Positive implicative ideals of BCK-algebras based on neutrosophic sets and falling shadows

Hashem Bordbar[1,*], Xiao Long Xin[2], Rajab Ali Borzooei[3], Young Bae Jun[4]

[1]Center for Information Technologies and Applied Mathematics, University of Nova Gorica, Slovenia.

E-mail: Hashem.bordbar@ung.si

[2] School of Mathematics, Northwest University, Xi'an 710127, China.

E-mail: xlxin@nwu.edu.cn

[3]Department of Mathematics, Shahid Beheshti University, Tehran 1983963113, Iran.

E-mail: borzooei@sbu.ac.ir

[4]Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea.

E-mail: skywine@gmail.com

*Correspondence: Hashem Bordbar (hashem.bordbar@ung.si)

Abstract: Neutrosophy is introduced by F. Smarandache in 1980 which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy considers a proposition, theory, event, concept, or entity, "A" in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A". Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics. In this article, we apply the notion of neutrosophic set theory to (positive implicative) ideals in BCK-algebras by using the concept of falling shadows. The notions of a positive implicative $(\in, \in)$-neutrosophic ideal and a positive implicative falling neutrosophic ideal are introduced, and several properties are investigated. Characterizations of a positive implicative $(\in, \in)$-neutrosophic ideal are considered, and relations between a positive implicative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal are discussed. Conditions for an $(\in, \in)$-neutrosophic ideal to be a positive implicative $(\in, \in)$-neutrosophic ideal are provided, and relations between a positive implicative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a positive implicative falling neutrosophic ideal are studied. Conditions for a falling neutrosophic ideal to be positive implicative are provided.

Keywords: neutrosophic random set, neutrosophic falling shadow, (positive implicative) $(\in, \in)$-neutrosophic ideal, (positive implicative) falling neutrosophic ideal.

1 Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [5] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [26] introduced the theory of falling shadows which directly relates probability concepts to the membership function of fuzzy sets. Falling shadow representation theory shows us a method of selection relaid on the joint degree distributions. It is a reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [27]. Tan et al. [24, 25] established a theoretical approach for defining a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Neutrosophic set (NS) developed by Smarandache [20, 21, 22] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun, Bordbar, Borumand Saeid and Öztürk studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [3], [4], [9], [11], [15], [17], [19] and [23]). It is a reasonable and convenient approach for the theoretical development and the practical applications of neutrosophic sets and neutrosophic logics. Jun et al. [12] introduced the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, they introduced the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. They discussed relations between falling neutrosophic subalgebra and falling neutrosophic ideal, and established a characterization of falling neutrosophic ideal [13]. Jun et al. [14] introduced the concepts of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal, and investigate several properties. They obtained characterizations of a commutative $(\in, \in)$-neutrosophic ideal, and discussed relations between a commutative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. They provided conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal, and considered relations between a commutative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a commutative falling neutrosophic ideal. They also gave conditions for a falling neutrosophic ideal to be commutative [18].

In this paper, we introduce the concepts of a positive implicative $(\in, \in)$-neutrosophic ideal and a positive implicative falling neutrosophic ideal, and investigate several properties. We obtain characterizations of a positive implicative $(\in, \in)$-neutrosophic ideal, and discuss relations between a positive implicative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a positive implicative $(\in, \in)$-neutrosophic ideal, and consider relations between a positive implicative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a positive implicative falling neutrosophic ideal. We give conditions for a falling neutrosophic ideal to be positive implicative.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [6] and [7]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\,(0 * x = 0)$,

then X is called a BCK-*algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be *positive implicative* if the following assertion is valid.

$$(\forall x, y, z \in X)\,((x * z) * (y * z) = (x * y) * z)\,. \tag{2.5}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.6}$$
$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.7}$$

A subset I of a BCK-algebra X is called a *positive implicative ideal* (see [16]) of X if it satisfies (2.6) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\; y * z \in I \;\Rightarrow\; x * z \in I)\,. \tag{2.8}$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [16]).

We refer the reader to the books [8, 16] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [21]) is a structure of the form:

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T, A_I, A_F)$ for the neutrosophic set

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$T_{\in}(A_{\sim};\alpha) := \{x \in X \mid A_T(x) \geq \alpha\},$$
$$I_{\in}(A_{\sim};\beta) := \{x \in X \mid A_I(x) \geq \beta\},$$
$$F_{\in}(A_{\sim};\gamma) := \{x \in X \mid A_F(x) \leq \gamma\}.$$

We say $T_{\in}(A_{\sim};\alpha)$, $I_{\in}(A_{\sim};\beta)$ and $F_{\in}(A_{\sim};\gamma)$ are *neutrosophic* $\in$*-subsets*.

A neutrosophic set $A_{\sim} = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic subalgebra* of X (see [9]) if the following assertions are valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x \in T_{\in}(A_{\sim};\alpha_x),\ y \in T_{\in}(A_{\sim};\alpha_y) \ \Rightarrow\ x * y \in T_{\in}(A_{\sim};\alpha_x \wedge \alpha_y), \\ x \in I_{\in}(A_{\sim};\beta_x),\ y \in I_{\in}(A_{\sim};\beta_y) \ \Rightarrow\ x * y \in I_{\in}(A_{\sim};\beta_x \wedge \beta_y), \\ x \in F_{\in}(A_{\sim};\gamma_x),\ y \in F_{\in}(A_{\sim};\gamma_y) \ \Rightarrow\ x * y \in F_{\in}(A_{\sim};\gamma_x \vee \gamma_y) \end{array}\right) \tag{2.9}$$

for all $\alpha_x,\ \alpha_y,\ \beta_x,\ \beta_y \in (0, 1]$ and $\gamma_x,\ \gamma_y \in [0, 1)$.

