NSS

Neutrosophic Sets and Systems, Vol. 42, 2021

University of New Mexico

Properties of Productional NeutroOrderedSemigroups

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Abstract. The introducing of NeutroAlgebra by Smarandache opened the door for researchers to define many related new concepts. NeutroOrderedAlgebra was one of these new related definitions. The aim of this paper is to study productional NeutroOrderedSemigroup. In this regard, we firstly present many examples and study subsets of productional NeutroOrderedSemigroups. Then, we find sufficient conditions for the productional NeutroSemigroup to be a NeutroOrderedSemigroup. Finally, we find sufficient conditions for subsets of the productional NeutroOrderedSemigroup to be NeutroOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters.

Keywords: NeutroSemigroup, NeutrosOrderedSemigroup, NeutroOrderedIdeal, NeutroOrderedFilter, Productional NeutroOrderedSemigroup.

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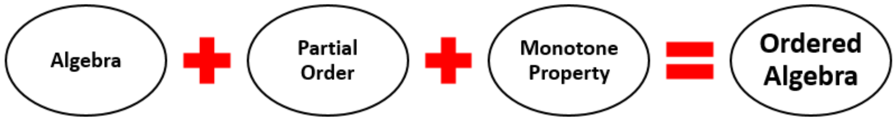
1. **Introduction**

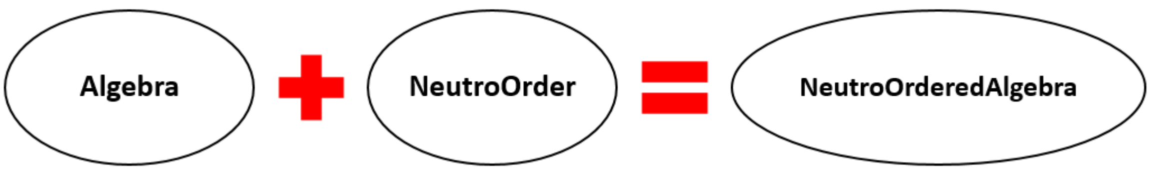
Smarandache [1–3] introduced NeutroAlgebra as a generalization of the known Algebra. It is known that in an Algebra, operations are well defined and axioms are always true whereas for NeutroAlgebra, operations and axioms are partially true, partially indeterminate, and partially false. The latter is considered as an extension of Partial Algebra where operations and axioms are partially true and partially false. Many researchers worked on special types of NeutroAlgebras by applying them to different types of algebraic structures such as semigroups, groups, rings, BE-Algebras, CI-Algebras, BCK-Algebras, etc. For more details about NeutroStructures, the reader may see [4–8]. l order on it that satisfies the monotone property, we get an Ordered Algebra (as illustrated in Figure 1). And starting with a partial order on a

Figure 1. Ordered Algebra



NeutroAlgebra, we get a NeutroStructure. The latter if it satisfies the conditions of **NeutroOrder**, it becomes a NeutroOrderedAlgebra (as illustrated in Figure 2). In [9], the authors

Figure 2. NeutroOrderedAlgebra



defined NeutroOrderedAlgebra and applied it to semigroups by studying NeutroOrderedSemigroups and their subsets such as NeutrosOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters.

Our paper is concerned about Cartesian product of NeutroOrderedSemigroups and the remainder part of it is as follows: In Section 2, we present some definitions and examples related to NeutroOrderedSemigroups. In Section 3, we define productional NeutroOrderedSemigroup and find sufficient conditions for the Cartesian product of NeutroSemigroups and semigroups to be NeutroOrderedSemigroups. Finally in Section 4, we find sufficient conditions for subsets of the productional NeutroOrderedSemigroup to be NeutroOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters.

2. NeutroOrderedSemigroups

In this section, we present some definitions and examples about NeutroOrderedSemigroups, introduced and studied by the authors in [9], that are used throughout the paper.

Definition 2.1. [10] Let $(S,\cdot)$ be a semigroup ("$\cdot$" is an associative and a binary closed operation) and "$\leq$" a partial order on S. Then $(S,\cdot,\leq)$ is an *ordered semigroup* if for every $x \leq y \in S$, $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $z \in S$.

Definition 2.2. [10] Let $(S, \cdot, \leq)$ be an ordered semigroup and $\emptyset \neq M \subseteq S$. Then

(1) M is an *ordered subsemigroup* of S if $(M, \cdot, \leq)$ is an ordered semigroup and $(x] \subseteq M$ for all $x \in M$. i.e., if $y \leq x$ then $y \in M$.
(2) M is an *ordered left ideal* of S if M is an ordered subsemigroup of S and for all $x \in M$, $r \in S$, we have $rx \in M$.
(3) M is an *ordered right ideal* of S if M is an ordered subsemigroup of S and for all $x \in M$, $r \in S$, we have $xr \in M$.
(4) M is an *ordered ideal* of S if M is both: an ordered left ideal of S and an ordered right ideal of S.
(5) M is an *ordered filter* of S if $(M, \cdot)$ is a semigroup and for all $x, y \in S$ with $x \cdot y \in M$, we have $x, y \in M$ and $[y) \subseteq M$ for all $y \in M$. i.e., if $y \in M$ with $y \leq x$ then $x \in M$.

