NSS

Neutrosophic Sets and Systems, Vol. 48, 2022

University of New Mexico

Pythagorean m-polar Fuzzy Neutrosophic Topology with Applications

Atiqa Siraj[1], Tehreem Fatima[2], Deeba Afzal[3], Khalid Naeem[4,*] and Faruk Karaaslan[5]

[1]Department of Mathematics and Statistics, The University of Lahore, Pakistan; atiqasiraj60@gmail.com

[2]Department of Mathematics and Statistics, The University of Lahore, Pakistan; tehreem161@gmail.com

[3]Department of Mathematics and Statistics, The University of Lahore, Pakistan; deebafzal@gmail.com

[4,*]Department of Mathematics, FG Degree College, Lahore Cantt., Pakistan; khalidnaeem333@gmail.com

[5]Department of Mathematics, Çankırı Karatekin University, Çankırı, Turkey; fkaraaslan@karatekin.edu.tr

*Correspondence: khalidnaeem333@gmail.com

Abstract. The overarching structures like intuitionistic fuzzy sets, Pythagorean fuzzy sets, m-polar fuzzy sets, and neutrosophic fuzzy sets etc. have their own inadequacies and impediments. These models are unable to do work because of their impediments in many real life situations. To overcome these deficiencies, in this paper, we introduce a set entitled Pythagorean m-polar fuzzy neutrosophic set (PmFNS), as a hybrid model of Pythagorean fuzzy set, m-polar fuzzy set and single-valued neutrosophic set. We define some notions related to PmFNS with the help of illustrations. We also present some concept of Pythagorean m-polar fuzzy neutrosophic topology alongside its leading characteristics. We render two applications of PmFNS of scarcity of water and uplifting economy ruined due to COVID-19 using TOPSIS.

Keywords: Pythagorean m-polar fuzzy neutrosophic set; Pythagorean m-polar fuzzy neutrosophic topology; TOPSIS; COVID-19

---

1. **Introduction**

The methods usually working in classical mathematics are not generally advantageous for the reason that uncertainties and unclearness being there, to tackle real world difficulties. There are numerous methods to handle such circumstances. Unfortunately, all these models have their own restrictions and drawbacks. In 1965, the thought of fuzzy sets as an augmentation of the conventional crisp set was inaugurated by Zadeh [18], to overcome these deficiencies, by associating the membership function $\mu_A : X \to [0, 1]$. Hence, in this new outline, we face the problems relating to topology, the study on them form the subjects of fuzzy topology. In 1968, Chang explained fuzzy topology, as a branch merging ordered structure with topological structure, on fuzzy set. Pao-Ming and Ying-Ming [10] defined the formation of neighborhood of fuzzy-point. In 1983, Atanassov [2, 3] provided the idea of intuitionistic fuzzy sets (IFSs). Later, intuitionistic fuzzy topological spaces via intuitionistic fuzzy sets were obtained by Çoker *et al.* [5]. Lee and Lee [7] gave the outlook of intuitionistic fuzzy points accompanied by the notion of intuitionistic fuzzy neighborhoods. They discovered the characteristics of continuous, open and closed maps in the intuitionistic fuzzy topological spaces. In 2013, Yager [15]- [17] presented Pythagorean fuzzy sets as an expansion of intuitionistic fuzzy sets with a wider scope of applications and presented Pythagorean membership grades with their practical implementations to the multi-criteria decision making (MCDM). Olgun *et al.* [9] introduced the idea of Pythagorean fuzzy topological space.

In 2005, the model of neutrosophic sets, which is the broad view of intuitionistic fuzzy sets, for handling with difficulties involving exaggeration, indeterminacy and irregularity was explored by Smarandache [13]. The notion of fuzzy neutrosophic sets was presented by Arockiarani *et al.* [1]. Recently, Jansi and Mohana [6] coined the notion of pairwise Pythagorean neutrosophic bitopological spaces treating truth and falsity membership functions as dependent neutrosophic components. Neutrosophic set was protracted to Plithogenic set [14] by Smarandache, which is a collection whose each element is regarded as by many attribute values and every attribute value has either a fuzzy, intuitionistic fuzzy or neutrosophic degree of appurtenance to the set. Chen *et al.* [4] expanded the view of bipolar fuzzy sets to m-polar fuzzy sets and provided some of its practical implementations in day-to-day situations. In 2019, Naeem *et al.* [8] explored the notions of Pythagorean m-polar fuzzy sets (PmFSs) along with some of their foremost features. They also gave an application of PmFSs in decision making difficulty of selection of most suitable manner of the advertisement using the conventional tool TOPSIS (Technique based on Order Preference by Similarity to Ideal Solution). Later, Riaz *et al.* [11] extended the notion to corresponding soft sets.

The main aspiration behind this article is to study some features of Pythagorean m-polar fuzzy neutrosophic sets and construct topology on it. There appear several circumstances where data contains multi-polar facts and figures. Pythagorean m-polar fuzzy neutrosophic sets (PmFNSs) is one of the utmost suitable tools for managing such conditions. It can be used to illustrate the ambiguous facts further satisfactorily and exactly. It has been used in many areas for example in aggregation operators, information measures, and decision making. Because of such an evolution, we present an outline on Pythagorean m-polar fuzzy neutrosophic sets with goal of offering a clear outlook on the different tools, concepts and trends related to their extensions. The rest of the paper is systemized as: Elementary notions are dealt with in Section 2. Section 3 presents some notions of Pythagorean m-polar fuzzy neutrosophic sets. The topological structure on our proposed model along with its prime attributes is presented in Section 4. Two applications of decision making are rendered in Section 5.