A neutrosophic set $A_{\sim} = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic ideal* of X (see [19]) if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_{\in}(A_{\sim};\alpha_x) \ \Rightarrow\ 0 \in T_{\in}(A_{\sim};\alpha_x) \\ x \in I_{\in}(A_{\sim};\beta_x) \ \Rightarrow\ 0 \in I_{\in}(A_{\sim};\beta_x) \\ x \in F_{\in}(A_{\sim};\gamma_x) \ \Rightarrow\ 0 \in F_{\in}(A_{\sim};\gamma_x) \end{array}\right) \tag{2.10}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_{\in}(A_{\sim};\alpha_x),\ y \in T_{\in}(A_{\sim};\alpha_y) \ \Rightarrow\ x \in T_{\in}(A_{\sim};\alpha_x \wedge \alpha_y) \\ x * y \in I_{\in}(A_{\sim};\beta_x),\ y \in I_{\in}(A_{\sim};\beta_y) \ \Rightarrow\ x \in I_{\in}(A_{\sim};\beta_x \wedge \beta_y) \\ x * y \in F_{\in}(A_{\sim};\gamma_x),\ y \in F_{\in}(A_{\sim};\gamma_y) \ \Rightarrow\ x \in F_{\in}(A_{\sim};\gamma_x \vee \gamma_y) \end{array}\right) \tag{2.11}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

In what follows, let X and $\mathcal{P}(X)$ denote a BCK/BCI-algebra and the power set of X, respectively, unless otherwise specified.

For each $x \in X$ and $D \in \mathcal{P}(X)$, let

$$\bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\}, \tag{2.12}$$

and

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{2.13}$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be a *hyper-measurable structure* on X if $\mathcal{B}$ is a σ-field in $\mathcal{P}(X)$ and $\bar{X} \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on X, a *neutrosophic random set* on X (see [12]) is defined to be a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which ξ_T, ξ_I and ξ_F are mappings from

Ω to $\mathcal{P}(X)$ which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(\forall C \in \mathcal{B}) \left(\begin{array}{l} \xi_T^{-1}(C) = \{\omega_T \in \Omega \mid \xi_T(\omega_T) \in C\} \in \mathcal{A} \\ \xi_I^{-1}(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\ \xi_F^{-1}(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A} \end{array} \right). \tag{2.14}$$

Given a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, consider functions:

$$\begin{aligned} &\tilde{H}_T : X \to [0,1],\ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I : X \to [0,1],\ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F : X \to [0,1],\ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)). \end{aligned}$$

Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a neutrosophic set on X, and we call it a *neutrosophic falling shadow* (see [12]) of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$, and $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *neutrosophic cloud* (see [12]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

For example, consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic set in X. Then a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\begin{aligned} &\xi_T : [0,1] \to \mathcal{P}(X), \alpha \mapsto T_\in(\tilde{H}; \alpha), \\ &\xi_I : [0,1] \to \mathcal{P}(X), \beta \mapsto I_\in(\tilde{H}; \beta), \\ &\xi_F : [0,1] \to \mathcal{P}(X), \gamma \mapsto F_\in(\tilde{H}; \gamma) \end{aligned}$$

is a neutrosophic random set and $\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. We will call $\xi := (\xi_T, \xi_I, \xi_F)$ defined above as the *neutrosophic cut-cloud* (see [12]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *falling neutrosophic subalgebra* (resp., *falling neutrosophic ideal*) of X (see [12]).

3 Positive implicative $(\in, \in)$-neutrosophic ideals

Definition 3.1. A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK-algebra X is called a *positive implicative $(\in, \in)$-neutrosophic ideal* of X if it satisfies the condition (2.10) and

$$\begin{array}{l} (x*y)*z \in T_\in(A_\sim; \alpha_x),\ y*z \in T_\in(A_\sim; \alpha_y) \ \Rightarrow\ x*z \in T_\in(A_\sim; \alpha_x \wedge \alpha_y) \\ (x*y)*z \in I_\in(A_\sim; \beta_x),\ y*z \in I_\in(A_\sim; \beta_y) \ \Rightarrow\ x*z \in I_\in(A_\sim; \beta_x \wedge \beta_y) \\ (x*y)*z \in F_\in(A_\sim; \gamma_x),\ y*z \in F_\in(A_\sim; \gamma_y) \ \Rightarrow\ x*z \in F_\in(A_\sim; \gamma_x \vee \gamma_y) \end{array} \tag{3.1}$$

for all $x, y, z \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Example 3.2. Consider a set $X = \{0,1,2,3,4\}$ with the binary operation $*$ which is given in Table 1 Then $(X;*,0)$ is a BCK-algebra (see [16]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

Table 2: Tabular representation of $A_\sim = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.8	0.6	0.1
1	0.7	0.6	0.4
2	0.6	0.5	0.4
3	0.4	0.2	0.6
4	0.2	0.3	0.9

Routine calculations show that $A_\sim = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of X.

Theorem 3.3. *Every positive implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X is an $(\in, \in)$-neutrosophic ideal of X.*

Proof. It is clear by taking $z = 0$ in (3.1) and using (2.1). □

Theorem 3.4. *For a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK-algebra X, the following are equivalent.*

(1) *The non-empty $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are positive implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(A_T(0) \geq A_T(x),\, A_I(0) \geq A_I(x),\, A_F(0) \leq A_F(x) \right) \tag{3.2}$$

and

$$(\forall x, y, z \in X)\begin{pmatrix} A_T(x * z) \geq A_T((x * y) * z) \wedge A_T(y * z) \\ A_I(x * z) \geq A_I((x * y) * z) \wedge A_I(y * z) \\ A_F(x * z) \leq A_F((x * y) * z) \vee A_F(y * z) \end{pmatrix} \tag{3.3}$$

Proof. Assume that the non-empty $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are positive implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. If $A_T(0) < A_T(a)$ for some $a \in X$, then $a \in T_\in(A_\sim; A_T(a))$ and $0 \notin T_\in(A_\sim; A_T(a))$. This is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Similarly, $A_I(0) \geq A_I(x)$ for all $x \in X$. Suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in F_\in(A_\sim; A_F(a))$ and

$0 \notin F_{\in}(A_{\sim}; A_F(a))$. This is a contradiction, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore (3.2) is valid. Assume that there exist $a, b, c \in X$ such that

$$A_T(a * c) < A_T((a * b) * c) \wedge A_T(b * c).$$

Taking $\alpha := A_T((a * b) * c) \wedge A_T(b * c)$ implies that $(a * b) * c \in T_{\in}(A_{\sim}; \alpha)$ and $b * c \in T_{\in}(A_{\sim}; \alpha)$ but $a * c \notin T_{\in}(A_{\sim}; \alpha)$, which is a contradiction. Hence

$$A_T(x * z) \geq A_T((x * y) * z) \wedge A_T(y * z)$$

for all $x, y, z \in X$. By the similar way, we can verify that

$$A_I(x * z) \geq A_I((x * y) * z) \wedge A_I(y * z)$$

for all $x, y, z \in X$. Now suppose there are $x, y, z \in X$ such that

$$A_F(x * z) > A_F((x * y) * z) \vee A_F(y * z) := \gamma.$$

Then $(x * y) * z \in F_{\in}(A_{\sim}; \gamma)$ and $y * z \in F_{\in}(A_{\sim}; \gamma)$ but $x * z \notin F_{\in}(A_{\sim}; \gamma)$, a contradiction. Thus

$$A_F(x * z) \leq A_F((x * y) * z) \vee A_F(y * z)$$

for all $x, y, z \in X$.

