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A New Approach to Operations on Bipolar Neutrosophic Soft Sets and Bipolar Neutrosophic Soft Topological Spaces

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Abstract: In this study, we re-define some operations on bipolar neutrosophic soft sets differently from the studies [2]. On this operations are given interesting examples and them basic properties. In the direction of these newly defined operations, we construct the bipolar neutrosophic soft topological spaces. Finally, we introduce basic definitions and theorems on bipolar neutrosophic soft topological spaces

Keywords: Bipolar neutrosophic soft set; bipolar neutrosophic soft operations; bipolar neutrosophic soft topological space; bipolar neutrosophic soft interior; bipolar neutrosophic soft closure.

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1. Introduction

Set theory which is inducted by Cantor is one of the main topic in mathematics and is frequently used while solving the problems with the mathematical methods. However the real life problems which we meet in several areas as medicine, economics, engineering and etc. include vagueness and this leads to break the precise of data and makes the mathematical solutions unusable. To overpass this lack alternative theories are developed as theory of fuzzy sets [25], theory of intuitionistic fuzzy sets[4], theory of soft sets [15] and etc. But all these approaches have their implicit crisis in solving the problems involving indeterminate and inconsistent data due to inadequacy of parameterization tools. Smarandache [20] studied the idea of neutrosophic set as an approach for solving issues that cover unreliable, indeterminacy and persistent data. Smarandache introduced the neutrosophic set theory as a generalization of many theories such as fuzzy set, intuitionistic fuzzy set etc. Neutrosophic set theory is still popular today. Researchers are working intensively on this set theory [1, 3, 14, 19].

Molodtsov [15] claimed that the theory of soft sets is free from the difficulties seen in the fuzzy set theory. Recently this new theory is used extensively both in mathmetics and in different areas. [6, 10, 21, 23, 24]. As it is known, in Boolean logic a property is either present or absent, i.e. it takes values in the set $\{0,1\}$ and also the theories developed for vagueness focus only on the existence of a property and so in these approaches coexistence of a property is ignored. Hence, it is impossible to model the coexistence of a property with these approaches. Coexistence is associated with bipolarity of an information. For this reason, bipolarity is also an important characteristic of the data which should be considered. In 2013, Shabir and Naz [22] defined bipolar soft sets and basic operations of union, intersection and complementation for bipolar soft sets. They gave examples of bipolar soft sets and an application of bipolar soft sets in a decision making problem. Many different studies have been conducted on bipolar soft set theory [11, 17]. The bipolar neutrosophic soft set theory was first presented by M. Ali at al.[2]. In their study, the structure of theory and the operations on this set structure are defined. However, when the study is examined carefully, one can see that some definitions need to be corrected and re-defined.

In our study, bipolar neutrosophic soft subset, empty bipolar neutrosophic soft set, absolute bipolar neutrosophic soft set, bipolar neutrosophic soft union and bipolar neutrosophic soft intersection are re-defined different from the paper written by M.Ali et al. [2] and also new algebraic operations are presented. Then the topology on the bipolar neutrosophic soft set is built. Closure and interior concepts of the obtained topological spaces are defined and basic theorems are presented. All of these presented notions are constructed with supporting examples.

2. Preliminary

In this section, we will give some preliminary information for the present study.

Definition 2.1 *[20] A neutrosophic set A on the universe of discourse X is defined as:*

$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where$T,\ I,\ F: X \to]^{-}0, 1^{+}[$ and $^{-}0 \le T_A(x) + I_A(x) + F_A(x) \le 3^{+}$.

Definition 2.2 *[15] Let X be an initial universe, E be a set of all parameters and $P(X)$ denotes the power set of X. A pair (F, E) is called a soft set over X, where F is a mapping given by $F: E \to P(X)$.*

In other words, the soft set is a parameterized family of subsets of the set X. For $e \in E$, $F(e)$ may be considered as the set of $e-$elements of the soft set (F, E), or as the set of $e-$approximate elements of the soft set, i.e.,

$$(F, E) = \{(e, F(e)) : e \in E, F: E \to P(X)\}.$$

Firstly, neutrosophic soft set defined by Maji [12] and later this concept has been modified by Deli and Bromi [9] as given below:

Definition 2.3 *Let X be an initial universe set and E be a set of parameters. Let $P(X)$ denote the set of all neutrosophic sets of X. Then, a neutrosophic soft set $(\tilde{F}, E)$ over X is a set defined by a set valued function $\tilde{F}$ representing a mapping $\tilde{F}: E \to P(X)$ where $\tilde{F}$ is called approximate function of the neutrosophic soft set $(\tilde{F}, E)$. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $P(X)$ and therefore it can be written as a set of ordered pairs,*

$$(\tilde{F}, E) = \{(e, \langle x, T_{\tilde{F}(e)}(x), I_{\tilde{F}(e)}(x), F_{\tilde{F}(e)}(x)\rangle : x \in X) : e \in E\}$$

where $T_{\tilde{F}(e)}(x)$, $I_{\tilde{F}(e)}(x)$, $F_{\tilde{F}(e)}(x) \in 0,1]$, respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $\tilde{F}(e)$. Since supremum of each T, I, F is 1 so the inequality $0 \le T_{\tilde{F}(e)}(x) + I_{\tilde{F}(e)}(x) + F_{\tilde{F}(e)}(x) \le 3$ is obvious.

Definition 2.4 *[16] Let $NSS(X, E)$ be the family of all neutrosophic soft sets over the universe set X and $\overset{NSS}{\tau} \subset NSS(X, E)$. Then $\overset{NSS}{\tau}$ is said to be a neutrosophic soft topology on X if*

1. $0_{(X,E)}$ and $1_{(X,E)}$ belongs to $\overset{NSS}{\tau}$

2. The union of any number of neutrosophic soft sets in $\overset{NSS}{\tau}$ belongs to $\overset{NSS}{\tau}$

3. The intersection of finite number of neutrosophic soft sets in $\overset{NSS}{\tau}$ belongs to $\overset{NSS}{\tau}$.

Then $\left(X, \overset{NSS}{\tau}, E\right)$ is said to be a neutrosophic soft topological space over X.

Definition 2.5 *[2] Let X be a universe and E be a set of parameters that are describing the elements of X. A bipolar neutrosophic soft set $(\tilde{B},E)$ in X is defined as;*

$$(\tilde{B},E)=\{(e,\langle x,(T^{+}_{B(e)}(x),I^{+}_{B(e)}(x),F^{+}_{B(e)}(x),T^{-}_{B(e)}(x),I^{-}_{B(e)}(x),F^{-}_{B(e)}(x))\rangle : x\in X): e\in E\}$$

where T^{+}_{B}, I^{+}_{B}, $F^{+}_{B}\to 0,1]$ and T^{-}_{B}, I^{-}_{B}, $F^{-}_{B}\to -1,0]$. The positive membership degree $T^{+}_{B(e)}(x)$, $I^{+}_{B(e)}(x)$, $F^{+}_{B(e)}(x)$ denotes the truth membership, indeterminate membership and false membership of an element corresponding to a bipolar neutrosophic soft set $(\tilde{B},E)$ and the negative membership degree $T^{-}_{B(e)}(x)$, $I^{-}_{B(e)}(x)$, $F^{-}_{B(e)}(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x\in X$ to some implicit counter-property corresponding to a bipolar neutrosophic soft set $(\tilde{B},E)$.

Definition 2.6 *[2] Let $(\tilde{B},E)$ be a bipolar neutrosophic soft set over X. Then, the complement of a bipolar neutrosophic soft set $(\tilde{B},E)$, is denoted by $(\tilde{B},E)^{c}$, is defined as;*

$$(\tilde{B},E)^{c}=\left\{\left(e,\left\langle x,\begin{pmatrix}F^{+}_{B(e)}(x),1-I^{+}_{B(e)}(x),T^{+}_{B(e)}(x),\\ F^{-}_{B(e)}(x),-1-I^{-}_{B(e)}(x),T^{-}_{B(e)}(x)\end{pmatrix}\right\rangle : x\in X\right): e\in E\right\}.$$

3. A New Approach to Operations on Bipolar Neutrosophic Soft Sets

In this section, we re-defined some concepts as absolute bipolar neutrosophic soft set, empty bipolar neutrosophic soft set, bipolar neutrosophic soft subset, bipolar neutrosophic soft union and intersection . In addition, basic properties of these operations was presented.

