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Products of Interval Neutrosophic Automata

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Abstract. In this paper, we introduced direct product, restricted direct product of interval neutrosophic automata and prove that direct, restricted direct product of cyclic and retrievable of interval neutrosophic automata are cyclic and retrievable interval neutrosophic automata.

Keywords: Cyclic, Retrievability, Direct product.

1. Introduction

Neutrosophic set is a part of neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set is a powerful general formal framework that has been recently proposed. The theory of neutrosophy and neutrosophic set was introduced by Florentin Smarandache in 1999 [18]. The neutrosophic set is the generalization of classical sets, fuzzy set [22] and so on. The concept of fuzzy set and intuitionstic fuzzy set unsuccessful when the relation is indeterminate. Neutrosophic sets are powerful logics designed to facilitate understanding of indeterminate and inconsistent information. A neutrosophic set consider truth-membership, in-determinacy-membership and falsity-membership which are completely independent. A neutrosophic set N is classified by a Truth membership T_N, Indeterminacy membership I_N, and Falsity membership function F_N, where $T_N, I_N,$ andF_N are real standard and non-standard subsets of $]0^-, 1^+[$.

Wang $etal.$ [20] introduced the notion of interval-valued neutrosophic sets. The interval neutrosophic set are characterized by an interval memebership degree, interval indeterminacy degree, and interval nonmembership degree.

Neutrosophic sets and methods have recently gained popularity in a variety of domains and it has lot of applications. For example, on similarity and entropy in neutrosophic sets were discussed in [16]. Subsequently, on entropy and similarity measure of interval valued neutrosophic sets was discussed in [1]. Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets were discussed in [19]. An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options were discussed in [13].

The concept of single valued and interval valued neutrosophic set applied in automata theory. It was introduced by Tahir Mahmood et. al in [14, 15]. Consequently, J. Kavikumar et.al were introduced neutrosophic general finite automata and composite neutrosophic finite automata [11, 12]. Later, the concept interval valued neutrosophic automata applied in retrievability, subsystem, strong subsystem and characterizations of submachines were discussed in [4–7].

Products is important concept in automata theory since it produce a new automata with the existing automata by taking products. The Cartesian composition of automata was discussed by W. Dorfler in 1977 [3]. Cartesian product of fuzzy automata was discussed by D. S. Malik et.al [17]. Later number of authors have worked in these lines. Generalized products of directable fuzzy automata were discussed in [8]. Generalized products of Δ-synchronized fuzzy automata were discussed in [9]. Cartesian products of interval neutrosophic automata were discussed in [10].

In this paper, we introduce direct and restricted direct prodcut of interval neutrosophic automata and prove that direct and restricted direct product of cyclic, retirevable of interval neutrosophic automata are cyclic, retirevable interval neutrosophic automata.

2. Preliminaries

Definition 2.1. [18] Let U be the universe of discourse. A neutrosophic set (NS) N in U is defined by a truth membership T_N, indeterminacy membership I_N and a falsity membership F_N, where T_N, I_N, and F_N are real standard or non-standard subsets of $]0^-, 1^+[$. That is
$N = \{\langle x, (T_N(x), I_N(x), F_N(x))\rangle, x \in U,\ T_N, I_N, F_N \in\ ]0^-, 1^+[\ \}$ and
$0^- \leq sup\ T_N(x) + sup\ I_N(x) + sup\ F_N(x) \leq 3^+$. We use the interval $[0, 1]$ instead of $]0^-, 1^+[$.

Definition 2.2. [20] Interval neutrosophic set (INS for short) is of the form $N = \{\langle \alpha_N(x), \beta_N(x), \gamma_N(x)\rangle\, | x \in U\}$
$= \{\langle x, [\inf\ \alpha_N(x), \sup\ \alpha_N(x)], [\inf\ \beta_N(x), \sup\ \beta_N(x)], [\inf\ \gamma_N(x), \sup \gamma_N(x)]\rangle\}$,
$x \in U$, $\alpha_N(x),\ \beta_N(x),\ \gamma_N(x) \subseteq [0, 1]$ and
$0 \leq \sup\ \alpha_N(x) + \sup\ \beta_N(x) + \sup\ \gamma_N(x) \leq 3$.