2. **Preliminaries**

Definition 2.1. [18] A collection of orderly pairs $(\hbar, T_{\mathcal{F}}(\hbar))$, $\hbar$ being an element of the underlying universe X and $T_{\mathcal{F}}$ (the affiliation, association or membership function) is a well-defined map, that drives members of X to $[0,1]$, is entitled as a *fuzzy set* (FS) $\mathcal{F}$ over X. In other words

$$T(\hbar) = \begin{cases} 1, & \text{if } \hbar \in \mathcal{F} \\ 0, & \text{if } \hbar \notin \mathcal{F} \\ ]0,1[, & \text{if } \hbar \text{ is partially in } \mathcal{F} \end{cases}$$

Definition 2.2. [2,3] An *intuitionistic fuzzy set* (IFS) G in X is an object having the form

$$G = \{\langle \hbar, T(\hbar), F(\hbar)\rangle : \hbar \in X\}$$

where the membership function $T(\hbar) : X \to [0,1]$ and the non-membership function $F(\hbar) : X \to [0,1]$ for every $x \in X$ obey the constraint

$$0 \leq T(\hbar) + F(\hbar) \leq 1.$$

Definition 2.3. [15,16] A *Pythagorean fuzzy set*, shortened as PFS, is a collection defined by

$$P = \Big\{ < \hbar, T_P(\hbar), F_P(\hbar) >: \hbar \in X \Big\}$$

where T_P and F_P are mappings from a set X to [0, 1] obeying the restriction $0 \leq T_P^2(\hbar) + F_P^2(\hbar) \leq 1$, representing correspondingly the affiliation and dissociation grades of $\hbar \in X$ to P. The ordered pair $p = (T_p, F_p)$ is accredited as Pythagorean fuzzy number (PFN). The quantity $\exists(\hbar) = \sqrt{1 - \{T^2(\hbar) + F^2(\hbar)\}}$ is famous as the hesitation margin.

Definition 2.4. [12,13] A *neutrosophic set* $\mathbb{N}$ on the underlying set X is defined as

$$\mathbb{N} = \{< \hbar, T_{\mathbb{N}}(\hbar), I_{\mathbb{N}}(\hbar), F_{\mathbb{N}}(\hbar) >: \hbar \in X\}$$

where $T, I, F : X \mapsto]^{-}0, 1^{+}[$ accompanied by the constraint $^{-}0 \leq T_{\mathbb{N}}(\hbar) + I_{\mathbb{N}}(\hbar) + F_{\mathbb{N}}(\hbar) \leq 3^{+}$. Here $T_{\mathbb{N}}(\hbar)$, $I_{\mathbb{N}}(\hbar)$ and $F_{\mathbb{N}}(\hbar)$ are the degrees of membership, indeterminacy and falsity (non-membership) of members of the given set, respectively. T, I and F are acknowledged as the neutrosophic components.

Definition 2.5. [1] A *fuzzy neutrosophic set* ($\mathfrak{fn}$-set) over X is delineated as

$$A = \{< \hbar, T_A(\hbar), I_A(\hbar), F_A(\hbar) >: \hbar \in X\}$$

where $T, I, F : X \mapsto [0,1]$ in such a way that $0 \le T_A(\hbar) + I_A(\hbar) + F_A(\hbar) \le 3$.

Definition 2.6. [8] Suppose that $m \in \mathbb{N}$. A *Pythagorean m-polar fuzzy set* (PmFS) $\mathcal{P}$ over X is regarded as by the mappings $T_{\mathcal{P}}^{(i)} : X \mapsto [0,1]$ (the membership functions) and $F_{\mathcal{P}}^{(i)} : X \mapsto [0,1]$ (the non-membership functions) with the limitation that

$$0 \le \left(T_{\mathcal{P}}^{(i)}(\hbar)\right)^2 + \left(F_{\mathcal{P}}^{(i)}(\hbar)\right)^2 \le 1$$

for integral values of i ranging from 1 to m.
A PmFS may be articulated as

$$\mathcal{P} = \left\{\left\langle \hbar, \left(\left(T_{\mathcal{P}}^{(1)}(\hbar), F_{\mathcal{P}}^{(1)}(\hbar)\right), \cdots, \left(T_{\mathcal{P}}^{(m)}(\hbar), F_{\mathcal{P}}^{(m)}(\hbar)\right)\right)\right\rangle : \hbar \in X\right\}$$

or more conveniently as

$$\begin{aligned}\mathcal{P} &= \left\{\frac{\hbar}{\left(\left(T_{\mathcal{P}}^{(1)}(\hbar), F_{\mathcal{P}}^{(1)}(\hbar)\right), \cdots, \left(T_{\mathcal{P}}^{(m)}(\hbar), F_{\mathcal{P}}^{(m)}(\hbar)\right)\right)} : \hbar \in X\right\} \\ &= \left\{\frac{\hbar}{\left(\left(T_{\mathcal{P}}^{(i)}(\hbar), F_{\mathcal{P}}^{(i)}(\hbar)\right)\right)} : \hbar \in X; i = 1, 2, \cdots, m\right\}\end{aligned}$$

