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Neutrosophic $\mathcal{N}-$structures on Sheffer stroke Hilbert algebras

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Abstract. In this study, a neutrosophic $\mathcal{N}-$subalgebra and a level set of a neutrosophic $\mathcal{N}-$structure are defined on Sheffer stroke Hilbert algebras. By determining a subalgebra on Sheffer stroke Hilbert algebras, it is proved that the level set of neutrosophic $\mathcal{N}-$subalgebras on this algebra is its subalgebra and vice versa. It is stated that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra forms a complete distributive lattice. Finally, a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is described and some of properties are given. Also, it is shown that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is generally not valid.

Keywords: Sheffer stroke (Hilbert algebra); ideal; neutrosophic $\mathcal{N}-$subalgebra; neutrosophic $\mathcal{N}-$ideal.

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1. Introduction

The Sheffer operation (or, Sheffer stroke) was originally introduced by H. M. Sheffer [29]. Because Sheffer stroke, which is also called NAND operator, is one of the two operators that can be used by itself without any other logical operators, to construct a logical system, any axiom of the system is restated by only this operation.. Thus, it is easy to control some properties of the new constructed system. Since the axioms of Boolean algebra, which is an algebraic counterpart of the well-known classical propositional calculi, can be written by only using the Sheffer operation [21], it causes that the Sheffer stroke is applied to many algebraic structures such as orthoimplication algebras [1], ortholattices [8], Sheffer stroke non-associative MV-algebras [9] and its filters [24], Sheffer stroke BL-algebras and (fuzzy) filters [25], Sheffer stroke UP-algebras [26] and Sheffer stroke BG-algebras [27]. Besides, Hilbert algebras, which were introduced by Henkin and Skolem [12], are algebraic parts of the propositional logic including the implication operator and the constant element 1 [28]. Also,these algebras are dual to positive implicative BCK-algebras [10], [13, 14]. Specially, Busneag and Diego widely studied on Hilbert algebras and the related notions [4–6] and [11]. Recently, Oner et al. presented Hilbert algebras with Sheffer operation and its (fuzzy) filters [22]- [23].

On the other side, Atanassov introduced the degree of nonmembership(or falsehood (f)) and intuitionistic fuzzy sets [2] which are generalizations of fuzzy sets [33] with the degree of membership (or truth (t)). Then Smarandache introduced the degree of indeteminacy/neutrality and neutrosophic sets which are generalizations of intuitionistic fuzzy sets with the degrees of membership and nonmembership [30,31]. In a sence, there exist three functions called membership (t), indeteminacy (i) and nonmembership (f) functions in neutrosophic sets. Particularly, Jun et al. applied neutrospohic sets to BCK/BCI-algebras and semigroups [3, 7, 15–20, 32, 34].

We give general definitions and notions of Sheffer stroke Hilbert algebras, $\mathcal{N}-$functions and neutrosophic $\mathcal{N}-$structures defined by these functions on a nonempty universe X. Then a neutrosophic $\mathcal{N}-$subalgebra and a $(\alpha, \beta, \gamma)-$level set are defined by means of $\mathcal{N}-$functions on Sheffer stroke Hilbert algebras. After describing a subalgebra of Sheffer stroke Hilbert algebras, we show that the $(\alpha, \beta, \gamma)-$level set of a neutrosophic $\mathcal{N}-$subalgebra defined by its $\mathcal{N}-$functions on this algebra is its subalgebra and the inverse is also valid. Also, it is proved that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra forms a complete distributive lattice. Some properties of neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra are investigated. Moreover, a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is defined by means of $\mathcal{N}-$functions and it is demonstrated that $\mathcal{N}-$functions which define a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra are order-preserving. It is stated that $(\alpha, \beta, \gamma)-$level set of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its ideal and the inverse holds. Besides, some features of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra are presented and it is shown that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse is not valid in general. Finally, new subsets of a Sheffer stroke Hilbert algebra are determined by $\mathcal{N}-$functions on the algebra and it is shown that these subsets are ideals of a Sheffer stroke Hilbert algebra for its neutrosophic $\mathcal{N}-$ideal. However, the validity of the inverse is satisfied under the special conditions.

2. **Preliminaries**

In this section, basic definitions and notions about Sheffer stroke Hilbert algebras and neutrosophic $\mathcal{N}-$structures.

Definition 2.1. [8] Let $\mathcal{H} = \langle H, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke operation* if it satisfies the following conditions:

$(S1)$ $x|y = y|x$,

$(S2)$ $(x|x)|(x|y) = x$,

$(S3)$ $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,

$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Definition 2.2. [22] A Sheffer stroke Hilbert algebra is a structure $\langle H, |\rangle$ of type (2), in which H is a non-empty set and $|$ is a Sheffer stroke operation on H such that the following identities are satisfied for all $x, y, z \in H$:

(SHa_1) $(x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|((x|(y|y))|((x|(z|z))|(x|(z|z))))) = x|(x|x)$,

(SHa_2) If $x|(y|y) = y|(x|x) = x|(x|x)$ then $x = y$.

Lemma 2.3. *[22] Let $\langle H, |\rangle$ be a Sheffer Stroke Hilbert algebra. Then the following identities hold for all $x \in H$:*

(i) $x|(x|x) = 1$,

(ii) $x|(1|1) = 1$,

(iii) $1|(x|x) = x$.