Definition 3.1 *An empty bipolar neutrosophic soft set $(\tilde{B}^{\emptyset},E)$ over X is defined by;*

$$(\tilde{B}^{\emptyset},E)=\{(e,\langle x,(0,0,1,-1,-1,0)\rangle : x\in X): e\in E\}.$$

An absolute bipolar neutrosophic soft set $(\tilde{B}^{X},E)$ over X is defined by;

$$(\tilde{B}^{X},E)=\{(e,\langle x,(1,1,0,0,0,-1)\rangle : x\in X): e\in E\}.$$

Clearly, $(\tilde{B}^{\emptyset},E)^{c}=(\tilde{B}^{X},E)$ and $(\tilde{B}^{X},E)^{c}=(\tilde{B}^{\emptyset},E)$.

Definition 3.2 *Let $(\tilde{B}_1,E)$ and $(\tilde{B}_2,E)$ be two bipolar neutrosophic soft sets over X. $(\tilde{B}_1,E)$ is said to be bipolar neutrosophic soft subset of $(\tilde{B}_2,E)$ if $T^{+}_{\tilde{B}_1(e)}(x)\le T^{+}_{\tilde{B}_2(e)}(x)$, $I^{+}_{\tilde{B}_1(e)}(x)\le I^{+}_{\tilde{B}_2(e)}(x)$, $F^{+}_{\tilde{B}_1(e)}(x)\ge F^{+}_{\tilde{B}_2(e)}(x)$, $T^{-}_{\tilde{B}_1(e)}(x)\le T^{-}_{\tilde{B}_2(e)}(x)$, $I^{-}_{\tilde{B}_1(e)}(x)\le I^{-}_{\tilde{B}_2(e)}(x)$ and $F^{-}_{\tilde{B}_1(e)}(x)\ge F^{-}_{\tilde{B}_2(e)}(x)$ for all $(e,x)\in E\times X$. It is denoted by $(\tilde{B}_1,E)\sqsubseteq(\tilde{B}_2,E)$.*

$(\tilde{B}_1,E)$ is said to be bipolar neutrosophic soft equal to $(\tilde{B}_2,E)$ if $(\tilde{B}_1,E)$ is bipolar neutrosophic soft subset of $(\tilde{B}_2,E)$ and $(\tilde{B}_2,E)$ is bipolar neutrosophic soft subset of $(\tilde{B}_1,E)$. It is denoted by $(\tilde{B}_1,E)=(\tilde{B}_2,E)$.

Example 3.3 *Let $X=\{x_1,x_2\}$ and $E=\{e_1,e_2\}$. If*

$$(\tilde{B}_1,E)=\begin{Bmatrix}(e_1,\langle x_1,(0.6,0.5,0.3,-0.4,-0.8,-0.4)\rangle,\langle x_2,(0.5,0.4,0.6,-0.4,-0.6,-0.3)\rangle),\\ (e_2,\langle x_1,(0.5,0.7,0.4,-0.3,-0.6,-0.5)\rangle,\langle x_2,(0.3,0.5,0.8,-0.3,-0.4,-0.2)\rangle)\end{Bmatrix}$$

and

$$(\tilde{B}_2,E)=\begin{Bmatrix}(e_1,\langle x_1,(0.7,0.8,0.1,-0.2,-0.5,-0.6)\rangle,\langle x_2,(0.6,0.6,0.3,-0.3,-0.5,-0.7)\rangle),\\ (e_2,\langle x_1,(0.6,0.9,0.2,-0.1,-0.4,-0.7)\rangle,\langle x_2,(0.4,0.7,0.6,-0.2,-0.3,-0.6)\rangle)\end{Bmatrix}$$

then, $(\tilde{B}_1,E)\sqsubseteq(\tilde{B}_2,E)$.

Definition 3.4 *Let* $(\tilde{B}_i, E) = \left\{\left(e, \left\langle x, \left(T^+_{B_i(e)}(x), I^+_{B_i(e)}(x), F^+_{B_i(e)}(x), T^-_{B_i(e)}(x), I^-_{B_i(e)}(x), F^-_{B_i(e)}(x)\right)\right\rangle : x \in X\right) : e \in E\right\}$ *for* $i = 1,2$ *be two bipolar neutrosophic soft sets over* X. *Then their union is denoted by* $(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)$ *and is defined as;*

$$\bigsqcup_{i=1}^{2} (\tilde{B}_i, E) = \left\{\left(e, \left\langle x, \begin{pmatrix} \max\{T^+_{B_i(e)}(x)\}, \max\{I^+_{B_i(e)}(x)\}, \min\{F^+_{B_i(e)}(x)\}, \\ \max\{T^-_{B_i(e)}(x)\}, \max\{I^-_{B_i(e)}(x)\}, \min\{F^-_{B_i(e)}(x)\} \end{pmatrix}\right\rangle : x \in X\right) : e \in E\right\}.$$

Definition 3.5 *Let* $(\tilde{B}_i, E) = \left\{\left(e, \left\langle x, \left(T^+_{B_i(e)}(x), I^+_{B_i(e)}(x), F^+_{B_i(e)}(x), T^-_{B_i(e)}(x), I^-_{B_i(e)}(x), F^-_{B_i(e)}(x)\right)\right\rangle : x \in X\right) : e \in E\right\}$ *for* $i = 1,2$ *be two bipolar neutrosophic soft sets over* X. *Then their intersection is denoted by* $(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)$ *and is defined as;*

$$\bigsqcap_{i=1}^{2} (\tilde{B}_i, E) = \left\{\left(e, \left\langle x, \begin{pmatrix} \min\{T^+_{B_i(e)}(x)\}, \min\{I^+_{B_i(e)}(x)\}, \max\{F^+_{B_i(e)}(x)\}, \\ \min\{T^-_{B_i(e)}(x)\}, \min\{I^-_{B_i(e)}(x)\}, \max\{F^-_{B_i(e)}(x)\} \end{pmatrix}\right\rangle : x \in X\right) : e \in E\right\}.$$

Definition 3.6 *Let* $(\tilde{B}_i, E) = \left\{\left(e, \left\langle x, \left(T^+_{B_i(e)}(x), I^+_{B_i(e)}(x), F^+_{B_i(e)}(x), T^-_{B_i(e)}(x), I^-_{B_i(e)}(x), F^-_{B_i(e)}(x)\right)\right\rangle : x \in X\right) : e \in E\right\}$ *for* $i \in I$ *be a family of bipolar neutrosophic soft sets over* X. *Then,*

$$\bigsqcup_{i \in I} (\tilde{B}_i, E) = \left\{\left(e, \left\langle x, \begin{pmatrix} \sup\{T^+_{B_i(e)}(x)\}, \sup\{I^+_{B_i(e)}(x)\}, \inf\{F^+_{B_i(e)}(x)\}, \\ \sup\{T^-_{B_i(e)}(x)\}, \sup\{I^-_{B_i(e)}(x)\}, \inf\{F^-_{B_i(e)}(x)\} \end{pmatrix}\right\rangle : x \in X\right) : e \in E\right\},$$

$$\bigsqcap_{i \in I} (\tilde{B}_i, E) = \left\{\left(e, \left\langle x, \begin{pmatrix} \inf\{T^+_{B_i(e)}(x)\}, \inf\{I^+_{B_i(e)}(x)\}, \sup\{F^+_{B_i(e)}(x)\}, \\ \inf\{T^-_{B_i(e)}(x)\}, \inf\{I^-_{B_i(e)}(x)\}, \sup\{F^-_{B_i(e)}(x)\} \end{pmatrix}\right\rangle : x \in X\right) : e \in E\right\}.$$

Proposition 3.7 *Let* $(\tilde{B}^{\emptyset}, E)$ *and* $(\tilde{B}^X, E)$ *be the empty bipolar neutrosophic soft set and absolute bipolar neutrosophic soft set over* X, *respectively. Then,*

1. $(\tilde{B}^{\emptyset}, E) \sqsubseteq (\tilde{B}^X, E)$,
2. $(\tilde{B}^{\emptyset}, E) \sqcup (\tilde{B}^X, E) = (\tilde{B}^X, E)$,
3. $(\tilde{B}^{\emptyset}, E) \sqcap (\tilde{B}^X, E) = (\tilde{B}^{\emptyset}, E)$.

