathcal{V}$ where $\mathcal{U} \times \mathcal{V}$ is a NC^{α}-OS in $\mathcal{S} \times \mathcal{J}$ (by theorem (2.7)). Hence there exists NC-OS $\mathcal{K}$ in $\mathcal{S} \times \mathcal{J}$ such that $(u, v) \in \mathcal{K} \sqsubseteq NCcl(\mathcal{K}) \sqsubseteq \mathcal{U} \times \mathcal{V}$ (since $\mathcal{S} \times \mathcal{J}$ is a $NC^{\alpha^{**}}$-regular). Then there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ in $\mathcal{S}$ and $\mathcal{J}$ such that $(u, v) \in \mathfrak{B} \times \mathfrak{D} \sqsubseteq NCcl(\mathfrak{B} \times \mathfrak{D}) = NCcl(\mathfrak{B}) \times NCcl(\mathfrak{D}) \sqsubseteq \mathcal{U} \times \mathcal{V}$. Hence $u \in \mathfrak{B} \sqsubseteq NCcl(\mathfrak{B}) \sqsubseteq \mathcal{U} \Longrightarrow \mathcal{S}$ is a $NC^{\alpha^{**}}$-regular space. Also, $v \in \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathcal{V} \Longrightarrow \mathcal{J}$ is a $NC^{\alpha^{**}}$-regular space. The proof (ii) is evident for others. ▪

Theorem 4.9:

If $(\mathcal{S}, \zeta)$ is a $NC^{\alpha^{*}}$-regular ($NC^{\alpha^{**}}$-regular respectively), then $\zeta = NC^{\alpha}O(\mathcal{S})$.

Proof:

It is clear that $\zeta \sqsubseteq NC^{\alpha}O(\mathcal{S})$. Let $\mathfrak{B}$ be a NC^{α}-OS in $\mathcal{S}$ containing u. Then there exists a NC^{α}-OS $\mathfrak{D}$ containing u such that $u \in \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$ ($\mathcal{S}$ is a $NC^{\alpha^{*}}$-regular). Therefore $NC^{\alpha}int(\mathfrak{D}) \sqsubseteq NC^{\alpha}int(NC^{\alpha}cl(\mathfrak{D})) \sqsubseteq \mathfrak{B}$. Thus $u \in \mathfrak{D} \sqsubseteq NCcl(NCint(\mathfrak{D})) \sqsubseteq \mathfrak{B}$ (since by proposition (2.9)). Hence $\mathfrak{B}$ is a NC-OS $\Longrightarrow NC^{\alpha}O(\mathcal{S}) \sqsubseteq \zeta$. Therefore $\zeta = NC^{\alpha}O(\mathcal{S})$. ▪

Proposition 4.10:

(i) If $\rho: \mathcal{S} \longrightarrow \mathcal{J}$ is a NC^{α}-CF and $\mathcal{S}$ is a $NC^{\alpha^{*}}$-regular, then ρ is a NC-CF.

(ii) If $\rho: \mathcal{S} \longrightarrow \mathcal{J}$ is a NC^{α}-CF and $\mathcal{J}$ is a $NC^{\alpha^{*}}$-regular, then ρ is a $NC^{\alpha^{*}}$-CF.

(iii) If $\rho: \mathcal{S} \longrightarrow \mathcal{J}$ is a $NC^{\alpha^{*}}$-CF and $\mathcal{S}$ is a $NC^{\alpha^{*}}$-regular, then ρ is a $NC^{\alpha^{**}}$-CF.

Proof:

(i) Let $\rho: \mathcal{S} \longrightarrow \mathcal{J}$ be a NC^{α}-CF, to prove that ρ is a NC-CF.

Let $\mathfrak{B}$ be a NC-OS in $\mathcal{J}$, then $\rho^{-1}(\mathfrak{B})$ is a NC^{α}-OS in $\mathcal{S}$ (since ρ is a NC^{α}-CF). But $\mathcal{S}$ is a $NC^{\alpha^{*}}$-regular space (by hypothesis). Hence $\rho^{-1}(\mathfrak{B})$ is a NC^{α}-OS in $\mathcal{S}$ (since by theorem (4.9)). Therefore ρ is a NC-CF. The proofs (ii), (iii) are evident for others. ▪

Definition 4.11:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) NC^{α}-T_3-space if $\mathcal{S}$ is a NC^{α}-T_1-space and NC^{α}-regular space.