Conversely, let $A_{\sim} = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying two conditions (3.2) and (3.3). Assume that $T_{\in}(A_{\sim}; \alpha)$, $I_{\in}(A_{\sim}; \beta)$ and $F_{\in}(A_{\sim}; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Let $x \in T_{\in}(A_{\sim}; \alpha)$, $a \in I_{\in}(A_{\sim}; \beta)$ and $u \in F_{\in}(A_{\sim}; \gamma)$ for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$ by (3.2). It follows that $0 \in T_{\in}(A_{\sim}; \alpha)$, $0 \in I_{\in}(A_{\sim}; \beta)$ and $0 \in F_{\in}(A_{\sim}; \gamma)$. Let $a, b, c \in X$ be such that $(a * b) * c \in T_{\in}(A_{\sim}; \alpha)$ and $b * c \in T_{\in}(A_{\sim}; \alpha)$ for $\alpha \in (0, 1]$. Then

$$A_T(a * c) \geq A_T((a * b) * c) \wedge A_T(b * c) \geq \alpha$$

by (3.3), and so $a * c \in T_{\in}(A_{\sim}; \alpha)$. If $(x * y) * z \in I_{\in}(A_{\sim}; \beta)$ and $y * z \in I_{\in}(A_{\sim}; \beta)$ for all $x, y, z \in X$ and $\beta \in (0, 1]$, then $A_I((x * y) * z) \geq \beta$ and $A_I(y * z) \geq \beta$. Hence the condition (3.3) implies that

$$A_I(x * z) \geq A_I((x * y) * z) \wedge A_I(y * z) \geq \beta,$$

that is, $x * z \in I_{\in}(A_{\sim}; \beta)$. Finally, suppose that $(x * y) * z \in F_{\in}(A_{\sim}; \gamma)$ and $y * z \in F_{\in}(A_{\sim}; \gamma)$ for all $x, y, z \in X$ and $\gamma \in (0, 1]$. Then $A_F((x * y) * z) \leq \gamma$ and $A_F(y * z) \leq \gamma$, which imply from the condition (3.3) that

$$A_F(x * z) \leq A_F((x * y) * z) \vee A_F(y * z) \leq \gamma.$$

Hence $x * z \in F_{\in}(A_{\sim}; \gamma)$. Therefore the non-empty $\in$-subsets $T_{\in}(A_{\sim}; \alpha)$, $I_{\in}(A_{\sim}; \beta)$ and $F_{\in}(A_{\sim}; \gamma)$ are positive implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. □

Theorem 3.5. *Let $A_{\sim} = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A_{\sim} = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of X if and only if the non-empty neutrosophic $\in$-subsets $T_{\in}(A_{\sim}; \alpha)$, $I_{\in}(A_{\sim}; \beta)$ and $F_{\in}(A_{\sim}; \gamma)$ are positive implicative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be a positive implicative $(\in, \in)$-neutrosophic ideal of X and assume that $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty for $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Then there exist $x, y, z \in X$ such that $x \in T_\in(A_\sim;\alpha)$, $y \in I_\in(A_\sim;\beta)$ and $z \in F_\in(A_\sim;\gamma)$. It follows from (2.10) that $0 \in T_\in(A_\sim;\alpha)$, $0 \in I_\in(A_\sim;\beta)$ and $0 \in F_\in(A_\sim;\gamma)$. Let $x, y, z, a, b, c, u, v, w \in X$ be such that $(x * y) * z \in T_\in(A_\sim;\alpha)$, $y * z \in T_\in(A_\sim;\alpha)$, $(a * b) * c \in I_\in(A_\sim;\beta)$, $b * c \in I_\in(A_\sim;\beta)$, $(u * v) * w \in F_\in(A_\sim;\gamma)$ and $v * w \in F_\in(A_\sim;\gamma)$. Then $x * z \in T_\in(A_\sim;\alpha \wedge \alpha) = T_\in(A_\sim;\alpha)$, $a * c \in I_\in(A_\sim;\beta \wedge \beta) = I_\in(A_\sim;\beta)$, and $u * w \in F_\in(A_\sim;\gamma \vee \gamma) = F_\in(A_\sim;\gamma)$ by (3.1). Hence the non-empty neutrosophic $\in$-subsets $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are positive implicative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.

Conversely, let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_\in(A_\sim;\alpha)$, $I_\in(A_\sim;\beta)$ and $F_\in(A_\sim;\gamma)$ are nonempty and are positive implicative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Obviously, (2.10) is valid. Let $x, y, z \in X$ and $\alpha_x, \alpha_y \in (0,1]$ be such that $(x*y)*z \in T_\in(A_\sim;\alpha_x)$ and $y*z \in T_\in(A_\sim;\alpha_y)$. Then $(x * y) * z \in T_\in(A_\sim;\alpha)$ and $y * z \in T_\in(A_\sim;\alpha)$ where $\alpha = \alpha_x \wedge \alpha_y$. Since $T_\in(A_\sim;\alpha)$ is a positive implicative ideal of X, it follows that $x * z \in T_\in(A_\sim;\alpha) = T_\in(A_\sim;\alpha_x \wedge \alpha_y)$. Similarly, if $(x * y) * z \in I_\in(A_\sim;\beta_x)$ and $y * z \in I_\in(A_\sim;\beta_y)$ for all $x, y, z \in X$ and $\beta_x, \beta_y \in (0,1]$, then $x * z \in I_\in(A_\sim;\beta_x \wedge \beta_y)$. Now, suppose that $(x * y) * z \in F_\in(A_\sim;\gamma_x)$ and $y * z \in F_\in(A_\sim;\gamma_y)$ for all $x, y, z \in X$ and $\gamma_x, \gamma_y \in [0,1)$. Then $(x*y)*z \in F_\in(A_\sim;\gamma)$ and $y*z \in F_\in(A_\sim;\gamma)$ where $\gamma = \gamma_x \vee \gamma_y$. Hence $x*z \in F_\in(A_\sim;\gamma) = F_\in(A_\sim;\gamma_x \vee \gamma_y)$ since $F_\in(A_\sim;\gamma)$ is a positive implicative ideal of X. Therefore $A_\sim = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of X. □

