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Generalized OWA Operator of Orthopair Neutrosophic Numbers and Their Application in Multiple Attribute Decision-Making Problems

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Abstract: A neutrosophic number (NN) is a useful mathematical tool in indeterminacy theory. As the mixed form of an intuitionistic fuzzy set and NN, an orthopair neutrosophic number (ONN) can express the true indeterminate degree and the false indeterminate degree. In view of generalized ordered weighted operators, this article presents two generalized ordered weighted operators of ONNs, including an orthopair neutrosophic number generalized ordered weighted average (ONNGOWA) operator and an orthopair neutrosophic number generalized ordered weighted geometric (ONNGOWG) operator, and their characteristics. A multi-attribute decision-making (MADM) model is established by the weighted operation of the ONNGOWA and ONNGOWG operators. Finally, an example on the selection problem of electric vehicle design schemes is given to reflect the effectivity of the proposed MADM model in the scenario of ONNs.

Keywords: orthopair neutrosophic number; generalized ordered weighted operator; multi-attribute decision-making

1. Introduction

In practical applications, it is difficult for decision makers to provide accurate evaluation values for complex decision-making problems in uncertain and incomplete circumstances. In this case, Zadeh presented the concept of fuzzy sets (FSs) [1]. On the basis of an extension of FS, Atanassov added a new parameter named a non-membership degree and defined an intuitionistic fuzzy set (IFS) [2]. Then, some scholars [3, 4] developed some intuitionistic fuzzy decision-making methods. Since various aggregation operators reveal important mathematical tools in multi-attribute decision-making (MADM) process, various aggregation operators of intuitionistic fuzzy numbers (IFNs) were proposed by many scholars. For example, Xu and Cai [5] and Xu and Yager [6] proposed intuitionistic fuzzy weighted aggregation operators, and then some researchers introduced the generalized aggregation operator of IFNs [7], the generalized geometric aggregation operator of IFNs [8], the induced generalized aggregation operators of IFNs [9], the power average operators of trapezoidal IFNs [10], and the Heronian aggregation operators of IFNs [11]. However, IFS/IFN cannot reasonably represent uncertain problems with uncertain membership and non-membership degrees. To express uncertain information, Smarandache proposed the concept of a neutrosophic number (NN) [12-14]. It is denoted by $N = g + hI$ for $I \in [I^-, I^+]$, where g is the determinate part and hI is the indeterminate part. Since NNs are very suitable for dealing with real problems with indeterminacy $I \in [I^-, I^+]$, they were currently used in production planning problems [15], fault diagnosis [16], medicine assessment [17], prediction of traffic volume [18]. Recently, Ye et al. [19] defined the concept of an orthopair neutrosophic number (ONN) as a mixed form of IFN and NN, which can represent the hybrid information of true and false indeterminate degrees, and then proposed the score and accuracy functions of ONN and the ONN weighted arithmetic and geometric averaging (ONNWAA and ONNWGA) operators for MADM.

With the complexity of the social and economic environment, it is difficult for a single decision-maker to consider all aspects of a MADM problem and to give a reasonable decision result. Accordingly, multiple decision makers are needed to provide decision information together and to construct a group decision-making result. Then, the aggregation algorithm of group decision information is very critical in group decision-making problems. Since the generalized ordered weighted averaging (GOWA) aggregation operators [20] consider not only the importance of parameters but also the importance of parameter positions, they reveal better aggregation algorithms in information aggregations. However, the GOWA operators have not been investigated for aggregating ONN information. On the basis of an extension of the GOWA operators, this article proposes the GOWA and generalized ordered weighted geometric (GOWG) operators of ONNs and a MADM model using the weighted operation of the GOWA and GOWG operators of ONNs.

The rest of the article consists of the following parts. The second part describes the related notions of ONNs, including the definition of ONN, the related operations of ONNs, as well as the score and accurate functions of ONNs and their sorting rules. The third part proposes an ONN generalized ordered weighted averaging (ONNGOWA) operator and an ONN generalized ordered weighted geometric (ONNGOWG) operator and indicates the characteristics of idempotency, boundedness, and monotonicity. The fourth part establishes a MADM model through the weighted operation of the ONNGOWA and ONNGOWG operators and addresses its decision steps. The fifth part applies the established MADM model to the choice problem of manufacturing schemes. The sixth part compares the established MADM model with the MADM model proposed in the previous literature [19]. The seventh part summarizes the conclusions and future research.