Lemma 2.4. *[22] Let $\langle H, |\rangle$ be a Sheffer stroke Hilbert algebra. Then the relation $x \leq y$ iff $x|(y|y) = 1$ is a partial order on H, that will be called natural ordering on H. With respect to this ordering, 1 is the largest element of H.*

If a Sheffer stroke Hilbert algebra $\langle H, |\rangle$ has the least element 0, then a unary operation * can be defined by $x^* = x|(0|0)$, for all x in H [22].

Lemma 2.5. *[22] Let $\langle H, |\rangle$ be a Sheffer stroke Hilbert algebra with 0. Then the followings hold, for all $x \in H$*

(i) $0|0 = 1$ *and* $1|1 = 0$,

(ii) $1^* = 0$ *and* $0^* = 1$,

(iii) $x|1 = x|x$,

(iv) $x^* = x|x$,

(v) $x|0 = 1$,

(vi) $(x^*)^* = x$,

(vii) $x|x^* = 1$.

Definition 2.6. [22] A non-empty subset I of H is called an ideal if

$(SSHI1)$ $0 \in I$,

$(SSHI2)$ $(x|(y|y))|(x|(y|y)) \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Theorem 2.7. *[22] Let I be a subset of H such that $0 \in I$. Then I is an ideal of H if and only if $x \leq y$ and $y \in I$ imply $x \in I$ for all $x \in H$.*

Definition 2.8. [15] $\mathcal{F}(X, [-1, 0])$ denotes the collection of functions from a set X to $[-1, 0]$ and a element of $\mathcal{F}(X, [-1, 0])$ is called a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}-$function on X). An $\mathcal{N}-$structure refers to an ordered pair (X, f) of X and $\mathcal{N}-$function f on X.

Definition 2.9. [20] A neutrosophic $\mathcal{N}-$structure over a nonempty universe X is defined by

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \{\frac{x}{(T_N(x), I_N(x), F_N(x))} : x \in X\},$$

where T_N, I_N and F_N are $\mathcal{N}-$function on X, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}-$structure X_N over X satisfies the condition

$$(\forall x \in X)(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0).$$

Definition 2.10. [16] Let X_N be a neutrosophic $\mathcal{N}-$structure on a set X and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$T_N^\alpha := \{x \in X : T_N(x) \leq \alpha\},$$

$$I_N^\beta := \{x \in X : I_N(x) \geq \beta\}$$

and

$$F_N^\gamma := \{x \in X : F_N(x) \leq \gamma\}.$$

The set

$$X_N(\alpha, \beta, \gamma) := \{x \in X : T_N(x) \leq \alpha, I_N(x) \geq \beta \text{ } and \text{ } T_N(x) \leq \gamma\}$$

is called the $(\alpha, \beta, \gamma)-$level set of X_N. Moreover, $X_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.

Consider sets

$$X_N^{w_t} := \{x \in X : T_N(x) \leq T_N(w_t),$$

$$X_N^{w_i} := \{x \in X : I_N(x) \geq I_N(w_i)$$

and

$$X_N^{w_f} := \{x \in X : F_N(x) \leq F_N(w_f),$$

for any $w_t, w_i, w_f \in X$. Obviously, $w_t \in X_N^{w_t}, w_i \in X_N^{w_i}$ and $w_f \in X_N^{w_f}$ [16].

3. Neutrosophic $\mathcal{N}-$structures

In this section, we present neutrosophic $\mathcal{N}-$subalgebras and neutrosophic $\mathcal{N}-$ideals on Sheffer stroke Hilbert algebras. Unless otherwise specified, H states a Sheffer stroke Hilbert algebra.

Definition 3.1. A neutrosophic $\mathcal{N}-$subalgebra H_N on a Sheffer stroke Hilbert algebra H is called a neutrosophic $\mathcal{N}-$structure of H satisfying the conditions

$$T_N((x(y|y))|(x(y|y))) \leq \bigvee\{T_N(x), T_N(y)\},$$

$$I_N((x(y|y))|(x(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\}$$

and

$$F_N((x(y|y))|(x(y|y))) \leq \bigvee\{F_N(x), F_N(y)\},$$

for all $x, y \in H$.

Example 3.2. Consider a Sheffer stroke Hilbert algebra $\langle H, |\rangle$, where the set $H = \{0, p, q, 1\}$ and the Sheffer operation $|$ on H has the Cayley table as below [22]:

TABLE 1

$\vert$	1	p	q	0
1	0	q	p	1
p	q	q	1	1
q	p	1	p	1
0	1	1	1	1

A neutrosophic $\mathcal{N}-$structure $H_N = \{\frac{0}{(-0.81, -0.13, -0.47)}, \frac{p}{(-0.69, -0.32, -0.35)}, \frac{q}{(-0.69, -0.32, -0.35)}, \frac{1}{(-0.56, -0.99, -0.42)}\}$ on H is a neutrosophic $\mathcal{N}-$subalgebra of H.

Definition 3.3. Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. For the sets

$$T_N^{\alpha} := \{x \in H : T_N(x) \leq \alpha\},$$

$$I_N^{\beta} := \{x \in H : I_N(x) \geq \beta\}$$

and

$$F_N^{\gamma} := \{x \in H : F_N(x) \leq \gamma\},$$

the set

$$H_N(\alpha, \beta, \gamma) := \{x \in H : T_N(x) \leq \alpha, I_N(x) \geq \beta \text{ and } F_N(x) \leq \gamma\}$$

is called the $(\alpha, \beta, \gamma)-$level set of H_N. Also, $H_N(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$.

Definition 3.4. A nonempty subset G of a Sheffer stroke Hilbert algebra H is called a subalgebra of H if $(x|(y|y))|(x|(y|y)) \in G$, for all $x, y \in G$.