For more details about semigroup theory and ordered algebraic structures, we refer to [10, 11].

Definition 2.3. [2] Let A be any non-empty set and "$\cdot$" be an operation on A. Then "$\cdot$" is called a *NeutroOperation* on A if the following conditions hold.

(1) There exist $x, y \in A$ with $x \cdot y \in A$. (This condition is called degree of truth, "T".)
(2) There exist $x, y \in A$ with $x \cdot y \notin A$. (This condition is called degree of falsity, "F".)
(3) There exist $x, y \in A$ with $x \cdot y$ is indeterminate in A. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical binary closed operation, and from $(0, 0, 1)$ that represents the AntiOperation.

Definition 2.4. [2] Let A be any non-empty set and "$\cdot$" be an operation on A. Then "$\cdot$" is called a *NeutroAssociative* on A if there exist $x, y, z, a, b, c, e, f, g \in A$ satisfying the following conditions.

(1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; (This condition is called degree of truth, "T".)
(2) $a \cdot (b \cdot c) \neq (a \cdot b) \cdot c$; (This condition is called degree of falsity, "F".)
(3) $e \cdot (f \cdot g)$ is indeterminate or $(e \cdot f) \cdot g$ is indeterminate or we can not find if $e \cdot (f \cdot g)$ and $(e \cdot f) \cdot g$ are equal. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical associative axiom, and from $(0, 0, 1)$ that represents the AntiAssociativeAxiom.

Definition 2.5. [2] Let A be any non-empty set and "$\cdot$" be an operation on A. Then $(A, \cdot)$ is called a *NeutroSemigroup* if "$\cdot$" is either a NeutroOperation or NeutroAssociative.

Definition 2.6. [9] Let $(S, \cdot)$ be a NeutroSemigroup and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on S. Then $(S, \cdot, \leq)$ is a *NeutroOrderedSemigroup* if the following conditions hold.

(1) There exist $x \leq y \in S$ with $x \neq y$ such that $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $z \in S$. (This condition is called degree of truth, "T".)

(2) There exist $x \leq y \in S$ and $z \in S$ such that $z \cdot x \nleq z \cdot y$ or $x \cdot z \nleq y \cdot z$. (This condition is called degree of falsity, "F".)

(3) There exist $x \leq y \in S$ and $z \in S$ such that $z \cdot x$ or $z \cdot y$ or $x \cdot z$ or $y \cdot z$ are indeterminate, or the relation between $z \cdot x$ and $z \cdot y$, or the relation between $x \cdot z$ and $y \cdot z$ are indeterminate. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical Ordered Semigroup, and from $(0, 0, 1)$ that represents the AntiOrderedSemigroup.

Definition 2.7. [9] Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup . If "$\leq$" is a total order on A then A is called *NeutroTotalOrderedSemigroup*.

Example 2.8. [9] Let $S_1 = \{s, a, m\}$ and $(S_1, \cdot_1)$ be defined by the following table.

$\cdot_1$	s	a	m
s	s	m	s
a	m	a	m
m	m	m	m

By defining the total order

$$\leq_1 = \{(m, m), (m, s), (m, a), (s, s), (s, a), (a, a)\}$$

on S_1, we get that $(S_1, \cdot_1, \leq_1)$ is a NeutroTotalOrderedSemigroup.

Example 2.9. Let $S_2 = \{0, 1, 2, 3\}$ and $(S_2, \cdot_2')$ be defined by the following table.

$\cdot_2'$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	3	2
3	0	1	3	2

By defining the partial order

$$\leq_2' = \{(0, 0), (0, 1), (0, 2), (1, 1), (2, 2), (3, 3)\}$$

on S_2, we get that $(S_2, \cdot_2', \leq_2')$ is a NeutroOrderedSemigroup.

Example 2.10. [9] Let $S_3 = \{0, 1, 2, 3, 4\}$ and $(S_3, \cdot_3)$ be defined by the following table.

$\cdot_3$	0	1	2	3	4
0	0	0	0	3	0
1	0	1	2	1	1
2	0	4	2	3	3
3	0	4	2	3	3
4	0	0	0	4	0

By defining the partial order

$$\leq_3 = \{(0,0),(0,1),(0,3),(0,4),(1,1),(1,3),(1,4),(2,2),(3,3),(3,4),(4,4)\}$$

on S_3, we get that $(S_3, \cdot_3, \leq_3)$ is a NeutroOrderedSemigroup.

Example 2.11. Let $\mathbb{Z}$ be the set of integers and define "$\star$" on $\mathbb{Z}$ as follows: $x \star y = xy - 2$ for all $x, y \in \mathbb{Z}$. We define the partial order "$\leq_\star$" on $\mathbb{Z}$ as $-2 \leq_\star x$ for all $x \in \mathbb{Z}$ and for $a, b \geq -2$, $a \leq_\star b$ is equivalent to $a \leq b$ and for $a, b < -2$, $a \leq_\star b$ is equivalent to $a \geq b$. In this way, we get $-2 \leq_\star -1 \leq_\star 0 \leq_\star 1 \leq_\star \dots$ and $-2 \leq_\star -3 \leq_\star -4 \leq_\star \dots$. Then $(\mathbb{Z}, \star, \leq_\star)$ is a NeutroOrderedSemigroup.