The tabular materialization of $\mathcal{P}$ is

$\mathcal{P}$			
$\hbar_1$	$(T_{\mathcal{P}}^{(1)}(\hbar_1), F_{\mathcal{P}}^{(1)}(\hbar_1))$	$\cdots$	$(T_{\mathcal{P}}^{(m)}(\hbar_1), F_{\mathcal{P}}^{(m)}(\hbar_1))$
$\hbar_2$	$(T_{\mathcal{P}}^{(1)}(\hbar_2), F_{\mathcal{P}}^{(1)}(\hbar_2))$	$\cdots$	$(T_{\mathcal{P}}^{(m)}(\hbar_2), F_{\mathcal{P}}^{(m)}(\hbar_2))$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\hbar_k$	$(T_{\mathcal{P}}^{(1)}(\hbar_k), F_{\mathcal{P}}^{(1)}(\hbar_k))$	$\cdots$	$(T_{\mathcal{P}}^{(m)}(\hbar_k), F_{\mathcal{P}}^{(m)}(\hbar_k))$

and in matrix format as

$$\mathcal{P} = \begin{bmatrix} (T_{\mathcal{P}}^{(1)}(\hbar_1), F_{\mathcal{P}}^{(1)}(\hbar_1)) & \cdots & (T_{\mathcal{P}}^{(m)}(\hbar_1), F_{\mathcal{P}}^{(m)}(\hbar_1)) \\ (T_{\mathcal{P}}^{(1)}(\hbar_2), F_{\mathcal{P}}^{(1)}(\hbar_2)) & \cdots & (T_{\mathcal{P}}^{(m)}(\hbar_2), F_{\mathcal{P}}^{(m)}(\hbar_2)) \\ \vdots & \ddots & \vdots \\ (T_{\mathcal{P}}^{(1)}(\hbar_k), F_{\mathcal{P}}^{(1)}(\hbar_k)) & \cdots & (T_{\mathcal{P}}^{(m)}(\hbar_k), F_{\mathcal{P}}^{(m)}(\hbar_k)) \end{bmatrix}$$

This matrix of order $k \times m$ is reckoned as *PmF-matrix*.

Definition 2.7. Let $X \neq \phi$ be a crisp set. A family τ of subsets of X is called a *topology* on X if

(i) ϕ and X itself belong to τ.
(ii) The union of any number of members of τ is again in τ.
(iii) The intersection of any finite number of members of τ belong to τ.

If τ is a topology on X, then (X, τ) is known as a *topological space*.

Example 2.8. Let $X = \{s, f\}$, then $\tau_1 = \{\phi, X\}$, $\tau_2 = \{\phi, \{s\}, X\}$, $\tau_3 = \{\phi, \{f\}, X\}$ and $\tau_4 = \{\phi, \{s\}, \{f\}, X\}$ are topologies on X.
Likewise, if τ is the union of all open intervals in the set $\mathbb{R}$ of reals, then τ is a topology (called *real topology*) on $\mathbb{R}$. $\mathbb{R}$ with this topology is called the *real line*.

3. Pythagorean m-Polar Fuzzy Neutrosophic Sets

In this section, we introduce the notion of Pythagorean m-polar fuzzy neutrosophic set along with its prime characteristics and illustrations.

Definition 3.1. A *Pythagorean m-polar fuzzy neutrosophic set* (PmFNS) $\Im$ over a basic set X is marked by three mappings $T_{\Im}^{(i)} : X \to [0,1]^m$, $I_{\Im}^{(i)} : X \to [0,1]^m$ and $F_{\Im}^{(i)} : X \to [0,1]^m$, where m is a natural number, $\forall i = 1, 2, \cdots, m$, with the limitation that

$$0 \le \left(T_{\Im}^{(i)}(\hbar)\right)^2 + \left(I_{\Im}^{(i)}(\hbar)\right)^2 + \left(F_{\Im}^{(i)}(\hbar)\right)^2 \le 2$$

for all $\hbar \in X$.
A PmFNS may be expressed as

$$\begin{aligned} \Im &= \left\{(\hbar, \left((T_{\Im}^{(1)}(\hbar), I_{\Im}^{(1)}(\hbar), F_{\Im}^{(1)}(\hbar)), \cdots, (T_{\Im}^{(m)}(\hbar), I_{\Im}^{(m)}(\hbar), F_{\Im}^{(m)}(\hbar))\right) : \hbar \in X\right\} \\ &= \left\{\frac{\hbar}{(T_{\Im}^{(1)}(\hbar), I_{\Im}^{(1)}(\hbar), F_{\Im}^{(1)}(\hbar)), \cdots, (T_{\Im}^{(m)}(\hbar), I_{\Im}^{(m)}(\hbar), F_{\Im}^{(m)}(\hbar))} : \hbar \in X\right\} \\ &= \left\{\frac{\hbar}{(T_{\Im}^{(i)}(\hbar), I_{\Im}^{(i)}(\hbar), F_{\Im}^{(i)}(\hbar))} : \hbar \in X, i = 1, 2, \cdots, m\right\} \end{aligned}$$

If cardinality of X is l, then tabular structure of $\Im$ is as in Table 1:

TABLE 1. Tabular representation of PmFNS $\Im$

$\Im$				
$\hbar_1$	$(T_{\Im}^{(1)}(\hbar_1), I_{\Im}^{(1)}(\hbar_1), F_{\Im}^{(1)}(\hbar_1))$	$(T_{\Im}^{(2)}(\hbar_1), I_{\Im}^{(2)}(\hbar_1), F_{\Im}^{(2)}(\hbar_1))$	$\cdots$	$(T_{\Im}^{(m)}(\hbar_1), I_{\Im}^{(m)}(\hbar_1), F_{\Im}^{(m)}(\hbar_1))$
$\hbar_2$	$(T_{\Im}^{(1)}(\hbar_2), I_{\Im}^{(1)}(\hbar_2), F_{\Im}^{(1)}(\hbar_2))$	$(T_{\Im}^{(2)}(\hbar_2), I_{\Im}^{(2)}(\hbar_2), F_{\Im}^{(2)}(\hbar_2))$	$\cdots$	$(T_{\Im}^{(m)}(\hbar_2), I_{\Im}^{(m)}(\hbar_2), F_{\Im}^{(m)}(\hbar_2))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\hbar_l$	$(T_{\Im}^{(1)}(\hbar_l), I_{\Im}^{(1)}(\hbar_l), F_{\Im}^{(1)}(\hbar_l))$	$(T_{\Im}^{(2)}(\hbar_l), I_{\Im}^{(2)}(\hbar_l), F_{\Im}^{(2)}(\hbar_l))$	$\cdots$	$(T_{\Im}^{(m)}(\hbar_l), I_{\Im}^{(m)}(\hbar_l), F_{\Im}^{(m)}(\hbar_l))$

The corresponding matrix format is

$$\Im = \begin{pmatrix} (T_{\Im}^{(1)}(\hbar_1), I_{\Im}^{(1)}(\hbar_1), F_{\Im}^{(1)}(\hbar_1)) & (T_{\Im}^{(2)}(\hbar_1), I_{\Im}^{(2)}(\hbar_1), F_{\Im}^{(2)}(\hbar_1)) & \cdots & (T_{\Im}^{(m)}(\hbar_1), I_{\Im}^{(m)}(\hbar_1), F_{\Im}^{(m)}(\hbar_1)) \\ (T_{\Im}^{(1)}(\hbar_2), I_{\Im}^{(1)}(\hbar_2), F_{\Im}^{(1)}(\hbar_2)) & (T_{\Im}^{(2)}(\hbar_2), I_{\Im}^{(2)}(\hbar_2), F_{\Im}^{(2)}(\hbar_2)) & \cdots & (T_{\Im}^{(m)}(\hbar_2), I_{\Im}^{(m)}(\hbar_2), F_{\Im}^{(m)}(\hbar_2)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{\Im}^{(1)}(\hbar_l), I_{\Im}^{(1)}(\hbar_l), F_{\Im}^{(1)}(\hbar_l)) & (T_{\Im}^{(2)}(\hbar_l), I_{\Im}^{(2)}(\hbar_l), F_{\Im}^{(2)}(\hbar_l)) & \cdots & (T_{\Im}^{(m)}(\hbar_l), I_{\Im}^{(m)}(\hbar_l), F_{\Im}^{(m)}(\hbar_l)) \end{pmatrix}$$

This $l \times m$ matrix is known as *PmFN matrix*. The assortment of each PmFNS characterized over universe would be designated by PmFNS(X).

Example 3.2. If X=$\{e, f\}$ be a crisp set, then

$$\Im = \left\{ \frac{e}{(0.57, 0.52, 0.91), (0.09, 0.37, 0.47), (0.00, 0.49, 0.81)}, \frac{f}{(0.79, 0.33, 0.67), (1.00, 0.00, 0.07), (0.77, 0.99, 1.00)} \right\}$$

is a P3FNS defined over X. The tabular form of this set is as in Table 2:

Table 2. Tabular representation of P3FNS $\Im$

$\Im$			
e	(0.57, 0.52, 0.91)	(0.09, 0.37, 0.47)	(0.00, 0.49, 0.81)
f	(0.79, 0.33, 0.67)	(1.00, 0.00, 0.07)	(0.77, 0.39, 1.00)

The matrix form of this set is

$$\Im = \begin{pmatrix} (0.57, 0.52, 0.91) & (0.09, 0.37, 0.47) & (0.00, 0.49, 0.81) \\ (0.79, 0.33, 0.67) & (1.00, 0.00, 0.07) & (0.77, 0.39, 1.00) \end{pmatrix}$$

Definition 3.3. Let $\Im_1$ and $\Im_2$ be PmFNSs over X. $\Im_1$ is acknowledged as a *subset* of $\Im_2$, written as $\Im_1 \subseteq \Im_2$, $\forall \Im \in X$ and each values of i ranging from 1 to m, if

1) $T^{(i)}_{\Im_1}(\hbar) \leq T^{(i)}_{\Im_2}(\hbar)$,
2) $I^{(i)}_{\Im_1}(\hbar) \geq I^{(i)}_{\Im_2}(\hbar)$,
3) $F^{(i)}_{\Im_1}(\hbar) \geq F^{(i)}_{\Im_2}(\hbar)$.

$\Im_1$ and $\Im_2$ are said to be *equal* if $\Im_1 \subseteq \Im_2 \subseteq \Im_1$ and is written as $\Im_1 = \Im_2$.

