On Q-Neutrosophic Soft Fields

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Abstract: As an extension of neutrosophic soft sets, Q-neutrosophic soft sets were established to deal with two-
dimensional indeterminate data. Different hybrid models of fuzzy sets were utilized to different algebraic structures,
for example groups, rings, fields and lie-algebras. A field is an essential algebraic structure, which is widely used
in algebra and several domains of mathematics. The motivation of the current work is to extend the thought of
Q-neutrosophic soft sets to fields. In this paper, we define the notion of Q-neutrosophic soft fields. Structural charac-
teristics of it are investigated. Moreover, the concepts of homomorphic image and pre-image of Q-neutrosophic soft
fields are discussed. Finally, the Cartesian product of Q-neutrosophic soft fields is defined and some related properties
are discussed.

Keywords: Neutrosophic soft field, Neutrosophic soft set, Q-neutrosophic soft field, Q-neutrosophic soft set.

1 Introduction

Fuzzy sets were established by Zadeh [1] as a tool to deal with uncertain data. Since then, fuzzy logic has
been utilized in several real-world problems in uncertain environments. Consequently, numerous analysts
discussed many results using distinct directions of fuzzy-set theory, for instance, interval valued fuzzy set [2]
and intuitionistic fuzzy set [3]. These extensions can deal with uncertain real-world problems but it does not
cope with indeterminate data. Thus, Smarandache [4] initiated the neutrosophic idea to overcome this problem.
A neutrosophic set (NS) [5] is a mathematical notion serving issues containing inconsistent, indeterminate,
and imprecise data. Molodtsov [6] introduced the concept of soft sets as another way to handle uncertainty.
Since its initiation, a plenty of hybrid models of soft set have been produced, for example, fuzzy soft sets [7],
eutrosophic soft sets (NSSs) [8]. Accordingly, NSSs became an important notion for more deep discussions
[9–17]. NSSs were extended to Q-neutrosophic soft sets (Q-NSSs) [18] a new model that deals with two-
dimensional uncertain data. Q-NSSs were further investigated and their basic operations and relations were
discussed in [18, 19].

Different hybrid models of fuzzy sets and soft sets were utilized in different branches of mathematics,
including algebra. This was started by Rosenfeld in 1971 [20] when he established the idea of fuzzy subgroup.
Since then, the theories and approaches of fuzzy soft sets on different algebraic structures developed rapidly.
Mukherjee and Bhattacharya [21] studied fuzzy groups, Sharma [22] discussed intuitionistic fuzzy groups.
Recently, many researchers have applied different hybrid models of fuzzy sets and soft sets to several algebraic
structures such as groups, semigroups, rings, fields and BCK/BCI-algebras [23–32]. NSs and NSSs have
received more attention in studying the algebraic structure of set theories dealing with uncertainty. Cetkin and Aygun [33] established the concept of neutrosophic subgroups. Bera and Mahapatra introduced the notion of neutrosophic soft group [34], neutrosophic soft fields [35]. Moreover, two-dimensional hybrid models of fuzzy sets and soft sets were also applied to different algebraic structures. Solairaju and Nagarajan [36] introduced the concept of neutrosophic subgroups. Thiruveni and Solairaju defined the concept of neutrosophic Q-fuzzy groups. Thiruveni and Solairaju defined the concept of neutrosophic Q-fuzzy fields [37]. Moreover, two-dimensional hybrid models of neutrosophic soft groups [33], neutrosophic soft fields [35].

In this section, we recall the basic definitions related to this work. Inspired by the above works and to utilize Q-NSSs to different algebraic structures, in the current paper, we continue the work presented in [41] about Q-neutrosophic soft fields (Q-NSFs) and investigate some of its structural characteristics; we give some theorems that simplifies the main definition, also we discuss the intersection and union of two Q-NSFs. The concepts of homomorphic image and pre-image of Q-NSFs are investigated. Also, we discuss the Cartesian product of Q-NSFs and discuss some related properties.

## 2 Preliminaries

In this section, we recall the basic definitions related to this work.

**Definition 2.1 ([18]).** Let \( X \) be a universal set, \( Q \) be a nonempty set and \( A \subseteq E \) be a set of parameters. Let \( \mu^{lQNS}(X) \) be the set of all multi Q-NSs on \( X \) with dimension \( l = 1 \). A pair \( (\Gamma_Q, A) \) is called a Q-NSS over \( X \), where \( \Gamma_Q : A \rightarrow \mu^{lQNS}(X) \) is a mapping, such that \( \Gamma_Q(e) = \phi \) if \( e \notin A \).

**Definition 2.2 ([19]).** The union of two Q-NSSs \( (\Gamma_Q, A) \) and \( (\Psi_Q, B) \) is the Q-NSS \( (\Lambda_Q, C) \) written as \( (\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, C) \), where \( C = A \cup B \) and for all \( c \in C \), \( (x, q) \in X \times Q \), the truth-membership, indeterminacy-membership and falsity-membership of \( (\Lambda_Q, C) \) are as follows:

\[
T_{\Lambda_Q(c)}(x, q) = \begin{cases}
T_{\Gamma_Q(c)}(x, q) & \text{if } c \in A - B, \\
T_{\Psi_Q(c)}(x, q) & \text{if } c \in B - A, \\
\max\{T_{\Gamma_Q(c)}(x, q), T_{\Psi_Q(c)}(x, q)\} & \text{if } c \in A \cap B,
\end{cases}
\]

\[
I_{\Lambda_Q(c)}(x, q) = \begin{cases}
I_{\Gamma_Q(c)}(x, q) & \text{if } c \in A - B, \\
I_{\Psi_Q(c)}(x, q) & \text{if } c \in B - A, \\
\min\{I_{\Gamma_Q(c)}(x, q), I_{\Psi_Q(c)}(x, q)\} & \text{if } c \in A \cap B,
\end{cases}
\]

\[
F_{\Lambda_Q(c)}(x, q) = \begin{cases}
F_{\Gamma_Q(c)}(x, q) & \text{if } c \in A - B, \\
F_{\Psi_Q(c)}(x, q) & \text{if } c \in B - A, \\
\min\{F_{\Gamma_Q(c)}(x, q), F_{\Psi_Q(c)}(x, q)\} & \text{if } c \in A \cap B.
\end{cases}
\]