2. Preliminaries of ONNs

This section introduces the relevant notions of ONNs presented by Ye et al. [19].

Definition 1 [19]. Each ONN n_j (j = 1, 2, ..., m) is given by

$$n_j = \langle A_j(I), B_j(I)\rangle = \langle e_j + f_jI, g_j + h_jI\rangle, \quad (1)$$

where $e_j + f_jI \subseteq [0, 1]$ and $g_j + h_jI \subseteq [0, 1]$ for $I \in [I^-, I^+]$ are the true indeterminate degree and the false indeterminate degree, such that the condition $0 \le \sup A_j(I) + \sup B_j(I) \le 1$.

Definition 2 [19]. Let $n_1 = \langle A_1(I), B_1(I)\rangle = \langle e_1 + f_1I, g_1 + h_1I\rangle$ and $n_2 = \langle A_2(I), B_2(I)\rangle = \langle e_2 + f_2I, g_2 + h_2I\rangle$ for $I \in [I^-, I^+]$ be two ONNs. Then the operation rules of ONNs are presented as follows:

(1) $n_1 \supseteq n_2 \Leftrightarrow A_1(I) \supseteq A_2(I)$ and $B_1(I) \subseteq B_2(I)$;

(2) $n_1 = n_2 \Leftrightarrow n_1 \subseteq n_2$ and $n_1 \supseteq n_2$;

(3) $(n_1)^c = \langle B_1(I), A_1(I)\rangle$ (Complement of n_1);

(4) $n_1 \oplus n_2 = \left\langle \begin{array}{l} [\inf A_1(I) + \inf A_2(I) - \inf A_1(I)\inf A_2(I), \\ \sup A_1(I) + \sup A_2(I) - \sup A_1(I)\sup A_2(I)], \\ [\inf B_1(I)\inf B_2(I), \sup B_1(I)\sup B_2(I)] \end{array} \right\rangle$;

(5) $n_1 \otimes n_2 = \left\langle \begin{matrix} [\inf A_1(I)\inf A_2(I), \sup A_1(I)\sup A_2(I)], \\ [\inf B_1(I) + \inf B_2(I) - \inf B_1(I)\inf B_2(I), \\ \sup B_1(I) + \sup B_2(I) - \sup B_1(I)\sup B_2(I)] \end{matrix} \right\rangle$;

(6) $\alpha n_1 = <[(1 - (1 - \inf A_1(I))^\alpha, 1 - (1 - \sup A_1(I))^\alpha], [(\inf B_1(I))^\alpha, (\sup B_1(I))^\alpha]>$ for $\alpha > 0$;

(7) $(n_1)^\alpha = <[(\inf A_1(I))^\alpha, (\sup A_1(I))^\alpha], [(1 - (1 - \inf B_1(I))^\alpha, 1 - (1 - \sup B_1(I))^\alpha]>$ for $\alpha > 0$.

To rank ONNs $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, g_j + h_jI>$ (j = 1, 2) with $I \in [I^-, I^+]$, the accuracy function of ONN is given as [19]

$$T(n_j) = \{\inf A_1(I) + \inf B_1(I) + \sup A_1(I) + \sup B_1(I)\}/2$$
$$= \{[2e_j + f_j(I^- + I^+)] + [2g_j + h_j(I^- + I^+)]\}/2, \text{ for } T(n_j) \in [0, 1]. \quad (2)$$

The score function of ONN is given as [19]

$$S(n_j) = \{\inf A_1(I) - \inf B_1(I) + \sup A_1(I) - \sup B_1(I)\}/2$$
$$= \{[2e_j + f_j(I^- + I^+)] - [2g_j + h_j(I^- + I^+)]\}/2, \text{ for } S(n_j) \in [-1, 1]. \quad (3)$$

The ranking rules are described as follows [19]:

(1) If $S(n_1) > S(n_2)$, then $n_1 > n_2$;

(2) If $S(n_1) = S(n_2)$ and $T(n_1) > T(n_2)$, then $n_1 > n_2$;

(3) If $S(n_1) = S(n_2)$ and $T(n_1) = T(n_2)$, then $n_1 = n_2$.