Example 3.4. Let

$$\Im_1 = \begin{pmatrix} (0.41, 0.29, 1.00) & (0.71, 0.09, 0.88) & (0.49, 0.23, 0.00) \\ (0.39, 0.76, 0.97) & (0.00, 1.00, 0.66) & (0.01, 0.59, 0.77) \\ (0.5, 0.02, 0.03) & (0.04, 0.43, 0.61) & (0.82, 0.03, 0.2) \end{pmatrix}$$

and

$$\Im_2 = \begin{pmatrix} (0.58, 0.06, 0.00) & (0.89, 0.04, 0.19) & (1.00, 0.21, 0.00) \\ (0.92, 0.04, 0.11) & (0.17, 0.00, 0.29) & (1.00, 0.33, 0.23) \\ (0.73, 0.02, 0.01) & (0.64, 0.22, 0.03) & (0.91, 0.01, 0.06) \end{pmatrix}$$

be PmFNSs over some set X, then $\Im_1 \subseteq \Im_2$.

Definition 3.5. A PmFNS $\Im$ over X is known as *null PmFNS* if $T^{(i)}_{\Im}(\hbar) = 0$, $I^{(i)}_{\Im}(\hbar) = 1$ and $F^{(i)}_{\Im}(\hbar) = 1$, $\forall \hbar \in X$ and all acceptable values of i. It is designated by Φ.

Thus,

$$\Phi = \begin{pmatrix} (0,1,1) & (0,1,1) & \cdots & (0,1,1) \\ (0,1,1) & (0,1,1) & \cdots & (0,1,1) \\ \vdots & \vdots & \ddots & \vdots \\ (0,1,1) & (0,1,1) & \cdots & (0,1,1) \end{pmatrix}.$$

Definition 3.6. A PmFNS $\Im$ over X is called an *absolute PmFNS* if $T^{(i)}_{\Im}(\hbar) = 1$, $I^{(i)}_{\Im}(\hbar) = 0$, and $F^{(i)}_{\Im}(\hbar) = 0$, $\forall \hbar \in X$. It is denoted by $\breve{\chi}$.

Thus,

$$\breve{\chi} = \begin{pmatrix} (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \end{pmatrix}.$$

Definition 3.7. The *complement* of a PmFNS

$$\Im = \left\{ \frac{\hbar}{(T^{(i)}_{\Im}(\hbar), I^{(i)}_{\Im}(\hbar), F^{(i)}_{\Im}(\hbar))} : \hbar \in X, i = 1, \cdots, m \right\}$$

over X is defined as

$$\Im^c = \left\{ \frac{\hbar}{(F^{(i)}_{\Im}(\hbar), 1 - I^{(i)}_{\Im}(\hbar), T^{(i)}_{\Im}(\hbar))} : \hbar \in X, i = 1, \cdots, m \right\}.$$

Example 3.8. The complement of the PmFNS $\Im$ given in example 3.2 is

$$\Im^c = \begin{pmatrix} (0.91, 0.48, 0.57) & (0.47, 0.63, 0.09) & (0.81, 0.51, 0.00) \\ (0.67, 0.67, 0.79) & (0.07, 1.00, 1.00) & (1.00, 0.01, 0.77) \end{pmatrix}.$$

Remark 3.9. It may be observed from the entry at $(2, 2)$ position of the matrix given in Example 3.8 that $0.07^2 + 1.00^2 + 1.00^2 \nleq 2$. Thus, we may infer that the complement of a PmFNS is not always a PmFNS. Further, the complement of a PmFNS will be a PmFNS iff the sum of squares of the three neutrosophic components does not exceed $2I^{(i)} + 1$ i.e. $(T^{(i)})^2 + (I^{(i)})^2 + (F^{(i)})^2 \leq 2I^{(i)} + 1$.

Definition 3.10. The *union* of any PmFNSs $\Im_1$ and $\Im_2$ expressed over the same universe X is represented as

$$\Im_1 \cup_{\mathfrak{M}} \Im_2 = \left\{ \frac{\hbar}{(\max(T^{(i)}_{\Im 1}(\hbar), T^{(i)}_{\Im 2}(\hbar)), \min(I^{(i)}_{\Im 1}(\hbar), I^{(i)}_{\Im 2}(\hbar)), \min(F^{(i)}_{\Im 1}(\hbar), F^{(i)}_{\Im 2}(\hbar))} : \hbar \in X, i = 1, \cdots, m \right\}$$

Definition 3.11. The *intersection* of any PmFNSs $\Im_1$ and $\Im_2$ expressed over the same universe X is represented as

$$\Im_1 \cap_{\mathfrak{M}} \Im_2 = \left\{ \frac{\hbar}{(\min(T^{(i)}_{\Im 1}(\hbar), T^{(i)}_{\Im 2}(\hbar)), \max(I^{(i)}_{\Im 1}(\hbar), I^{(i)}_{\Im 2}(\hbar), \max(F^{(i)}_{\Im 1}(\hbar), F^{(i)}_{\Im 2}(\hbar))} : \hbar \in X, i = 1, \cdots, m \right\}$$