Definition 2.12. [9] Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup and $\emptyset \neq M \subseteq S$. Then

(1) M is a *NeutroOrderedSubSemigroup* of S if $(M, \cdot, \leq)$ is a NeutroOrderedSemigroup and there exist $x \in M$ with $(x] = \{y \in S : y \leq x\} \subseteq M$.

(2) M is a *NeutroOrderedLeftIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $r \cdot x \in M$ for all $r \in S$.

(3) M is a *NeutroOrderedRightIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $x \cdot r \in M$ for all $r \in S$

(4) M is a *NeutroOrderedIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $r \cdot x \in M$ and $x \cdot r \in M$ for all $r \in S$.

(5) M is a *NeutroOrderedFilter* of S if $(M, \cdot, \leq)$ is a NeutroOrderedSemigroup and there exists $x \in S$ such that for all $y, z \in S$ with $x \cdot y \in M$ and $z \cdot x \in M$, we have $y, z \in M$ and there exists $y \in M$ $[y) = \{x \in S : y \leq x\} \subseteq M$.

Definition 2.13. [9] Let $(A, \star, \leq_A)$ and $(B, \circledast, \leq_B)$ be NeutroOrderedSemigroups and $\phi : A \to B$ be a function. Then

(1) ϕ is called *NeutroOrderedHomomorphism* if $\phi(x \star y) = \phi(x) \circledast \phi(y)$ for some $x, y \in A$ and there exist $a \leq_A b \in A$ with $a \neq b$ such that $\phi(a) \leq_B \phi(b)$.

(2) ϕ is called *NeutroOrderedIsomomorphism* if ϕ is a bijective NeutroOrderedHomomorphism.

(3) ϕ is called *NeutroOrderedStrongHomomorphism* if $\phi(x \star y) = \phi(x) \circledast \phi(y)$ for all $x, y \in A$ and $a \leq_A b \in A$ is equivalent to $\phi(a) \leq_B \phi(b) \in B$.

(4) ϕ is called *NeutroOrderedStrongIsomomorphism* if ϕ is a bijective NeutroOrdered-StrongHomomorphism.

Example 2.14. Let $(S_3, \cdot_3, \leq_3)$ be the NeutroOrderedSemigroup presented in Example 2.10. Then $I = \{0, 1, 2\}$ is both: a NeutroOrderedLefttIdeal and a NeutroOrderedRightIdeal of S_3.

Example 2.15. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NeutroOrderedSemigroup presented in Example 2.11. Then $I = \{-2, -1, 0, 1, -2, -3, -4, \ldots\}$ is a NeutroOrderedIdeal of $\mathbb{Z}$.

Example 2.16. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NeutroOrderedSemigroup presented in Example 2.11. Then $F = \{-2, -1, 0, 1, 2, 3, 4, \ldots\}$ is a NeutroOrderedFilter of $\mathbb{Z}$.

3. Productional NeutroOrderedSemigroups

Let $(A_\alpha, \leq_\alpha)$ be a partial ordered set for all $\alpha \in \Gamma$. We define "$\leq$" on $\prod_{\alpha \in \Gamma} A_\alpha$ as follows: For all $(x_\alpha), (y_\alpha) \in \prod_{\alpha \in \Gamma} A_\alpha$,

$$(x_\alpha) \leq (y_\alpha) \Longleftrightarrow x_\alpha \leq_\alpha y_\alpha \text{ for all } \alpha \in \Gamma.$$

One can easily see that $(\prod_{\alpha \in \Gamma} A_\alpha, \leq)$ is a partial ordered set.

Let A_α be any non-empty set for all $\alpha \in \Gamma$ and "$\cdot_\alpha$" be an operation on A_α. We define "$\cdot$" on $\prod_{\alpha \in \Gamma} A_\alpha$ as follows: For all $(x_\alpha), (y_\alpha) \in \prod_{\alpha \in \Gamma} A_\alpha$, $(x_\alpha) \cdot (y_\alpha) = (x_\alpha \cdot_\alpha y_\alpha)$.

Throughout the paper, we write NOS instead of NeutroOrderedSemigroup.

Theorem 3.1. *Let $(G_1, \leq_1), (G_2, \leq_2)$ be partially ordered sets with operations $\cdot_1, \cdot_2$ respectively. Then $(G_1 \times G_2, \cdot, \leq)$ is an NOS if one of the following statements is true.*

(1) *G_1 and G_2 are NeutroSemigroups with at least one of them is an NOS.*

(2) *One of G_1, G_2 is an NOS and the other is a semigroup.*

Proof. Without loss of generality, let G_1 be an NOS. We prove 1. and 2. is done similarly. We have three cases for "$\cdot_1$" and "$\cdot_2$": Case "$\cdot_1$" is a NeutroOperation, Case "$\cdot_2$" is a NeutroOperation, and Case "$\cdot_1$" and "$\cdot_2$" are NeutroAssociative.