**Definition 2.3 ([19]).** The intersection of two Q-NSSs \( (\Gamma_Q, A) \) and \( (\Psi_Q, B) \) is the Q-NSS \( (\Lambda_Q, C) \) written as \( (\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, C) \), where \( C = A \cap B \) and for all \( c \in C \) and \( (x, q) \in X \times Q \) the truth-membership,
3 Q-Neutrosophic Soft Fields

In this section, we define the notion of Q-NSF and discuss several related properties.

**Definition 3.1.** Let \((\Gamma_Q, A)\) be a Q-NSS over a field \((F, +, \cdot)\). Then, \((\Gamma_Q, A)\) is said to be a Q-NSF over \((F, +, \cdot)\) if for all \(e \in A\), \(\Gamma_Q(e)\) is a Q-neutrosophic subfield of \((F, +, \cdot)\), where \(\Gamma_Q(e)\) is a mapping given by \(\Gamma_Q(e) : F \times Q \rightarrow [0, 1]^3\).

**Definition 3.2.** Let \((F, +, \cdot)\) be a field and \((\Gamma_Q, A)\) be a Q-NSS over \((F, +, \cdot)\). Then, \((\Gamma_Q, A)\) is called a Q-NSF over \((F, +, \cdot)\) if for all \(x, y \in F, q \in Q\) and \(e \in A\) it satisfies:

1. \(T_{\Gamma_Q(e)}(x + y, q) \geq \min \{T_{\Gamma_Q(e)}(x, q), T_{\Gamma_Q(e)}(y, q)\}\) and \(F_{\Gamma_Q(e)}(x + y, q) \leq \max \{F_{\Gamma_Q(e)}(x, q), F_{\Gamma_Q(e)}(y, q)\}\).
2. \(T_{\Gamma_Q(e)}(-x, q) \geq T_{\Gamma_Q(e)}(x, q), I_{\Gamma_Q(e)}(-x, q) \leq I_{\Gamma_Q(e)}(x, q)\) and \(F_{\Gamma_Q(e)}(-x, q) \leq F_{\Gamma_Q(e)}(x, q)\).
3. \(T_{\Gamma_Q(e)}(x, y, q) \geq \min \{T_{\Gamma_Q(e)}(x, q), T_{\Gamma_Q(e)}(y, q)\}\) and \(F_{\Gamma_Q(e)}(x, y, q) \leq \max \{F_{\Gamma_Q(e)}(x, q), F_{\Gamma_Q(e)}(y, q)\}\).
4. \(T_{\Gamma_Q(e)}(x^{-1}, q) \geq T_{\Gamma_Q(e)}(x, q), I_{\Gamma_Q(e)}(x^{-1}, q) \leq I_{\Gamma_Q(e)}(x, q)\) and \(F_{\Gamma_Q(e)}(x^{-1}, q) \leq F_{\Gamma_Q(e)}(x, q)\).

**Example 3.3.** Let \(F = (\mathbb{R}, +, \cdot)\) be the field of real numbers and \(A = \mathbb{N}\) the set of natural numbers be the parametric set. Define a Q-NSS \((\Gamma_Q, A)\) as follows for \(q \in Q, x \in \mathbb{R}\) and \(m \in \mathbb{N}\):

\[
T_{\Gamma_Q(m)}(x, q) = \begin{cases} 
0 & \text{if } x \text{ is rational} \\
\frac{1}{9m} & \text{if } x \text{ is irrational} 
\end{cases}
\]

\[
I_{\Gamma_Q(m)}(x, q) = \begin{cases} 
1 - \frac{1}{3m} & \text{if } x \text{ is rational} \\
0 & \text{if } x \text{ is irrational} 
\end{cases}
\]

\[
F_{\Gamma_Q(m)}(x, q) = \begin{cases} 
1 + \frac{3}{m} & \text{if } x \text{ is rational} \\
0 & \text{if } x \text{ is irrational} 
\end{cases}
\]

It is clear that \((\Gamma_Q, \mathbb{N})\) is a Q-NSF over \(F\).

**Proposition 3.4.** Let \((\Gamma_Q, A)\) be a Q-NSF over \((F, +, \cdot)\). Then, for the additive identity \(0_F\) and the multiplicative identity \(1_F\), for all \(x \in F, q \in Q\) and \(e \in A\) the following hold:

1. \(T_{\Gamma_Q(e)}(0_F, q) \geq T_{\Gamma_Q(e)}(x, q), I_{\Gamma_Q(e)}(0_F, q) \leq I_{\Gamma_Q(e)}(x, q)\) and \(F_{\Gamma_Q(e)}(0_F, q) \leq F_{\Gamma_Q(e)}(x, q)\).
2. $T_{\Gamma_Q}(1_F, q) \geq T_{\Gamma_Q}(x, q), I_{\Gamma_Q}(1_F, q) \leq I_{\Gamma_Q}(x, q)$ and $F_{\Gamma_Q}(1_F, q) \leq F_{\Gamma_Q}(x, q)$ for $x \neq 0_F$.

3. $T_{\Gamma_Q}(0_F, q) \geq T_{\Gamma_Q}(1_F, q), I_{\Gamma_Q}(0_F, q) \leq I_{\Gamma_Q}(1_F, q)$ and $F_{\Gamma_Q}(0_F, q) \leq F_{\Gamma_Q}(1_F, q)$.