Example 3.12. If

$$\Im_1 = \begin{pmatrix} (0.57, 0.61, 0.19) & (0.74, 0.61, 0.00) & (0.00, 0.55, 0.22) \\ (0.11, 0.88, 1.00) & (0.49, 0.99, 0.10) & (0.92, 0.67, 0.80) \\ (0.00, 0.36, 0.29) & (0.70, 0.20, 1.00) & (1.00, 0.00, 0.46) \end{pmatrix}$$

and

$$\Im_2 = \begin{pmatrix} (1.00, 0.59, 0.32) & (0.50, 0.72, 1.00) & (0.33, 1.00, 0.70) \\ (0.78, 0.09, 0.50) & (0.00, 0.66, 0.11) & (0.54, 0.61, 0.00) \\ (0.60, 0.00, 0.85) & (0.28, 0.43, 0.90) & (0.83, 0.40, 0.14) \end{pmatrix}$$

are two PmFNSs defined over the same universe of discourse X, then

$$\Im_1 \cup_{\mathfrak{M}} \Im_2 = \begin{pmatrix} (1.00, 0.59, 0.19) & (0.74, 0.61, 0.00) & (0.33, 0.55, 0.22) \\ (0.78, 0.09, 0.50) & (0.49, 0.66, 0.10) & (0.92, 0.61, 0.00) \\ (0.60, 0.00, 0.29) & (0.70, 0.20, 0.90) & (1.00, 0.00, 0.14) \end{pmatrix}$$

and

$$\Im_1 \cap_{\mathfrak{M}} \Im_2 = \begin{pmatrix} (0.57, 0.61, 0.32) & (0.50, 0.72, 1.00) & (0.00, 1.00, 0.70) \\ (0.11, 0.88, 1.00) & (0.00, 0.99, 0.11) & (0.54, 0.67, 0.80) \\ (0.00, 0.36, 0.85) & (0.28, 0.43, 1.00) & (0.83, 0.40, 0.46) \end{pmatrix}$$

Proposition 3.13. *If $\Im, \Im_1, \Im_2, \Im_3$ are PmFNSs over X, then*

(1) $\Phi \cup_{\mathfrak{M}} \Im = \Im$
(2) $\Phi \cap_{\mathfrak{M}} \Im = \Phi$
(3) $\breve{\chi} \cup_{\mathfrak{M}} \Im = \breve{\chi}$
(4) $\breve{\chi} \cap_{\mathfrak{M}} \Im = \Im$
(5) $\Im \cup_{\mathfrak{M}} \Im = \Im$
(6) $\Im \cap_{\mathfrak{M}} \Im = \Im$
(7) $\Im_1 \cup_{\mathfrak{M}} \Im_2 = \Im_2 \cup_{\mathfrak{M}} \Im_1$
(8) $\Im_1 \cap_{\mathfrak{M}} \Im_2 = \Im_2 \cap_{\mathfrak{M}} \Im_1$
(9) $\Im_1 \cup_{\mathfrak{M}} (\Im_2 \cup_{\mathfrak{M}} \Im_3) = (\Im_1 \cup_{\mathfrak{M}} \Im_2) \cup_{\mathfrak{M}} \Im_3$
(10) $\Im_1 \cap_{\mathfrak{M}} (\Im_2 \cap_{\mathfrak{M}} \Im_3) = (\Im_1 \cap_{\mathfrak{M}} \Im_2) \cap_{\mathfrak{M}} \Im_3$
(11) $\Im_1 \cup_{\mathfrak{M}} (\Im_2 \cap_{\mathfrak{M}} \Im_3) = (\Im_1 \cup_{\mathfrak{M}} \Im_2) \cap_{\mathfrak{M}} (\Im_1 \cup_{\mathfrak{M}} \Im_3)$
(12) $\Im_1 \cap_{\mathfrak{M}} (\Im_2 \cup_{\mathfrak{M}} \Im_3) = (\Im_1 \cap_{\mathfrak{M}} \Im_2) \cup_{\mathfrak{M}} (\Im_1 \cap_{\mathfrak{M}} \Im_3)$

Proof. Here, we prove only (11). We may assume, without losing the generality, that $\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)) = T^{(i)}_{\Im_1}(\hbar)$, $\max(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)) = I^{(i)}_{\Im_1}(\hbar)$ and $\max(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)) =$

$F^{(i)}_{\Im_1}(\hbar)$. Then, $\forall \hbar \in X$ and $i = 1, 2, \cdots, m$.

$$\begin{aligned}
\Im_2 \cap_{\mathfrak{M}} \Im_3 &= \left\{ \frac{\hbar}{\left(\min(T^{(i)}_{\Im_2}(\hbar), T^{(i)}_{\Im_3}(\hbar)), \max(I^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_3}(\hbar)), \max(F^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_3}(\hbar)) \right)} \right\} \\
&= \left\{ \frac{\hbar}{\left(T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_2}(\hbar)\right)} \right\} \\
\therefore \Im_1 \cup_{\mathfrak{M}} (\Im_2 \cap_{\mathfrak{M}} \Im_3) &= \left\{ \frac{\hbar}{\left(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar)\right)} \right\} \cup_{\mathfrak{M}} \left\{ \frac{\hbar}{\left(T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_2}(\hbar)\right)} \right\} \\
&= \left\{ \frac{\hbar}{\left(\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)), \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)), \min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)) \right)} \right\} \\
&= \left\{ \frac{\hbar}{\left(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar)\right)} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\Im_1 \cup_{\mathfrak{M}} \Im_2 &= \left\{ \frac{\hbar}{\left(\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)), \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)), \min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)) \right)} \right\} \\
&= \left\{ \frac{\hbar}{\left(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar)\right)} \right\} \\
\Im_1 \cup_{\mathfrak{M}} \Im_3 &= \left\{ \frac{\hbar}{\left(\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_3}(\hbar)), \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_3}(\hbar)), \min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_3}(\hbar)) \right)} \right\} \\
&= \left\{ \frac{\hbar}{\left(T^{(i)}_{\Im_3}(\hbar), I^{(i)}_{\Im_3}(\hbar), F^{(i)}_{\Im_3}(\hbar)\right)} \right\} \\
\therefore (\Im_1 \cup_{\mathfrak{M}} \Im_2) \cap_{\mathfrak{M}} (\Im_1 \cup_{\mathfrak{M}} \Im_3) &= \left\{ \frac{\hbar}{\left(\min(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_3}(\hbar)), \max(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_3}(\hbar)), \max(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_3}(\hbar)) \right)} \right\} \\
&= \left\{ \frac{\hbar}{\left(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar)\right)} \right\}
\end{aligned}$$