Case "$\cdot_1$" is a NeutroOperation. There exist $x_1, y_1, a_1, b_1 \in G_1$ such that $x_1 \cdot_1 y_1 \in G_1$ and $a_1 \cdot_1 b_1 \notin G_1$ or $x_1 \cdot_1 y_1$ is indeterminate in G_1. Since G_2 is a NeutroSemigroup, it follows that there exist $x_2, y_2 \in G_2 \neq \emptyset$ such that $x_2 \cdot_2 y_2 \in G_2$ or $x_2 \cdot_2 y_2$ is indeterminate in G_2 (If no such elements exist then G_2 will be an AntiSemigroup.). Then $(x_1, x_2) \cdot (y_1, y_2) \in G_1 \times G_2$ and $(a_1, x_2) \cdot (b_1, y_2) \notin G_1 \times G_2$ or $(x_1, x_2) \cdot (y_1, y_2)$ is indeterminate in $G_1 \times G_2$. Thus "$\cdot$" is a NeutroOperation.

Case "$\cdot_2$" is a NeutroOperation. This case can be done in a similar way to Case "$\cdot_1$" is a

NeutroOperation.

Case "$\cdot_1$" and "$\cdot_2$" are NeutroAssociative. There exist $x_1, y_1, z_1, a_1, b_1, c_1 \in G_1$ and $x_2, y_2, z_2, a_2, b_2, c_2 \in G_2$ such that

$$x_1 \cdot_1 (y_1 \cdot_1 z_1) = (x_1 \cdot_1 y_1) \cdot_1 z_1 \ , \ a_1 \cdot_1 (b_1 \cdot_1 c_1) \neq (a_1 \cdot_1 b_1) \cdot_1 c_1,$$

$$x_2 \cdot_2 (y_2 \cdot_2 z_2) = (x_2 \cdot_2 y_2) \cdot_2 z_2 \ , \text{ and } a_2 \cdot_2 (b_2 \cdot_2 c_2) \neq (a_2 \cdot_2 b_2) \cdot_2 c_2.$$

The latter implies that

$$(x_1, x_2) \cdot ((y_1, y_2) \cdot (z_1, z_2)) = ((x_1, x_2) \cdot (y_1, y_2)) \cdot (z_1, z_2)$$

and

$$(a_1, a_2) \cdot ((b_1, b_2) \cdot (c_1, c_2)) = ((a_1, a_2) \cdot (b_1, b_2)) \cdot (c_1, c_2).$$

Thus, "$\cdot$" is NeutroAssociative.

Having "$\leq_1$" a NeutroOrder on G_1 implies that:

(1) There exist $x \leq_1 y \in G_1$ with $x \neq y$ such that $z \cdot_1 x \leq_1 z \cdot_1 y$ and $x \cdot_1 z \leq_1 y \cdot_1 z$ for all $z \in G_1$.

(2) There exist $x \leq_1 y \in G_1$ and $z \in G_1$ such that $z \cdot_1 x \not\leq z \cdot_1 y$ or $x \cdot_1 z \not\leq y \cdot_1 z$.

(3) There exist $x \leq_1 y \in G_1$ and $z \in G_1$ such that $z \cdot_1 x$ or $z \cdot_1 y$ or $x \cdot_1 z$ or $y \cdot_1 z$ are indeterminate, or the relation between $z \cdot_1 x$ and $z \cdot_1 y$, or the relation between $x \cdot_1 z$ and $y \cdot_1 z$ are indeterminate.

Where (T, I, F) is different from $(1, 0, 0)$ and from $(0, 0, 1)$.

Having $b \leq_2 b$ for all $b \in G_2$ implies that:

By (1), we get that there exist $(x, b) \leq (y, b) \in G_1 \times G_2$ with $(x, b) \neq (y, b)$. For all $(z, a) \in G_1 \times G_2$, we have either $a \cdot_2 b \in G_2$ or $a \cdot_2 b \notin G_2$ or $a \cdot_2 b$ is indeterminate in G_2. Similarly for $b \cdot_2 a$. The latter implies that $(z, a) \cdot (x, b) \leq (z, a) \cdot (y, b)$ and $(x, b) \cdot (z, a) \leq (y, b) \cdot (z, a)$ or $(z, a) \cdot (x, b) \leq (z, a) \cdot (y, b)$ is indeterminate in $G_1 \times G_2$ or $(x, b) \cdot (z, a) \leq (y, b) \cdot (z, a)$ is indeterminate in $G_1 \times G_2$.

By (2), we get that there exist $(x, b) \leq (y, b) \in G_1 \times G_2$ and $(z, a) \in G_1 \times G_2$ such that $(z, a) \cdot (x, b) \not\leq (z, a) \cdot (y, b)$ or $(x, b) \cdot (z, a) \not\leq (y, b) \cdot (z, a)$ or $(z, a) \cdot (x, b) \leq (z, a) \cdot (y, b)$ is indeterminate in $G_1 \times G_2$ or $(x, b) \cdot (z, a) \leq (y, b) \cdot (z, a)$ is indeterminate in $G_1 \times G_2$.