(ii) $NC^{\alpha^{*}}$-T_3-space if $\mathcal{S}$ is NC^{α}-T_1-space and $NC^{\alpha^{*}}$-regular space.

(iii) $NC^{\alpha^{**}}$-T_3-space if $\mathcal{S}$ is NC^{α}-T_1-space and $NC^{\alpha^{**}}$-regular space.

Definition 4.12:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) $NC^{S\alpha}$-T_3-space if $\mathcal{S}$ is a $NC^{S\alpha}$-T_1-space and $NC^{S\alpha}$-regular space.

(ii) $NC^{S\alpha^{*}}$-T_3-space if $\mathcal{S}$ is $NC^{S\alpha}$-T_1-space and $NC^{S\alpha^{*}}$-regular space.

(iii) $NC^{S\alpha^{**}}$-T_3-space if $\mathcal{S}$ is $NC^{S\alpha}$-T_1-space and $NC^{S\alpha^{**}}$-regular space.

Remark 4.13:

(i) $NC^{\alpha^{*}}$-T_3 ($NC^{S\alpha^{*}}$-T_3 respectively) property is a $NC^{\alpha^{*}}$ ($NC^{S\alpha^{*}}$ respectively) topological property.

(ii) $NC^{\alpha^{**}}$-T_3 ($NC^{S\alpha^{**}}$-T_3 respectively) property is a $NC^{\alpha^{**}}$ ($NC^{S\alpha^{**}}$ respectively) topological property.

Remark 4.14:

(i) Every NC-T_3-space is a NC^{α}-T_3-space and $NC^{S\alpha}$-T_3-space.

(ii) Every NC^{α}-T_3-space is a $NC^{S\alpha}$-T_3-space.

(iii) Every $NC^{\alpha^{**}}$-T_3-space ($NC^{S\alpha^{**}}$-T_3-space respectively) is a $NC^{\alpha^{*}}$-T_3-space ($NC^{S\alpha^{*}}$-T_3-space, respectively).

(iv) Every $NC^{\alpha^{*}}$-T_3-space ($NC^{S\alpha^{*}}$-T_3-space respectively) is a NC^{α}-T_2-space ($NC^{S\alpha}$-T_2-space, respectively).

Proposition 4.15:

$\mathcal{S} \times \mathcal{I}$ is a $NC^{\alpha^{**}}$-T_3-space if and only if both $\mathcal{S}$ and $\mathcal{I}$ are $NC^{\alpha^{**}}$-T_3-spaces.

Proof:

Follow directly from proposition (3.12) part (i) and proposition (4.8) part (i). ▪

5. Some New Concepts of Weakly Neutrosophic Crisp Normality

Definition 5.1:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) NC^{α}-normal ($NC^{S\alpha}$-normal respectively) if for every two NC-CSs $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$There exist two NC^{α}-OSs ($NC^{S\alpha}$-OSs respectively) $\mathfrak{B}$ and $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{B}$ and $\mathcal{Q}_2 \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

(ii) $NC^{\alpha^{*}}$-normal ($NC^{S\alpha^{*}}$-normal respectively) if for every two NC^{α}-CSs ($NC^{S\alpha}$-CSs respectively) $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$There exist two NC^{α}-OSs ($NC^{S\alpha}$-OSs respectively) $\mathfrak{B}$ and $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{B}$ and $\mathcal{Q}_2 \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

(iii) $NC^{\alpha^{**}}$-normal ($NC^{S\alpha^{**}}$-normal respectively) if for every two NC^{α}-CSs ($NC^{S\alpha}$-CSs respectively) $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$, there exist two NC-OSs $\mathfrak{B}$ and $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{B}$ and $\mathcal{Q}_2 \sqsubseteq \mathfrak{D}$ and $\mathfrak{B} \sqcap \mathfrak{D} = \emptyset_{\mathcal{N}}$.