0.1cm□

Corollary 3.14. (1) $\Phi \cup_{\mathfrak{M}} \breve{\chi} = \breve{\chi}$

(2) $\Phi \cap_{\mathfrak{M}} \breve{\chi} = \Phi$

Proposition 3.15. *If $\Im_1$ and $\Im_2$ are PmFNSs over X, then*

(1) $\Im_1 \cap_{\mathfrak{M}} \Im_2 \subseteq \Im_1 \subseteq \Im_1 \cup_{\mathfrak{M}} \Im_2$

(2) $\Im_1 \cap_{\mathfrak{M}} \Im_2 \subseteq \Im_2 \subseteq \Im_1 \cup_{\mathfrak{M}} \Im_2$

Proof. The results are easy consequences of properties of max and min. 0.1cm□

Proposition 3.16. *Let $\Im_1$, $\Im_2$ be PmFNSs over universe set X, then De Morgan laws hold i.e.*

(1) $(\Im_1 \cup_{\mathfrak{M}} \Im_2)^c = \Im_1^c \cap_{\mathfrak{M}} \Im_2^c$.

(2) $(\Im_1 \cap_{\mathfrak{M}} \Im_2)^c = \Im_1^c \cup_{\mathfrak{M}} \Im_2^c$.

Proof. : Here, we demonstrate only (1). The verification of (2) perhaps provided in the same way. We may assume, without losing the generality, that $\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)) = T^{(i)}_{\Im_1}(\hbar)$, $\max(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)) = I^{(i)}_{\Im_1}(\hbar)$ and $\max(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)) = F^{(i)}_{\Im_1}(\hbar)$. Then, $\forall\ \hbar \in X$ and $i = 1, 2, \cdots, m$.

$$\begin{aligned}
(\Im_1 \cup_{\mathfrak{M}} \Im_2)^c &= \left\{\frac{\hbar}{(\max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar)), \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)), \min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)))}\right\}^c \\
&= \left\{\frac{\hbar}{(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar))}\right\}^c \\
&= \left\{\frac{\hbar}{(F^{(i)}_{\Im_1}(\hbar), 1 - I^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_1}(\hbar))}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\Im_1^c \cap_{\mathfrak{M}} \Im_2^c &= \left\{\frac{\hbar}{(T^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_1}(\hbar))}\right\}^c \cap_{\mathfrak{M}} \left\{\frac{\hbar}{(T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_2}(\hbar))}\right\}^c \\
&= \left\{\frac{\hbar}{(F^{(i)}_{\Im_1}(\hbar), 1 - I^{(i)}_{\Im_2}(\hbar), T^{(i)}_{\Im_1}(\hbar))}\right\} \cap_{\mathfrak{M}} \left\{\frac{\hbar}{(F^{(i)}_{\Im_2}(\hbar), 1 - I^{(i)}_{\Im_2}(\hbar), T^{(i)}_{\Im_2}(\hbar))}\right\} \\
&= \left\{\frac{\hbar}{(\min(F^{(i)}_{\Im_1}(\hbar), F^{(i)}_{\Im_2}(\hbar)), 1 - \min(I^{(i)}_{\Im_1}(\hbar), I^{(i)}_{\Im_2}(\hbar)), \max(T^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_2}(\hbar))}\right\} \\
&= \left\{\frac{\hbar}{(F^{(i)}_{\Im_1}(\hbar), 1 - I^{(i)}_{\Im_1}(\hbar), T^{(i)}_{\Im_1}(\hbar))}\right\}
\end{aligned}$$

0.1cm□

Remark 3.17. Let $\Im$ is a PmFNS over universe set X. Then

(1) $\Im \cup_{\mathfrak{M}} \Im^c \neq \breve{\chi}$

(2) $\Im \cap_{\mathfrak{M}} \Im^c \neq \Phi$

Proposition 3.18. (1) $\Phi^c = \breve{\chi}$

(2) $\breve{\chi}^c = \Phi$

(3) $(\Im^c)^c = \Im$

Proof. Straight forward. 0.1cm□

Definition 3.19. The *difference* of two PmFNS $\Im_1$ and $\Im_2$ expressed over the same universe X is represented as

$$\Im_1 \setminus \Im_2 = \left\{\frac{\hbar}{(\min(T^{(i)}_{\Im 1}(\hbar), F^{(i)}_{\Im 2}(\hbar)), \min(I^{(i)}_{\Im 1}(\hbar), I^{(i)}_{\Im 2}(\hbar)), \max(F^{(i)}_{\Im 1}(\hbar), T^{(i)}_{\Im 2}(\hbar))} : \hbar \in X, i = 1, 2, \cdots, m\right\}$$