By (3), we get that there exist $(x, b) \leq (y, b) \in G_1 \times G_2$ and $(z, a) \in G_1 \times G_2$ such that $(z, a) \cdot (x, b) \leq (z, a) \cdot (y, b)$ is indeterminate in $G_1 \times G_2$ or $(x, b) \cdot (z, a) \leq (y, b) \cdot (z, a)$ is indeterminate in $G_1 \times G_2$ or $(z, a) \cdot (x, b)$ is indeterminate in $G_1 \times G_2$ or $(x, b) \cdot (z, a)$ is indeterminate in $G_1 \times G_2$. Therefore, $(G_1 \times G_2, \cdot, \leq)$ is an NOS. □

Theorem 3.1 implies that $G_1 \times G_2$ is an NOS if either G_1, G_2 are both NOS, G_1 is an NOS and G_2 is a NeutroSemigroup, G_1 is an NOS and G_2 is a semigroup (or odered semigroup),

G_1 is a NeutroSemigroup and G_2 is an NOS, or G_1 is a semigroup (or ordered semigroup) and G_2 is an NOS.

We present a generalization of Theorem 3.1.

Theorem 3.2. *Let $(G_\alpha, \leq_\alpha)$ be a partially ordered set with operation "$\cdot_\alpha$" for all $\alpha \in \Gamma$. Then $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ is an NOS if there exist $\alpha_0 \in \Gamma$ such that $(G_{\alpha_0}, \cdot_{\alpha_0}, \leq_{\alpha_0})$ is an NOS and $(G_\alpha, \cdot_\alpha)$ is a semigroup or NeutroSemigroup for all $\alpha \in \Gamma - \{\alpha_0\}$.*

Notation 1. *Let $(G_\alpha, \leq_\alpha)$ be a partially ordered set with operation "$\cdot_\alpha$" for all $\alpha \in \Gamma$. If $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ is an NOS then we call it the **productional NOS**.*

Proposition 3.3. *Let $(G_1, \cdot_1, \leq_1)$ and $(G_2, \cdot_2, \leq_2)$ be NeutroTotalOrderedSemigroups with $|G_1|, |G_2| \geq 2$. Then $(G_1 \times G_2, \cdot, \leq)$ is not a NeutroTotalOrderedSemigroup.*

Proof. Since $(G_1, \cdot_1, \leq_1)$ and $(G_2, \cdot_2, \leq_2)$ are NeutroTotalOrderedSemigroups with $|G_1| \geq 2$ and $|G_2| \geq 2$, it follows that there exist $a \leq_1 b \in G_1$, $c \leq_2 d \in G_2$ with $a \neq b$ and $c \neq d$. One can easily see that $(a, d) \nleq (b, c) \in G_1 \times G_2$ and $(b, c) \nleq (a, d) \in G_1 \times G_2$. Therefore, $(G_1 \times G_2, \cdot, \leq)$ is not a NeutroTotalOrderedSemigroup. □

Corollary 3.4. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be NeutroTotalOrderedSemigroups for all $\alpha \in \Gamma$ with $|G_{\alpha_0}|, |G_{\alpha_1}| \geq 2$ for $\alpha_0 \neq \alpha_1 \in \Gamma$. Then $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ is not a NeutroTotalOrderedSemigroup.*

Proof. The proof follows from Proposition 3.3. □

Example 3.5. Let $S_1 = \{s, a, m\}$, $(S_1, \cdot_1, \leq_1)$ be the NOS presented in Example 2.8, and "$\leq_1'$" be the trivial order on S_1. Theorem 3.1 asserts that Cartesian product $(S_1 \times S_1, \cdot, \leq)$ resulting from $(S_1, \cdot_1, \leq_1)$ and $(S_1, \cdot_1, \leq_1')$ is an NOS of order 9.

Example 3.6. Let $S_1 = \{s, a, m\}$, $(S_1, \cdot_1, \leq_1)$ be the NOS presented in Example 2.8, and $(\mathbb{R}, \cdot_s, \leq_u)$ be the semigroup of real numbers under standard multiplication and usual order. Theorem 3.1 asserts that Cartesian product $(\mathbb{R} \times S_1, \cdot, \leq)$ is an NOS of infinite order.

Example 3.7. Let $S_1 = \{s, a, m\}$ and $(S_1, \cdot_1, \leq_1)$ be the NOS presented in Example 2.8. Theorem 3.2 asserts that $(S_1 \times S_1 \times S_1, \cdot, \leq)$ is an NOS of order 27. Moreover, by means of Proposition 3.3, $(S_1 \times S_1 \times S_1, \cdot, \leq)$ is not a NeutroTotalOrderedSemigroup.

Example 3.8. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NOS presented in Example 2.11 and $(\mathbb{Z}_n, \odot, \leq_t)$ be the semigroup under standard multiplication of integers modulo n and "$\leq_t$" is defined as follows. For all $\overline{x}, \overline{y} \in \mathbb{Z}_n$ with $0 \leq x, y \leq n-1$,

$$\overline{x} \leq_t \overline{y} \Longleftrightarrow x \leq y \in \mathbb{Z}.$$

Then $(\mathbb{Z}_n \times \mathbb{Z}, \cdot, \leq)$ is an NOS.