Proof. Straightforward.

Remark 3.8 *When we consider the definitions of absolute bipolar neutrosophic soft set, empty bipolar neutrosophic soft set, bipolar neutrosophic soft subset, bipolar neutrosophic soft union and intersection presented by M.Ali et al. in [1] then Proposition 3.7 does not hold.*

Definition 3.9 *Let* $(\tilde{B}_1, E)$ *and* $(\tilde{B}_2, E)$ *be two bipolar neutrosophic soft sets over* X. *Then "*$(\tilde{B}_1, E)$ *difference* $(\tilde{B}_2, E)$*" operation on them is denoted by* $(\tilde{B}_1, E) \setminus (\tilde{B}_2, E) = (\tilde{B}_3, E)$ *and is defined by* $(\tilde{B}_3, E) = (\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)^c$ *as follows:*

$$(\tilde{B}_3, E) = \left\{\left(e, \left\langle x, \begin{pmatrix} T^+_{B_3(e)}(x), I^+_{B_3(e)}(x), F^+_{B_3(e)}(x), \\ T^-_{B_3(e)}(x), I^-_{B_3(e)}(x), F^-_{B_3(e)}(x) \end{pmatrix}\right\rangle : x \in X\right) : e \in E\right\}$$

where

$$T^+_{B_3(e)}(x) = \min\{T^+_{B_1(e)}(x), F^+_{B_2(e)}(x)\}, T^-_{B_3(e)}(x) = \min\{T^-_{B_1(e)}(x), F^-_{B_2(e)}(x)\},$$
$$I^+_{B_3(e)}(x) = \min\{I^+_{B_1(e)}(x), 1 - I^+_{B_2(e)}(x)\}, I^-_{B_3(e)}(x) = \min\{I^-_{B_1(e)}(x), -1 - I^-_{B_2(e)}(x)\},$$
$$F^+_{B_3(e)}(x) = \max\{F^+_{B_1(e)}(x), T^+_{B_2(e)}(x)\}, F^-_{B_3(e)}(x) = \max\{F^-_{B_1(e)}(x), T^-_{B_2(e)}(x)\}.$$

Definition 3.10 *Let* $(\tilde{B}_1, E)$ *and* $(\tilde{B}_2, E)$ *be two bipolar neutrosophic soft sets over* X. *Then "AND" operation on them is denoted by* $(\tilde{B}_1, E) \wedge (\tilde{B}_2, E) = (\tilde{B}_3, E \times E)$ *and is defined by:*

$$(\tilde{B}_3, E \times E) = \left\{\left((e_1, e_2), \left\langle x, \begin{pmatrix} T^+_{B_3(e_1,e_2)}(x), I^+_{B_3(e_1,e_2)}(x), F^+_{B_3(e_1,e_2)}(x), \\ T^-_{B_3(e_1,e_2)}(x), I^-_{B_3(e_1,e_2)}(x), F^-_{B_3(e_1,e_2)}(x) \end{pmatrix}\right\rangle : x \in X\right) : (e_1, e_2) \in E \times E\right\}$$

where

$$T^+_{B_3(e_1,e_2)}(x) = \min\{T^+_{B_1(e_1)}(x), T^+_{B_2(e_2)}(x)\}, T^-_{B_3(e_1,e_2)}(x) = \min\{T^-_{B_1(e_1)}(x), T^-_{B_2(e_2)}(x)\},$$
$$I^+_{B_3(e_1,e_2)}(x) = \min\{I^+_{B_1(e_1)}(x), I^+_{B_2(e_2)}(x)\}, I^-_{B_3(e_1,e_2)}(x) = \min\{I^-_{B_1(e_1)}(x), I^-_{B_2(e_2)}(x)\},$$
$$F^+_{B_3(e_1,e_2)}(x) = \max\{F^+_{B_1(e_1)}(x), F^+_{B_2(e_2)}(x)\}, F^-_{B_3(e_1,e_2)}(x) = \max\{F^-_{B_1(e_1)}(x), F^-_{B_2(e_2)}(x)\}.$$

Definition 3.11 *Let $(\tilde{B}_1, E)$ and $(\tilde{B}_2, E)$ be two bipolar neutrosophic soft sets over X. Then "OR" operation on them is denoted by $(\tilde{B}_1, E) \vee (\tilde{B}_2, E) = (\tilde{B}_3, E \times E)$ and is defined by:*

$$(\tilde{B}_3, E \times E) = \left\{\left((e_1, e_2), \left\langle x, \begin{pmatrix} T^+_{B_3(e_1,e_2)}(x), I^+_{B_3(e_1,e_2)}(x), F^+_{B_3(e_1,e_2)}(x), \\ T^-_{B_3(e_1,e_2)}(x), I^-_{B_3(e_1,e_2)}(x), F^-_{B_3(e_1,e_2)}(x) \end{pmatrix} \right\rangle : x \in X \right) : (e_1, e_2) \in E \times E \right\}$$

where

$$T^+_{B_3(e_1,e_2)}(x) = \max\{T^+_{B_1(e_1)}(x), T^+_{B_2(e_2)}(x)\}, T^-_{B_3(e_1,e_2)}(x) = \max\{T^-_{B_1(e_1)}(x), T^-_{B_2(e_2)}(x)\},$$
$$I^+_{B_3(e_1,e_2)}(x) = \max\{I^+_{B_1(e_1)}(x), I^+_{B_2(e_2)}(x)\}, I^-_{B_3(e_1,e_2)}(x) = \max\{I^-_{B_1(e_1)}(x), I^-_{B_2(e_2)}(x)\},$$
$$F^+_{B_3(e_1,e_2)}(x) = \min\{F^+_{B_1(e_1)}(x), F^+_{B_2(e_2)}(x)\}, F^-_{B_3(e_1,e_2)}(x) = \min\{F^-_{B_1(e_1)}(x), F^-_{B_2(e_2)}(x)\}.$$

Example 3.12 *Let $X = \{x_1, x_2\}$ and $E = \{e_1, e_2\}$. If*

$$(\tilde{B}_1, E) = \begin{Bmatrix} (e_1, \langle x_1, (0.3,0.5,0.7,-0.6,-0.5,-0.7)\rangle, \langle x_2, (0.3,0.5,0.4,-0.2,-0.5,-0.8)\rangle), \\ (e_2, \langle x_1, (0.4,0.4,0.3,-0.7,-0.4,-0.3)\rangle, \langle x_2, (0.5,0.8,0.9,-0.1,-0.9,-0.7)\rangle) \end{Bmatrix}$$

and

$$(\tilde{B}_2, E) = \begin{Bmatrix} (e_1, \langle x_1, (0.4,0.6,0.8,-0.5,-0.3,-0.9)\rangle, \langle x_2, (0.4,0.6,0.2,-0.3,-0.2,-0.3)\rangle), \\ (e_2, \langle x_1, (0.3,0.3,0.5,-0.3,-0.6,-0.8)\rangle, \langle x_2, (0.4,0.5,0.3,-0.6,-0.1,-0.3)\rangle) \end{Bmatrix}$$

then

$$(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E) = \begin{Bmatrix} (e_1, \langle x_1, (0.4,0.6,0.7,-0.5,-0.3,-0.9)\rangle, \langle x_2, (0.4,0.6,0.2,-0.2,-0.2,-0.8)\rangle), \\ (e_2, \langle x_1, (0.4,0.4,0.3,-0.3,-0.4,-0.8)\rangle, \langle x_2, (0.5,0.8,0.3,-0.1,-0.1,-0.7)\rangle) \end{Bmatrix},$$

$$(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E) = \begin{Bmatrix} (e_1, \langle x_1, (0.3,0.5,0.8,-0.6,-0.5,-0.7)\rangle, \langle x_2, (0.3,0.5,0.4,-0.3,-0.5,-0.3)\rangle), \\ (e_2, \langle x_1, (0.3,0.3,0.5,-0.7,-0.6,-0.3)\rangle, \langle x_2, (0.4,0.5,0.9,-0.6,-0.9,-0.3)\rangle) \end{Bmatrix},$$