Definition 2.3. [20] An INS N is empty if $\inf\,\alpha_N(x) = \sup\,\alpha_N(x) = 0,\ \inf\,\beta_N(x) = \sup\,\beta_N(x) = 1,\ \inf\,\gamma_N(x) = \sup\,\gamma_N(x) = 1$ forall $x \in U$.

Definition 2.4. [14] Interval neutrosophic automaton $M = (Q,\ \Sigma,\ N)$ $(INAforshort)$, where Q and Σ are non-empty finite sets called the set of states and input symbols respectively, and $N = \{\langle \alpha_N(x), \beta_N(x), \gamma_N(x)\rangle\}$ is an INS in $Q \times \Sigma \times Q$.

The set of all words of finite length of Σ is denoted by Σ^*. The empty word is denoted by ϵ, and the length of each $x \in \Sigma^*$ is denoted by $|x|$.

Definition 2.5. [14] Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton and extended interval neutrosophic set is defined as $N^* = \{\langle \alpha_{N^*}(x), \beta_{N^*}(x), \gamma_{N^*}(x)\rangle\}$ in $Q \times \Sigma^* \times Q$ by

$$\alpha_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [1,1] & \text{if } q_i = q_j \\ [0,0] & \text{if } q_i \neq q_j \end{cases}$$

$$\beta_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [0,0] & \text{if } q_i = q_j \\ [1,1] & \text{if } q_i \neq q_j \end{cases}$$

$$\gamma_{N^*}(q_i,\ \epsilon,\ q_j) = \begin{cases} [0,0] & \text{if } q_i = q_j \\ [1,1] & \text{if } q_i \neq q_j \end{cases}$$

$\alpha_{N^*}(q_i, w, q_j) = \alpha_{N^*}(q_i, xy, q_j) = \vee_{q_r \in Q}[\alpha_{N^*}(q_i, x, q_r) \wedge \alpha_{N^*}(q_r, y, q_j)]$,
$\beta_{N^*}(q_i, w, q_j) = \beta_{N^*}(q_i, xy, q_j) = \wedge_{q_r \in Q}[\beta_{N^*}(q_i, x, q_r) \vee \beta_{N^*}(q_r, y, q_j)]$,
$\gamma_{N^*}(q_i, w, q_j) = \gamma_{N^*}(q_i, xy, q_j) = \wedge_{q_r \in Q}[\gamma_{N^*}(q_i, x, q_r) \vee \gamma_{N^*}(q_r, y, q_j)], \forall q_i, q_j \in Q$,
$w = xy, x \in \Sigma^*$ and $y \in \Sigma$.

3. Products of Interval Neutrosophic Automata

Definition 3.1. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, N_1 \times N_2)$, where
$(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), (a, b), (q_k, q_l)) = \alpha_{N_1}(q_i, a, q_k) \wedge \alpha_{N_2}(q_j, b, q_l)$
$(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), (a, b), (q_k, q_l)) = \beta_{N_1}(q_i, a, q_k) \vee \beta_{N_2}(q_j, b, q_l)$
$(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), (a, b), (q_k, q_l)) = \gamma_{N_1}(q_i, a, q_k) \vee \gamma_{N_2}(q_j, b, q_l)$.
$\forall (q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2, (a, b) \in \Sigma_1 \times \Sigma_2$. Then $M_1 \times M_2$ is called direct product of interval neutrosophic automata.

Definition 3.2. Let $M_i = (Q_i, \Sigma, N_i), i = 1, 2$ be interval neutrosophic automata. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma, N_1 \times N_2)$, where
$(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), a, (q_k, q_l)) = \alpha_{N_1}(q_i, a, q_k) \wedge \alpha_{N_2}(q_j, a, q_l)$
$(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), a, (q_k, q_l)) = \beta_{N_1}(q_i, a, q_k) \vee \beta_{N_2}(q_j, a, q_l)$
$(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), a, (q_k, q_l)) = \gamma_{N_1}(q_i, a, q_k) \vee \gamma_{N_2}(q_j, a, q_l)$.
$\forall (q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2, a, \in \Sigma$. Then $M_1 \times M_2$ is called restricted direct product of interval neutrosophic automata.