Remark 5.2:

The following diagram shows the relation between the different types of weakly NC-normal and weakly NC^{α}-normal ($NC^{S\alpha}$-normal respectively) spaces:

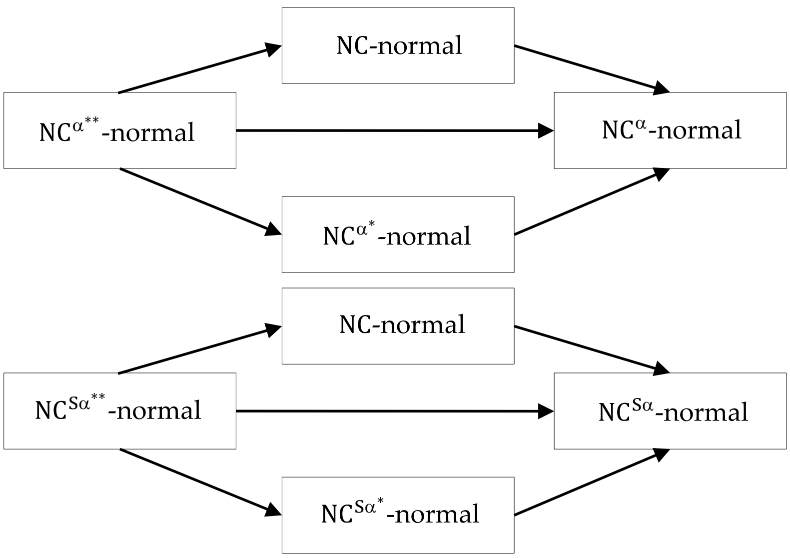


Fig. 5.1

Theorem 5.3:

Let $(\mathcal{S},\zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a NC^{α}-normal space if and only if for every NC-CS $\mathcal{Q}$ and every NC-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC^{α}-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(ii) $\mathcal{S}$ is a NC^{α^*}-normal space if and only if for every NC^{α}-CS $\mathcal{Q}$ and every NC^{α}-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC^{α}-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(iii) $\mathcal{S}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space if and only if for every NC^{α}-CS $\mathcal{Q}$ and every NC^{α}-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

Proof:

(i) $\Rightarrow$ Let $\mathcal{S}$ be a NC^{α}-normal space. Let $\mathcal{Q} \sqsubseteq \mathfrak{B}$, where $\mathcal{Q}$ is a NC-CS and $\mathfrak{B}$ is a NC-OS $\Longrightarrow \mathcal{Q} \sqcap \mathfrak{B}^c = \emptyset_{\mathcal{N}}$, where $\mathfrak{B}^c$ is a NC-CS. Hence there exist two NC^{α}-OSs $\mathfrak{D}_1, \mathfrak{D}_2$ such that $\mathcal{Q} \sqsubseteq \mathfrak{D}_1$ and $\mathfrak{B}^c \sqsubseteq \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$ (since $\mathcal{S}$ is a NC^{α}-normal space). Therefore $\mathcal{Q} \sqsubseteq \mathfrak{D}_1 \sqsubseteq {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B} \Longrightarrow NC^{\alpha}cl(\mathcal{Q}) \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq NC^{\alpha}cl({\mathfrak{D}_2}^c) = {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B}$. Hence $\mathcal{Q} \sqsubseteq \mathfrak{D}_1 \sqsubseteq NC^{\alpha}cl(\mathfrak{D}_1) \sqsubseteq \mathfrak{B}$, where $\mathfrak{D}_1$ is a NC^{α}-OS in $\mathcal{S}$.