$$(\tilde{B}_1, E) \backslash (\tilde{B}_2, E) = \begin{Bmatrix} (e_1, \langle x_1, (0.3,0.4,0.7,-0.9,-0.7,-0.5)\rangle, \langle x_2, (0.2,0.4,0.4,-0.3,-0.8,-0.3)\rangle), \\ (e_2, \langle x_1, (0.4,0.4,0.3,-0.8,-0.4,-0.3)\rangle, \langle x_2, (0.3,0.5,0.9,-0.3,-0.9,-0.6)\rangle) \end{Bmatrix},$$

$$(\tilde{B}_1, E) \wedge (\tilde{B}_2, E) = \begin{Bmatrix} ((e_1, e_1), \langle x_1, (0.3,0.5,0.8,-0.6,-0.5,-0.7)\rangle, \langle x_2, (0.3,0.5,0.4,-0.3,-0.5,-0.3)\rangle), \\ ((e_1, e_2), \langle x_1, (0.3,0.3,0.7,-0.6,-0.6,-0.7)\rangle, \langle x_2, (0.3,0.5,0.4,-0.6,-0.5,-0.3)\rangle), \\ ((e_2, e_1), \langle x_1, (0.4,0.4,0.8,-0.7,-0.4,-0.3)\rangle, \langle x_2, (0.4,0.6,0.9,-0.3,-0.9,-0.3)\rangle), \\ ((e_2, e_2), \langle x_1, (0.3,0.3,0.5,-0.7,-0.6,-0.3)\rangle, \langle x_2, (0.4,0.5,0.9,-0.6,-0.9,-0.3)\rangle) \end{Bmatrix},$$

$$(\tilde{B}_1, E) \vee (\tilde{B}_2, E) = \begin{Bmatrix} ((e_1, e_1), \langle x_1, (0.4,0.6,0.7,-0.5,-0.3,-0.9)\rangle, \langle x_2, (0.4,0.6,0.2,-0.2,-0.2,-0.8)\rangle), \\ ((e_1, e_2), \langle x_1, (0.3,0.5,0.5,-0.3,-0.5,-0.8)\rangle, \langle x_2, (0.4,0.5,0.3,-0.2,-0.1,-0.8)\rangle), \\ ((e_2, e_1), \langle x_1, (0.4,0.6,0.3,-0.5,-0.3,-0.9)\rangle, \langle x_2, (0.5,0.8,0.2,-0.1,-0.2,-0.7)\rangle), \\ ((e_2, e_2), \langle x_1, (0.4,0.4,0.3,-0.3,-0.4,-0.8)\rangle, \langle x_2, (0.5,0.8,0.3,-0.1,-0.1,-0.7)\rangle) \end{Bmatrix}.$$

Proposition 3.13 *Let $(\tilde{B}_1, E)$, $(\tilde{B}_2, E)$ and $(\tilde{B}_3, E)$ be bipolar neutrosophic soft sets over X. Then,*

1. $(\tilde{B}_1, E) \sqcup [(\tilde{B}_2, E) \sqcup (\tilde{B}_3, E)] = [(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)] \sqcup (\tilde{B}_3, E)$ and
$(\tilde{B}_1, E) \sqcap [(\tilde{B}_2, E) \sqcap (\tilde{B}_3, E)] = [(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)] \sqcap (\tilde{B}_3, E)$;

2. $(\tilde{B}_1, E) \sqcup [(\tilde{B}_2, E) \sqcap (\tilde{B}_3, E)] = [(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)] \sqcap [(\tilde{B}_1, E) \sqcup (\tilde{B}_3, E)]$ and
$(\tilde{B}_1, E) \sqcap [(\tilde{B}_2, E) \sqcup (\tilde{B}_3, E)] = [(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)] \sqcup [(\tilde{B}_1, E) \sqcap (\tilde{B}_3, E)]$;

3. $(\tilde{B}_1, E) \sqcup (\tilde{B}^{\emptyset}, E) = (\tilde{B}_1, E)$ and $(\tilde{B}_1, E) \sqcap (\tilde{B}^{\emptyset}, E) = (\tilde{B}^{\emptyset}, E)$;

4. $(\tilde{B}_1, E) \sqcup (\tilde{B}^{X}, E) = (\tilde{B}^{X}, E)$ and $(\tilde{B}_1, E) \sqcap (\tilde{B}^{X}, E) = (\tilde{B}_1, E)$;

5. $(\tilde{B}^{\emptyset}, E) \backslash (\tilde{B}^{X}, E) = (\tilde{B}^{\emptyset}, E)$ and $(\tilde{B}^{X}, E) \backslash (\tilde{B}^{\emptyset}, E) = (\tilde{B}^{X}, E)$

Proof. Straightforward.

Proposition 3.14 *Let* $(\tilde{B}_1, E)$ *and* $(\tilde{B}_2, E)$ *be two bipolar neutrosophic soft sets over* X. *Then,*

1. $[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]^c = (\tilde{B}_1, E)^c \sqcap (\tilde{B}_2, E)^c$;
2. $[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]^c = (\tilde{B}_1, E)^c \sqcup (\tilde{B}_2, E)^c$.

Proof. 1. For all $e \in E$ and $x \in X$,

$$\sqcup_{i=1}^{2} (\tilde{B}_i, E) = \left\{ e, \left\langle x, \begin{pmatrix} \max\{T^+_{B_1(e)}(x), T^+_{B_2(e)}(x)\}, \max\{I^+_{B_1(e)}(x), I^+_{B_2(e)}(x)\}, \min\{F^+_{B_1(e)}(x), F^+_{B_2(e)}(x)\}, \\ \max\{T^-_{B_1(e)}(x), T^-_{B_2(e)}(x)\}, \max\{I^-_{B_1(e)}(x), I^-_{B_2(e)}(x)\}, \min\{F^-_{B_1(e)}(x), F^-_{B_2(e)}(x)\} \end{pmatrix} \right\rangle \right\}$$

$$\left[\sqcup_{i=1}^{2} (\tilde{B}_i, E)\right]^c = \left\{ e, \left\langle x, \begin{pmatrix} \min\{F^+_{B_1(e)}(x), F^+_{B_2(e)}(x)\}, 1 - \max\{I^+_{B_1(e)}(x), I^+_{B_2(e)}(x)\}, \max\{T^+_{B_1(e)}(x), T^+_{B_2(e)}(x)\}, \\ \min\{F^-_{B_1(e)}(x), F^-_{B_2(e)}(x)\}, -1 - \max\{I^-_{B_1(e)}(x), I^-_{B_2(e)}(x)\}, \max\{T^-_{B_1(e)}(x), T^-_{B_2(e)}(x)\} \end{pmatrix} \right\rangle \right\}.$$

Now,

$$(\tilde{B}_1, E)^c = \{e, \langle x, (F^+_{B_1(e)}(x), 1 - I^+_{B_1(e)}(x), T^+_{B_1(e)}(x), F^-_{B_1(e)}(x), -1 - I^-_{B_1(e)}(x), T^-_{B_1(e)}(x))\rangle\},$$
$$(\tilde{B}_2, E)^c = \{e, \langle x, (F^+_{B_2(e)}(x), 1 - I^+_{B_2(e)}(x), T^+_{B_2(e)}(x), F^-_{B_2(e)}(x), -1 - I^-_{B_2(e)}(x), T^-_{B_2(e)}(x))\rangle\}.$$

Then,

$$\sqcap_{i=1}^{2} (\tilde{B}_i, E)^c = \left\{ e, \left\langle x, \begin{pmatrix} \min\{F^+_{B_1(e)}(x), F^+_{B_2(e)}(x)\}, \min\{(1 - I^+_{B_1(e)}(x)), (1 - I^+_{B_2(e)}(x))\}, \max\{T^+_{B_1(e)}(x), T^+_{B_2(e)}(x)\} \\ \min\{F^-_{B_1(e)}(x), F^-_{B_2(e)}(x)\}, \min\{(-1 - I^-_{B_1(e)}(x)), (-1 - I^-_{B_2(e)}(x))\}, \max\{T^-_{B_1(e)}(x), T^-_{B_2(e)}(x)\} \end{pmatrix} \right\rangle \right\}$$
$$= \left\{ e, \left\langle x, \begin{pmatrix} \min\{F^+_{B_1(e)}(x), F^+_{B_2(e)}(x)\}, 1 - \max\{I^+_{B_1(e)}(x), I^+_{B_2(e)}(x)\}, \max\{T^+_{B_1(e)}(x), T^+_{B_2(e)}(x)\}, \\ \min\{F^-_{B_1(e)}(x), F^-_{B_2(e)}(x)\}, -1 - \max\{I^-_{B_1(e)}(x), I^-_{B_2(e)}(x)\}, \max\{T^-_{B_1(e)}(x), T^-_{B_2(e)}(x)\} \end{pmatrix} \right\rangle \right\}.$$

Thus, $[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]^c = (\tilde{B}_1, E)^c \sqcap (\tilde{B}_2, E)^c$.