Definition 3.3. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton. M is said to be cyclic if $\exists\ q_i \in Q$ such that $Q = S(q_i)$.

Definition 3.4. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton. M is said to be connected if $\forall\ q_j, q_i$ and $\exists\ a \in \Sigma$ such that either
$\alpha_N(q_i, a, q_j) > [0, 0],\ \beta_N(q_i, a, q_j) < [1, 1],\ \gamma_N(q_i, a, q_j) < [1, 1]$ or
$\alpha_N(q_j, a, q_i) > [0, 0],\ \beta_N(q_j, a, q_i) < [1, 1],\ \gamma_N(q_j, a, q_i) < [1, 1]$.

Definition 3.5. Let $M = (Q,\ \Sigma,\ N)$ be an interval neutrosophic automaton. M is said to be strongly connected if for every $q_i,\ q_j \in Q$, there exists $u \in \Sigma^*$ such that $\alpha^*_N(q_i,\ u,\ q_j) > [0, 0]$, $\beta^*_N(q_i,\ u,\ q_j) < [1, 1], \gamma^*_N(q_i,\ u,\ q_j) < [1, 1]$. M is strongly connected if it has no proper subautomaton.

4. Properties of Products of Interval Neutrosophic Automata

Theorem 4.1. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma_1 \times \Sigma_2, N_1 \times N_2)$ be the full direct product of M_1 and M_2. Then $\forall x_1 \in \Sigma_1^*, x_2 \in \Sigma_2^*, x_1, x_2 \neq \epsilon$
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x_1, x_2)(q_k, q_l)) = \alpha_{N_1{}^*}(q_i, x_1, q_k) \wedge \alpha_{N_2{}^*}(q_j, x_2, q_l)$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x_1, x_2)(q_k, q_l)) = \beta_{N_1{}^*}(q_i, x_1, q_k) \vee \beta_{N_2{}^*}(q_j, x_2, q_l)$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x_1, x_2)(q_k, q_l)) = \gamma_{N_1{}^*}(q_i, x_1, q_k) \vee \gamma_{N_2{}^*}(q_j, x_2, q_l)$
$\forall (q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2$.

Proof. Let $x_1 \in \Sigma_1^*,\ x_2 \in \Sigma_2^*, x_1, x_2 \neq \epsilon$. Let $|x_1| = |x_2| = m$. The result is trivial if $m = 1$. Suppose the result is true $\forall u_1 \in \Sigma_1^*,\ u_2 \in \Sigma_2^*, |u_1| = |u_2| = m - 1, m > 1$. Let $x_1 = a_1 u_1,\ x_2 = a_2 u_2$ where $a_1 \in \Sigma_1,\ a_2 \in \Sigma_2$ and $u_1 \in \Sigma_1^*, u_2 \in \Sigma^*$. Now,
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (a_1u_1, a_2u_2)(q_k, q_l)) =$
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), a_1u_1, (q_k, q_l)) \wedge (\alpha_1 \times \alpha_2)^*((q_i, q_j), a_2u_2, (q_k, q_l))$
$= \{\vee_{(q_r,q_s)\in Q_1\times Q_2}(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), a_1, (q_r, q_s)) \wedge (\alpha_{N_1} \times \alpha_{N_2})^*((q_r, q_s), u_1, (q_k, q_l))\} \wedge$
$\{\vee_{(q_u,q_v)\in Q_1\times Q_2}\{(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), a_2, (q_u, q_v)) \wedge (\alpha_{N_1} \times \alpha_{N_2})^*((q_u, q_v), u_2, (q_k, q_l))\}$
$= \{\vee_{q_r\in Q_1}\{\alpha_{N_1}(q_i, a_1, q_r) \wedge \alpha^*_{N_1}(q_r, u_1, q_k)\}\} \wedge \{\vee_{q_u\in Q_2}\{\alpha_{N_2}(q_j, a_2, q_u) \wedge \alpha^*_{N_2}(q_u, u_2, q_l)\}\}$
$= \{\alpha^*_{N_1}(q_i, a_1u_1, q_k \wedge \alpha^*_{N_2}(q_j, a_2u_2, q_l)\}$
$= \{\alpha^*_{N_1}(q_i, x_1, q_k \wedge \alpha^*_{N_2}(q_j, x_2, q_l)\}$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (a_1u_1, a_2u_2)(q_k, q_l)) =$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), a_1u_1, (q_k, q_l)) \vee (\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), a_2u_2, (q_k, q_l))$
$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), a_1, (q_r, q_s)) \vee (\beta_{N_1} \times \beta_{N_2})^*((q_r, q_s), u_1, (q_k, q_l))\} \vee$
$\{\wedge_{(q_u,q_v)\in Q_1\times Q_2}\{(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), a_2, (q_u, q_v)) \vee (\beta_{N_1} \times \beta_{N_2})^*((q_u, q_v), u_2, (q_k, q_l))\}$
$= \{\wedge_{q_r\in Q_1}\{\beta_{N_1}(q_i, a_1, q_r) \vee \beta^*_{N_1}(q_r, u_1, q_k)\}\} \vee \{\wedge_{q_u\in Q_2}\{\beta_{N_2}(q_j, a_2, q_u) \vee \beta^*_{N_2}(q_u, u_2, q_l)\}\}$
$= \{\beta^*_{N_1}(q_i, a_1u_1, q_k \vee \beta^*_{N_2}(q_j, a_2u_2, q_l)\}$