$\Leftarrow$ To prove $\mathcal{S}$ is a NC^{α}-normal space. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be NC-CSs in $\mathcal{S}$ such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$. Hence $\mathcal{Q}_1 \sqsubseteq {\mathcal{Q}_2}^c$, where ${\mathcal{Q}_2}^c$ is a NC-OS. Then there exists a NC^{α}-OS $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D}) \sqsubseteq {\mathcal{Q}_2}^c$ (by hypothesis). Hence $\mathcal{Q}_1 \sqsubseteq \mathfrak{D}$, $\mathcal{Q}_2 \sqsubseteq (NC^{\alpha}cl(\mathfrak{D}))^c$. On the other hand $NC^{\alpha}cl(\mathfrak{D}) \sqcap \left(NC^{\alpha}cl(\mathfrak{D})\right)^c = \emptyset_{\mathcal{N}}$. Hence $\mathfrak{D} \sqcap \left(NC^{\alpha}cl(\mathfrak{D})\right)^c = \emptyset_{\mathcal{N}}$ (since $\mathfrak{D} \sqsubseteq NC^{\alpha}cl(\mathfrak{D})$). Therefore $\mathcal{S}$ is a NC^{α}-normal space. The proofs (ii), (iii) are evident for others. ▪

Theorem 5.4:

Let $(\mathcal{S},\zeta)$ be a NCTS, then:

(i) $\mathcal{S}$ is a $\mathrm{NC}^{S\alpha}$-normal space if and only if for every NC-CS $\mathcal{Q}$ and every NC-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists $\mathrm{NC}^{S\alpha}$-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(ii) $\mathcal{S}$ is a $\mathrm{NC}^{S\alpha^*}$-normal space if and only if for every $\mathrm{NC}^{S\alpha}$-CS $\mathcal{Q}$ and every $\mathfrak{B}$ $\mathrm{NC}^{S\alpha}$-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC^{α}-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

(iii) $\mathcal{S}$ is a $\mathrm{NC}^{S\alpha^{**}}$-normal space if and only if for every $\mathrm{NC}^{S\alpha}$-CS $\mathcal{Q}$ and every $\mathrm{NC}^{S\alpha}$-OS $\mathfrak{B}$ containing $\mathcal{Q}$, there exists NC-OS say $\mathfrak{D}$, such that $\mathcal{Q} \sqsubseteq \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}$.

Proof:

(i) $\Rightarrow$ Let $\mathcal{S}$ be a $\mathrm{NC}^{S\alpha}$-normal space. Let $\mathcal{Q} \sqsubseteq \mathfrak{B}$, where $\mathcal{Q}$ is a NC-CS and $\mathfrak{B}$ is a NC-OS $\Longrightarrow \mathcal{Q} \sqcap \mathfrak{B}^c = \emptyset_{\mathcal{N}}$, where $\mathfrak{B}^c$ is a NC-CS. Hence there exist two $\mathrm{NC}^{S\alpha}$-OSs $\mathfrak{D}_1, \mathfrak{D}_2$ such that $\mathcal{Q} \sqsubseteq \mathfrak{D}_1$ and $\mathfrak{B}^c \sqsubseteq \mathfrak{D}_2$ and $\mathfrak{D}_1 \sqcap \mathfrak{D}_2 = \emptyset_{\mathcal{N}}$ (since $\mathcal{S}$ is a $\mathrm{NC}^{S\alpha}$-normal space). Therefore $\mathcal{Q} \sqsubseteq \mathfrak{D}_1 \sqsubseteq {\mathfrak{D}_2}^c \sqsubseteq \mathfrak{B} \Longrightarrow NC^{S\alpha}cl(\mathcal{Q}) \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}_1) \sqsubseteq NC^{S\alpha}cl({\mathfrak{D}_2}^c) \sqsubseteq NC^{S\alpha}cl(\mathfrak{B})$. Hence $\mathcal{Q} \sqsubseteq \mathfrak{D}_1 \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}_1) \sqsubseteq \mathfrak{B}$, where $\mathfrak{D}_1$ is a $\mathrm{NC}^{S\alpha}$-OS in $\mathcal{S}$.