3. Two Generalized Ordered Weighted Aggregation Operators of ONNs

This section proposes the ONNGOWA and ONNGOWG operators through the operation rules in Definition 2.

3.1. ONNGOWA Operator

The ONNGOWA operator for a group of ONNs can be derived from the operation rules in Definition 2.

Definition 3. Set $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, g_j + h_jI>$ (j = 1, 2, ..., m) as a group of ONNs. Thus, the ONNGOWA operator is defined below:

$$\text{ONNGOWA}(n_1, n_2, ..., n_m) = \left(\sum_{j=1}^{m} v_j n_j^{\delta}\right)^{\frac{1}{\delta}}, \quad (4)$$

where v_j (j = 1, 2, ..., m) is the weight of n_j for $0 \le v_j \le 1$ and $\sum_{j=1}^{m} v_j = 1$.

Theorem 1. Set $n_j = <A_j(I), B_j(I)> = <e_j + f_jI, gj + h_jI>$ (j = 1, 2, ..., m) as a group of ONNs. Thus, the value of the ONNGOWA operator is still ONN, which is obtained by the following formula:

$$\text{ONNGOWA}(n_1, n_2, ..., n_m) = \left(\sum_{j=1}^{m} v_j n_j^{\delta}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{matrix} \left[\left(1 - \prod_{j=1}^{m}\left(1 - \left(e_j + f_j I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1 - \prod_{j=1}^{m}\left(1 - \left(e_j + f_j I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}} \right], \\ \left[1 - \left(1 - \prod_{j=1}^{m}\left(1 - \left(1 - g_j - h_j I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1 - \left(1 - \prod_{j=1}^{m}\left(1 - \left(1 - g_j - h_j I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}} \right] \end{matrix} \right\rangle, \quad (5)$$

where v_j (j = 1, 2, ..., m) is the weight of n_j for $0 \le v_j \le 1$ and $\sum_{j=1}^{m} v_j = 1$.

Proof:

According to the relevant operation rules in Definition 2, Eq. (5) can be verified below.

$$v_j n_j^{\delta} = \left\langle \begin{array}{l} \left[\left(1-\left(1-\left(e_j+f_j I^-\right)^{\delta}\right)^{v_j}\right), \left(1-\left(1-\left(e_j+f_j I^+\right)^{\delta}\right)^{v_j}\right)\right], \\ \left[1-\left(1-\left(1-\left(1-g_j-h_j I^-\right)^{\delta}\right)^{v_j}\right), 1-\left(1-\left(1-\left(1-g_j-h_j I^+\right)^{\delta}\right)^{v_j}\right)\right] \end{array} \right\rangle. \quad (6)$$

Then, we get the following equation:

$$\sum_{j=1}^{m} v_j n_j^{\delta} = \left\langle \begin{array}{l} \left[\left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_j I^-\right)^{\delta}\right)^{v_j}\right), \left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_j I^+\right)^{\delta}\right)^{v_j}\right)\right], \\ \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_j I^-\right)^{\delta}\right)^{v_j}\right), 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_j I^+\right)^{\delta}\right)^{v_j}\right)\right] \end{array} \right\rangle. \quad (7)$$

We can further get the result:

$$\left(\sum_{j=1}^{m} v_j n_j^{\delta}\right)^{\frac{1}{\delta}} =$$

$$\left\langle \begin{array}{l} \left[\left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_j I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_j I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_j I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_j I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle. \quad (8)$$

So, the proof of the ONNGOWA operator is completed.

Theorem 2. The ONNGOWA operator expressed by Eq. (5) has the following properties:

(a) Idempotency: Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs. If $n_j = n$ (j = 1, 2, …, m), then ONNGOWA(n_1, n_2, ..., n_m) = n.

(b) Boundedness: Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs, and then let the maximum and minimum ONNs be the following values:

$$n_{\max} = \left\langle \left[\max_j\left(e_j+f_jI^-\right), \max_j\left(e_j+f_jI^+\right)\right], \left[\min_j\left(g_j+h_jI^-\right), \min_j\left(g_j+h_jI^+\right)\right] \right\rangle,$$

$$n_{\min} = \left\langle \left[\min_j\left(e_j+f_jI^-\right), \min_j\left(e_j+f_jI^+\right)\right], \left[\max_j\left(g_j+h_jI^-\right), \max_j\left(g_j+h_jI^+\right)\right] \right\rangle. \quad (9)$$

Thus, the inequality $n_{\min}$ ≤ ONNGOWA(n_1, n_2, ..., n_m) ≤ $n_{\max}$ exists.