Proposition 3.9. *Let $(G_\alpha, \leq_\alpha)$ be a partially ordered set with operation "$\cdot_\alpha$" for all $\alpha \in \Gamma$ and $(G_{\alpha_0}, \cdot_{\alpha_0}, \leq_{\alpha_0})$ be an NOS for some $\alpha_0 \in \Gamma$. Then $\phi : (\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq) \to G_{\alpha_0}$ with $\phi((x_\alpha)) = x_{\alpha_0}$ is a NeutroOrderedHomomorphism.*

Proof. The proof is straightforward. □

Remark 3.10. If $|\Gamma| \geq 2$ and there exist $\alpha \neq \alpha_0 \in \Gamma$ with $|G_\alpha| \geq 2$ then the NeutroOrderedHomomorphism ϕ in Proposition 3.9 is not a NeutroOrderedIsomorphism.

Remark 3.11. If $|\Gamma| \geq 2$ and there exist $\alpha \neq \alpha_0 \in \Gamma$ with $|G_\alpha| \geq 2$ then $G_{\alpha_0} \ncong_s \prod_{\alpha\in\Gamma} G_\alpha$. This is clear as there exist no bijective function from G_{α_0} to $\prod_{\alpha\in\Gamma} G_\alpha$.

Proposition 3.12. *There are infinite non-isomorphic NOS.*

Proof. Let $(G, \cdot_G, \leq_G)$ be an NOS with $|G| \geq 2$, $\Gamma \subseteq \mathbb{R}$, and $|\Gamma| \geq 2$. Theorem 3.2 asserts that $(\prod_{\alpha\in\Gamma} G, \cdot, \leq)$ is an NOS for every $\Gamma \subseteq \mathbb{R}$. For all $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}$ with $|\Gamma_1| \neq |\Gamma_2|$, Remark 3.11 asserts that $\prod_{\alpha\in\Gamma_1} G \ncong_s \prod_{\alpha\in\Gamma_2} G$. Therefore, there are infinite non-isomorphic NOS. □

Example 3.13. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NOS presented in Example 2.11. Then for every $n \in \mathbb{N}$, we have $(\prod_{i=1}^{n} \mathbb{Z}, \cdot, \leq)$ is an NOS. Moreover, we have infinite such non-isomorphic NOS.

Theorem 3.14. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ and $(G'_\alpha, \cdot'_\alpha, \leq'_\alpha)$ be NOS for all $\alpha \in \Gamma$. Then the following statements hold.*

(1) *If there is a NeutroOrderedHomomorphism from G_α to G'_α for all $\alpha \in \Gamma$ then there is a NeutroOrderedHomomorphism from $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ to $(\prod_{\alpha\in\Gamma} G'_\alpha, \cdot', \leq')$.*
(2) *If there is a NeutroOrderedStrongHomomorphism from G_α to G'_α for all $\alpha \in \Gamma$ then there is a NeutroOrderedStrongHomomorphism from $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq)$ to $(\prod_{\alpha\in\Gamma} G'_\alpha, \cdot', \leq')$.*
(3) *If $G_\alpha \cong G'_\alpha$ for all $\alpha \in \Gamma$ then $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq) \cong (\prod_{\alpha\in\Gamma} G'_\alpha, \cdot', \leq')$.*
(4) *If $G_\alpha \cong_s G'_\alpha$ for all $\alpha \in \Gamma$ then $(\prod_{\alpha\in\Gamma} G_\alpha, \cdot, \leq) \cong_s (\prod_{\alpha\in\Gamma} G'_\alpha, \cdot', \leq')$.*

Proof. We prove 1. and the proof of 2., 3., and 4. are done similarly. Let $\phi_\alpha : G_\alpha \to G'_\alpha$ be a NeutroOrderedHomomorphism and define $\phi : \prod_{\alpha\in\Gamma} G_\alpha \to \prod_{\alpha\in\Gamma} G'_\alpha$ as follows: For all $(x_\alpha) \in \prod_{\alpha\in\Gamma} G_\alpha$,

$$\phi((x_\alpha)) = (\phi_\alpha(x_\alpha)).$$

one can easily see that ϕ is a NeutroOrderedHomomorphism. □

4. Subsets of productional NeutroOrderedSemigroups

In this section, we find some sufficient conditions for subsets of the productional NOS to be NeutroOrderedSubSemigroups, NeutroOrderedIdeals, and NeutroOrderedFilters. Moreover, we present some related examples.