Example 3.5. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Then $\{0, 1\}$ is a subalgebra of H.

Theorem 3.6. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If H_N is a neutrosophic $\mathcal{N}-$subalgebra of H, then the nonempty $(\alpha, \beta, \gamma)-$level set of H_N is a subalgebra of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$subalgebra of H and x, y be any elements of $H_N(\alpha, \beta, \gamma)$. Then $T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma$ and $T_N(y) \leq \alpha, I_N(y) \geq \beta, F_N(y) \leq \gamma$. Thus, it is obtained that

$$T_N((x(y|y))|(x(y|y))) \leq \bigvee\{T_N(x), T_N(y)\} \leq \alpha,$$
$$I_N((x(y|y))|(x(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\} \geq \beta$$

and

$$F_N((x(y|y))|(x(y|y))) \leq \bigvee\{F_N(x), F_N(y)\} \leq \gamma,$$

for all $x, y \in H$. So, $(x|(y|y))|(x|(y|y)) \in H_N(\alpha, \beta, \gamma)$ which means that $H_N(\alpha, \beta, \gamma)$ is a subalgebra of H. □

Theorem 3.7. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and T_N^α, I_N^β and F_N^γ be subalgebras of H, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then H_N is a neutrosophic $\mathcal{N}-$subalgebra of H.*

Proof. Let T_N^α, I_N^β and F_N^γ be subalgebras of H, for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Suppose that x and y be any elements of H such that $a = T_N((x(y|y))|(x(y|y))) > \bigvee\{T_N(x), T_N(y)\} = b$. Then $b < \alpha_1 < a$ where $\alpha_1 = \frac{1}{2}(a + b) \in [-1, 0)$. Thus, $x, y \in T_N^{\alpha_1}$ but $(x(y|y))|(x(y|y)) \notin T_N^{\alpha_1}$ which is a contradiction. Hence, $T_N((x(y|y))|(x(y|y))) \leq \bigvee\{T_N(x), T_N(y)\}$, for all $x, y \in H$.

Assume that x and y be any elements of H such that $u = I_N((x(y|y))|(x(y|y))) < \bigwedge\{I_N(x), I_N(y)\} = v$. Then $u < \beta_1 < v$ in which $\beta_1 = \frac{1}{2}(u + v) \in [-1, 0)$. So, $x, y \in I_N^{\beta_1}$ while $(x(y|y))|(x(y|y)) \notin I_N^{\beta_1}$ which is a contradiction. Thus, $I_N((x(y|y))|(x(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\}$, for all $x, y \in H$.

Suppose that x and y be any elements of H such that $m = F_N((x(y|y))|(x(y|y))) > \bigvee\{F_N(x), F_N(y)\} = n$. Then $n < \gamma_1 < m$ where $\gamma_1 = \frac{1}{2}(m + n) \in [-1, 0)$. Hence, $x, y \in F_N^{\gamma_1}$ but $(x(y|y))|(x(y|y)) \notin F_N^{\gamma_1}$ which is a contradiction. Therefore, $F_N((x(y|y))|(x(y|y))) \leq \bigvee\{F_N(x), F_N(y)\}$, for all $x, y \in H$.

Thereby, H_N is a neutrosophic $\mathcal{N}-$subalgebra of H. □

Theorem 3.8. *Let $\{H_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra H. Then $\{H_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive lattice.*

Proof. Let G be a nonempty subset of $\{H_{N_i} : i \in \mathbb{N}\}$. Since H_{N_i} is a neutrosophic $\mathcal{N}-$subalgebra of H, for all $H_{N_i} \in G$, it satisfies

$$T_N((x(y|y))|(x(y|y))) \leq \bigvee\{T_N(x), T_N(y)\},$$

$$I_N((x(y|y))|(x(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\}$$

and

$$F_N((x(y|y))|(x(y|y))) \leq \bigvee\{F_N(x), F_N(y)\},$$

for all $x, y \in H$. Then $\bigcap G$ satisfies these inequalities, which means that $\bigcap G$ is a neutrosophic $\mathcal{N}-$subalgebra of H.

Let P be a family of all neutrosophic $\mathcal{N}-$subalgebras of H containing $\bigcup\{H_{N_i} : i \in \mathbb{N}\}$. Then $\bigcap P$ is a neutrosophic $\mathcal{N}-$subalgebra of H.

If $\bigwedge_{i\in\mathbb{N}} H_{N_i} = \bigcap_{i\in\mathbb{N}} H_{N_i}$ and $\bigvee_{i\in\mathbb{N}} H_{N_i} = \bigcap P$, then $(\{H_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$ is a complete lattice. Also, it is distibutive by the definitions of $\bigvee$ and $\bigwedge$. □

Proposition 3.9. *If a neutrosophic $\mathcal{N}-$structure H_N on a Sheffer stroke Hilbert algebra H is a neutrosophic $\mathcal{N}-$subalgebra of H, then $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$, for all $x \in H$.*

Proof. By substituting $[y := 0]$ in the inequalities in Definition 3.1, we have from Lemma 2.3 (i) and Lemma 2.5 (i) that

$$T_N(0) = T_N(1|1) = T_N((x(x|x))|(x(x|x))) \leq \bigvee\{T_N(x), T_N(x)\} = T_N(x),$$

$$I_N(0) = I_N(1|1) = I_N((x(x|x))|(x(x|x))) \geq \bigwedge\{I_N(x), I_N(x)\} = I_N(x)$$

and

$$F_N(0) = F_N(1|1) = F_N((x(x|x))|(x(x|x))) \leq \bigvee\{F_N(x), F_N(x)\} = F_N(x),$$

for all $x \in H$. □

The inverse of Proposition 3.9 is generally not true.