Corollary 3.6. *Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A_\sim = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of X if and only if it satisfies two conditions (3.2) and (3.3).*

Lemma 3.7 ([18])**.** *Every $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the following assertion.*

$$(\forall x, y \in X)\left(x \le y \;\Rightarrow \left\{\begin{array}{l} A_T(x) \ge A_T(y) \\ A_I(x) \ge A_I(y) \\ A_F(x) \le A_F(y) \end{array}\right.\right). \tag{3.4}$$

Lemma 3.8 ([18])**.** *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the following assertions are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.*

(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X)\left(\; A_T(0) \ge A_T(x),\, A_I(0) \ge A_I(x),\, A_F(0) \le A_F(x) \;\right) \tag{3.5}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x) \ge A_T(x * y) \wedge A_T(y) \\ A_I(x) \ge A_I(x * y) \wedge A_I(y) \\ A_F(x) \le A_F(x * y) \vee A_F(y) \end{array}\right) \tag{3.6}$$

Proposition 3.9. *Every positive implicative $(\in, \in)$-neutrosophic ideal $A_\sim = (A_T, A_I, A_F)$ of a BCK-algebra*

X satisfies the following assertions.

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x*y) \geq A_T((x*y)*y) \\ A_I(x*y) \geq A_I((x*y)*y) \\ A_F(x*y) \leq A_F((x*y)*y) \end{array}\right), \tag{3.7}$$

$$(\forall x, y \in X)\left(\begin{array}{l} A_T((x*z)*(y*z)) \geq A_T((x*y)*z) \\ A_I((x*z)*(y*z)) \geq A_I((x*y)*z) \\ A_F((x*z)*(y*z)) \leq A_F((x*y)*z) \end{array}\right), \tag{3.8}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x*y) \geq A_T(((x*y)*y)*z) \wedge A_T(z) \\ A_I(x*y) \geq A_I(((x*y)*y)*z) \wedge A_I(z) \\ A_F(x*y) \leq A_F(((x*y)*y)*z) \vee A_F(z) \end{array}\right). \tag{3.9}$$

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be a positive implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Then $A_\sim = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of a BCK-algebra X (see Theorem 3.3). Since $x*x = 0$ for all $x \in X$, putting $z = y$ in (3.3) and using (3.2) induce (3.7). Since

$$((x*(y*z))*z)*z = ((x*z)*(y*z))*z \leq (x*y)*z$$

for all $x, y, z \in X$, we have

$$\begin{aligned} A_T((x*z)*(y*z)) &= A_T((x*(y*z))*z) \\ &\geq A_T(((x*(y*z))*z)*z) \\ &\geq A_T((x*y)*z), \end{aligned}$$

$$\begin{aligned} A_I((x*z)*(y*z)) &= A_I((x*(y*z))*z) \\ &\geq A_I(((x*(y*z))*z)*z) \\ &\geq A_I((x*y)*z) \end{aligned}$$

and

$$\begin{aligned} A_F((x*z)*(y*z)) &= A_F((x*(y*z))*z) \\ &\leq A_F(((x*(y*z))*z)*z) \\ &\leq A_F((x*y)*z) \end{aligned}$$

by (2.3), (3.7) and Lemma 3.7. Thus (3.8) is valid. Note that

$$(x*y)*z = ((x*z)*y)*(y*y)$$

for all $x, y \in X$. It follows from Lemma 3.8, (3.8) and (2.3) that

$$\begin{aligned} A_T(x*y) &\geq A_T((x*y)*z) \wedge A_T(z) \\ &= A_T(((x*z)*y)*(y*y)) \wedge A_T(z) \\ &\geq A_T(((x*z)*y)*y) \wedge A_T(z) \\ &= A_T(((x*y)*y)*z) \wedge A_T(z), \end{aligned}$$

$$\begin{aligned} A_I(x*y) &\geq A_I((x*y)*z) \wedge A_I(z) \\ &= A_I(((x*z)*y)*(y*y)) \wedge A_I(z) \\ &\geq A_I(((x*z)*y)*y) \wedge A_I(z) \\ &= A_I(((x*y)*y)*z) \wedge A_I(z), \end{aligned}$$

and

$$\begin{aligned} A_F(x*y) &\leq A_F((x*y)*z) \vee A_F(z) \\ &= A_F(((x*z)*y)*(y*y)) \vee A_F(z) \\ &\leq A_F(((x*z)*y)*y) \vee A_F(z) \\ &= A_F(((x*y)*y)*z) \vee A_F(z) \end{aligned}$$

for all $x, y, z \in X$. Therefore (3.9) is valid. □

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.10. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 3

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [16]). Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 4

Routine calculations show that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Neutrosophic $\in$-

Table 4: Tabular representation of $A_{\sim} = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.9	0.3
1	0.4	0.7	0.5
2	0.5	0.6	0.4
3	0.4	0.7	0.5
4	0.1	0.4	0.6

subsets are given as follows.

$$T_{\in}(A_{\sim};\alpha) = \begin{cases} \emptyset & \text{if } \alpha \in (0.7, 1], \\ \{0\} & \text{if } \alpha \in (0.5, 0.7], \\ \{0, 2\} & \text{if } \alpha \in (0.4, 0.5], \\ \{0, 1, 2, 3\} & \text{if } \alpha \in (0.1, 0.4], \\ X & \text{if } \alpha \in (0, 0.1], \end{cases}$$

$$I_{\in}(A_{\sim};\beta) = \begin{cases} \emptyset & \text{if } \beta \in (0.9, 1], \\ \{0\} & \text{if } \beta \in (0.7, 0.9], \\ \{0, 1, 3\} & \text{if } \beta \in (0.6, 0.7], \\ \{0, 1, 2, 3\} & \text{if } \beta \in (0.4, 0.6], \\ X & \text{if } \beta \in (0, 0.4], \end{cases}$$

and

$$F_{\in}(A_{\sim};\gamma) = \begin{cases} X & \text{if } \gamma \in [0.6, 1), \\ \{0, 1, 2, 3\} & \text{if } \gamma \in [0.5, 0.6), \\ \{0, 2\} & \text{if } \gamma \in [0.4, 0.5), \\ \{0\} & \text{if } \gamma \in [0.3, 0.4), \\ \emptyset & \text{if } \gamma \in [0, 0.3). \end{cases}$$

If $\alpha \in (0.4, 0.5]$ and $\gamma \in [0.4, 0.5)$, then $T_{\in}(A_{\sim};\alpha)$ and $F_{\in}(A_{\sim};\gamma)$ are not positive implicative ideals of X. Thus $A_{\sim} = (A_T, A_I, A_F)$ is not a positive implicative $(\in, \in)$-neutrosophic ideal of X by Theorems 3.4 and 3.5.