2. It is obtained in a similar way.

Proposition 3.15 *Let* $(\tilde{B}_1, E)$ *and* $(\tilde{B}_2, E)$ *be two bipolar neutrosophic soft sets over* X. *Then,*

1. $[(\tilde{B}_1, E) \vee (\tilde{B}_2, E)]^c = (\tilde{B}_1, E)^c \wedge (\tilde{B}_2, E)^c$;
2. $[(\tilde{B}_1, E) \wedge (\tilde{B}_2, E)]^c = (\tilde{B}_1, E)^c \vee (\tilde{B}_2, E)^c$.

Proof. 1. For all $(e_1, e_2) \in E \times E$ and $x \in X$,

$$\vee_{i=1}^{2} (\tilde{B}_i, E) = \left\{ (e_1, e_2), \left\langle x, \begin{pmatrix} \max\{T^+_{B_1(e_1)}(x), T^+_{B_2(e_2)}(x)\}, \max\{I^+_{B_1(e_1)}(x), I^+_{B_2(e_2)}(x)\}, \min\{F^+_{B_1(e_1)}(x), F^+_{B_2(e_2)}(x)\}, \\ \max\{T^-_{B_1(e_1)}(x), T^-_{B_2(e_2)}(x)\}, \max\{I^-_{B_1(e_1)}(x), I^-_{B_2(e_2)}(x)\}, \min\{F^-_{B_1(e_1)}(x), F^-_{B_2(e_2)}(x)\} \end{pmatrix} \right\rangle \right\},$$

$$\left[\vee_{i=1}^{2} (\tilde{B}_i, E)\right]^c = \left\{ (e_1, e_2), \left\langle x, \begin{matrix} \min\{F^+_{B_1(e_1)}(x), F^+_{B_2(e_2)}(x)\}, 1 - \max\{I^+_{B_1(e_1)}(x), I^+_{B_2(e_2)}(x)\}, \max\{T^+_{B_1(e_1)}(x), T^+_{B_2(e_2)}(x)\}, \\ \min\{F^-_{B_1(e_1)}(x), F^-_{B_2(e_2)}(x)\}, -1 - \max\{I^-_{B_1(e_1)}(x), I^-_{B_2(e_2)}(x)\}, \max\{T^-_{B_1(e_1)}(x), T^-_{B_2(e_2)}(x)\} \end{matrix} \right\rangle \right\}.$$

On the other hand,

$$(\tilde{B}_1, E)^c = \{e_1, \langle x, F^+_{B_1(e_1)}(x), 1 - I^+_{B_1(e_1)}(x), T^+_{B_1(e_1)}(x), F^-_{B_1(e_1)}(x), -1 - I^-_{B_1(e_1)}(x), T^-_{B_1(e_1)}(x)\rangle : e_1 \in E\},$$
$$(\tilde{B}_2, E)^c = \{e_2, \langle x, F^+_{B_2(e_2)}(x), 1 - I^+_{B_2(e_2)}(x), T^+_{B_2(e_2)}(x), F^-_{B_2(e_2)}(x), -1 - I^-_{B_2(e_2)}(x), T^-_{B_2(e_2)}(x)\rangle : e_2 \in E\}.$$

Then,

$$\wedge_{i=1}^{2} (\tilde{B}_i, E)^c = \left\{ (e_1, e_2), \left\langle \begin{matrix} x, \min\{F^+_{B_1(e_1)}(x), F^+_{B_2(e_2)}(x)\}, \min\{(1 - I^+_{B_1(e_1)}(x)), (1 - I^+_{B_2(e_2)}(x))\}, \max\{T^+_{B_1(e_1)}(x), T^+_{B_2(e_2)}(x)\} \\ \min\{F^-_{B_1(e_1)}(x), F^-_{B_2(e_2)}(x)\}, \min\{(-1 - I^-_{B_1(e_1)}(x)), (-1 - I^-_{B_2(e_2)}(x))\}, \max\{T^-_{B_1(e_1)}(x), T^-_{B_2(e_2)}(x)\} \end{matrix} \right\rangle \right\}$$
$$= \left\{ (e_1, e_2), \left\langle x, \begin{matrix} \min\{F^+_{B_1(e_1)}(x), F^+_{B_2(e_2)}(x)\}, 1 - \max\{I^+_{B_1(e_1)}(x), I^+_{B_2(e_2)}(x)\}, \max\{T^+_{B_1(e_1)}(x), T^+_{B_2(e_2)}(x)\}, \\ \min\{F^-_{B_1(e_1)}(x), F^-_{B_2(e_2)}(x)\}, -1 - \max\{I^-_{B_1(e_1)}(x), I^-_{B_2(e_2)}(x)\}, \max\{T^-_{B_1(e_1)}(x), T^-_{B_2(e_2)}(x)\} \end{matrix} \right\rangle \right\}.$$

Hence, $[(\tilde{B}_1, E) \vee (\tilde{B}_2, E)]^c = (\tilde{B}_1, E)^c \wedge (\tilde{B}_2, E)^c$.

2. It is obtained in a similar way.

4. Bipolar Neutrosophic Soft Topological Spaces

In this section we defined neutrosophic soft topology by the revised form of neutrosophic soft sets and also we gave the basic structures of the bipolar neutrosophic soft topological spaces.

Definition 4.1 *Let $BNSS(X,E)$ be the family of all bipolar neutrosophic soft sets over X and $\tau^{BN} \subset BNSS(X,E)$. Then τ^{BN} is said to be a bipolar neutrosophic soft topology on X if*

1. $(\tilde{B}^{\emptyset},E)$ and $(\tilde{B}^{X},E)$ belongs to τ^{BN}
2. the union of any number of bipolar neutrosophic soft sets in τ^{BN} belongs to τ^{BN}
3. the intersection of finite number of bipolar neutrosophic soft sets in τ^{BN} belongs to τ^{BN}.

Then (X,τ^{BN},E) is said to be a bipolar neutrosophic soft topological space over X. Each members of τ^{BN} is said to be bipolar neutrosophic soft open set.

Definition 4.2 *Let (X,τ^{BN},E) be a bipolar neutrosophic soft topological space over X and $(\tilde{B},E)$ be a bipolar neutrosophic soft set over X. Then $(\tilde{B},E)$ is said to be bipolar neutrosophic soft closed set iff its complement is a bipolar neutrosophic soft open set.*

Proposition 4.3 *Let (X,τ^{BN},E) be a bipolar neutrosophic soft topological space over X. Then*

1. $(\tilde{B}^{\emptyset},E)$ and $(\tilde{B}^{X},E)$ are bipolar neutrosophic soft closed sets over X
2. the intersection of any number of bipolar neutrosophic soft closed sets is a bipolar neutrosophic soft closed set over X
3. the union of finite number of bipolar neutrosophic soft closed sets is a bipolar neutrosophic soft closed set over X.

Proof. It is easily obtained from the definition bipolar neutrosophic soft topological space and Proposition 2.