$= \{\beta^*_{N_1}(q_i, x_1, q_k \vee \beta^*_{N_2}(q_j, x_2, q_l)\}$

$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (a_1u_1, a_2u_2)(q_k, q_l)) =$

$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), a_1u_1, (q_k, q_l)) \vee (\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), a_2u_2, (q_k, q_l))$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), a_1, (q_r, q_s)) \vee (\gamma_{N_1} \times \gamma_{N_2})^*((q_r, q_s), u_1, (q_k, q_l))\} \vee$
$\{\wedge_{(q_u,q_v)\in Q_1\times Q_2}\{(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), a_2, (q_u, q_v)) \vee (\gamma_{N_1} \times \gamma_{N_2})^*((q_u, q_v), u_2, (q_k, q_l))\}$

$= \{\wedge_{q_r\in Q_1}\{\gamma_{N_1}(q_i, a_1, q_r) \vee \gamma^*_{N_1}(q_r, u_1, q_k)\}\} \vee \{\wedge_{q_u\in Q_2}\{\gamma_{N_2}(q_j, a_2, q_u) \vee \gamma^*_{N_2}(q_u, u_2, q_l)\}\}$

$= \{\gamma^*_{N_1}(q_i, a_1u_1, q_k \vee \gamma^*_{N_2}(q_j, a_2u_2, q_l)\}$

$= \{\gamma^*_{N_1}(q_i, x_1, q_k \vee \gamma^*_{N_2}(q_j, x_2, q_l)\}$

Theorem 4.2. Let $M_i = (Q_i, \Sigma, N_i), i = 1, 2$ be interval neutrosophic automata. Let $M_1 \times M_2 = (Q_1 \times Q_2, \Sigma, N_1 \times N_2)$ be the restricted direct product of M_1 and M_2. Then $\forall x \in \Sigma^*$

$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), x(q_k, q_l)) = \alpha_{N_1{}^*}(q_i, x, q_k) \wedge \alpha_{N_2{}^*}(q_j, x, q_l)$

$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), x(q_k, q_l)) = \beta_{N_1{}^*}(q_i, x, q_k) \vee \beta_{N_2{}^*}(q_j, x, q_l)$

$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), x(q_k, q_l)) = \gamma_{N_1{}^*}(q_i, x, q_k) \vee \gamma_{N_2{}^*}(q_j, x, q_l)$

$\forall(q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2$.