Example 3.10. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$H_N = \{\frac{0}{(-1,0,-1)}, \frac{p}{(-0.2,-0.2,-0.2)}, \frac{q}{(-0.3,-0.3,-0.3)}, \frac{1}{(-0.4,-0.4,-0.4)}\}$$

on H is not a neutrosophic $\mathcal{N}-$subalgebra of H since

$$T_N((1(q|q))|(1(q|q))) = T_N(p) = -0.2 > -0.3 = \bigvee\{-0.3,-0.4\} = \bigvee\{T_N(1), T_N(q)\}.$$

Lemma 3.11. *Let H_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke Hilbert algebra H. If there exists a sequence $\{a_n\}$ in H such that $\lim_{n\longrightarrow\infty} T_N(a_n) = -1 = \lim_{n\longrightarrow\infty} F_N(a_n)$ and $\lim_{n\longrightarrow\infty} I_N(a_n) = 0$, then $T_N(0) = -1 = F_N(0)$ and $I_N(0) = 0$.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke Hilbert algebra H. Assume that there exists a sequence $\{a_n\}$ in H such that $\lim_{n\longrightarrow\infty} T_N(a_n) = -1 = \lim_{n\longrightarrow\infty} F_N(a_n)$ and $\lim_{n\longrightarrow\infty} I_N(a_n) = 0$. Since $T_N(0) \leq T_N(a_n)$, $I_N(0) \geq I_N(a_n)$ and $F_N(0) \leq F_N(a_n)$, for every $n \in \mathbb{Z}^+$ from Proposition 3.9, it follows that

$$-1 = \lim_{n\longrightarrow\infty} -1 \leq \lim_{n\longrightarrow\infty} T_N(0) = T_N(0) \leq \lim_{n\longrightarrow\infty} T_N(a_n) = -1,$$

$$0 = \lim_{n\longrightarrow\infty} 0 \geq \lim_{n\longrightarrow\infty} I_N(0) = I_N(0) \geq \lim_{n\longrightarrow\infty} I_N(a_n) = 0$$

and

$$-1 = \lim_{n\longrightarrow\infty} -1 \leq \lim_{n\longrightarrow\infty} F_N(0) = F_N(0) \leq \lim_{n\longrightarrow\infty} F_N(a_n) = -1.$$

Hence, $T_N(0) = -1 = F_N(0)$ and $I_N(0) = 0$. □

Proposition 3.12. *Every neutrosophic $\mathcal{N}-$subalgebra H_N of a Sheffer stroke Hilbert algebra H satisfies*

$$T_N((x(y|y))|(x(y|y))) \leq T_N(y),$$

$$I_N((x(y|y))|(x(y|y))) \geq I_N(y)$$

and

$$F_N((x(y|y))|(x(y|y))) \leq F_N(y),$$

for all $x, y \in H$ if and only if T_N, I_N and F_N are constant.

Proof. $(\Rightarrow)$ Since

$$\begin{aligned} T_N(x) &= T_N((x|x)|(x|x)) \\ &= T_N((1|((x|x)|(x|x)))|(1|((x|x)|(x|x)))) \\ &= T_N((x|1)|(x|1)) \\ &= T_N((x|(0|0))|(x|(0|0))) \\ &\leq T_N(0), \end{aligned}$$

and similarly, $I_N(0) \leq I_N(x)$, $F_N(x) \leq F_N(0)$ from (S1), (S2), Lemma 2.3 (iii) and Lemma 2.5 (i), we have from Proposition 3.9 that $T_N(x) = T_N(0)$, $I_N(x) = I_N(0)$ and $F_N(x) = F_N(0)$, for all $x \in X$.

$(\Leftarrow)$ It is obvious by the fact that H_N is a neutrosophic $\mathcal{N}-$subalgebra of H and T_N, I_N and F_N are constant. □

Definition 3.13. A neutrosophic $\mathcal{N}-$structure H_N on H is called a neutrosophic $\mathcal{N}-$ideal of H if

$$T_N(0) \leq T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\},$$

$$I_N(0) \geq I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\}$$

and

$$F_N(0) \leq F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\},$$

for all $x, y \in H$.

Example 3.14. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Then a neutrosophic $\mathcal{N}-$structure

$$H_N = \{\frac{0}{(-1,0,-0.21)}, \frac{p}{(-1,0,-0.21)}, \frac{q}{(-0.71,-0.55,-0.11)}, \frac{1}{(-0.71,-0.55,-0.11)}\}$$

on H is a neutrosophic $\mathcal{N}-$ideal of H.

Proposition 3.15. *Let H_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H. Then $x \leq y$ implies $T_N(x) \leq T_N(y)$, $I_N(x) \geq I_N(y)$ and $F_N(x) \leq F_N(y)$, for all $x, y \in H$.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H and $x \leq y$. Then $x|(y|y) = 1$ from Lemma 2.4, and so, $(x|(y|y))|(x|(y|y)) = 1|1 = 0$ from Lemma 2.5 (i). Thus,

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} = \bigvee\{T_N(0), T_N(y)\} = T_N(y)$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} = \bigwedge\{I_N(0), I_N(y)\} = I_N(y)$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} = \bigvee\{F_N(0), F_N(y)\} = F_N(y).$$

□

The inverse of Proposition 3.15 does not hold in general.

Example 3.16. Consider the neutrosophic $\mathcal{N}-$ideal of H in Example 3.14. Then $p \nleq q$ when $T_N(p) = -1 \leq -0.71 = T_N(q)$.