Definition 4.4 *Let $BNSS(X,E)$ be the family of all bipolar neutrosophic soft sets over the universe set X.*

1. If $\tau^{BN} = \{(\tilde{B}^{\emptyset},E),(\tilde{B}^{X},E)\}$, then τ^{BN} is said to be the bipolar neutrosophic soft indiscrete topology and (X,τ^{BN},E) is said to be a bipolar neutrosophic soft indiscrete topological space over X.
2. If $\tau^{BN} = BNSS(X,E)$, then τ^{BN} is said to be the bipolar neutrosophic soft discrete topology and (X,τ^{BN},E) is said to be a bipolar neutrosophic soft discrete topological space over X.

Proposition 4.5 *Let (X,τ_1^{BN},E) and (X,τ_2^{BN},E) be two bipolar neutrosophic soft topological spaces over the same universe set X. Then $(X,\tau_1^{BN} \cap \tau_2^{BN},E)$ is bipolar neutrosophic soft topological space over X.*

Proof. 1. Since $(\tilde{B}^{\emptyset},E)$, $(\tilde{B}^{X},E) \in \tau_1^{BN}$ and $(\tilde{B}^{\emptyset},E)$, $(\tilde{B}^{X},E) \in \tau_2^{BN}$, then $(\tilde{B}^{\emptyset},E)$, $(\tilde{B}^{X},E) \in \tau_1^{BN} \cap \tau_2^{BN}$.
2. Suppose that $\{(\tilde{B}_i,E) | i \in I\}$ be a family of bipolar neutrosophic soft sets in $\tau_1^{BN} \cap \tau_2^{BN}$. Then $(\tilde{B}_i,E) \in \tau_1^{BN}$ and $(\tilde{B}_i,E) \in \tau_2^{BN}$ for all $i \in I$, so $\sqcup_{i \in I} (\tilde{B}_i,E) \in \tau_1^{BN}$ and $\sqcup_{i \in I} (\tilde{B}_i,E) \in \tau_2^{BN}$. Thus $\sqcup_{i \in I} (\tilde{B}_i,E) \in \tau_1^{BN} \cap \tau_2^{BN}$.
3. Let $\{(\tilde{B}_i,E) | i = \overline{1,n}\}$ be a family of the finite number of bipolar neutrosophic soft sets in $\tau_1^{BN} \cap \tau_2^{BN}$. Then $(\tilde{B}_i,E) \in \tau_1^{BN}$ and $(\tilde{B}_i,E) \in \tau_2^{BN}$ for $i = \overline{1,n}$, so $\sqcap_{i=1}^{n} (\tilde{B}_i,E) \in \tau_1^{BN}$ and $\sqcap_{i=1}^{n} (\tilde{B}_i,E) \in \tau_2^{BN}$. Thus $\sqcap_{i=1}^{n} (\tilde{B}_i,E) \in \tau_1^{BN} \cap \tau_2^{BN}$.

Remark 4.6 *The union of two bipolar neutrosophic soft topologies over X may not be a bipolar neutrosophic soft topology on X.*

Example 4.7 *Let $X=\{x_1,x_2\}$ be an initial universe set, $E=\{e_1,e_2\}$ be a set of parameters and $\tau_1^{BN}=\{(\tilde{B}^{\emptyset},E),(\tilde{B}^{U},E),(\tilde{B}_1,E),(\tilde{B}_2,E),(\tilde{B}_3,E)\}$ and $\tau_2^{BN}=\{(\tilde{B}^{\emptyset},E),(\tilde{B}^{U},E),(\tilde{B}_2,E),(\tilde{B}_4,E)\}$ be two bipolar neutrosophic soft topologies over X. Here, the bipolar neutrosophic soft sets $(\tilde{B}_1,E)$, $(\tilde{B}_2,E)$, $(\tilde{B}_3,E)$ and $(\tilde{B}_4,E)$ over X are defined as following:*

$$(\tilde{B}_1,E)=\begin{Bmatrix} e_1,\langle x_1,(0.9,0.4,0.3,-0.2,-0.3,-0.7)\rangle,\langle x_2,(0.5,0.6,0.5,-0.1,-0.2,-0.8)\rangle \\ e_2,\langle x_1,(0.7,0.3,0.4,-0.4,-0.5,-0,4)\rangle,\langle x_2,(0.6,0.6,0.2,-0.6,-0.7,-0.5)\rangle \end{Bmatrix},$$
$$(\tilde{B}_2,E)=\begin{Bmatrix} e_1,\langle x_1,(0.7,0.4,0.5,-0.3,-0.4,-0.6)\rangle,\langle x_2,(0.4,0.5,0.5,-0.2,-0.3,-0.7)\rangle \\ e_2,\langle x_1,(0.6,0.2,0.4,-0.5,-0.6,-0.3)\rangle,\langle x_2,(0.5,0.4,0.3,-0.7,-0.8,-0.4)\rangle \end{Bmatrix},$$
$$(\tilde{B}_3,E)=\begin{Bmatrix} e_1,\langle x_1,(0.5,0.3,0.6,-0.4,-0.5,-0.5)\rangle,\langle x_2,(0.3,0.4,0.7,-0.3,-0.4,-0.6)\rangle \\ e_2,\langle x_1,(0.4,0.1,0.5,-0.6,-0.7,-0.2)\rangle,\langle x_2,(0.4,0.3,0.4,-0.8,-0.9,-0.3)\rangle \end{Bmatrix},$$
$$(\tilde{B}_4,E)=\begin{Bmatrix} e_1,\langle x_1,(0.8,0.5,0.4,-0.1,-0.2,-0.8)\rangle,\langle x_2,(0.5,0.6,0.3,-0.1,-0.1,-0.9)\rangle \\ e_2,\langle x_1,(0.7,0.3,0.3,-0.3,-0.4,-0.5)\rangle,\langle x_2,(0.6,0.5,0.1,-0.5,-0.6,-0.6)\rangle \end{Bmatrix}.$$

Since $(\tilde{B}_1,E)\cup(\tilde{B}_4,E)\notin\tau_1^{BN}\sqcup\tau_2^{BN}$, then $\tau_1^{BN}\sqcup\tau_2^{BN}$ is not a bipolar neutrosophic soft topology over X.

Proposition 4.8 *Let (X,τ^{BN},E) be a bipolar neutrosophic soft topological space over X and $\tau^{BN}=\{(\tilde{B}_i,E):(\tilde{B}_i,E)\in BNSS(X,E)\}$ where*

$(\tilde{B}_i,E)=\{(e,\langle x,(T^{+}_{B_i(e)}(x),I^{+}_{B_i(e)}(x),F^{+}_{B_i(e)}(x),T^{-}_{B_i(e)}(x),I^{-}_{B_i(e)}(x),F^{-}_{B_i(e)}(x))\rangle:x\in X):e\in E\}$ for $i\in I$.

Then

$$\overset{NSS}{\tau}=\{(\tilde{B}_i^{+},E)=\{(e,\langle x,(T^{+}_{B_i(e)}(x),I^{+}_{B_i(e)}(x),F^{+}_{B_i(e)}(x))\rangle:x\in X):e\in E\}:(\tilde{B}_i^{+},E)\in NSS(X,E)\}$$

define neutrosophic soft topology on X.

Proof. Straightforward.

Definition 4.9 *Let (X,τ^{BN},E) be a bipolar neutrosophic soft topological space over X and $(\tilde{B},E)\in BNSS(X,E)$ be a bipolar neutrosophic soft set. Then, bipolar neutrosophic soft interior of $(\tilde{B},E)$, denoted $(\tilde{B},E)^{\circ}$, is defined as the bipolar neutrosophic soft union of all bipolar neutrosophic soft open subsets of $(\tilde{B},E)$.* Clearly, $(\tilde{B},E)^{\circ}$ is the biggest bipolar neutrosophic soft open set contained by $(\tilde{B},E)$.