Proof. We prove the result by induction on $|x| = n$. If $n = 1$ then the result is obivious. Suppose the result is true for all $x \in \Sigma^*$. Let $x = au$, where $a \in \Sigma, u \in \Sigma^*, |u| = m - 1, m > 1$. Then

$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = (\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), au, (q_k, q_l))$

$= \{\vee_{(q_r,q_s)\in Q_1\times Q_2}\{(\alpha_{N_1} \times \alpha_{N_2})((q_i, q_j), a, (q_k, q_l))\} \wedge \{(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), u, (q_k, q_l))\}\}$

$= \{\vee_{(q_r,q_s)\in Q_1\times Q_2}\{\alpha_{N_1}(q_i, a, q_r) \wedge \alpha_{N_2}(q_j, a, q_s) \wedge \alpha^*_{N_1}(q_r, u, q_k) \wedge \alpha^*_{N_2}(q_s, u, q_l)\}\}$

$= \alpha^*_{N_1}(q_i, au, q_k) \wedge \alpha^*_{N_2}(q_j, au, q_l)$

$= \alpha^*_{N_1}(q_i, x, q_k) \wedge \alpha^*_{N_2}(q_j, x, q_l)$

$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = (\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), au, (q_k, q_l))$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}\{(\beta_{N_1} \times \beta_{N_2})((q_i, q_j), a, (q_k, q_l))\} \vee \{(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), u, (q_k, q_l))\}\}$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}\{\beta_{N_1}(q_i, a, q_r) \vee \beta_{N_2}(q_j, a, q_s) \vee \beta^*_{N_1}(q_r, u, q_k) \vee \beta^*_{N_2}(q_s, u, q_l)\}\}$

$= \beta^*_{N_1}(q_i, au, q_k) \vee \beta^*_{N_2}(q_j, au, q_l)$

$= \beta^*_{N_1}(q_i, x, q_k) \vee \beta^*_{N_2}(q_j, x, q_l)$

$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = (\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), au, (q_k, q_l))$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}\{(\gamma_{N_1} \times \gamma_{N_2})((q_i, q_j), a, (q_k, q_l))\} \vee \{(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), u, (q_k, q_l))\}\}$

$= \{\wedge_{(q_r,q_s)\in Q_1\times Q_2}\{\gamma_{N_1}(q_i, a, q_r) \vee \gamma_{N_2}(q_j, a, q_s) \vee \gamma^*_{N_1}(q_r, u, q_k) \vee \gamma^*_{N_2}(q_s, u, q_l)\}\}$

$= \gamma^*_{N_1}(q_i, au, q_k) \vee \gamma^*_{N_2}(q_j, au, q_l)$

$= \gamma^*_{N_1}(q_i, x, q_k) \vee \gamma^*_{N_2}(q_j, x, q_l)$

Theorem 4.3. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata. Then full direct product of $M_1 \times M_2$ is cyclic if and only if M_1 and M_2 are cyclic.

Proof. Let $\times$ be full direct product.Suppose M_1 and M_2 are cyclic, say $Q_1 = S(q_i)$ and $Q_2 = S(p_j)$ for some $q_i \in Q_1$, $p_j \in Q_2$. Let $(q_k, p_l) \in Q_1 \times Q_2$. Then $\exists x \in \Sigma_1^*$ and such

that $\alpha^*_{N_1}(q_i, x, q_k) > [0,0], \beta^*_{N_1}(q_i, x, q_k) < [1,1], \gamma^*_{N_1}(q_i, x, q_k) < [1,1]$ and
$\alpha^*_{N_2}(q_j, y, q_l) > [0,0], \beta^*_{N_2}(q_j, y, q_l) < [1,1], \gamma^*_{N_2}(q_j, y, q_l) < [1,1]$, Thus
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \alpha^*_{N_1}(q_i, x, q_k) \wedge \alpha^*_{N_2}(q_j, y, q_l) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \beta^*_{N_1}(q_i, x, q_k) \vee \beta^*_{N_2}(q_j, y, q_l) < [1,1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \gamma^*_{N_1}(q_i, x, q_k) \vee \gamma^*_{N_2}(q_j, y, q_l) < [1,1]$.
Hence $(q_k, p_l) \in S((q_i, p_j))$. Thus $Q_1 \times Q_2 = S((q_i, p_j))$. Hence $M_1 \times M_2$ is cyclic. Conversely, suppose $M_1 \times M_2$ is cyclic. Let $Q_1 \times Q_2 = S((q_i, p_j))$ for some $(q_i, p_j) \in Q_1 \times Q_2$. Let $q_k \in Q_1$ and $q_l \in Q_2$.
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \alpha^*_{N_1}(q_i, x, q_k) \wedge \alpha^*_{N_2}(q_j, y, q_l) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \beta^*_{N_1}(q_i, x, q_k) \vee \beta^*_{N_2}(q_j, y, q_l) < [1,1]$ and
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) = \beta^*_{N_1}(q_i, x, q_k) \vee \beta^*_{N_2}(q_j, y, q_l) < [1,1]$.