Lemma 3.17. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and α, β, γ be any elements of $[-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If H_N is a neutrosophic $\mathcal{N}-$ideal of H, then the nonempty set $H_N(\alpha, \beta, \gamma)$ is an ideal of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H and $H_N(\alpha,\beta,\gamma) \neq \emptyset$, for any $\alpha,\beta,\gamma \in [-1,0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Since $T_N(0) \leq T_N(x) \leq \alpha, I_N(0) \geq I_N(x) \geq \beta$ and $F_N(0) \leq F_N(x) \leq \gamma$, for any $x \in H_N(\alpha,\beta,\gamma)$, we have $0 \in H_N(\alpha,\beta,\gamma)$. Let $(x|(y|y))|(x|(y|y)), y \in H_N(\alpha,\beta,\gamma)$. Then $T_N((x|(y|y))|(x|(y|y))) \leq \alpha$, $I_N((x|(y|y))|(x|(y|y))) \geq \beta$, $F_N((x|(y|y))|(x|(y|y))) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. Since

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \leq \bigvee\{\alpha,\alpha\} = \alpha,$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \geq \bigwedge\{\beta,\beta\} = \beta$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq \bigvee\{\gamma,\gamma\} = \gamma,$$

for all $x, y \in H$, we get $x \in H_N(\alpha,\beta,\gamma)$ which means that $H_N(\alpha,\beta,\gamma)$ is an ideal of H. □

Lemma 3.18. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and $T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ be ideals of H, for all $\alpha,\beta,\gamma \in [-1,0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Then H_N is a neutrosophic $\mathcal{N}-$ideal of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H and $T_N^{\alpha}, I_N^{\beta}, F_N^{\gamma}$ be ideals of H, for all $\alpha,\beta,\gamma \in [-1,0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Suppose that x_0, y_0 and z_0 be any elements of H such that $T_N(0) > T_N(x_0)$, $I_N(0) < I_N(y_0)$ and $F_N(0) > F_N(z_0)$. If $\alpha = \frac{1}{2}(T_N(0)+T_N(x_0))$, $\beta = \frac{1}{2}(I_N(0)+I_N(y_0))$ and $\gamma = \frac{1}{2}(F_N(0)+F_N(z_0))$, for $\alpha,\beta,\gamma \in [-1,0)$, then $T_N(0) > \alpha > T_N(x_0)$, $I_N(0) < \beta < I_N(y_0)$ and $F_N(0) > \gamma > F_N(z_0)$ which imply that $0 \notin T_N^{\alpha}, 0 \notin I_N^{\beta}$ and $0 \notin F_N^{\gamma}$, respectively. This contradicts with (SSHI1). So, $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$, for all $x \in H$. Assume that x_1, x_2, x_3, y_1, y_2 and y_3 be any elements of H such that

$$a_1 = T_N(x_1) > \bigvee\{T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))), T_N(y_1)\} = b_1,$$

$$a_2 = I_N(x_2) < \bigwedge\{I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))), I_N(y_2)\} = b_2$$

and

$$a_3 = F_N(x_3) > \bigvee\{F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))), F_N(y_3)\} = b_3.$$

If $\alpha' = \frac{1}{2}(a_1+b_1)$, $\beta' = \frac{1}{2}(a_2+b_2)$ and $\gamma' = \frac{1}{2}(a_3+b_3)$, then $b_1 < \alpha' < a_1$, $a_2 < \beta' < b_2$ and $b_3 < \gamma' < a_3$. Thus, $(x_1|(y_1|y_1))|(x_1|(y_1|y_1)), y_1 \in T_N^{\alpha'}$, $(x_2|(y_2|y_2))|(x_2|(y_2|y_2)), y_2 \in I_N^{\beta'}$ and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3)), y_3 \in F_N^{\gamma'}$, and so, $x_1 \in T_N^{\alpha'}$, $x_2 \in I_N^{\beta'}$ and $x_3 \in F_N^{\gamma'}$ which contradicts with the assumption. Therefore,

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\},$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\}$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\},$$

for all $x, y \in H$. Thereby, H_N is a neutrosophic $\mathcal{N}-$ideal of H. □

Lemma 3.19. *Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H. Then H_N is a neutrosophic $\mathcal{N}-$ideal of H if and only if $(x|(y|y))|(x|(y|y)) \leq z$ implies*

$$T_N(x) \leq \bigvee\{T_N(y), T_N(z)\},$$

$$I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \bigvee\{F_N(y), F_N(z)\},$$

for all $x, y, z \in H$.

Proof. $(\Rightarrow)$ Let H_N be a neutrosophic $\mathcal{N}-$ideal of H and $(x|(y|y))|(x|(y|y)) \leq z$. Then $(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)) = 1|1 = 0$ from Lemma 2.4 and Lemma 2.5 (i). Since

$$\begin{aligned} T_N((x|(y|y))|(x|(y|y))) &\leq \bigvee\{T_N((((x|(y|y))|(x|(y|y)))|(z|z))| \\ &\quad (((x|(y|y))|(x|(y|y)))|(z|z))), T_N(z)\} \\ &= \bigvee\{T_N(0), T_N(z)\} \\ &= T_N(z), \end{aligned}$$

$$\begin{aligned} I_N((x|(y|y))|(x|(y|y))) &\geq \bigwedge\{I_N((((x|(y|y))|(x|(y|y)))|(z|z))| \\ &\quad (((x|(y|y))|(x|(y|y)))|(z|z))), I_N(z)\} \\ &= \bigwedge\{I_N(0), I_N(z)\} \\ &= I_N(z) \end{aligned}$$

and

$$\begin{aligned} F_N((x|(y|y))|(x|(y|y))) &\leq \bigvee\{F_N((((x|(y|y))|(x|(y|y)))|(z|z))| \\ &\quad (((x|(y|y))|(x|(y|y)))|(z|z))), F_N(z)\} \\ &= \bigvee\{F_N(0), F_N(z)\} \\ &= F_N(z), \end{aligned}$$

we have

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \leq \bigvee\{T_N(y), T_N(z)\},$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \geq \bigwedge\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \leq \bigvee\{F_N(y), F_N(z)\},$$

for all $x, y, z \in H$.