$\Leftarrow$ To prove $\mathcal{S}$ is a $\mathrm{NC}^{S\alpha}$-normal space. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be NC-CSs in $\mathcal{S}$, such that $\mathcal{Q}_1 \sqcap \mathcal{Q}_2 = \emptyset_{\mathcal{N}}$. Hence $\mathcal{Q}_1 \sqsubseteq {\mathcal{Q}_2}^c$, where ${\mathcal{Q}_2}^c$ is a NC-OS. Then there exists a $\mathrm{NC}^{S\alpha}$-OS $\mathfrak{D}$ such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D}) \sqsubseteq {\mathcal{Q}_2}^c$ (by hypothesis). Hence $\mathcal{Q}_1 \sqsubseteq \mathfrak{D}$, $\mathcal{Q}_2 \sqsubseteq (NC^{S\alpha}cl(\mathfrak{D}))^c$. On the other hand $NC^{S\alpha}cl(\mathfrak{D}) \sqcap \left(NC^{S\alpha}cl(\mathfrak{D})\right)^c = \emptyset_{\mathcal{N}}$. Hence $\mathfrak{D} \sqcap \left(NC^{S\alpha}cl(\mathfrak{D})\right)^c = \emptyset_{\mathcal{N}}$ (since $\mathfrak{D} \sqsubseteq NC^{S\alpha}cl(\mathfrak{D})$). Therefore $\mathcal{S}$ is a $\mathrm{NC}^{S\alpha}$-normal space. The proofs (ii), (iii) are evident for others. ▪

Remark 5.5:

(i) NC^{α}-normal property is a $\mathrm{NC}^{\alpha^{**}}$-topological property.

(ii) NC^{α^*}-normal property is a NC^{α^*}-topological property.

(iii) $\mathrm{NC}^{\alpha^{**}}$-normal property is a $\mathrm{NC}^{\alpha^{**}}$-topological property.

(iv) $\mathrm{NC}^{S\alpha}$-normal property is a $\mathrm{NC}^{S\alpha^*}$-topological property.

(v) $\mathrm{NC}^{S\alpha^*}$-normal property is a $\mathrm{NC}^{S\alpha^*}$-topological property.

(vi) $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-normal property is a $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-topological property.

Proposition 5.6:

(i) If $\mathcal{S} \times \mathcal{J}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space, then both $\mathcal{S}$ and $\mathcal{J}$ are $\mathrm{NC}^{\alpha^{**}}$-normal spaces.

(ii) If $\mathcal{S} \times \mathcal{J}$ is a $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-normal space, then both $\mathcal{S}$ and $\mathcal{J}$ are $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-normal spaces.

Proof:

(i) Suppose that $\mathcal{S} \times \mathcal{J}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space, to prove that $\mathcal{S}$ and $\mathcal{J}$ are $\mathrm{NC}^{\alpha^{**}}$-normal spaces. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two NC^{α}-OSs in $\mathcal{S}$ and $\mathcal{J}$ respectively, such that $\mathcal{Q}_1 \sqsubseteq \mathfrak{B}_1$ and $\mathcal{Q}_2 \sqsubseteq \mathfrak{B}_2$, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are NC^{α}-CSs in $\mathcal{S}$ and $\mathcal{J}$ respectively. Hence $\mathcal{Q}_1 \times \mathcal{Q}_2 \sqsubseteq \mathfrak{B}_1 \times \mathfrak{B}_2$ where $\mathcal{Q}_1 \times \mathcal{Q}_2$ is a NC^{α}-CS and $\mathfrak{B}_1 \times \mathfrak{B}_2$ is a NC^{α}-OS in $\mathcal{S} \times \mathcal{J}$ (by theorem (2.7) and corollary (2.8)). But $\mathcal{S} \times \mathcal{J}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space. Then there exists a NC-OS say $\mathfrak{D}$ in $\mathcal{S} \times \mathcal{J}$ such that $\mathcal{Q}_1 \times \mathcal{Q}_2 \sqsubseteq \mathfrak{D} \sqsubseteq NCcl(\mathfrak{D}) \sqsubseteq \mathfrak{B}_1 \times \mathfrak{B}_2$. Then there exist NC-OSs $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{S} \times \mathcal{J}$ such that $\mathcal{Q}_1 \times \mathcal{Q}_2 \sqsubseteq \mathcal{U}_1 \times \mathcal{U}_2 \sqsubseteq NCcl(\mathcal{U}_1 \times \mathcal{U}_2) = NCcl(\mathcal{U}_1) \times NCcl(\mathcal{U}_2) \sqsubseteq \mathfrak{B}_1 \times \mathfrak{B}_2$. Hence $\mathcal{Q}_1 \sqsubseteq \mathcal{U}_1 \sqsubseteq NCcl(\mathcal{U}_1) \sqsubseteq \mathfrak{B}_1 \Longrightarrow \mathcal{S}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space. Also, $\mathcal{Q}_2 \sqsubseteq \mathcal{U}_2 \sqsubseteq NCcl(\mathcal{U}_2) \sqsubseteq \mathfrak{B}_2 \Longrightarrow \mathcal{J}$ is a $\mathrm{NC}^{\alpha^{**}}$-normal space. The proof (ii) is evident for others. ▪