Example 4.10 *Let us consider the bipolar neutrosophic soft topology τ_1^{BN} given in Example 4.7. Suppose that an any $(\tilde{B},E)\in BNSS(X,E)$ is defined as following:*

$$(\tilde{B},E)=\begin{Bmatrix} e_1,\langle x_1,(0.8,0.4,0.2,-0.1,-0.2,-0.6)\rangle,\langle x_2,(0.4,0.7,0.3,-0.2,-0.1,-0.9)\rangle \\ e_2,\langle x_1,(0.9,0.2,0.3,-0.3,-0.6,-0.5)\rangle,\langle x_2,(0.7,0.5,0.1,-0.4,-0.6,-0.6)\rangle \end{Bmatrix}.$$

Then $(\tilde{B}^{\emptyset},E)$, $(\tilde{B}_2,E)$, $(\tilde{B}_3,E)\sqsubseteq(\tilde{B},E)$. Therefore, $(\tilde{B},E)^{\circ}=(\tilde{B}^{\emptyset},E)\sqcup(\tilde{B}_2,E)\sqcup(\tilde{B}_3,E)=(\tilde{B}_2,E)$.

Theorem 4.11 *Let (X,τ^{BN},E) be a bipolar neutrosophic soft topological space over X and $(\tilde{B},E)\in BNSS(X,E)$. $(\tilde{B},E)$ is a bipolar neutrosophic soft open set iff $(\tilde{B},E)=(\tilde{B},E)^{\circ}$.*

Proof. Let $(\tilde{B},E)$ be a bipolar neutrosophic soft open set. Then the biggest bipolar neutrosophic soft open set that is contained by $(\tilde{B},E)$ is equal to $(\tilde{B},E)$. Hence, $(\tilde{B},E)=(\tilde{B},E)^{\circ}$.

Conversely, it is known that $(\tilde{B},E)^{\circ}$ is a bipolar neutrosophic soft open set and if $(\tilde{B},E)=(\tilde{B},E)^{\circ}$, then $(\tilde{B},E)$ is a bipolar neutrosophic soft open set.

Theorem 4.12 *Let (X, τ^{BN}, E) be a bipolar neutrosophic soft topological space over X and $(\tilde{B}_1, E), (\tilde{B}_2, E) \in BNSS(X, E)$. Then,*

1. $[(\tilde{B}_1, E)^\circ]^\circ = (\tilde{B}_1, E)^\circ$,
2. $(\tilde{B}^\emptyset, E)^\circ = (\tilde{B}^\emptyset, E)$ and $(\tilde{B}^X, E)^\circ = (\tilde{B}^X, E)$,
3. $(\tilde{B}_1, E) \sqsubseteq (\tilde{B}_2, E) \Rightarrow (\tilde{B}_1, E)^\circ \sqsubseteq (\tilde{B}_2, E)^\circ$,
4. $[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]^\circ = (\tilde{B}_1, E)^\circ \sqcap (\tilde{B}_2, E)^\circ$,
5. $(\tilde{B}_1, E)^\circ \sqcup (\tilde{B}_2, E)^\circ \sqsubseteq [(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]^\circ$.

Proof. 1. Let $(\tilde{B}_1, E)^\circ = (\tilde{B}_2, E)$. Then $(\tilde{B}_2, E) \in \tau^{BN}$ iff $(\tilde{B}_2, E) = (\tilde{B}_2, E)^\circ$. So, $[(\tilde{B}_1, E)^\circ]^\circ = (\tilde{B}_1, E)^\circ$.
2. Straighforward.
3. It is known that $(\tilde{B}_1, E)^\circ \sqsubseteq (\tilde{B}_1, E) \sqsubseteq (\tilde{B}_2, E)$ and $(\tilde{B}_2, E)^\circ \sqsubseteq (\tilde{B}_2, E)$. Since $(\tilde{B}_2, E)^\circ$ is the biggest bipolar neutrosophic soft open set contained in $(\tilde{B}_2, E)$ and so, $(\tilde{B}_1, E)^\circ \sqsubseteq (\tilde{B}_2, E)^\circ$.
4. Since $(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E) \sqsubseteq (\tilde{B}_1, E)$ and $(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E) \sqsubseteq (\tilde{B}_2, E)$, then $[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]^\circ \sqsubseteq (\tilde{B}_1, E)^\circ$ and $[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]^\circ \sqsubseteq (\tilde{B}_2, E)^\circ$ and so, $[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]^\circ \sqsubseteq (\tilde{B}_1, E)^\circ \sqcap (\tilde{B}_2, E)^\circ$.
On the other hand, since $(\tilde{B}_1, E)^\circ \sqsubseteq (\tilde{B}_1, E)$ and $(\tilde{B}_2, E)^\circ \sqsubseteq (\tilde{B}_2, E)$, then $(\tilde{B}_1, E)^\circ \sqcap (\tilde{B}_2, E)^\circ \sqsubseteq (\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)$. Besides, $[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]^\circ \sqsubseteq (\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)$ and it is the biggest bipolar neutrosophic soft open set. Therefore, $(\tilde{B}_1, E)^\circ \sqcap (\tilde{B}_2, E)^\circ \sqsubseteq [(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]^\circ$.
Thus, $[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]^\circ = (\tilde{B}_1, E)^\circ \sqcap (\tilde{B}_2, E)^\circ$.
5. Since $(\tilde{B}_1, E) \sqsubseteq (\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)$ and $(\tilde{B}_2, E) \sqsubseteq (\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)$, then $(\tilde{B}_1, E)^\circ \sqsubseteq [(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]^\circ$ and $(\tilde{B}_2, E)^\circ \sqsubseteq [(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]^\circ$. Therefore, $(\tilde{B}_1, E)^\circ \sqcup (\tilde{B}_2, E)^\circ \sqsubseteq [(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]^\circ$.

Definition 4.13 *Let (X, τ^{BN}, E) be a bipolar neutrosophic soft topological space over X and $(\tilde{B}, E) \in BNSS(X, E)$ be a bipolar neutrosophic soft set. Then, bipolar neutrosophic soft closure of $(\tilde{B}, E)$, denoted $\overline{(\tilde{B}, E)}$, is defined as the bipolar neutrosophic soft intersection of all bipolar neutrosophic soft closed supersets of $(\tilde{B}, E)$.*
Clearly, $\overline{(\tilde{B}, E)}$ is the smallest bipolar neutrosophic soft closed set that containing $(\tilde{B}, E)$.

Example 4.14 *Let us consider the bipolar neutrosophic soft topology τ_1^{BN} given in Example 4.7. Suppose that an any $(\tilde{B}, E) \in BNSS(X, E)$ is defined as following:*

$$(\tilde{B}, E) = \begin{Bmatrix} e_1, \langle x_1, (0.1,0.4,0.9,-0.8,-0.9,-0.1)\rangle, \langle x_2, (0.4,0.2,0.7,-0.9,-0.8,-0.1)\rangle \\ e_2, \langle x_1, (0.2,0.3,0.8,-0.6,-0.7,-0,2)\rangle, \langle x_2, (0.1,0.2,0.8,-0.6,-0.7,-0.4)\rangle \end{Bmatrix}.$$

Obviously, $(\tilde{B}^\emptyset, E)$, $(\tilde{B}^U, E)$, $(\tilde{B}_1, E)^c$, $(\tilde{B}_2, E)^c$ and $(\tilde{B}_3, E)^c$ are all bipolar neutrosophic soft closed sets over (X, τ_1^{BN}, E). They are given as following:

$$(\tilde{B}^\emptyset, E)^c = (\tilde{B}^U, E), (\tilde{B}^U, E)^c = (\tilde{B}^\emptyset, E)$$
$$(\tilde{B}_1, E)^c = \begin{Bmatrix} e_1, \langle x_1, (0.3,0.6,0.9,-0.7,-0.7,-0.2)\rangle, \langle x_2, (0.5,0.4,0.5,-0.8,-0.8,-0.1)\rangle \\ e_2, \langle x_1, (0.4,0.7,0.7,-0.4,-0.5,-0,4)\rangle, \langle x_2, (0.2,0.4,0.6,-0.5,-0.3,-0.6)\rangle \end{Bmatrix},$$
$$(\tilde{B}_2, E)^c = \begin{Bmatrix} e_1, \langle x_1, (0.5,0.6,0.7,-0.6,-0.6,-0.3)\rangle, \langle x_2, (0.5,0.5,0.4,-0.7,-0.7,-0.2)\rangle \\ e_2, \langle x_1, (0.4,0.8,0.6,-0.3,-0.4,-0,5)\rangle, \langle x_2, (0.3,0.6,0.5,-0.4,-0.2,-0.7)\rangle \end{Bmatrix},$$
$$(\tilde{B}_3, E)^c = \begin{Bmatrix} e_1, \langle x_1, (0.6,0.7,0.5,-0.5,-0.5,-0.4)\rangle, \langle x_2, (0.7,0.6,0.3,-0.6,-0.6,-0.3)\rangle \\ e_2, \langle x_1, (0.5,0.9,0.4,-0.2,-0.3,-0,6)\rangle, \langle x_2, (0.4,0.7,0.4,-0.3,-0.1,-0.8)\rangle \end{Bmatrix}.$$

Then $(\tilde{B}^U, E)^c$, $(\tilde{B}_1, E)^c$, $(\tilde{B}_2, E)^c, (\tilde{B}_3, E)^c \sqsupseteq (\tilde{B}, E)$. Therefore, $\overline{(\tilde{B}, E)} = (\tilde{B}^U, E)^c \sqcap (\tilde{B}_1, E)^c \sqcap (\tilde{B}_2, E)^c \sqcap (\tilde{B}_3, E)^c = (\tilde{B}_1, E)^c$.