We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a positive implicative $(\in, \in)$-neutrosophic ideal.

Theorem 3.11. *Given a neutrosophic set $A_{\sim} = (A_T, A_I, A_F)$ in a BCK-algebra X, the following assertions are equivalent.*

(1) *$A_{\sim} = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of X.*

(2) *$A_{\sim} = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X that satisfies the condition (3.7).*

(3) $A_\sim = (A_T, A_I, A_F)$ *is an* $(\in, \in)$*-neutrosophic ideal of* X *that satisfies the condition* (3.8).

(4) $A_\sim = (A_T, A_I, A_F)$ *satisfies two conditions* (3.2) *and* (3.9).

Proof. Assume that $A_\sim = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of X. Then $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X by Theorem 3.3. If we take $z = y$ in (3.3) and use (3.2), then we get the condition (3.7). Suppose that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X satisfying the condition (3.7). Note that

$$((x * (y * z)) * z) * z = ((x * z) * (y * z)) * z \leq (x * y) * z$$

for all $x, y, z \in X$. It follows from (2.3), (3.7) and Lemma 3.7 that

$$\begin{aligned} A_T((x * z) * (y * z)) &= A_T((x * (y * z)) * z) \\ &\geq A_T(((x * (y * z)) * z) * z) \\ &\geq A_T((x * y) * z), \end{aligned}$$

$$\begin{aligned} A_I((x * z) * (y * z)) &= A_I((x * (y * z)) * z) \\ &\geq A_I(((x * (y * z)) * z) * z) \\ &\geq A_I((x * y) * z), \end{aligned}$$

and

$$\begin{aligned} A_F((x * z) * (y * z)) &= A_F((x * (y * z)) * z) \\ &\leq A_F(((x * (y * z)) * z) * z) \\ &\leq A_F((x * y) * z). \end{aligned}$$

Hence (3.8) is valid. Assume that $A_\sim = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X satisfying the condition (3.8). It is clear that $A_\sim = (A_T, A_I, A_F)$ satisfies the condition (3.2). Using (3.6), (III), (2.3) and (3.8), we have

$$\begin{aligned} T_\in(x * y) &\geq T_\in((x * y) * z) \wedge T_\in(z) \\ &= T_\in(((x * z) * y) * (y * y)) \wedge T_\in(z) \\ &\geq T_\in(((x * z) * y) * y) \wedge T_\in(z) \\ &= T_\in(((x * y) * y) * z) \wedge T_\in(z), \end{aligned}$$

$$\begin{aligned} I_\in(x * y) &\geq I_\in((x * y) * z) \wedge I_\in(z) \\ &= I_\in(((x * z) * y) * (y * y)) \wedge I_\in(z) \\ &\geq I_\in(((x * z) * y) * y) \wedge I_\in(z) \\ &= I_\in(((x * y) * y) * z) \wedge I_\in(z), \end{aligned}$$

and

$$\begin{aligned} F_{\in}(x*y) &\leq F_{\in}((x*y)*z) \vee F_{\in}(z) \\ &= F_{\in}(((x*z)*y)*(y*y)) \vee F_{\in}(z) \\ &\leq F_{\in}(((x*z)*y)*y) \vee F_{\in}(z) \\ &= F_{\in}(((x*y)*y)*z) \vee F_{\in}(z) \end{aligned}$$

for all $x, y, z \in X$. Thus (3.9) is valid. Finally suppose that $A_{\sim} = (A_T, A_I, A_F)$ satisfies two conditions (3.2) and (3.9). Using (2.1) and (3.9), we get

$$\begin{aligned} A_T(x) &= A_T(x*0) \\ &\geq A_T(((x*0)*0)*y) \wedge A_T(y) \\ &= A_T(x*y) \wedge A_T(y), \end{aligned}$$

$$\begin{aligned} A_I(x) &= A_I(x*0) \\ &\geq A_I(((x*0)*0)*y) \wedge A_I(y) \\ &= A_I(x*y) \wedge A_I(y), \end{aligned}$$

and

$$\begin{aligned} A_F(x) &= A_F(x*0) \\ &\leq A_F(((x*0)*0)*y) \vee A_F(y) \\ &= A_F(x*y) \vee A_F(y) \end{aligned}$$

for all $x, y \in X$. Hence $A_{\sim} = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. Since

$$((x*z)*z)*(y*z) \leq (x*z)*y = (x*y)*z$$

for all $x, y, z \in X$, it follows from (3.9) and (3.4) that

$$\begin{aligned} A_T(x*z) &\geq A_T(((x*z)*z)*(y*z)) \wedge A_T(y*z) \\ &\geq A_T((x*y)*z) \wedge A_T(y*z), \end{aligned}$$

$$\begin{aligned} A_I(x*z) &\geq A_I(((x*z)*z)*(y*z)) \wedge A_I(y*z) \\ &\geq A_I((x*y)*z) \wedge A_I(y*z), \end{aligned}$$

and

$$\begin{aligned} A_F(x*z) &\leq A_F(((x*z)*z)*(y*z)) \vee A_F(y*z) \\ &\leq A_F((x*y)*z) \vee A_F(y*z) \end{aligned}$$

for all $x, y, z \in X$. Therefore $A_{\sim} = (A_T, A_I, A_F)$ is a positive implicative $(\in, \in)$-neutrosophic ideal of X. □

4 Positive implicative falling neutrosophic ideals

Definition 4.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, that is,

$$\begin{aligned} &\tilde{H}_T(x_T) = P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I(x_I) = P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F(x_F) = 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)) \end{aligned} \tag{4.1}$$

is called a *positive implicative falling neutrosophic ideal* of X.