**Proof.**

Proof. \( \forall x, f, q \in Q \) and \( e \in A \)

1. $T_{\Gamma_Q}(0_F, q) = T_{\Gamma_Q}(x, x) \geq \min \{ T_{\Gamma_Q}(x, q), T_{\Gamma_Q}(x, q) \} = T_{\Gamma_Q}(x, q)$

2. $T_{\Gamma_Q}(1_F, q) = T_{\Gamma_Q}(x, x^{-1}) \geq \min \{ T_{\Gamma_Q}(x, q), T_{\Gamma_Q}(x, q) \} = T_{\Gamma_Q}(x, q)$

3. Follows directly by applying 1.

**Theorem 3.5.** A Q-NSS \((\Gamma_Q, A)\) over the field \((F, +, \cdot)\) is a Q-NSF if and only if for all \(x, y, q \in F, q \in Q\) and \(e \in A\)

1. $T_{\Gamma_Q}(x - y, q) \geq \min \{ T_{\Gamma_Q}(x, q), T_{\Gamma_Q}(y, q) \}, I_{\Gamma_Q}(x - y, q) \leq \max \{ I_{\Gamma_Q}(x, q), I_{\Gamma_Q}(y, q) \}, F_{\Gamma_Q}(x - y, q) \leq \max \{ F_{\Gamma_Q}(x, q), F_{\Gamma_Q}(y, q) \}$

Proof. Suppose that \((\Gamma_Q, A)\) is a Q-NSF over \((F, +, \cdot)\). Then,

\[
T_{\Gamma_Q}(x - y, q) \geq \min \{ T_{\Gamma_Q}(x, q), T_{\Gamma_Q}(\cdot - y, q) \} \geq \min \{ T_{\Gamma_Q}(x, q), T_{\Gamma_Q}(y, q) \},
\]

\[
I_{\Gamma_Q}(x - y, q) \leq \max \{ I_{\Gamma_Q}(x, q), I_{\Gamma_Q}(-y, q) \} \leq \max \{ I_{\Gamma_Q}(x, q), I_{\Gamma_Q}(y, q) \},
\]

\[
F_{\Gamma_Q}(x - y, q) \leq \max \{ F_{\Gamma_Q}(x, q), F_{\Gamma_Q}(-y, q) \} \leq \max \{ F_{\Gamma_Q}(x, q), F_{\Gamma_Q}(y, q) \}.
\]

Also,

\[
T_{\Gamma_Q}(x, y^{-1}, q) \geq \min \{ T_{\Gamma_Q}(x, q), T_{\Gamma_Q}(y^{-1}, q) \} \geq \min \{ T_{\Gamma_Q}(x, q), T_{\Gamma_Q}(y, q) \},
\]

\[
I_{\Gamma_Q}(x, y^{-1}, q) \leq \max \{ I_{\Gamma_Q}(x, q), I_{\Gamma_Q}(y^{-1}, q) \} \leq \max \{ I_{\Gamma_Q}(x, q), I_{\Gamma_Q}(y, q) \},
\]

\[
F_{\Gamma_Q}(x, y^{-1}, q) \leq \max \{ F_{\Gamma_Q}(x, q), F_{\Gamma_Q}(y^{-1}, q) \} \leq \max \{ F_{\Gamma_Q}(x, q), F_{\Gamma_Q}(y, q) \}.
\]

Conversely, Suppose that conditions 1 and 2 are satisfied. We show that for each \(e \in A\), \((\Gamma_Q, A)\) is a Q-neutrosophic subfield

\[
T_{\Gamma_Q}(e^{-1}, q) = T_{\Gamma_Q}(0_F - x, q) \geq \min \{ T_{\Gamma_Q}(0_F, q), T_{\Gamma_Q}(x, q) \}
\]

\[
\geq \min \{ T_{\Gamma_Q}(x, q), T_{\Gamma_Q}(e^{-1}, q) \} = T_{\Gamma_Q}(e^{-1}, q),
\]

\[
I_{\Gamma_Q}(e^{-1}, q) = I_{\Gamma_Q}(0_F - x, q) \leq \max \{ I_{\Gamma_Q}(0_F, q), I_{\Gamma_Q}(x, q) \}
\]

\[
\leq \max \{ I_{\Gamma_Q}(x, q), I_{\Gamma_Q}(e^{-1}, q) \} = I_{\Gamma_Q}(e^{-1}, q),
\]

\[
F_{\Gamma_Q}(e^{-1}, q) = F_{\Gamma_Q}(0_F - x, q) \leq \max \{ F_{\Gamma_Q}(0_F, q), F_{\Gamma_Q}(x, q) \}
\]

\[
\leq \max \{ F_{\Gamma_Q}(x, q), F_{\Gamma_Q}(e^{-1}, q) \} = F_{\Gamma_Q}(e^{-1}, q).
\]

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also,

\[ T_{\Gamma_{Q}(e)}(x + y, q) = T_{\Gamma_{Q}(e)}(x - (-y), q) \geq \min \{ T_{\Gamma_{Q}(e)}(x, q), T_{\Gamma_{Q}(e)}(y, q) \}, \]

\[ I_{\Gamma_{Q}(e)}(x + y, q) = I_{\Gamma_{Q}(e)}(x - (-y), q) \leq \max \{ I_{\Gamma_{Q}(e)}(x, q), I_{\Gamma_{Q}(e)}(y, q) \}, \]

\[ F_{\Gamma_{Q}(e)}(x + y, q) = F_{\Gamma_{Q}(e)}(x - (-y), q) \leq \max \{ F_{\Gamma_{Q}(e)}(x, q), F_{\Gamma_{Q}(e)}(y, q) \}. \]