($\Leftarrow$) Let H_N be a neutrosophic $\mathcal{N}-$structure on H such that $(x|(y|y))|(x|(y|y)) \leq z$ implies

$$T_N(x) \leq \bigvee\{T_N(y), T_N(z)\},$$

$$I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\}$$

and

$$F_N(x) \leq \bigvee\{F_N(y), F_N(z)\},$$

for all $x, y, z \in H$. Since $(0|(x|x))|(0|(x|x)) = ((x|x)|(1|1))|((x|x)|(1|1)) = 1|1 = 0 \leq z$ from (S1), Lemma 2.3 (ii) and Lemma 2.5 (i), we get $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$, for all $x \in H$. Since

$$((x|(x|(y|y)))|(x|(x|(y|y))))|(y|y) = (x|(y|y))|((x|(y|y))|(x|(y|y))) = 1$$

from (S1), (S3) and Lemma 2.3 (i), it follows from Lemma 2.4 that $(x|(x|(y|y)))|(x|(x|(y|y))) \leq y$. Since $(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y)))))|(x|(((x|(y|y))|(x|(y|y)))|((x|(y|y))|(x|(y|y))))) = (x|(x|(y|y)))|(x|(x|(y|y))) \leq y$ from (S2), it is obtained that

$$T_N(x) \leq \bigvee\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\},$$

$$I_N(x) \geq \bigwedge\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\}$$

and

$$F_N(x) \leq \bigvee\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\},$$

for all $x, y, z \in H$. Thus, H_N is a neutrosophic $\mathcal{N}-$ideal of H. □

Theorem 3.20. *Every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H is a neutrosophic $\mathcal{N}-$subalgebra of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$ideal of H. Then it follows from (S1), (S3), Lemma 2.3 (i)-(ii), Lemma 2.5 (i) and Definition 3.13 that

$$\begin{aligned}
T_N((x|(y|y))|(x|(y|y))) &\leq \bigvee\{T_N((((x|(y|y))|(x|(y|y)))|(x|x))| \\
&\quad (((x|(y|y))|(x|(y|y)))|(x|x))), T_N(x)\} \\
&= \bigvee\{T_N(((y|y)|((x|(x|x))|(x|(x|x))))| \\
&\quad ((y|y)|((x|(x|x))|(x|(x|x))))), T_N(x)\} \\
&= \bigvee\{T_N(((y|y)|(1|1))|((y|y)|(1|1))), T_N(x)\} \\
&= \bigvee\{T_N(1|1), T_N(x)\} \\
&= \bigvee\{T_N(0), T_N(x)\} \\
&= T_N(x) \\
&\leq \bigvee\{T_N(x), T_N(y)\},
\end{aligned}$$

and similarly,

$$I_N((x|(y|y))|(x|(y|y))) \geq \bigwedge\{I_N(x), I_N(y)\},$$

$$F_N((x|(y|y))|(x|(y|y))) \leq \bigvee\{F_N(x), F_N(y)\},$$

for all $x, y \in H$. Hence, H_N is a neutrosophic $\mathcal{N}-$subalgebra of H. □

The inverse of Theorem 3.20 is mostly not true.

Example 3.21. The neutrosophic $\mathcal{N}-$subalgebra H_N of H in Example 3.2 is not a neutrosophic $\mathcal{N}-$ideal of H since $T_N(1) = -0.56 > -0.69 = T_N(p) = \bigvee\{T_N(p), T_N(q)\} = \bigvee\{T_N((1|(q|q))|(1|(q|q))), T_N(q)\}$.

Definition 3.22. Let H be a Sheffer stroke Hilbert algebra. We define

$$H_N^{x_t} := \{x \in H : T_N(x) \leq T_N(x_t)\},$$

$$H_N^{x_i} := \{x \in H : I_N(x) \geq I_N(x_i)\}$$

and

$$H_N^{x_f} := \{x \in H : F_N(x) \leq F_N(x_f)\},$$

for all $x_t, x_i, x_f \in H$. Obviously, $x_t \in H_N^{x_t}, x_i \in H_N^{x_i}$ and $x_f \in H_N^{x_f}$.