Definition 5.7:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) NC^{α}-T_4-space if $\mathcal{S}$ is a NC^{α}-T_1-space and NC^{α}-normal space.

(ii) $\mathrm{NC}^{\alpha^{*}}$-T_4-space if $\mathcal{S}$ is NC^{α}-T_1-space and $\mathrm{NC}^{\alpha^{*}}$-normal space.

(iii) $\mathrm{NC}^{\alpha^{**}}$-$T_4$-space if $\mathcal{S}$ is NC^{α}-T_1-space and $\mathrm{NC}^{\alpha^{**}}$-normal space.

Definition 5.8:

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$-space if $\mathcal{S}$ is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space and $\mathrm{NC}^{\mathrm{S}\alpha}$-normal space.

(ii) $\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-$T_4$-space if $\mathcal{S}$ is $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space and $\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-normal space.

(iii) $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-$T_4$-space if $\mathcal{S}$ is $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_1$-space and $\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-normal space.

Remark 5.9:

(i) NC^{α}-T_4 ($\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$ respectively) property is a $\mathrm{NC}^{\alpha^{**}}$ ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$ respectively) topological property.

(ii) $\mathrm{NC}^{\alpha^{*}}$-T_4 ($\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-$T_4$ respectively) property is a $\mathrm{NC}^{\alpha^{*}}$ ($\mathrm{NC}^{\mathrm{S}\alpha^{*}}$ respectively) topological property.

(iii) $\mathrm{NC}^{\alpha^{**}}$-$T_4$ ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-$T_4$ respectively) property is a $\mathrm{NC}^{\alpha^{**}}$ ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$ respectively) topological property.

Remark 5.10:

(i) Every NC-T_4-space is a NC^{α}-T_4-space and $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$-space.

(ii) Every NC^{α}-T_4-space is a $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$-space.

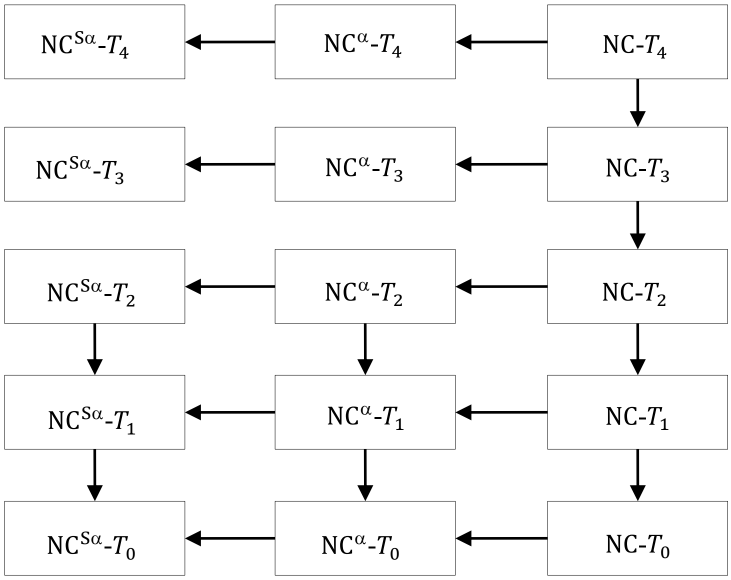
(iii) Every $\mathrm{NC}^{\alpha^{**}}$-$T_4$-space is a $\mathrm{NC}^{\alpha^{*}}$-T_4-space and $\mathrm{NC}^{\mathrm{S}\alpha}$-$T_4$-space.