Theorem 4.4. Let $M_i = (Q_i, \Sigma, N_i), i = 1,2$ be interval neutrosophic automata. If restricted direct product of interval neutrosophic automata $M_1 \times M_2$ is cyclic, then M_1 and M_2 are cyclic.

Proof. Let $\times$ be restricted direct product. Suppose $M_1 \times M_2$ are cyclic. $Q_1 \times Q_2 = S((q_i, q_j))$ for some $q_i, q_j \in Q_1 \times Q_2$. Let $q_k \in Q_1, q_l \in Q_2$. Then $(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = \alpha^*_{N_1}(q_i, x, q_k) \wedge \alpha_{N_2})^*(q_j, x, q_l) > [0,0]$ $(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = \beta^*_{N_1}(q_i, x, q_k) \vee \beta_{N_2})^*(q_j, x, q_l) < [1,1]$ $(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), x, (q_k, q_l)) = \gamma^*_{N_1}(q_i, x, q_k) \vee \gamma_{N_2})^*(q_j, x, q_l) < [1,1]$. Thus $q_k \in S(q_i)$ and $q_l \in S(q_j)$. Therefore $Q_1 = S(q_i)$ for some $q_i \in Q_1$ and $Q_2 = S(q_j)$. Hence M_1 and M_2 are cyclic.

Theorem 4.5. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1,2$ be interval neutrosophic automata. Then the full direct product of interval neutrosophic automata $M_1 \times M_2$ is retrievable if and only if M_1 and M_2 are interval neutrosophic retrievable automata.

Proof. Let $\times$ be full direct product. Suppose that M_1 and M_2 are interval neutrosophic retrievable.
Let $(q_i, q_j), (t_k, s_l) \in Q_1 \times Q_2$ and $(x, y) \in (\Sigma_1 \times \Sigma_2)^*$ be such that
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x, y), (t_k, s_l)) = \alpha^*_{N_1}(q_i, x, t_k) \wedge \alpha^*_{N_2}(q_j, y, s_l) > [0,0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x, y), (t_k, s_l)) = \beta^*_{N_1}(q_i, x, t_k) \vee \beta^*_{N_2}(q_j, y, s_l) < [1,1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x, y), (t_k, s_l)) = \gamma^*_{N_1}(q_i, x, t_k) \vee \gamma^*_{N_2}(q_j, y, s_l) < [1,1]$
Since M_1 and M_2 are interval neutrosophic retrievable $\exists u_1 \in \Sigma_1^*, u_2 \in \Sigma_2^*$ such that
$\alpha^*_{N_1}(q_k, u_1, q_i) > [0,0], \beta^*_{N_1}(q_k, u_1, q_i) < [1,1], \gamma^*_{N_1}(q_k, u_1, q_i) < [1,1]$
$\alpha^*_{N_2}(q_l, u_2, q_j) > [0,0], \beta^*_{N_2}(q_l, u_2, q_l) < [1,1], \gamma^*_{N_2}(q_l, u_2, q_j) < [1,1]$.
$\alpha^*_{N_1}(q_k, u_1, q_i) \wedge \alpha^*_{N_2}(q_l, u_2, q_j)$ $(\alpha_{N_1} \times \alpha_{N_2})^*((q_k, q_l), (u_1, u_2), (q_i, q_j)) > [0,0]$ $\beta^*_{N_1}(q_k, u_1, q_i) \vee \beta^*_{N_2}(q_l, u_2, q_j)$ $(\beta_{N_1} \times \beta_{N_2})^*((q_k, q_l), (u_1, u_2), (q_i, q_j)) < [1,1]$ $\gamma^*_{N_1}(q_k, u_1, q_i) \vee \gamma^*_{N_2}(q_l, u_2, q_j)$ $(\gamma_{N_1} \times \gamma_{N_2})^*((q_k, q_l), (u_1, u_2), (q_i, q_j)) < [1,1]$. Thus, $M_1 \times M_2$ are interval neutrosophic retrievable.
Conversely, suppose $M_1 \times M_2$ are interval neutrosophic retrievable. Let $(q_i, q_j) \in Q_1 \times Q_2$ and