Next,

\[ T_{\Gamma_{Q}(e)}(x^{-1}, q) = T_{\Gamma_{Q}(e)}(1_F, x^{-1}, q) \geq \min \{ T_{\Gamma_{Q}(e)}(1_F, q), T_{\Gamma_{Q}(e)}(x, q) \} \]

\[ \geq \min \{ T_{\Gamma_{Q}(e)}(1_F, q), T_{\Gamma_{Q}(e)}(x, q) \} = T_{\Gamma_{Q}(e)}(x, q), \]

\[ I_{\Gamma_{Q}(e)}(x^{-1}, q) = I_{\Gamma_{Q}(e)}(1_F, x^{-1}, q) \leq \max \{ I_{\Gamma_{Q}(e)}(1_F, q), I_{\Gamma_{Q}(e)}(x, q) \} \]

\[ \leq \max \{ I_{\Gamma_{Q}(e)}(1_F, q), I_{\Gamma_{Q}(e)}(x, q) \} = I_{\Gamma_{Q}(e)}(x, q), \]

\[ F_{\Gamma_{Q}(e)}(x^{-1}, q) = F_{\Gamma_{Q}(e)}(1_F, x^{-1}, q) \leq \max \{ F_{\Gamma_{Q}(e)}(1_F, q), F_{\Gamma_{Q}(e)}(x, q) \} \]

\[ \leq \max \{ F_{\Gamma_{Q}(e)}(1_F, q), F_{\Gamma_{Q}(e)}(x, q) \} = F_{\Gamma_{Q}(e)}(x, q) \]

and

\[ T_{\Gamma_{Q}(e)}(x, y, q) = T_{\Gamma_{Q}(e)}(x(y^{-1}), q) \geq \min \{ T_{\Gamma_{Q}(e)}(x, q), T_{\Gamma_{Q}(e)}(y, q) \}, \]

\[ I_{\Gamma_{Q}(e)}(x, y, q) = I_{\Gamma_{Q}(e)}(x(y^{-1}), q) \leq \max \{ I_{\Gamma_{Q}(e)}(x, q), I_{\Gamma_{Q}(e)}(y, q) \}, \]

\[ F_{\Gamma_{Q}(e)}(x, y, q) = F_{\Gamma_{Q}(e)}(x(y^{-1}), q) \leq \max \{ F_{\Gamma_{Q}(e)}(x, q), F_{\Gamma_{Q}(e)}(y, q) \}. \]

This completes the proof. \(\square\)

**Theorem 3.6.** Let \((\Gamma_Q, A)\) and \((\Psi_Q, B)\) be two Q-NSFs over \((F, +, .)\). Then, \((\Gamma_Q, A) \cap (\Psi_Q, B)\) is also Q-NSF over \((F, +, .)\).

**Proof.** Let \((\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, A \cap B)\). Now, \(\forall x, y \in F, q \in Q\) and \(e \in A \cap B\),

\[ T_{\Lambda_{Q}(e)}(x - y, q) = \min \left\{ T_{\Gamma_{Q}(e)}(x - y, q), T_{\Psi_{Q}(e)}(x - y, q) \right\} \]

\[ \geq \min \left\{ \min \left\{ T_{\Gamma_{Q}(e)}(x, q), T_{\Gamma_{Q}(e)}(y, q) \right\}, \min \left\{ T_{\Psi_{Q}(e)}(x, q), T_{\Psi_{Q}(e)}(y, q) \right\} \right\} \]

\[ = \min \left\{ \min \left\{ T_{\Gamma_{Q}(e)}(x, q), T_{\Psi_{Q}(e)}(x, q) \right\}, \min \left\{ T_{\Gamma_{Q}(e)}(y, q), T_{\Psi_{Q}(e)}(y, q) \right\} \right\} \]

\[ = \min \left\{ T_{\Lambda_{Q}(e)}(x, q), T_{\Lambda_{Q}(e)}(y, q) \right\}, \]

also,

\[ I_{\Lambda_{Q}(e)}(x - y, q) = \max \left\{ I_{\Gamma_{Q}(e)}(x - y, q), I_{\Psi_{Q}(e)}(x - y, q) \right\} \]

\[ \leq \max \left\{ \max \left\{ I_{\Gamma_{Q}(e)}(x, q), I_{\Gamma_{Q}(e)}(y, q) \right\}, \max \left\{ I_{\Psi_{Q}(e)}(x, q), I_{\Psi_{Q}(e)}(y, q) \right\} \right\} \]

\[ = \max \left\{ \max \left\{ I_{\Gamma_{Q}(e)}(x, q), I_{\Psi_{Q}(e)}(x, q) \right\}, \max \left\{ I_{\Gamma_{Q}(e)}(y, q), I_{\Psi_{Q}(e)}(y, q) \right\} \right\} \]

\[ = \max \left\{ I_{\Lambda_{Q}(e)}(x, q), I_{\Lambda_{Q}(e)}(y, q) \right\}. \]
similarly, \( F_{\Lambda_Q}(x - y, q) \leq \max \left\{ F_{\Lambda_Q}(x, q), F_{\Lambda_Q}(y, q) \right\} \). Next,

\[
T_{\Lambda_Q}(x, y^{-1}, q) = \min \left\{ T_{\Gamma_Q}(x, y^{-1}, q), T_{\Psi_Q}(x, y^{-1}, q) \right\}
\geq \min \left\{ \min \left\{ T_{\Gamma_Q}(x, q), T_{\Psi_Q}(y, q) \right\}, \min \left\{ T_{\Psi_Q}(x, q), T_{\Psi_Q}(y, q) \right\} \right\}
= \min \left\{ \min \left\{ T_{\Gamma_Q}(x, q), T_{\Psi_Q}(x, q) \right\}, \min \left\{ T_{\Gamma_Q}(y, q), T_{\Psi_Q}(y, q) \right\} \right\}
= \min \left\{ T_{\Lambda_Q}(x, q), T_{\Lambda_Q}(y, q) \right\},
\]
also,