Example 3.23. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Let $T_N(0) = -0.11, T_N(p) = -0.14, T_N(q) = -0.17, T_N(1) = -0.2, I_N(0) = -0.12, I_N(p) = -0.15$, $I_N(q) = -0.13, I_N(1) = -0.21, F_N(0) = -0.22, F_N(p) = -0.19, F_N(q) = -0.2, F_N(1) = -0.23, x_t = 1, x_i = p$ and $x_f = q$. Then

$$H_N^{x_t} = \{x \in H : T_N(x) \leq T_N(1)\} = \{1\},$$

$$H_N^{x_i} = \{x \in H : I_N(x) \geq I_N(p)\} = \{0, p, q\}$$

and

$$H_N^{x_f} = \{x \in H : F_N(x) \leq F_N(q)\} = \{0, q, 1\}.$$

Theorem 3.24. *Let x_t, x_i and x_f be any elements of a Sheffer stroke Hilbert algebra H. If H_N is a neutrosophic $\mathcal{N}-$ideal of H, then $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H.*

Proof. Let H_N be a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra H. Since $T_N(0) \leq T_N(x_t)$, $I_N(0) \geq I_N(x_i)$ and $F_N(0) \leq F_N(x_f)$, for any $x_t, x_i, x_f \in H$, it follows that $0 \in H_N^{x_t}$, $0 \in H_N^{x_i}$ and $0 \in H_N^{x_f}$. Let $(x_1|(y_1|y_1))|(x_1|(y_1|y_1)), y_1 \in H_N^{x_t}$, $(x_2|(y_2|y_2))|(x_2|(y_2|y_2))$, $y_2 \in H_N^{x_i}$ and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3))$, $y_3 \in H_N^{x_f}$. Then $T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))) \leq T_N(x_t)$, $T_N(y_1) \leq T_N(x_t)$, $I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))) \geq I_N(x_i)$, $I_N(y_2) \geq I_N(x_i)$ and $F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))) \leq F_N(x_f)$, $F_N(y_3) \leq F_N(x_f)$. Since

$$T_N(x_1) \leq \bigvee\{T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))), T_N(y_1)\} \leq T_N(x_t),$$

$$I_N(x_2) \geq \bigwedge\{I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))), I_N(y_2)\} \geq I_N(x_i)$$

and

$$F_N(x_3) \leq \bigvee\{F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))), F_N(y_3)\} \leq F_N(x_f),$$

we get $x_1 \in H_N^{x_t}$, $x_2 \in H_N^{x_i}$ and $x_3 \in H_N^{x_f}$. Therefore, $H_N^{x_t}$, $H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H. □

Example 3.25. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. For a neutrosophic $\mathcal{N}-$ideal

$$H_N = \{\frac{0}{(-0.69,-0.1,-0.41)}, \frac{p}{(-0.57,-0.27,-0.38)}, \frac{q}{(-0.69,-0.1,-0.41)}, \frac{1}{(-0.57,-0.27,-0.38)}\}$$

of H, $x_t = p$ and $x_i = x_f = q \in H$, the subsets

$$H_N^{x_t} = \{x \in H : T_N(x) \leq T_N(p)\} = H,$$

$$H_N^{x_i} = \{x \in H : I_N(x) \geq I_N(q)\} = \{0, q\}$$

and

$$H_N^{x_f} = \{x \in H : F_N(x) \leq F_N(q)\} = \{0, q\}$$

of H are ideals of H.

Theorem 3.26. *Let x_t, x_i and x_f be any elements of a Sheffer stroke Hilbert algebra H and H_N be a neutrosophic $\mathcal{N}-$structure on H.*

(1) *If $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H, then the following condition is satisfied:*

$$T_N(x) \geq \bigvee\{T_N((y|(z|z))(y|(z|z))), T_N(z)\} \Rightarrow T_N(x) \geq T_N(y),$$

$$I_N(x) \leq \bigwedge\{I_N((y|(z|z))(y|(z|z))), I_N(z)\} \Rightarrow I_N(x) \leq I_N(y) \quad \text{and} \tag{1}$$

$$F_N(x) \geq \bigvee\{F_N((y|(z|z))(y|(z|z))), F_N(z)\} \Rightarrow F_N(x) \geq F_N(y),$$

for all $x, y, z \in H$.

(2) *If H_N satisfies the condition (1) and*

$$T_N(0) \leq T_N(x), \quad I_N(0) \geq I_N(x) \quad \text{and} \quad F_N(0) \leq F_N(x), \quad \text{for all } x \in H, \tag{2}$$

then $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H, for all $x_t \in T_N^{-1}$, $x_i \in I_N^{-1}$ and $x_f \in F_N^{-1}$.

Proof. Let H_N be a neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra H.

(1) $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H, for any $x_t, x_i, x_f \in H$, and x, y, z be any elements of H such that $T_N(x) \geq \bigvee\{T_N((y|(z|z))(y|(z|z))), T_N(z)\}$, $I_N(x) \leq \bigvee\{I_N((y|(z|z))(y|(z|z))), I_N(z)\}$ and $F_N(x) \geq \bigvee\{F_N((y|(z|z))(y|(z|z))), F_N(z)\}$. Then $(y|(z|z))(y|(z|z))), z \in H_N^{x_t} \cap H_N^{x_i} \cap H_N^{x_f}$ where $x_t = x_i = x_f = x$. So, it is obtained from

(SSHI2) that $y \in H_N^{x_t} \cap H_N^{x_i} \cap H_N^{x_f}$ where $x_t = x_i = x_f = x$. Thus, $T_N(x) \geq T_N(y)$, $I_N(x) \leq I_N(y)$ and $F_N(x) \geq F_N(y)$.