Proposition 4.1. *Let $(A_\alpha, \leq_\alpha)$ be a partial ordered set for all $\alpha \in \Gamma$ and $(x_\alpha) \in \prod_{\alpha\in\Gamma} A_\alpha$. Then $((x_\alpha)] = \prod_{\alpha\in\Gamma}(x_\alpha]$.*

Proof. Let $(y_\alpha) \in ((x_\alpha)]$. Then $(y_\alpha) \leq (x_\alpha)$. The latter implies that $y_\alpha \leq_\alpha x_\alpha$ for all $\alpha \in \Gamma$ and hence, $y_\alpha \in (x_\alpha]$ for all $\alpha \in \Gamma$. We get now that $(y_\alpha) \in \prod_{\alpha\in\Gamma}(x_\alpha]$. Thus, $((x_\alpha)] \subseteq \prod_{\alpha\in\Gamma}(x_\alpha]$. Similarly, we can prove that $\prod_{\alpha\in\Gamma}(x_\alpha] \subseteq ((x_\alpha)]$. □

Proposition 4.2. *Let $(A_\alpha, \leq_\alpha)$ be a partial ordered set for all $\alpha \in \Gamma$ and $(x_\alpha) \in \prod_{\alpha\in\Gamma} A_\alpha$. Then $[(x_\alpha)) = \prod_{\alpha\in\Gamma}[x_\alpha)$.*

Proof. The proof is similar to that of Proposition 4.1. □

Theorem 4.3. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If S_α is a NeutroOrderedSubSemigroup of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} S_\alpha$ is a NeutroOrderedSubSemigroup of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. For all $\alpha \in \Gamma$, we have S_α an NOS (as it is NeutroOrderedSubSemigroup of G_α). Theorem 3.2 asserts that $\prod_{\alpha\in\Gamma} S_\alpha$ is an NOS. Since S_α is a NeutroOrderedSubSemigroup of G_α for every $\alpha \in \Gamma$, it follows that for every $\alpha \in \Gamma$ there exist $x_\alpha \in S_\alpha$ with $(x_\alpha] \subseteq S_\alpha$. Using Proposition 4.1, we get that there exist $(x_\alpha) \in \prod_{\alpha\in\Gamma} S_\alpha$ such that $((x_\alpha)] = \prod_{\alpha\in\Gamma}(x_\alpha] \subseteq \prod_{\alpha\in\Gamma} S_\alpha$. Therefore, $\prod_{\alpha\in\Gamma} S_\alpha$ is a NeutroOrderedSubSemigroup of $\prod_{\alpha\in\Gamma} G_\alpha$. □

Corollary 4.4. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that S_{α_0} is a NeutroOrderedSubSemigroup of G_{α_0} then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times S_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedSubSemigroup of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.3 and having G_α a NeutroOrderedSubSemigroup of itself. □

Theorem 4.5. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If I_α is a NeutroOrderedLeftIdeal of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} I_\alpha$ is a NeutroOrderedLeftIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. Having every NeutroOrderedLeftIdeal a NeutroOrderedSubSemigroup and that I_α is a NeutroOrderedLeftIdeal of G_α for all $\alpha \in \Gamma$ implies, by means of Theorem 4.3, that $\prod_{\alpha\in\Gamma} I_\alpha$ is a NeutroOrderedSubSemigroup of $\prod_{\alpha\in\Gamma} G_\alpha$. Since I_α is a NeutroOrderedLeftIdeal of G_α for all $\alpha \in \Gamma$, it follows that for every $\alpha \in \Gamma$ there exist $x_\alpha \in I_\alpha$ such that $r_\alpha \cdot_\alpha x_\alpha \in I_\alpha$ for all $r_\alpha \in G_\alpha$. The latter implies that there exist $(x_\alpha) \in \prod_{\alpha\in\Gamma} I_\alpha$ such that $(r_\alpha)\cdot(x_\alpha) = (r_\alpha \cdot_\alpha x_\alpha) \in \prod_{\alpha\in\Gamma} I_\alpha$ for all $(r_\alpha) \in \prod_{\alpha\in\Gamma} G_\alpha$. Therefore, $\prod_{\alpha\in\Gamma} I_\alpha$ is a NeutroOrderedLeftIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$. □

Corollary 4.6. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that I_{α_0} is a NeutroOrderedLeftIdeal of G_{α_0} and for $\alpha \neq \alpha_0$ there exist $x_\alpha \in G_\alpha$ such that $r_\alpha \cdot_\alpha x_\alpha \in G_\alpha$ for all $r_\alpha \in G_\alpha$ then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times I_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedLeftIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.5 and having G_α a NeutroOrderedLeftIdeal of itself. □

Theorem 4.7. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If I_α is a NeutroOrderedRightIdeal of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} I_\alpha$ is a NeutroOrderedRightIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof is similar to that of Theorem 4.5. □

Corollary 4.8. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that I_{α_0} is a NeutroOrderedRightIdeal of G_{α_0} and for $\alpha \neq \alpha_0$ there exist $x_\alpha \in G_\alpha$ such that $x_\alpha \cdot_\alpha r_\alpha \in G_\alpha$ for all $r_\alpha \in G_\alpha$ then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times I_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedRightIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.7 and having G_α a NeutroOrderedRightIdeal of itself. □

Theorem 4.9. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If I_α is a NeutroOrderedIdeal of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} S_\alpha$ is a NeutroOrderedIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof is similar to that of Theorem 4.5. □

Corollary 4.10. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that I_{α_0} is a NeutroOrderedIdeal of G_{α_0} and for $\alpha \neq \alpha_0$ there exist $x_\alpha \in G_\alpha$ such that $r_\alpha \cdot_\alpha x_\alpha, x_\alpha \cdot_\alpha r_\alpha \in G_\alpha$ for all $r_\alpha \in G_\alpha$ then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times I_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedIdeal of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.9 and having G_α a NeutroOrderedIdeal of itself. □