\[
I_{\Lambda_Q}(x, y^{-1}, q) = \max \left\{ I_{\Gamma_Q}(x, y^{-1}, q), I_{\Psi_Q}(x, y^{-1}, q) \right\}
\leq \max \left\{ \max \left\{ I_{\Gamma_Q}(x, q), I_{\Gamma_Q}(y, q) \right\}, \max \left\{ I_{\Psi_Q}(x, q), I_{\Psi_Q}(y, q) \right\} \right\}
= \max \left\{ \max \left\{ I_{\Gamma_Q}(x, q), I_{\Psi_Q}(x, q) \right\}, \max \left\{ I_{\Gamma_Q}(y, q), I_{\Psi_Q}(y, q) \right\} \right\}
= \max \left\{ I_{\Lambda_Q}(x, q), I_{\Lambda_Q}(y, q) \right\}
\]
similarly, we can show \( F_{\Lambda_Q}(x, y^{-1}, q) \leq \max \left\{ F_{\Lambda_Q}(x, q), F_{\Lambda_Q}(y, q) \right\} \). This completes the proof. \( \square \)

**Remark 3.7.** For two Q-NSFs \((\Gamma_Q, A)\) and \((\Psi_Q, B)\) over \((F, +, .)\), \((\Gamma_Q, A) \cup (\Psi_Q, B)\) is not generally a Q-NSF.

For example, let \( F = (\mathbb{Q}, +, .)\), \( E = 2\mathbb{Z} \). Consider two Q-NSFs \((\Gamma_Q, E)\) and \((\Psi_Q, E)\) over \( F \) as follows: for \( x \in \mathbb{Q}, q \in Q \) and \( m \in \mathbb{Z} \)

\[
T_{\Gamma_Q}(x, q) = \begin{cases} 0.50 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
I_{\Gamma_Q}(x, q) = \begin{cases} 0 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0.25 & \text{otherwise}, \end{cases}
\]

\[
F_{\Gamma_Q}(x, q) = \begin{cases} 0.40 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0.10 & \text{otherwise}, \end{cases}
\]

and

\[
T_{\Psi_Q}(x, q) = \begin{cases} 0.70 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
I_{\Psi_Q}(x, q) = \begin{cases} 0 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0.50 & \text{otherwise}, \end{cases}
\]

\[
F_{\Psi_Q}(x, q) = \begin{cases} 0.20 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0.40 & \text{otherwise}. \end{cases}
\]
Let \((\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, E)\). For \(m = 2, x = 8, y = 12\) we have
\[
T_{\Lambda_Q}(8 - 12, q) = T_{\Lambda_Q}(8, -4) = \max \left\{ T_{\Gamma_Q}(8, -4, q), T_{\Psi_Q}(8, -4, q) \right\} = \max \{0, 0\} = 0
\]
and
\[
\min \left\{ T_{\Lambda_Q}(8, q), T_{\Lambda_Q}(12, q) \right\} = \min \left\{ \max \left\{ T_{\Gamma_Q}(8, q), T_{\Psi_Q}(8, q) \right\}, \max \left\{ T_{\Gamma_Q}(12, q), T_{\Psi_Q}(12, q) \right\} \right\}
\]
\[
= \min \left\{ \max \{0.50, 0\}, \max \{0, 0.7\} \right\}
\]
\[
= \min \{0.50, 0.70\} = 0.50.
\]
Hence, \(T_{\Lambda_Q}(8 - 12, q) < \min \left\{ T_{\Lambda_Q}(8, q), T_{\Lambda_Q}(12, q) \right\}\). Thus, the union is not a Q-NSF.

4 Q-Neutrosophic Soft Homomorphism

In this section, we define the Q-neutrosophic soft function, then define the image and pre-image of a Q-NSS under a Q-neutrosophic soft function. In continuation, we introduce the notion of Q-neutrosophic soft homomorphism along with some of its properties.

**Definition 4.1.** Let \(g : X \times Q \rightarrow Y \times Q\) and \(h : A \rightarrow B\) be two functions where \(A\) and \(B\) are parameter sets. Then, the pair \((g, h)\) is called a Q-neutrosophic soft function from \(X \times Q\) to \(Y \times Q\).

**Definition 4.2.** Let \((\Gamma_Q, A)\) and \((\Psi_Q, B)\) be two Q-NSs defined over \(X \times Q\) and \(Y \times Q\), respectively, and \((g, h)\) be a Q-neutrosophic soft function from \(X \times Q\) to \(Y \times Q\). Then,

1. The image of \((\Gamma_Q, A)\) under \((g, h)\), denoted by \((g, h)\Gamma_Q, A)\), is a Q-NS over \(Y \times Q\) and is defined by:
\[
(g, h)\Gamma_Q, A) = \left\{ (g(x), h(A)) : b \in h(A) \right\}
\]

where for all \(b \in h(A), y \in Y\) and \(q \in Q\),
\[
T_{g(x),q}(b)(y, q) = \begin{cases} 
\max_{g(x, q) = (y, q)} \max_{h(a) = b} [T_{\Gamma_Q}(a)(x, q)] & \text{if } (x, q) \in g^{-1}(y, q), \\
0 & \text{otherwise},
\end{cases}
\]
\[
I_{g(x),q}(b)(y, q) = \begin{cases} 
\min_{g(x, q) = (y, q)} \min_{h(a) = b} [I_{\Gamma_Q}(a)(x, q)] & \text{if } (x, q) \in g^{-1}(y, q), \\
1 & \text{otherwise},
\end{cases}
\]
\[
F_{g(x),q}(b)(y, q) = \begin{cases} 
\min_{g(x, q) = (y, q)} \min_{h(a) = b} [F_{\Gamma_Q}(a)(x, q)] & \text{if } (x, q) \in g^{-1}(y, q), \\
1 & \text{otherwise},
\end{cases}
\]

2. The preimage of \((\Psi_Q, B)\) under \((g, h)\), denoted by \((g, h)^{-1}(\Psi_Q, B)\), is a Q-NS over \(X\) and is defined
Proof. Let $b \in \mathbb{N}$ and $y_1, y_2 \in \mathbb{N}$, then $g^{-1}(y_1, q) = \phi$ or $g^{-1}(y_2, q) = \phi$, the proof is straight forward.