$(x, y) \in (\Sigma_1 \times \Sigma_2)^*, \exists(q_k, q_l) \in Q_1 \times Q_2$ such that
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) > [0, 0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) < [1, 1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), (x, y), (q_k, q_l)) < [1, 1]$.
Then $\exists(u_1, u_2) \in (\Sigma_1 \times \Sigma_2)^*$ such that $(\alpha_{N_1} \times \alpha_{N_2})^*((q_k, q_l), (u_1, u_2), (q_k, q_l)) > [0, 0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_k, q_l), (u_1, u_2), (q_k, q_l)) < [1, 1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_k, q_l), (u_1, u_2), (q_k, q_l)) < [1, 1]$
$(\alpha^*_{N_1}(q_k, u_1, q_i) \wedge (\alpha^*_{N_2}(q_l, u_2, q_j) > [0, 0]$
$(\beta^*_{N_1}(q_k, u_1, q_i) \vee (\beta^*_{N_2}(q_l, u_2, q_j) < [1, 1]$
$(\gamma^*_{N_1}(q_k, u_1, q_i) \vee (\gamma^*_{N_2}(q_l, u_2, q_j) < [1, 1]$.
Hence, M_1 and M_2 are interval neutrosophic retrievable.

Theorem 4.6. Let $M_i = (Q_i, \Sigma_i, N_i), i = 1, 2$ be interval neutrosophic automata. Then restricted direct product of interval neutrosophic automata $M_1 \times M_2$ is interval neutrosophic retrievable then M_1 and M_2 are interval neutrosophic retrievable.

Proof. Let $\times$ be interval neutrosophic resticted direct product. suppose $M_1 \times M_2$ is interval neutrosophic retrievable. Let $(q_i, q_j), (q_k, q_l) \in Q_1 \times Q_2, x \in \Sigma$ such that
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_i, q_j), x, (q_k, q_l)) > [0, 0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_i, q_j), x, (q_k, q_l)) < [1, 1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_i, q_j), x, (q_k, q_l)) < [1, 1]$ Then $\exists u \in \Sigma^*$ such that
$(\alpha_{N_1} \times \alpha_{N_2})^*((q_k, q_l), u, (q_i, q_j)) = \alpha^*_{N_1}(q_k, u, q_i) \wedge \alpha^*_{N_2}(q_l, u, q_j) > [0, 0]$
$(\beta_{N_1} \times \beta_{N_2})^*((q_k, q_l), u, (q_i, q_j)) = \beta^*_{N_1}(q_k, u, q_i) \vee \beta^*_{N_2}(q_l, u, q_j) < [1, 1]$
$(\gamma_{N_1} \times \gamma_{N_2})^*((q_k, q_l), u, (q_i, q_j)) = \gamma^*_{N_1}(q_k, u, q_i) \vee \gamma^*_{N_2}(q_l, u, q_j) < [1, 1]$.
Hence M_1 and M_2 are interval neutrosophic retrievable.

5. **Conclusions**

In this paper, we introduced direct product, restricted direct product of interval neutrosophic automata and prove that direct, restricted direct product of cyclic and retrievable of interval neutrosophic automata are cyclic and retrievable interval neutrosophic automata.

Conflicts of Interest: "The authors declare no conflict of interest."

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