Example 4.2. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 5

Table 5: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [16]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.25), \\ \{0,1\} & \text{if } t \in [0.25, 0.55), \\ \{0,1,2\} & \text{if } t \in [0.55, 0.85), \\ \{0,1,3\} & \text{if } t \in [0.85, 0.95), \\ X & \text{if } t \in [0.95, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1,2\} & \text{if } t \in [0, 0.45), \\ \{0,1,2,3\} & \text{if } t \in [0.45, 0.75), \\ \{0,1,2,4\} & \text{if } t \in [0.75, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0,3\} & \text{if } t \in (0.7, 0.9], \\ \{0,1,2\} & \text{if } t \in (0.5, 0.7], \\ \{0,1,2,3\} & \text{if } t \in (0.3, 0.5], \\ \{0,1,2,4\} & \text{if } t \in [0, 0.3]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are positive implicative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a positive implicative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.75 & \text{if } x = 1, \\ 0.35 & \text{if } x = 2, \\ 0.4 & \text{if } x = 3, \\ 0.05 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x \in \{0,1,2\}, \\ 0.3 & \text{if } x = 3, \\ 0.25 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.7 & \text{if } x \in \{1,2\}, \\ 0.4 & \text{if } x = 3, \\ 0.3 & \text{if } x = 4. \end{cases}$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$. For $x \in X$, let

$$\begin{aligned} &\Omega(x;\xi_T) := \{\omega_T \in \Omega \mid x \in \xi_T(\omega_T)\}, \\ &\Omega(x;\xi_I) := \{\omega_I \in \Omega \mid x \in \xi_I(\omega_I)\}, \\ &\Omega(x;\xi_F) := \{\omega_F \in \Omega \mid x \in \xi_F(\omega_F)\}. \end{aligned}$$

Then $\Omega(x;\xi_T), \Omega(x;\xi_I), \Omega(x;\xi_F) \in \mathcal{A}$ (see [12]).

Proposition 4.3. *Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a positive implicative falling neutrosophic ideal of X, then*

$$(\forall x,y,z \in X) \begin{pmatrix} \Omega((x*y)*z;\xi_T) \cap \Omega(y*z;\xi_T) \subseteq \Omega(x*z;\xi_T) \\ \Omega((x*y)*z;\xi_I) \cap \Omega(y*z;\xi_I) \subseteq \Omega(x*z;\xi_I) \\ \Omega((x*y)*z;\xi_F) \cap \Omega(y*z;\xi_F) \subseteq \Omega(x*z;\xi_F) \end{pmatrix}, \tag{4.2}$$

$$(\forall x,y,z \in X) \begin{pmatrix} \Omega(x*z;\xi_T) \subseteq \Omega((x*y)*z;\xi_T) \\ \Omega(x*z;\xi_I) \subseteq \Omega((x*y)*z;\xi_I) \\ \Omega(x*z;\xi_F) \subseteq \Omega((x*y)*z;\xi_F) \end{pmatrix}. \tag{4.3}$$

Proof. Let $\omega_T \in \Omega((x*y)*z;\xi_T) \cap \Omega(y*z;\xi_T)$, $\omega_I \in \Omega((x*y)*z;\xi_I) \cap \Omega(y*z;\xi_I)$ and $\omega_F \in \Omega((x*y)*z;\xi_F) \cap \Omega(y*z;\xi_F)$ for all $x,y,z \in X$. Then
$(x*y)*z \in \xi_T(\omega_T)$ and $y*z \in \xi_T(\omega_T)$,
$(x*y)*z \in \xi_I(\omega_I)$ and $y*z \in \xi_I(\omega_I)$,

$(x * y) * z \in \xi_F(\omega_F)$ and $y * z \in \xi_F(\omega_F)$.
Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals of X, it follows from (2.8) that $x * z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$ and so that $\omega_T \in \Omega(x * z; \xi_T)$, $\omega_I \in \Omega(x * z; \xi_I)$ and $\omega_F \in \Omega(x * z; \xi_F)$. Hence (4.2) is valid. Now let $x, y, z \in X$ be such that $\omega_T \in \Omega(x * z; \xi_T)$, $\omega_I \in \Omega(x * z; \xi_I)$, and $\omega_F \in \Omega(x * z; \xi_F)$. Then $x * z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$. Note that

$$\begin{aligned}
&((x * y) * z) * (x * z) = ((x * y) * (x * z)) * z \\
&\leq (z * y) * z = (z * z) * y \\
&= 0 * y = 0,
\end{aligned}$$

which yields

$$((x * y) * z) * (x * z) = 0 \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals and hence ideals of X, it follows that $(x * y) * z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$. Hence $\omega_T \in \Omega((x*y)*z; \xi_T)$, $\omega_I \in \Omega((x*y)*z; \xi_I)$, and $\omega_F \in \Omega((x*y)*z; \xi_F)$. Therefore (4.3) is valid. □

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{4.4}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\,((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{4.5}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a BCK/BCI-algebra (see [10]) where θ is given as follows:

$$\theta : \Omega \to X,\ \omega \mapsto 0.$$

For any subset A of X and $g_T, g_I, g_F \in \mathcal{F}(X)$, consider the followings:

$$\begin{aligned}
A_T^g &:= \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\}, \\
A_I^g &:= \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\}, \\
A_F^g &:= \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\}
\end{aligned}$$

and

$$\begin{aligned}
&\xi_T : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_T \mapsto \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in A\}, \\
&\xi_I : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_I \mapsto \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in A\}, \\
&\xi_F : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_F \mapsto \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in A\}.
\end{aligned}$$

Then $A_T^g, A_I^g, A_F^g \in \mathcal{A}$ (see [12]).

Theorem 4.4. *If K is a positive implicative ideal of a BCK-algebra X, then*

$$\begin{aligned}\xi_T(\omega_T) = \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in K\},\\ \xi_I(\omega_I) = \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in K\},\\ \xi_F(\omega_F) = \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in K\}\end{aligned}$$

are positive implicative ideals of $\mathcal{F}(X)$.