Definition 4.3. Let $(g, h)$ be a Q-neutrosophic soft function from $X \times Q$ to $Y \times Q$. If $g$ is a homomorphism from $X \times Q$ to $Y \times Q$, then $(g, h)$ is said to be a Q-neutrosophic soft homomorphism. If $g$ is an isomorphism from $X \times Q$ to $Y \times Q$ and $h$ is a one-to-one mapping from $A$ to $B$, then $(g, h)$ is said to be a Q-neutrosophic soft isomorphism.

Example 4.4. Let $A = \mathbb{N}$ (the set of natural numbers) be the parametric set and $F = (\mathbb{Z}_5, +, \cdot)$ be a field. Define a Q-NSS $(\Gamma_Q, A)$ as follows, for any $a \in A$, $q \in Q$ and $x \in \mathbb{Z}_5$,

$$T_{\Gamma_Q(a)}(x, q) = \begin{cases} 
0 & \text{if } x \in \{1, 3\} \\
\frac{1}{3a} & \text{if } x \in \{0, 2, 4\} 
\end{cases},$$

$$I_{\Gamma_Q(a)}(x, q) = \begin{cases} 
1 - \frac{1}{a} & \text{if } x \in \{1, 3\} \\
0 & \text{if } x \in \{0, 2, 4\} 
\end{cases},$$

$$F_{\Gamma_Q(a)}(x, q) = \begin{cases} 
\frac{3}{a+1} & \text{if } x \in \{1, 3\} \\
0 & \text{if } x \in \{0, 2, 4\} 
\end{cases}.$$

Now, let $g : \mathbb{Z}_5 \times Q \rightarrow \mathbb{Z}_5 \times Q$ and $h : \mathbb{N} \rightarrow \mathbb{N}$ be given by $g(x, q) = 3x + 1$ and $h(a) = a^2$. Then for $b \in \mathbb{N}^2$, $y \in 3\mathbb{Z}_5 + 1$, the image of $(\Gamma_Q, A)$ under $(g, h)$ as follows:

$$T_{g(\Gamma_Q)}(b)(y, q) = \begin{cases} 
0 & \text{if } y \in \{0, 2, 4\} \\
\frac{1}{3\sqrt{b}} & \text{if } y \in \{1, 3\} 
\end{cases},$$

$$I_{g(\Gamma_Q)}(b)(y, q) = \begin{cases} 
1 - \frac{1}{\sqrt{b}} & \text{if } y \in \{0, 2, 4\} \\
0 & \text{if } y \in \{1, 3\} 
\end{cases},$$

$$F_{g(\Gamma_Q)}(b)(y, q) = \begin{cases} 
\frac{1}{1+\sqrt{b}} & \text{if } y \in \{0, 2, 4\} \\
0 & \text{if } y \in \{1, 3\} 
\end{cases}.$$

Theorem 4.5. Let $(\Gamma_Q, A)$ be a Q-NSF over $F_1$ and $(g, h) : F_1 \times Q \rightarrow F_2 \times Q$ be a Q-neutrosophic soft homomorphism. Then, $(g, h)(\Gamma_Q, A)$ is a Q-NSF over $F_2$.

Proof. Let $b \in h(A)$ and $y_1, y_2 \in F_2$. For $g^{-1}(y_1, q) = \phi$ or $g^{-1}(y_2, q) = \phi$, the proof is straight forward.
Since, the inequality is satisfied for each $x_1, x_2 \in F_1$ such that $g(x_1, q) = (y_1, q)$ and $g(x_2, q) = (y_2, q)$. Then,

$$T_{g(\Gamma_Q)}(b)(y_1 - y_2, q) = \max_{g(x,q)=(y_1 - y_2, q)} \max_{h(a)=b} \left[ T_{\Gamma_Q(a)}(x, q) \right] \geq \max_{h(a)=b} \left[ T_{\Gamma_Q(a)}(x_1 - x_2, q) \right] \geq \max_{h(a)=b} \left[ \min \left\{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(-x_2, q) \right\} \right] \geq \max_{h(a)=b} \left[ \min \left\{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(x_2, q) \right\} \right] = \min \left\{ \max_{h(a)=b} \left[ T_{\Gamma_Q(a)}(x_1, q) \right], \max_{h(a)=b} \left[ T_{\Gamma_Q(a)}(x_2, q) \right] \right\}$$

Thus, the inequality is satisfied for each $x_1, x_2 \in F_1$, satisfying $g(x_1, q) = (y_1, q)$ and $g(x_2, q) = (y_2, q)$. Then,

$$T_{g(\Gamma_Q)}(b)(y_1 - y_2, q) \geq \min \left\{ \max_{g(x,q)=(y_1 - y_2, q)} \max_{h(a)=b} \left[ T_{\Gamma_Q(a)}(x, q) \right], \max_{g(x_2,q)=(y_1 - y_2, q)} \max_{h(a)=b} \left[ T_{\Gamma_Q(a)}(x_2, q) \right] \right\} = \min \left\{ T_{g(\Gamma_Q)}(b)(y_1, q), T_{g(\Gamma_Q)}(b)(y_2, q) \right\}.$$ 