(2) Let $x_t \in T_N^{-1}$, $x_i \in I_N^{-1}$ and $x_f \in F_N^{-1}$ and H_N be a neutrosophic $\mathcal{N}-$structure on H satisfying the conditions (1) and (2). Then $0 \in H_N^{x_t}$, $0 \in H_N^{x_i}$ and $0 \in H_N^{x_f}$ from the condition (2). Let $(x_1|(y_1|y_1))|(x_1|(y_1|y_1)), y_1 \in H_N^{x_t}$, $(x_2|(y_2|y_2))|(x_2|(y_2|y_2)), y_2 \in H_N^{x_i}$ and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3)), y_3 \in H_N^{x_f}$. Thus, $T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))) \leq T_N(x_t)$, $T_N(y_1) \leq T_N(x_t)$, $I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))) \geq I_N(x_i)$, $I_N(y_2) \geq I_N(x_i)$ and $F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))) \leq F_N(x_f)$, $F_N(y_3) \leq F_N(x_f)$. Since

$$\bigvee\{T_N((x_1|(y_1|y_1))|(x_1|(y_1|y_1))), T_N(y_1)\} \leq T_N(x_t),$$

$$\bigwedge\{I_N((x_2|(y_2|y_2))|(x_2|(y_2|y_2))), I_N(y_2)\} \geq I_N(x_i)$$

and

$$\bigvee\{F_N((x_3|(y_3|y_3))|(x_3|(y_3|y_3))), F_N(y_3)\} \leq F_N(x_f),$$

we have from the condition (1) that $T_N(x_1) \leq T_N(x_t)$, $I_N(x_2) \geq I_N(x_i)$ and $F_N(x_3) \leq F_N(x_f)$ which imply that $x_1 \in H_N^{x_t}$, $x_2 \in H_N^{x_i}$ and $x_3 \in H_N^{x_f}$. Thereby, $H_N^{x_t}$, $H_N^{x_i}$ and $H_N^{x_f}$ are ideals of H. □

Example 3.27. Consider the Sheffer stroke Hilbert algebra H in Example 3.2. Let $T_N(0) = T_N(q) = -0.997, T_N(p) = T_N(1) = 0, I_N(0) = I_N(q) = -0.08, I_N(p) = I_N(1) = -1, F_N(0) = F_N(q) = -0.8, F_N(p) = F_N(1) = -0.7$. Then the ideals $H_N^{x_t} = \{0, q\}, H_N^{x_i} = \{0\}$ and $H_N^{x_f} = H$ of H satisfy the condition (1), for $x_t = q, x_i = 0$ and $x_f = p \in H$.

Also, let

$$H_N = \{\frac{0}{(-0.7, -0.13, -0.6)}, \frac{p}{(-0.7, -0.13, -0.6)}, \frac{q}{(-0.41, -0.87, -0.52)}, \frac{1}{(-0.41, -0.87, -0.52)}\}$$

be a neutrosophic $\mathcal{N}-$structure on H satisfyinh the conditions (1) and (2). For $x_t = p, x_i = 1$ and $x_f = q \in H$, the subsets

$$H_N^{x_t} = \{x \in H : T_N(x) \leq T_N(p)\} = \{0, p\},$$

$$H_N^{x_i} = \{x \in H : I_N(x) \geq I_N(1)\} = H$$

and

$$H_N^{x_f} = \{x \in H : F_N(x) \leq F_N(q)\} = H$$

of H are ideals of H.

4. Conclusion

In this study, we have studied neutrosophic $\mathcal{N}-$structures defined by $\mathcal{N}-$functions on Sheffer stroke Hilbert algebras. By giving basic definitions and notions about Sheffer stroke Hilbert algebras and neutrosophic $\mathcal{N}-$structures defined by $\mathcal{N}-$functions on a nonempty universe X, a neutrosophic $\mathcal{N}-$subalgebra and a $(\alpha,\beta,\gamma)-$level set of a neutrosophic $\mathcal{N}-$structure are described by $\mathcal{N}-$functions on Sheffer stroke Hilbert algebras. It is proved that the $(\alpha,\beta,\gamma)-$level set of a neutrosophic $\mathcal{N}-$subalgebra defined by the $\mathcal{N}-$functions on this algebra is its subalgebra and also the inverse is valid. We show that the family of all neutrosophic $\mathcal{N}-$subalgebras of a Sheffer stroke Hilbert algebra forms a complete distributive lattice. Besides, it is demonstrated that every neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke Hilbert algebra satisfies $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$, for all $x \in H$ but a neutrosophic $\mathcal{N}-$structure of a Sheffer stroke Hilbert algebra satisfying the property is mostly not its neutrosophic $\mathcal{N}-$subalgebra. Also, it is comprehensively examined the statement which $\mathcal{N}-$functions defining a neutrosophic $\mathcal{N}-$subalgebra of a Sheffer stroke Hilbert algebra are constant. After describing a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra by means of $\mathcal{N}-$functions, we demonstrate that $\mathcal{N}-$functions defining a neutrosophic $\mathcal{N}-$ideal of the algebra are order-preserving whereas the inverse does not hold in general. In fact, $(\alpha,\beta,\gamma)-$level set of a neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its ideal and vice versa. we present that a lemma is equivalent to the definition of the neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra, and that every neutrosophic $\mathcal{N}-$ideal of a Sheffer stroke Hilbert algebra is its neutrosophic $\mathcal{N}-$subalgebra but the inverse does not usually hold. Moreover, new three subsets $H_N^{x_t}, H_N^{x_i}$ and $H_N^{x_f}$ of a Sheffer stroke Hilbert algebra are described by $\mathcal{N}-$functions and certain elements x_t, x_i and x_f of the algebra. It is proved that these subsets are ideals of a Sheffer stroke Hilbert algebra for its neutrosophic $\mathcal{N}-$ideal defined by the $\mathcal{N}-$functions. A neutrosophic $\mathcal{N}-$structure on a Sheffer stroke Hilbert algebra is generally not the $\mathcal{N}-$ideal in the case which these subsets are its ideals.

In the future works, we wish to study on plithogenic sets of Sheffer stroke Hilbert algebras and neutrosophic structures of other Sheffer stroke algebras.

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