(c) Monotonicity: Let n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> and n_j^* = <$A_j^*(I)$, $B_j^*(I)$> (j = 1, 2, ..., m) be two groups of ONNs. If $n_j \le n_j^*$, then there is the inequality ONNGOWA(n_1, n_2, ..., n_m) ≤ ONNGOWA(n_1^*, n_2^*, ..., n_m^*).

Proof:

(a) When $n_j = n$ (j = 1, 2, ..., m), the result of Eq. (5) is obtained below:

$$\text{ONNGOWA}(n_1, n_2, ..., n_m) = \left(\sum_{j=1}^{m} v_j n_j^{\delta}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[\left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_jI^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(e_j+f_jI^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_jI^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_j-h_jI^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(1-\left(1-\left(e+fI^-\right)^{\delta}\right)^{\sum_{j=1}^{m}v_j}\right)^{\frac{1}{\delta}}, \left(1-\left(1-\left(e+fI^+\right)^{\delta}\right)^{\sum_{j=1}^{m}v_j}\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(1-\left(1-\left(1-g-hI^-\right)^{\delta}\right)^{\sum_{j=1}^{m}v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\left(1-\left(1-g-hI^+\right)^{\delta}\right)^{\sum_{j=1}^{m}v_j}\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(1-\left(1-\left(e+fI^-\right)^{\delta}\right)\right)^{\frac{1}{\delta}}, \left(1-\left(1-\left(e+fI^+\right)^{\delta}\right)\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(1-\left(1-\left(1-g-hI^-\right)^{\delta}\right)\right)^{\frac{1}{\delta}}, 1-\left(1-\left(1-\left(1-g-hI^+\right)^{\delta}\right)\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(1-1+\left(e+fI^-\right)^{\delta}\right)^{\frac{1}{\delta}}, \left(1-1+\left(e+fI^+\right)^{\delta}\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(1-1+\left(1-g-hI^-\right)^{\delta}\right)^{\frac{1}{\delta}}, 1-\left(1-1+\left(1-g-hI^+\right)^{\delta}\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(\left(e+fI^-\right)^{\delta}\right)^{\frac{1}{\delta}}, \left(\left(e+fI^+\right)^{\delta}\right)^{\frac{1}{\delta}} \right], \\ \left[1-\left(\left(1-g-hI^-\right)^{\delta}\right)^{\frac{1}{\delta}}, 1-\left(\left(1-g-hI^+\right)^{\delta}\right)^{\frac{1}{\delta}} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[\left(e+fI^-\right),\left(e+fI^+\right)\right], \\ \left[\left(g+hI^-\right),\left(g+hI^+\right)\right] \end{array} \right\rangle = n. \tag{10}$$

(b) Since $n_{\max}$ and $n_{\min}$ are the maximum and minimum ONNs, there is $n_{\min} \le n_j \le n_{\max}$. Hence, the inequality $\sum_{j=1}^{m} v_j n_{\min} \le \sum_{j=1}^{m} v_j n_j \le \sum_{j=1}^{m} v_j n_{\max}$ is established. According to the property (a), there exists $n_{min} \le \sum_{j=1}^{m} v_j n_j \le n_{max}$, i.e., $n_{\min} \le$ ONNGOWA(n_1, n_2, ..., n_m) $\le n_{\max}$.

(c) If $n_j \le n_j^*$, then the inequality $\sum_{j=1}^{m} v_j n_j \le \sum_{j=1}^{m} v_j n_j^*$ is established, i.e., the inequality ONNGOWA(n_1, n_2, ..., n_m) $\le$ ONNGOWA(n_1^*, n_2^*, ..., n_m^*) holds.

Thus, we complete the proof of Theorem 2.

3.2. ONNGOWG Operator

The ONNGOWG operator for a group of ONNs can be derived from the operation rules in Definition 2.

Definition 4. Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs. Thus, the ONNGOWG operator is defined below:

$$\text{ONNGOWG}(n_1, n_2, ..., n_m) = \left(\prod_{j=1}^{m} n_j^{\delta v_j}\right)^{\frac{1}{\delta}}, \quad (11)$$

where v_j (j = 1, 2, ..., m) is the weight of n_j for $0 \le v_j \le 1$ and $\sum_{j=1}^{m} v_j = 1$.