(iv) Every $\mathrm{NC}^{\alpha^{*}}$-T_4-space ($\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-$T_4$-space respectively) is a $\mathrm{NC}^{\alpha^{*}}$-T_3-space ($\mathrm{NC}^{\mathrm{S}\alpha^{*}}$-$T_3$-space, respectively).

(v) Every $\mathrm{NC}^{\alpha^{**}}$-$T_4$-space ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-$T_4$-space respectively) is a $\mathrm{NC}^{\alpha^{**}}$-$T_3$-space ($\mathrm{NC}^{\mathrm{S}\alpha^{**}}$-$T_3$-space, respectively).

Remark 5.11:

The following diagram explains the relationships between usual NC-separation axioms, NC^{α}-separation axioms and $\mathrm{NC}^{\mathrm{S}\alpha}$-separation axioms:



Also, we have the following diagram:

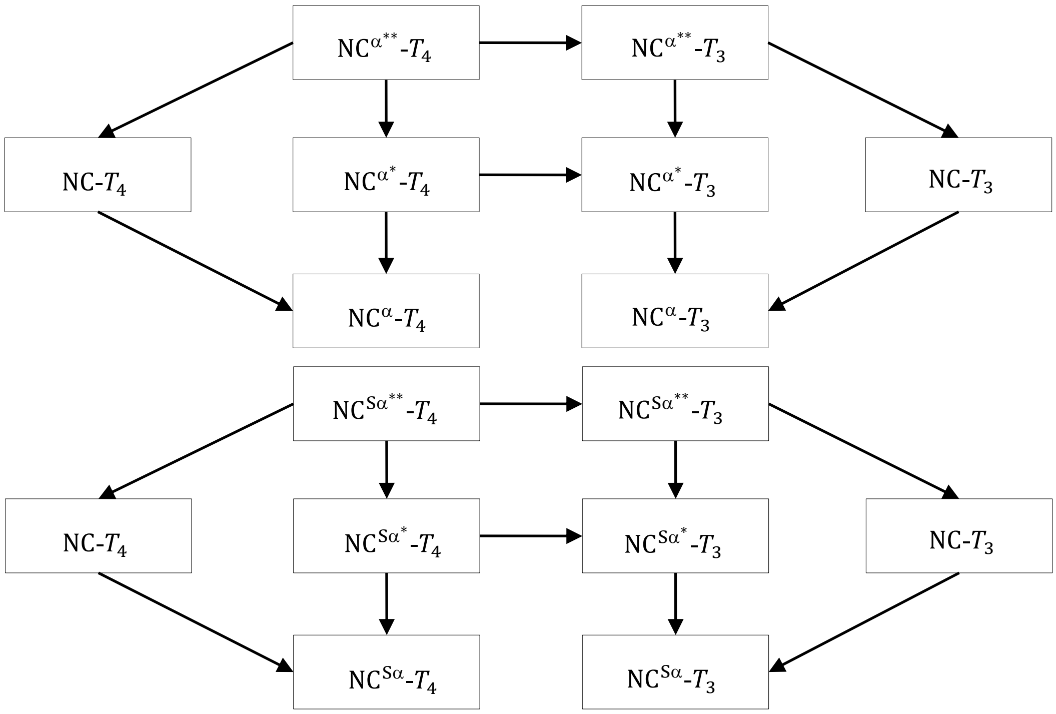


Fig. 5.2

6. Conclusions

We have provided some new concepts of weakly neutrosophic crisp separation axioms. Some characterizations have been provided to illustrate how far topological structures are conserved by the new neutrosophic crisp notion defined. Furthermore, some new concepts of weakly neutrosophic crisp regularity are also studied. The study demonstrated some new concepts of weakly neutrosophic crisp normality and proved some of their related attributes.

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