Example 4.11. Let $(S_3, \cdot_3, \leq_3)$ be the NeutroOrderedSemigroup presented in Example 2.10. Example 2.14 asserts that $I = \{0, 1, 2\}$ is both: a NeutroOrderedLefttIdeal and a NeutroOrderedRightIdeal of S_3. Theorem 4.5 and Theorem 4.7 imply that $I \times I = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$ is both: a NeutroOrderedLefttIdeal and a NeutroOrderedRightIdeal of $S_3 \times S_3$. Moreover, $I \times S_3$ and $S_3 \times I$ are both: NeutroOrderedLefttIdeals and NeutroOrderedRightIdeals of $S_3 \times S_3$.

Example 4.12. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NeutroOrderedSemigroup presented in Example 2.11. Example 2.15 asserts that $I = \{-2, -1, 0, 1, -2, -3, -4, \ldots\}$ is a NeutroOrderedIdeal of $\mathbb{Z}$. Theorem 4.9 asserts that $I \times I \times I$ is NeutroOrderedIdeal of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Theorem 4.13. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If F_α is a NeutroOrderedFilter of G_α for all $\alpha \in \Gamma$ then $\prod_{\alpha\in\Gamma} F_\alpha$ is a NeutroOrderedFilter of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. For all $\alpha \in \Gamma$, we have F_α an NOS (as it is NeutroOrderedFilter of G_α). Theorem 3.2 asserts that $\prod_{\alpha\in\Gamma} S_\alpha$ is an NOS. Having F_α is a NeutroOrderedFilter of G_α for all $\alpha \in \Gamma$ implies that for every $\alpha \in \Gamma$ there exist $x_\alpha \in F_\alpha$ such that for all $y_\alpha, z_\alpha \in F_\alpha$, $x_\alpha \cdot_\alpha y_\alpha \in F_\alpha$ and $z_\alpha \cdot_\alpha x_\alpha \in F_\alpha$ imply that $y_\alpha, z_\alpha \in F_\alpha$. We get now that there exist $(x_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$ such that for all $(y_\alpha), (z_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$, $(x_\alpha)\cdot(y_\alpha) = (x_\alpha \cdot_\alpha y_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$ and $(z_\alpha)\cdot(x_\alpha) = (z_\alpha \cdot_\alpha x_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$ imply that $(y_\alpha), (z_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$. Since F_α is a NeutroOrderedFilter of G_α for every $\alpha \in \Gamma$, it follows that for every $\alpha \in \Gamma$ there exist $x_\alpha \in F_\alpha$ with $[x_\alpha) \subseteq F_\alpha$. Using Proposition 4.2, we get that there exist $(x_\alpha) \in \prod_{\alpha\in\Gamma} F_\alpha$ such that $[(x_\alpha)) = \prod_{\alpha\in\Gamma}[x_\alpha) \subseteq \prod_{\alpha\in\Gamma} F_\alpha$. Therefore, $\prod_{\alpha\in\Gamma} F_\alpha$ is a NeutroOrderedFilter of $\prod_{\alpha\in\Gamma} G_\alpha$. □

Corollary 4.14. *Let $(G_\alpha, \cdot_\alpha, \leq_\alpha)$ be an NOS for all $\alpha \in \Gamma$. If there exists $\alpha_0 \in \Gamma$ such that F_{α_0} is a NeutroOrderedFilter of G_{α_0} then $\prod_{\alpha\in\Gamma,\alpha<\alpha_0} G_\alpha \times F_{\alpha_0} \times \prod_{\alpha\in\Gamma,\alpha>\alpha_0}$ is a NeutroOrderedFilter of $\prod_{\alpha\in\Gamma} G_\alpha$.*

Proof. The proof follows from Theorem 4.13 and having G_α a NeutroOrderedFilter of itself. □

Example 4.15. Let $(\mathbb{Z}, \star, \leq_\star)$ be the NeutroOrderedSemigroup presented in Example 2.11. Example 2.16 asserts that $F = \{-2, -1, 0, 1, 2, 3, 4, \ldots\}$ is a NeutroOrderedFilter of $\mathbb{Z}$. Theorem 4.13 implies that $F \times F \times F \times F$ is a NeutroOrderedFilter of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Moreover, $\mathbb{Z} \times \mathbb{Z} \times F \times \mathbb{Z}$ is a NeutroOrderedFilter of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

5. **Conclusion**

The class of NeutroAlgebras is very large. This paper considered NeutroOrderedSemigroups (introduced by the authors in [9]) as a subclass of NeutroAlgebras. Results related to productional NOS and its subsets were investigated and some examples were elaborated.

For future work, it will be interesting to investigate the following.

(1) Find necessary conditions for the productional NeutroSemigroup to be NeutroOrdered-Semigroup.

(2) Check the possibility of introducing the quotient NeutroOrderedSemigroup and investigate its properties.

(3) Study other types of productional NetroOrderedStructures.

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Received : 08-09-2020 Accepted: 20-04-2021