Similarly, we show that

$$I_{g(\Gamma_Q)}(b)(y_1 - y_2, q) \leq \max \left\{ I_{g(\Gamma_Q)}(b)(y_1 - y_2, q), I_{g(\Gamma_Q)}(b)(y_2, q) \right\},$$

$$I_{g(\Gamma_Q)}(b)(y_1 - y_2, q) \leq \max \left\{ I_{g(\Gamma_Q)}(b)(y_1 - y_2, q), I_{g(\Gamma_Q)}(b)(y_2, q) \right\},$$

$$F_{g(\Gamma_Q)}(b)(y_1 - y_2, q) \leq \max \left\{ F_{g(\Gamma_Q)}(b)(y_1 - y_2, q), F_{g(\Gamma_Q)}(b)(y_2, q) \right\},$$

$$F_{g(\Gamma_Q)}(b)(y_1 - y_2, q) \leq \max \left\{ F_{g(\Gamma_Q)}(b)(y_1 - y_2, q), F_{g(\Gamma_Q)}(b)(y_2, q) \right\}.$$
Theorem 4.6. Let \((\Psi_Q, B)\) be a Q-NSF over \(F_2\) and \((g, h)\) be a Q-neutrosophic soft homomorphism from \(F_1 \times Q\) to \(F_2 \times Q\). Then, \((g, h)^{-1}(\Psi_Q, B)\) is a Q-NSF over \(F_1\).

\[
T_{g^{-1}(\Psi_Q)}(x_1 - x_2, q) = T_{\Psi_Q[h(a)]}(g(x_1 - x_2, q))
= T_{\Psi_Q[h(a)]}(g(x_1, q) - g(x_2, q))
\geq \min \left\{ T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(-g(x_2, q)) \right\}
\geq \min \left\{ T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q)) \right\}
= \min \left\{ T_{g^{-1}(\Psi_Q)}(x_1, q), T_{g^{-1}(\Psi_Q)}(x_2, q) \right\}
\]

and

\[
T_{g^{-1}(\Psi_Q)}(x_1, x_2^{-1}, q) = T_{\Psi_Q[h(a)]}(g(x_1, x_2^{-1}, q))
= T_{\Psi_Q[h(a)]}(g(x_1, q) \cdot g(x_2^{-1}, q))
\geq \min \left\{ T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q)^{-1}) \right\}
\geq \min \left\{ T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q)) \right\}
= \min \left\{ T_{g^{-1}(\Psi_Q)}(x_1, q), T_{g^{-1}(\Psi_Q)}(x_2, q) \right\}
\]

Similarly, we can obtain

\[
I_{g^{-1}(\Psi_Q)}(x_1 - x_2, q) \leq \max \left\{ I_{g^{-1}(\Psi_Q)}(x_1, q), I_{g^{-1}(\Psi_Q)}(x_2, q) \right\};
I_{g^{-1}(\Psi_Q)}(x_1, x_2^{-1}, q) \leq \max \left\{ I_{g^{-1}(\Psi_Q)}(x_1, q), I_{g^{-1}(\Psi_Q)}(x_2, q) \right\};
F_{g^{-1}(\Psi_Q)}(x_1 - x_2, q) \leq \max \left\{ F_{g^{-1}(\Psi_Q)}(x_1, q), F_{g^{-1}(\Psi_Q)}(x_2, q) \right\};
F_{g^{-1}(\Psi_Q)}(x_1, x_2^{-1}, q) \leq \max \left\{ F_{g^{-1}(\Psi_Q)}(x_1, q), F_{g^{-1}(\Psi_Q)}(x_2, q) \right\}.
\]

Thus, the theorem is proved.

\[
\square
\]

5 Cartesian Product of Q-Neutrosophic Soft Fields

In this section, we define the Cartesian product of Q-NSFs and prove that it is also a Q-NSF.

Definition 5.1. Let \((\Gamma_Q, A)\) and \((\Psi_Q, B)\) be two Q-NSFs over \((F_1, +, )\) and \((F_2, +, )\), respectively. Then, their Cartesian product \((\Lambda_Q, A \times B) = (\Gamma_Q, A) \times (\Psi_Q, B)\), where \(\Lambda_Q(a, b) = \Gamma_Q(a) \times \Psi_Q(b)\) for \((a, b) \in A \times B\). Analytically, for \(x \in F_1, y \in F_2\) and \(q \in Q\)

\[
\Lambda_Q(a, b) = \left\{ ((x, y), q), T_{\Lambda_Q(a, b)}((x, y), q), I_{\Lambda_Q(a, b)}((x, y), q), F_{\Lambda_Q(a, b)}((x, y), q) \right\},
\]

where

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Similarly, also, let 

\[
T_{\Lambda_Q(a,b)}((x,y), q) = \min \{ T_{\Gamma_Q(a)}(x,q), T_{\Psi_Q(b)}(y,q) \},
\]

\[
I_{\Lambda_Q(a,b)}((x,y), q) = \max \{ I_{\Gamma_Q(a)}(x,q), I_{\Psi_Q(b)}(y,q) \},
\]

\[
F_{\Lambda_Q(a,b)}((x,y), q) = \max \{ F_{\Gamma_Q(a)}(x,q), F_{\Psi_Q(b)}(y,q) \}.
\]

**Theorem 5.2.** Let \((\Gamma_Q, A)\) and \((\Psi_Q, B)\) be two Q-NSFs over \((F_1, +, .)\) and \((F_2, +, .)\), respectively. Then, their Cartesian product \((\Gamma_Q, A) \times (\Psi_Q, B)\) is a Q-NSF over \((F_1 \times F_2)\).