Proof. Assume that K is a positive implicative ideal of a BCK-algebra X. Since $\theta(\omega_T) = 0 \in K$, $\theta(\omega_I) = 0 \in K$ and $\theta(\omega_F) = 0 \in K$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, we have

$$\theta \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Let $f_T, g_T, h_T \in \mathcal{F}(X)$ be such that $(f_T \circledast g_T) \circledast h_T \in \xi_T(\omega_T)$ and $g_T \circledast h_T \in \xi_T(\omega_T)$. Then

$$(f_T(\omega_T) * g_T(\omega_T)) * h_T(\omega_T) = ((f_T \circledast g_T) \circledast h_T)(\omega_T) \in K$$

and $g_T(\omega_T) * h_T(\omega_T) \in K$. Since K is a positive implicative ideal of X, it follows from (2.8) that

$$(f_T \circledast h_T)(\omega_T) = f_T(\omega_T) * h_T(\omega_T) \in K,$$

that is, $f_T \circledast h_T \in \xi_T(\omega_T)$. Hence $\xi_T(\omega_T)$ is a positive implicative ideal of $\mathcal{F}(X)$. Similarly, we can verify that $\xi_I(\omega_I)$ is a positive implicative ideal of $\mathcal{F}(X)$. Now, let $f_F, g_F, h_F \in \mathcal{F}(X)$ be such that $(f_F \circledast g_F) \circledast h_F \in \xi_F(\omega_F)$ and $g_F \circledast h_F \in \xi_F(\omega_F)$. Then

$$(f_F(\omega_F) * g_F(\omega_F)) * h_F(\omega_F) = ((f_F \circledast g_F) \circledast h_F)(\omega_F) \in K$$

and $g_F(\omega_F) * h_F(\omega_F) \in K$. Then

$$(f_F \circledast h_F)(\omega_F) = f_F(\omega_F) * h_F(\omega_F) \in K,$$

and so $f_F \circledast h_F \in \xi_F(\omega_F)$. Hence $\xi_F(\omega_F)$ is a positive implicative ideal of $\mathcal{F}(X)$. This completes the proof. □

Theorem 4.5. *If we consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, then every positive implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra is a positive implicative falling neutrosophic ideal.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a positive implicative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Then $T_\in(\tilde{H}; \alpha)$, $I_\in(\tilde{H}; \beta)$ and $F_\in(\tilde{H}; \gamma)$ are positive implicative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ by Theorem 3.5. Hence a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\begin{aligned}&\xi_T : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto T_\in(\tilde{H}; \alpha),\\ &\xi_I : [0,1] \to \mathcal{P}(X),\ \beta \mapsto I_\in(\tilde{H}; \beta),\\ &\xi_F : [0,1] \to \mathcal{P}(X),\ \gamma \mapsto F_\in(\tilde{H}; \gamma)\end{aligned}$$

is a neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$, and so $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a positive implicative falling neutrosophic ideal of X. □

The converse of Theorem 4.5 is not true as seen in the following example.

Table 6: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Example 4.6. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 6

Then $(X; *, 0)$ is a BCK-algebra (see [16]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 1\} & \text{if } t \in [0, 0.2), \\ \{0, 2\} & \text{if } t \in [0.2, 0.55), \\ \{0, 2, 4\} & \text{if } t \in [0.55, 0.75), \\ \{0, 1, 2, 3\} & \text{if } t \in [0.75, 1], \end{cases}$$

$$\xi_I : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, 2\} & \text{if } t \in [0, 0.26), \\ \{0, 4\} & \text{if } t \in [0.26, 0.68), \\ \{0, 1, 2, 3\} & \text{if } t \in [0.68, 1] \end{cases}$$

and

$$\xi_F : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.77, 1], \\ \{0, 1\} & \text{if } t \in (0.66, 0.77], \\ \{0, 2\} & \text{if } t \in (0.48, 0.66], \\ \{0, 2, 4\} & \text{if } t \in (0.23, 0.48], \\ \{0, 1, 2, 3\} & \text{if } t \in [0, 0.23]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are positive implicative ideals of X for all $t \in [0, 1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a positive implicative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, \\ 0.8 & \text{if } x = 2, \\ 0.25 & \text{if } x = 3, \\ 0.2 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.32 & \text{if } x \in \{1, 3\}, \\ 0.58 & \text{if } x = 2, \\ 0.42 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.66 & \text{if } x = 1, \\ 0.34 & \text{if } x = 2, \\ 0.77 & \text{if } x = 3, \\ 0.75 & \text{if } x = 4. \end{cases}$$

But $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not a positive implicative $(\in, \in)$-neutrosophic ideal of X since

$$\tilde{H}_T(3 * 4) = \tilde{H}_T(3) = 0.25 < 0.8 = \tilde{H}_T((3 * 2) * 4) \wedge \tilde{H}_T(2 * 4)$$

and/or

$$\tilde{H}_T(3 * 4) = \tilde{H}_T(3) = 0.77 > 0.66 = \tilde{H}_T((3 * 1) * 4) \vee \tilde{H}_T(1 * 4).$$

We provide relations between a falling neutrosophic ideal and a positive implicative falling neutrosophic ideal .

Theorem 4.7. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a positive implicative falling neutrosophic ideal of X, then it is a falling neutrosophic ideal of X.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a positive implicative falling neutrosophic ideal of a BCK-algebra X. Then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are positive implicative ideals of X, and so $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X. □

The following example shows that the converse of Theorem 4.7 is not true in general.

Example 4.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 7

Table 7: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	3	2	0	3
4	4	4	4	4	0

Then $(X;*,0)$ is a BCK-algebra (see [16]). Consider $(\Omega,\mathcal{A},P)=([0,1],\mathcal{A},m)$ and let $\xi:=(\xi_T,\xi_I,\xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T:[0,1]\rightarrow\mathcal{P}(X),\ \ x\mapsto\begin{cases}\{0,4\} & \text{if } t\in[0,0.37),\\ \{0,1,2,3\} & \text{if } t\in[0.37,0.67),\\ \{0,1,4\} & \text{if } t\in[0.67,1],\end{cases}$$

$$\xi_I:[0,1]\rightarrow\mathcal{P}(X),\ \ x\mapsto\begin{cases}\{0,4\} & \text{if } t\in[0,0.45),\\ \{0,1,2,3\} & \text{if } t\in[0.45,1],\end{cases}$$

and

$$\xi_F:[0,1]\rightarrow\mathcal{P}(X),\ \ x\mapsto\begin{cases}\{0\} & \text{if } t\in(0.74,1],\\ \{0,1\} & \text{if } t\in(0.66,0.74],\\ \{0,4\} & \text{if } t\in(0.48,0.66],\\ \{0,1,2,3\} & \text{if } t\in[0,0.48].\end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are ideals of X for all $t\in[0,1]$. Hence the neutrosophic falling shadow $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ of $\xi:=(\xi_T,\xi_I,\xi_F)$ is a falling neutrosophic ideal of X. But it is not a positive implicative falling neutrosophic ideal of X because if $\alpha\in[0.67,1]$, $\beta\in[0,0.45)$ and $\gamma\in(0.66,0.74]$, then $\xi_T(\alpha)=\{0,1,4\}$, $\xi_I(\beta)=\{0,4\}$ and $\xi_F(\gamma)=\{0,1\}$ are not positive implicative ideals of X respectively.