Theorem 3. Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs. Thus, the value of the ONNGOWG operator is still ONN, which is obtained by the following formula:

$$\text{ONNGOWG}(n_1, n_2, ..., n_m) = \left(\prod_{j=1}^{m} n_j^{\delta v_j}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-e_j-f_jI^{-}\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-e_j-f_jI^{+}\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[\left(1-\prod_{j=1}^{m}\left(1-\left(g_j+h_jI^{-}\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(g_j+h_jI^{+}\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle, \quad (12)$$

where v_j (j = 1, 2, …, m) is the weight of n_j for $0 \le v_j \le 1$ and $\sum_{j=1}^{m} v_j = 1$.

The verification process of Eq. (12) is similar to that of Theorem 1, so it is omitted.

Theorem 4. The ONNGOWG operator of Eq. (12) has the following properties:

(a) Idempotency: Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs. If $n_j = n$ (j = 1, 2, ..., m), then ONNGOWG(n_1, n_2, ..., n_m) = n.

(b) Boundedness: Set n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> (j = 1, 2, ..., m) as a group of ONNs, and then let the maximum and minimum ONNs:

$$n_{\max} = \left\langle \left(\max_j\left(e_j + f_jI^{-}\right), \max_j\left(e_j + f_jI^{+}\right)\right), \left(\min_j\left(g_j + h_jI^{-}\right), \min_j\left(g_j + h_jI^{+}\right)\right) \right\rangle,$$

$$n_{\min} = \left\langle \left(\min_j\left(e_j + f_jI^{-}\right), \min_j\left(e_j + f_jI^{+}\right)\right), \left(\max_j\left(g_j + h_jI^{-}\right), \max_j\left(g_j + h_jI^{+}\right)\right) \right\rangle. \quad (13)$$

Thus, $n_{min} \le$ ONNGOWG(n_1, n_2, ..., n_m) $\le n_{max}$.

(c) Monotonicity: Let n_j = <$A_j(I)$, $B_j(I)$> = <$e_j + f_jI$, $g_j + h_jI$> and n_j^* = <$A_j^*(I)$, $B_j^*(I)$> (j = 1, 2, ..., m) be two groups of ONNs. If $n_j \le n_j^*$, then ONNGOWG(n_1, n_2, ..., n_m) $\le$ ONNGOWG(n_1^*, n_2^*, ..., n_m^*).

4. MADM Model Based on the ONNGOWA and ONNGOWG Operators

In this section, a MADM model are established based on the weighted operation of the ONNGOWA and ONNGOWG operators to perform MADM problems with ONNs.

For a MADM problem, $D = \{D_1, D_2, ..., D_q\}$ represents a set of q alternatives and then $F = \{f_1, f_2, ..., f_m\}$ represents a set of m attributes. The importance of each attribute f_j (j = 1, 2, ..., m) is determined by the weight v_j. Experts/decision makers evaluate the satisfactory levels of each alternative D_i (i = 1, 2, ..., q) relative to the attributes f_j (j = 1, 2, ..., m) through true and falsity indeterminate degrees, which are expressed as the ONNs $n_{ij} = \langle A_{ij}(I), B_{ij}(I)\rangle = \langle e_{ij} + f_{ij}I, g_{ij} + h_{ij}I\rangle$ for $A_{ij}(I), B_{ij}(I) \in [0, 1]$, $I \in [I^-, I^+]$, and $0 \le \sup A_{ij}(I) + \sup B_{ij}(I) \le 1$. Thus, the decision matrix of ONNs can be expressed as $N = (n_{ij})_{q\times m}$. Therefore, the MADM model according to the weighted operation of the ONNGOWA and ONNGOWG operators is established through the following steps:

Step 1: Based on Eqs. (5) and (12), the aggregated ONNs n_{1i} and n_{2i} are obtained by the following equations:

$$n_{1i} = \text{ONNGOWA}(n_{i1}, n_{i2}, ..., n_{im}) = \left(\sum_{j=1}^{m} v_j n_{ij}^{\ \delta}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[\left(1-\prod_{j=1}^{m}\left(1-\left(e_{ij}+f_{ij}I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(e_{ij}+f_{ij}I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_{ij}-h_{ij}I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-g_{ij}-h_{ij}I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle, \quad (14)$$

$$n_{2i} = \text{ONNGOWG}(n_{i1}, n_{i2}, ..., n_{im}) = \left(\prod_{j=1}^{m} n_{ij}^{\ \delta v_j}\right)^{\frac{1}{\delta}}$$

$$= \left\langle \begin{array}{l} \left[1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-e_{ij}-f_{ij}I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, 1-\left(1-\prod_{j=1}^{m}\left(1-\left(1-e_{ij}-f_{ij}I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right], \\ \left[\left(1-\prod_{j=1}^{m}\left(1-\left(g_{ij}+h_{ij}I^-\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}, \left(1-\prod_{j=1}^{m}\left(1-\left(g_{ij}+h_{ij}I^+\right)^{\delta}\right)^{v_j}\right)^{\frac{1}{\delta}}\right] \end{array} \right\rangle.$$

(15)

Step 2: The weighted operation of the ONNGOWA and ONNGOWG operators with the weights ψ_1 and $\psi_2 = 1 - \psi_1$ for $\psi_1 \in [0, 1]$ is obtained by the following equation:

$$H_i = \psi_1 n_{1i} \oplus \psi_2 n_{2i} = \psi_1 n_{1i} \oplus (1-\psi_1) n_{2i}$$

$$= \left\langle \begin{array}{l} \left[1-\left(1-\inf A_{1i}(I)\right)^{\psi_1}, 1-\left(1-\sup A_{1i}(I)\right)^{\psi_1}\right], \\ \left[\inf(B_{1i}(I))^{\psi_1}, \sup(B_{1i}(I))^{\psi_1}\right] \end{array} \right\rangle \oplus \left\langle \begin{array}{l} \left[1-\left(1-\inf A_{2i}(I)\right)^{1-\psi_1}, 1-\left(1-\sup A_{2i}(I)\right)^{1-\psi_1}\right], \\ \left[\inf(B_{2i}(I))^{1-\psi_1}, \sup(B_{2i}(I))^{1-\psi_1}\right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \left[1-\left(1-\inf A_{1i}(I)\right)^{\psi_1}\left(1-\inf A_{2i}(I)\right)^{1-\psi_1}, 1-\left(1-\sup A_{1i}(I)\right)^{\psi_1}\left(1-\sup A_{2i}(I)\right)^{1-\psi_1}\right], \\ \left[\inf(B_{1i}(I))^{\psi_1}\cdot \inf(B_{2i}(I))^{1-\psi_1}, \sup(B_{1i}(I))^{\psi_1}\cdot \sup(B_{2i}(I))^{1-\psi_1}\right] \end{array} \right\rangle. \quad (16)$$

Step 3: The values of $S(H_i)$ and $T(H_i)$ (i = 1, 2, ..., q) are obtained by Eqs. (2) and (3).

Step 4: The alternatives are sorted according to the sorting rules and the best one is chosen.

Step 5: End.

5. Illustrative Example

In this section, the MADM model based on the weighted operation of the ONNGOWA and ONNGOWG operators is applied to the selection of electric vehicle design schemes.

A manufacturing company needs to choose the best design scheme of electric vehicles, the technique department preliminarily provides four design schemes of electric vehicles as a set of alternatives $D = \{D_1, D_2, D_3, D_4\}$. Each alternative is satisfactorily assessed by the three attributes: charging rate (f_1), driving range (f_2), and manufacturing cost (f_3). The weight vector of the three attributes is specified by v = (0.36, 0.3, 0.34). Therefore, experts/decision makers evaluate the four alternatives that satisfy these attributes by ONNs $n_{ij} = <A_{ij}(I), B_{ij}(I)> = <e_{ij} + f_{ij}I, g_{ij} + h_{ij}I>$ (i = 1, 2, 3, 4 and j= 1, 2, 3) for $A_{ij}(I), B_{ij}(I) \in [0, 1]$, $I \in [I^-, I^+]$, and $0 \le \sup A_{ij}(I) + \sup B_{ij}(I) \le 1$. Thus, the ONN decision matrix is listed in Table 1.