**Proof.** Let \((\Lambda_Q, A \times B) = (\Gamma_Q, A) \times (\Psi_Q, B)\), where \(\Lambda_Q(a, b) = \Gamma_Q(a) \times \Psi_Q(b)\) for \((a, b) \in A \times B\). Then, for \(((x_1, y_1), q), ((x_2, y_2), q) \in (F_1 \times F_2) \times Q\) we have,

\[
T_{\Lambda_Q(a,b)}\left(\left( (x_1, y_1) - (x_2, y_2), q \right) \right)
= T_{\Lambda_Q(a,b)}\left( (x_1 - x_2, y_1 - y_2), q \right)
= \min \left\{ T_{\Gamma_Q(a)}\left( (x_1 - x_2), q \right), T_{\Psi_Q(b)}\left( (y_1 - y_2), q \right) \right\}
\geq \min \left\{ \min \{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(-x_2, q) \}, \min \{ T_{\Psi_Q(b)}(y_1, q), T_{\Psi_Q(b)}(-y_2, q) \} \right\}
\geq \min \left\{ \min \{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(x_2, q) \}, \min \{ T_{\Psi_Q(b)}(y_1, q), T_{\Psi_Q(b)}(y_2, q) \} \right\}
= \min \left\{ \min \{ T_{\Gamma_Q(a)}(x_1, q), T_{\Psi_Q(b)}(y_1, q) \}, \min \{ T_{\Gamma_Q(a)}(x_2, q), T_{\Psi_Q(b)}(y_2, q) \} \right\}
= \min \left\{ T_{\Lambda_Q(a,b)}((x_1, y_1), q), T_{\Lambda_Q(a,b)}((x_2, y_2), q) \right\}
\]

also,

\[
I_{\Lambda_Q(a,b)}\left(\left( (x_1, y_1) - (x_2, y_2), q \right) \right)
= I_{\Lambda_Q(a,b)}\left( (x_1 - x_2, y_1 - y_2), q \right)
= \max \left\{ I_{\Gamma_Q(a)}\left( (x_1 - x_2), q \right), I_{\Psi_Q(b)}\left( (y_1 - y_2), q \right) \right\}
\leq \max \left\{ \max \{ I_{\Gamma_Q(a)}(x_1, q), I_{\Gamma_Q(a)}(-x_2, q) \}, \max \{ I_{\Psi_Q(b)}(y_1, q), I_{\Psi_Q(b)}(-y_2, q) \} \right\}
\leq \max \left\{ \max \{ I_{\Gamma_Q(a)}(x_1, q), I_{\Gamma_Q(a)}(x_2, q) \}, \max \{ I_{\Psi_Q(b)}(y_1, q), I_{\Psi_Q(b)}(y_2, q) \} \right\}
= \max \left\{ \max \{ I_{\Gamma_Q(a)}(x_1, q), I_{\Psi_Q(b)}(y_1, q) \}, \max \{ I_{\Gamma_Q(a)}(x_2, q), I_{\Psi_Q(b)}(y_2, q) \} \right\}
= \max \left\{ I_{\Lambda_Q(a,b)}((x_1, y_1), q), I_{\Lambda_Q(a,b)}((x_2, y_2), q) \right\},
\]

similarly, \(F_{\Lambda_Q(a,b)}\left(\left( (x_1, y_1) - (x_2, y_2), q \right) \right) \leq \max \left\{ F_{\Lambda_Q(a,b)}((x_1, y_1), q), F_{\Lambda_Q(a,b)}((x_2, y_2), q) \right\} \). Next,

\[
T_{\Lambda_Q(a,b)}\left(\left( (x_1, y_1), (x_2, y_2)^{-1}, q \right) \right)
= T_{\Lambda_Q(a,b)}\left( (x_1, x_2^{-1}, y_1, y_2^{-1}), q \right)
= \min \left\{ T_{\Gamma_Q(a)}\left( (x_1, x_2^{-1}), q \right), T_{\Psi_Q(b)}\left( (y_1, y_2^{-1}), q \right) \right\}
\geq \min \left\{ \min \{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(x_2^{-1}, q) \}, \min \{ T_{\Psi_Q(b)}(y_1, q), T_{\Psi_Q(b)}(y_2^{-1}, q) \} \right\}
\[ I_{\Lambda_Q(a,b)}((x_1, y_1), (x_2, y_2)^{-1}, q) \]
\[ = I_{\Lambda_Q(a,b)}((x_1, x_2^{-1}, y_1, y_2^{-1}), q) \]
\[ = \max \left\{ I_{\Gamma_Q(a)}((x_1, x_2^{-1}), q), I_{\Psi_Q(b)}((y_1, y_2^{-1}), q) \right\} \]
\[ \leq \max \left\{ \max \left\{ I_{\Gamma_Q(a)}(x_1, q), I_{\Gamma_Q(a)}(x_2^{-1}, q) \right\}, \max \left\{ I_{\Psi_Q(b)}(y_1, q), I_{\Psi_Q(b)}(y_2^{-1}, q) \right\} \right\} \]
\[ \leq \max \left\{ \max \left\{ I_{\Gamma_Q(a)}(x_1, q), I_{\Gamma_Q(a)}(x_2, q) \right\}, \max \left\{ I_{\Psi_Q(b)}(y_1, q), I_{\Psi_Q(b)}(y_2, q) \right\} \right\} \]
\[ = \max \left\{ I_{\Lambda_Q(a,b)}((x_1, y_1), q), I_{\Lambda_Q(a,b)}((x_2, y_2), q) \right\}, \]

similarly, \( F_{\Lambda_Q(a,b)}((x_1, y_1), q), (x_2, y_2)^{-1}, q) \) \( \leq \max \left\{ F_{\Lambda_Q(a,b)}((x_1, y_1), q), F_{\Lambda_Q(a,b)}((x_2, y_2), q) \right\} \). This completes the proof. \( \square \)

6 Conclusions

In this study, we have introduced the concept of Q-neutrosophic soft fields. We have investigated some of its structural characteristics. Also, we have discussed the concepts of homomorphic image and pre-image of Q-neutrosophic soft fields. Moreover, we have defined the Cartesian product of Q-neutrosophic soft fields and discussed some related properties. The proposed notion enriches knowledge on neutrosophic sets in the branch of algebra. Also, it illuminates the way for more further deep discussion in algebra under neutrosophic and Q-neutrosophic soft environment for example, by establishing the notions of n-valued neutrosophic soft fields Q-neutrosophic soft modules and more.

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