Theorem 4.15 *Let* (X, τ^{BN}, E) *be a bipolar neutrosophic soft topological space over* X *and* $(\tilde{B}, E) \in BNSS(X, E)$. $(\tilde{B}, E)$ *is bipolar neutrosophic soft closed set iff* $(\tilde{B}, E) = \overline{(\tilde{B}, E)}$.

Proof. Straightforward.

Theorem 4.16 *Let* (X, τ^{BN}, E) *be a bipolar neutrosophic soft topological space over* X *and* $(\tilde{B}_1, E), (\tilde{B}_2, E) \in BNSS(X, E)$. *Then,*

1. $\overline{\left[\overline{(\tilde{B}_1, E)}\right]} = \overline{(\tilde{B}_1, E)}$,
2. $\overline{(\tilde{B}^{\emptyset}, E)} = (\tilde{B}^{\emptyset}, E)$ and $\overline{(\tilde{B}^{X}, E)} = (\tilde{B}^{X}, E)$
3. $(\tilde{B}_1, E) \sqsubseteq (\tilde{B}_2, E) \Rightarrow \overline{(\tilde{B}_1, E)} \sqsubseteq \overline{(\tilde{B}_2, E)}$,
4. $\overline{[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]} = \overline{(\tilde{B}_1, E)} \sqcup \overline{(\tilde{B}_2, E)}$,
5. $\overline{[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]} \sqsubseteq \overline{(\tilde{B}_1, E)} \sqcap \overline{(\tilde{B}_2, E)}$.

Proof. 1. Let $\overline{(\tilde{B}_1, E)} = (\tilde{B}_2, E)$. Then, $(\tilde{B}_2, E)$ is a bipolar neutrosophic soft closed set. Hence, $(\tilde{B}_2, E)$ and $\overline{(\tilde{B}_2, E)}$ are equal. Therefore, $\overline{\left[\overline{(\tilde{B}_1, E)}\right]} = \overline{(\tilde{B}_1, E)}$.

2. Straightforward.

3. It is known that $(\tilde{B}_1, E) \sqsubseteq \overline{(\tilde{B}_1, E)}$ and $(\tilde{B}_2, E) \sqsubseteq \overline{(\tilde{B}_2, E)}$ and so, $(\tilde{B}_1, E) \sqsubseteq (\tilde{B}_2, E) \sqsubseteq \overline{(\tilde{B}_2, E)}$. Since $\overline{(\tilde{B}_1, E)}$ is the smallest bipolar neutrosophic soft closed set containing $(\tilde{B}_1, E)$, then $\overline{(\tilde{B}_1, E)} \sqsubseteq \overline{(\tilde{B}_2, E)}$.

4. Since $(\tilde{B}_1, E) \sqsubseteq (\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)$ and $(\tilde{B}_2, E) \sqsubseteq (\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)$, then $\overline{(\tilde{B}_1, E)} \sqsubseteq \overline{[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]}$ and $\overline{(\tilde{B}_2, E)} \sqsubseteq \overline{[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]}$ and so, $\overline{(\tilde{B}_1, E)} \sqcup \overline{(\tilde{B}_2, E)} \sqsubseteq \overline{[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]}$.
Conversely, since $(\tilde{B}_1, E) \sqsubseteq \overline{(\tilde{B}_1, E)}$ and $(\tilde{B}_2, E) \sqsubseteq \overline{(\tilde{B}_2, E)}$, then $(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E) \sqsubseteq \overline{(\tilde{B}_1, E)} \sqcup \overline{(\tilde{B}_2, E)}$. Besides, $\overline{[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]}$ is the smallest bipolar neutrosophic soft closed set that containing $(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)$. Therefore, $\overline{[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]} \sqsubseteq \overline{(\tilde{B}_1, E)} \sqcup \overline{(\tilde{B}_2, E)}$. Thus, $\overline{[(\tilde{B}_1, E) \sqcup (\tilde{B}_2, E)]} = \overline{(\tilde{B}_1, E)} \sqcup \overline{(\tilde{B}_2, E)}$.

5. Since $(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E) \sqsubseteq \overline{(\tilde{B}_1, E)} \sqcap \overline{(\tilde{B}_2, E)}$ and $\overline{[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]}$ is the smallest bipolar neutrosophic soft closed set that containing $(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)$, then $\overline{[(\tilde{B}_1, E) \sqcap (\tilde{B}_2, E)]} \sqsubseteq \overline{(\tilde{B}_1, E)} \sqcap \overline{(\tilde{B}_2, E)}$.

Theorem 4.17 *Let* (X, τ^{BN}, E) *be a bipolar neutrosophic soft topological space over* X *and* $(\tilde{B}, E) \in BNSS(X, E)$. *Then,*

1. $\left[\overline{(\tilde{B}, E)}\right]^c = \left[(\tilde{B}, E)^c\right]^{\circ}$,
2. $\left[(\tilde{B}, E)^{\circ}\right]^c = \overline{\left[(\tilde{B}, E)^c\right]}$.

Proof. 1. $\overline{(\tilde{B}, E)} = \sqcap_{i \in I} \{(\tilde{B}_i, E) \in (\tau^{BN})^c : (\tilde{B}_i, E) \sqsupseteq (\tilde{B}, E)\}$

$$\Rightarrow \left[\overline{(\tilde{B}, E)}\right]^c = \left[\sqcap_{i \in I} \{(\tilde{B}_i, E) \in (\tau^{BN})^c : (\tilde{B}_i, E) \sqsupseteq (\tilde{B}, E), \forall i \in I\}\right]^c$$

$$= \sqcup_{i \in I} \left\{(\tilde{B}_i, E)^c \in \overset{NSS}{\tau} : (\tilde{B}_i, E)^c \sqsubseteq (\tilde{B}, E)^c\right\} = \left[(\tilde{B}, E)^c\right]^{\circ}.$$

2. $(\tilde{B}, E)^{\circ} = \sqcup_{i \in I} \{(\tilde{B}_i, E) \in \tau^{BN} : (\tilde{B}_i, E) \sqsubseteq (\tilde{B}, E)\}$

$$\Rightarrow \left[(\tilde{B}, E)^{\circ}\right]^c = \left[\sqcup_{i \in I} \left\{(\tilde{B}_i, E) \in \overset{NSS}{\tau} : (\tilde{B}_i, E) \sqsubseteq (\tilde{F}, E)\right\}\right]^c$$

$$= \sqcap_{i \in I} \{(\tilde{B}_i, E)^c \in (\tau^{BN})^c : (\tilde{B}_i, E)^c \sqsupseteq (\tilde{B}, E)^c\} = \overline{\left[(\tilde{B}, E)^c\right]}.$$

5. Conclusions

Re-defined operations in this study are placed on a suitable system to present topological structure on bipolar neutrosophic soft sets. Later, bipolar neutrosophic soft topological spaces are defined and their structural properties are presented. Since this study is the basic characteristic of bipolar neutrosophic soft set theory, it will be able to lead the study of bipolar neutrosophic soft set structure in all sub-branches of mathematics. It can be also considered as a preliminary study of the theory mentioned in topology.

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Conflicts of Interest

The authors declare no conflict of interest.

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