Since every ideal is positive implicative in a positive implicative BCK-algebra, we have the following theorem.

Theorem 4.9. *Let $(\Omega,\mathcal{A},P)$ be a probability space and let $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi:=(\xi_T,\xi_I,\xi_F)$ on a positive implicative BCK-algebra. If $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is a positive implicative falling neutrosophic ideal of X.*

Corollary 4.10. *Let $(\Omega,\mathcal{A},P)$ be a probability space. For any BCK-algebra X which satisfies one of the following assertions*

$$\begin{aligned}&(\forall x,y\in X)(x*y=(x*y)*y),\\&(\forall x,y\in X)((x*(x*y))*(y*x)=x*(x*(y*(y*x)))),\\&(\forall x,y\in X)(x*y=(x*y)*(x*(x*y))),\\&(\forall x,y\in X)(x*(x*y)=(x*(x*y))*(x*y)),\\&(\forall x,y\in X)((x*(x*y))*(y*x)=(y*(y*x))*(x*y)),\end{aligned}$$

let $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi:=(\xi_T,\xi_I,\xi_F)$ on X. If $\tilde{H}:=(\tilde{H}_T,\tilde{H}_I,\tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is a positive implicative falling neutrosophic ideal of X.

References

[1] H. Bordbar, BCK codes, Advances in Computational Intelligence, Vol. 2, No. 1 (2022), page 1–9.

[2] H. Bordbar, I. Cristea, Divisible hypermodules, Analele Universităţii" Ovidius" Constanţa, Vol. 30, No. 1 (2022), page 57–74.

[3] H. Bordbar, M. Mohseni Takallo, R.A. Borzooei, Y.B. Jun, BMBJ-neutrosophic subalgebra in BCK/BCI-algebras, Neutrosophic Sets and Systems, Vol. 31, No. 1 (2019), page 3.

[4] A. Borumand Saeid and Y.B. Jun, Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points, Ann. Fuzzy Math. Inform. **14** (2017), no. 1, 87–97.

[5] I.R. Goodman, Fuzzy sets as equivalence classes of random sets, in "Recent Developments in Fuzzy Sets and Possibility Theory"(R. Yager, Ed.), Pergamon, New York 1982, pp. 327–343.

[6] K. Iséki, On BCI-algebras, Math. Seminar Notes **8** (1980), 125–130.

[7] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. **23** (1978), 1–26.

[8] Y. Huang, *BCI-algebra,* Science Press, Beijing, 2006.

[9] Y.B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, Ann. Fuzzy Math. Inform. **14** (2017), no. 1, 75–86.

[10] Y.B. Jun and C.H. Park, Falling shadows applied to subalgebras and ideals of BCK/BCI-algebras, Honam Math. J. **34** (2012), no. 2, 135–144.

[11] Y.B. Jun, F. Smarandache and H. Bordbar, Neutrosophic N-structures applied to BCK/BCI-algebras Information 8 (4) (2017), 128.

[12] Y.B. Jun, F. Smarandache and H. Bordbar, Neutrosophic falling shadows applied to subalgebras and ideals in BCK/BCI-algebras, Ann. Fuzzy Math. Inform, 15 (3), 253-263.

[13] Y.B. Jun, F. Smarandache, S.Z. Song and H. Bordbar, Neutrosophic Permeable Values and Energetic Subsets with Applications in BCK/BCI-Algebras, Mathematics 5(6) (2018), 74–90.

[14] Y.B. Jun, F. Smarandache and Mehmat Ali Öztürk, Commutative falling neutrosophic ideals in BCK-algebras, Neutrosophic Sets and Systems, **20** (2018), 44–53.

[15] Y.B. Jun, S.Z.Song, F. Smarandache, H. Bordbar, Neutrosophic Quadruple BCK/BCI-Algebras, Axioms 7 (2) (2018), 41.

[16] J. Meng and Y. B. Jun, *BCK-algebras,* Kyungmoonsa Co. Seoul, Korea 1994.

[17] M. Mohseni Takallo, H. Bordbar, R.A. Borzooei, Y.B. Jun, BMBJ-neutrosophic ideals in BCK/BCI-algebras, Neutrosophic Sets and Systems, 27 (1) (2019), 2.

[18] G. Muhiuddin, H. Bordbar, F. Smarandache and Y.B. Jun, Further results on $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras, Neutrosophic Sets and Systems, **20** (2018), 36–43.

[19] M.A. Öztürk and Y.B. Jun, Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points, J. Inter. Math. Virtual Inst. **8** (2018), 1–17.

[20] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998. http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).

[21] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Reserch Press, Rehoboth, NM, 1999.

[22] F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, Int. J. Pure Appl. Math. **24** (2005), no.3, 287–297.

[23] F. Smarandache, H. Bordbar, R. A. Borzooei, Y. B. Jun, A General Model of Neutrosophic Ideals in BCK/BCI-algebras Based on Neutrosophic Points, Bulletin of the Section of Logic, (2020),

[24] S.K. Tan, P.Z. Wang and E.S. Lee, Fuzzy set operations based on the theory of falling shadows, J. Math. Anal. Appl. **174** (1993), 242–255.

[25] S.K. Tan, P.Z. Wang and X.Z. Zhang, Fuzzy inference relation based on the theory of falling shadows, Fuzzy Sets and Systems **53** (1993), 179–188.

[26] P.Z. Wang and E. Sanchez, Treating a fuzzy subset as a projectable random set, in: M.M. Gupta, E. Sanchez (Eds.), Fuzzy Information and Decision, Pergamon, New York, 1982, pp. 212–219.

[27] P.Z. Wang, Fuzzy Sets and Falling Shadows of Random Sets, Beijing Normal Univ. Press, People's Republic of China, 1985. [In Chinese]

Received: July 17, 2021 Accepted: : Feb 1, 2022.