Table 1. The decision matrix of ONNs.

	f_1	f_2	f_3
D_1	$<0.5 + 0.2I, 0.1 + 0.1I>$	$<0.6 + 0.1I, 0.1 + 0.1I>$	$<0.5 + 0.1I, 0.1 + 0.2I>$
D_2	$<0.6 + 0.1I, 0.1 + 0.1I>$	$<0.6 + 0.2I, 0.1 + 0.1I>$	$<0.6 + 0.1I, 0.1 + 0.1I>$
D_3	$<0.6 + 0.1I, 0.1 + 0.1I>$	$<0.6 + 0.1I, 0.1 + 0.2I>$	$<0.5 + 0.2I, 0.1 + 0.2I>$
D_4	$<0.5 + 0.2I, 0.1 + 0.1I>$	$<0.6 + 0.2I, 0.1 + 0.1I>$	$<0.7 + 0.1I, 0.1 + 0.1I>$

Regarding the MADM problem in an ONN environment, the MADM steps are given below.

Step 1: Using Eqs. (14) and (15) with δ = 0.5 and $I \in [I^-, I^+] = [0, 0.3]$, the aggregated ONNs n_{1i} and n_{2i} (i = 1, 2, 3, 4) are obtained below:

$$\begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \\ n_{14} \end{bmatrix} = \begin{bmatrix} <0.5318 + 0.5724I, 0.1000 + 0.1395I> \\ <0.6000 + 0.6392I, 0.1000 + 0.1300I> \\ <0.5679 + 0.6073I, 0.1000 + 0.1485I> \\ <0.6053 + 0.6540I, 0.1000 + 0.1300I> \end{bmatrix},$$

$$\begin{bmatrix} n_{21} \\ n_{22} \\ n_{23} \\ n_{24} \end{bmatrix} = \begin{bmatrix} <0.5288 + 0.5700I, 0.1000 + 0.1401I> \\ <0.6000 + 0.6389I, 0.1000 + 0.1300I> \\ <0.5647 + 0.6057I, 0.1000 + 0.1491I> \\ <0.5948 + 0.6461I, 0.1000 + 0.1300I> \end{bmatrix}.$$

Step 2: By Eq. (16) for ψ_1 = 0.5 and $I \in [I^-, I^+] = [0, 0.3]$, the values of H_i are given below:
$H_1 = <0.5303 + 0.7017I, 0.1000 + 0.1419I>$, $H_2 = <0.6000 + 0.7917I, 0.1000 + 0.1390I>$,
$H_3 = <0.5663 + 0.7483I, 0.1000 + 0.1446I>$, and $H_4 = <0.6001 + 0.7952I, 0.1000 + 0.1390I>$.

Step 3: Using Eq. (3), the values of $S(H_i)$ for the alternatives D_i (i = 1, 2, 3, 4) are given as follows: $S(H_1) = 0.495$, $S(H_2) = 0.5763$, $S(H_3) = 0.535$, and $S(H_4) = 0.5781$.

Step 4: Since $S(H_4) > S(H_2) > S(H_3) > S(H_1)$, the sorting order of the four alternatives is $D_4 > D_2 > D_3 > D_1$, then the best one is D_4.

In order to reflect the influence of δ *and* ψ_1 on the decision results of the proposed MADM model, the corresponding ranking results are shown in Table 2.

In view of the ranking results shown in Table 2, different parameter values of δ and different weight values of ψ_1 can influence the ranking order of the four alternatives, which reveals the flexibility of the decision results.

Table 2. Values of $S(H_i)$ and ranking orders corresponding to δ = 0.3, 0.7, 1 *and* ψ_1 = 0, 0.1, 0.3, 0.5, 0.7, 1.

δ	ψ_1	$[I^-, I^+]$	$S(H_1), S(H_2), S(H_3), S(H_4)$	Ranking order	The best one
δ = 0.3	ψ_1 = 0.0	[0, 0.3]	0.4935, 0.5763, 0.5335, 0.5733	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.1	[0, 0.3]	0.4939, 0.5763, 0.5338, 0.5744	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.3	[0, 0.3]	0.4944, 0.5763, 0.5344, 0.5764	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 0.5	[0, 0.3]	0.4950, 0.5763, 0.5350, 0.5783	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 0.7	[0, 0.3]	0.4956, 0.5763, 0.5356, 0.5803	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 1.0	[0, 0.3]	0.4965, 0.5764, 0.5365, 0.5832	$D_4 > D_2 > D_3 > D_1$	D_4
δ = 0.7	ψ_1 = 0.0	[0, 0.3]	0.4929, 0.5763, 0.5328, 0.5711	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.1	[0, 0.3]	0.4933, 0.5763, 0.5333, 0.5725	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.3	[0, 0.3]	0.4941, 0.5763, 0.5340, 0.5752	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.5	[0, 0.3]	0.4949, 0.5763, 0.5349, 0.5779	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 0.7	[0, 0.3]	0.4958, 0.5763, 0.5357, 0.5806	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 1.0	[0, 0.3]	0.4969, 0.5764, 0.5369, 0.5846	$D_4 > D_2 > D_3 > D_1$	D_4
δ = 1	ψ_1 = 0.0	[0, 0.3]	0.4924, 0.5763, 0.5323, 0.5691	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.1	[0, 0.3]	0.4929, 0.5763, 0.5328, 0.5709	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.3	[0, 0.3]	0.4939, 0.5763, 0.5338, 0.5743	$D_2 > D_4 > D_3 > D_1$	D_2
	ψ_1 = 0.5	[0, 0.3]	0.4949, 0.5763, 0.5348, 0.5775	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 0.7	[0, 0.3]	0.4959, 0.5764, 0.5358, 0.5808	$D_4 > D_2 > D_3 > D_1$	D_4
	ψ_1 = 1.0	[0, 0.3]	0.4974, 0.5764, 0.5374, 0.5856	$D_4 > D_2 > D_3 > D_1$	D_4

6. Comparative Analysis

To prove the effectiveness of the proposed model, the proposed MADM model based on the weighted operation of the ONNGOWA and ONNGOWG operators is compared with the MADM model proposed in [19]. The decision results of the existing MADM model for the above example are summarized in Table 3.

Table 3. The best one and ranking order corresponding to the existing MADM model [19].

Aggregation operator	Aggregated value	Score value	Ranking order	The best one
ONNWGA operator for I = [0, 0.3] [19]	0.5281, 0.6989, 0.1000, 0.1448 0.6342, 0.8363, 0.1000, 0.1390 0.5639, 0.7480, 0.1000, 0.1453 0.6323, 0.8344, 0.1000, 0.1390	$S(H_i)$ = (0.4924, 0.5763, 0.5323, 0.5691)	$D_2 > D_4 > D_3 > D_1$	D_2
ONNWAA operator for I = [0, 0.3] [19]	0.5324, 0.6274, 0.1000, 0.1911 0.6000, 0.6928, 0.1000, 0.1700 0.5685, 0.6601, 0.1000, 0.2120 0.6373, 0.7506, 0.1000, 0.1700	$S(p_j, I)$ = (0.4974, 0.5764, 0.5374, 0.5857)	$D_4 > D_2 > D_3 > D_1$	D_4

Regarding the decision results in Tables 2 and 3, the ranking results of the design schemes and the best one based on the proposed MADM model with δ = 1, ψ_1 = 0, 1, and I = [0, 0.3] are the same as those based on the existing MADM model [19] because the ONNWAA and ONNWGA operators [19] are the special cases of the ONNGOWA and ONNGOWG operators with $\delta = 1$ and $\psi_1 = 0, 1$. However, the proposed MADM model contains the advantage of flexible decision making, while the existing MADM model [19] lacks flexibility in the decision process. Therefore, the proposed MADM model reveals the obvious superiority over the existing MADM model [19] in an ONN circumstance.

7. Conclusions

In this paper, we presented the ONNGOWA and ONNGOWG operators based on the concepts of ONNs and the GOWA operators to reach more flexible aggregation operations than the existing ONNWAA and ONNWGA operators [19]. Then, the proposed MADM model based on the weighted operation of the ONNGOWA and ONNGOWG operators was established to solve flexible MAGM problems in an uncertain circumstance. However, the application of the proposed MADM model in an illustrative example demonstrated its effectivity, and then the comparative results reflected that the proposed MADM model revealed the advantage of flexible decision making in an ONN circumstance.

However, there are still many aggregation operators of ONNs for MADM to need further research and to apply them in practical areas, including supplier selection, fault diagnosis, medical diagnosis, etc.

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