Example 3.20. For $\Im_1$ and $\Im_2$ given in Example 3.12, we have

$$\Im_1 \setminus \Im_2 = \begin{pmatrix} (0.32, 0.59, 1.00) & (0.74, 0.61, 0.50) & (0.00, 0.55, 0.33) \\ (0.11, 0.09, 1.00) & (0.11, 0.66, 0.10) & (0.00, 0.61, 0.80) \\ (0.00, 0.00, 0.60) & (0.70, 0.20, 1.00) & (0.14, 0.00, 0.83) \end{pmatrix}$$

Definition 3.21. The *symmetric difference* of two PmFNSs $\Im_1$ and $\Im_2$ is set of elements which are either in $\Im_1$ or in $\Im_2$ but not in both i.e.

$$\Im_1 \triangle \Im_2 = (\Im_1 \setminus \Im_2) \cup_{\mathfrak{M}} (\Im_2 \setminus \Im_1)$$

Example 3.22. Let

$$\Im_1 = \begin{pmatrix} (0.57, 0.61, 0.19) & (0.74, 0.61, 0.00) & (0.00, 0.55, 0.22) \\ (0.11, 0.88, 1.00) & (0.49, 0.99, 0.10) & (0.92, 0.67, 0.80) \\ (0.00, 0.36, 0.29) & (0.70, 0.20, 1.00) & (1.00, 0.00, 0.46) \end{pmatrix}$$

and

$$\Im_2 = \begin{pmatrix} (1.00, 0.59, 0.32) & (0.50, 0.72, 1.00) & (0.33, 1.00, 0.70) \\ (0.78, 0.09, 0.50) & (0.00, 0.66, 0.11) & (0.54, 0.61, 0.00) \\ (0.60, 0.00, 0.85) & (0.28, 0.43, 0.90) & (0.83, 0.40, 0.14) \end{pmatrix}$$

so that

$$\Im_1 \setminus \Im_2 = \begin{pmatrix} (0.32, 0.59, 1.00) & (0.74, 0.61, 0.50) & (0.00, 0.55, 0.33) \\ (0.11, 0.09, 1.00) & (0.11, 0.66, 0.10) & (0.00, 0.61, 0.80) \\ (0.00, 0.00, 0.60) & (0.70, 0.20, 1.00) & (0.14, 0.00, 0.83) \end{pmatrix}$$

and

$$\Im_2 \setminus \Im_1 = \begin{pmatrix} (0.19, 0.59, 0.57) & (0.00, 0.61, 1.00) & (0.22, 0.55, 0.70) \\ (0.78, 0.09, 0.50) & (0.00, 0.66, 0.49) & (0.54, 0.61, 0.92) \\ (0.29, 0.00, 0.85) & (0.28, 0.20, 0.90) & (0.46, 0.00, 1.00) \end{pmatrix}$$

$$\begin{aligned} \therefore (\Im_1 \setminus \Im_2) \cup_{\mathfrak{M}} (\Im_2 \setminus \Im_1) &= \begin{pmatrix} (0.32, 0.59, 0.57) & (0.74, 0.61, 0.50) & (0.22, 0.55, 0.33) \\ (0.78, 0.09, 0.91) & (0.11, 0.66, 0.10) & (0.54, 0.61, 0.80) \\ (0.29, 0.00, 0.60) & (0.70, 0.20, 0.90) & (0.46, 0.00, 0.83) \end{pmatrix} \\ &= \Im_1 \triangle \Im_2 \end{aligned}$$

Definition 3.23. The *sum* of two PmFNSs $\Im_1$ and $\Im_2$ chosen from same universe X is represented as

$$\Im_1 \oplus \Im_2 = \Big\{ \frac{\hbar}{\big(\sqrt{(T^{(i)}_{\Im_1}(\hbar))^2 + (T^{(i)}_{\Im_2}(\hbar))^2 - (T^{(i)}_{\Im_1}(\hbar) T^{(i)}_{\Im_2}(\hbar))^2}, I^{(i)}_{\Im_1}(\hbar) I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_1}(\hbar) F^{(i)}_{\Im_2}(\hbar)\big)} \Big\}$$

where $\hbar \in X$ and i runs from 1 to m.

Example 3.24. For $\Im_1$ and $\Im_2$ given in Example 3.12, we have

$$\Im_1 \oplus \Im_2 = \begin{pmatrix} (1.00, 0.36, 0.06) & (0.81, 0.44, 0.00) & (0.33, 0.55, 0.15) \\ (0.78, 0.08, 0.50) & (0.49, 0.65, 0.01) & (0.94, 0.41, 0.00) \\ (0.60, 0.00, 0.25) & (0.73, 0.09, 0.90) & (1.00, 0.00, 0.06) \end{pmatrix}$$

Definition 3.25. The *product* of two PmFNSs $\Im_1$ & $\Im_2$ take off the same universe X is explained as

$$\Im_1 \otimes \Im_2 = \left\{ \frac{\hbar}{(T^{(i)}_{\Im_1}(\hbar)T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_1}(\hbar)I^{(i)}_{\Im_2}(\hbar), \sqrt{(F^{(i)}_{\Im_1}(\hbar))^2 + (F^{(i)}_{\Im_2}(\hbar))^2 - (F^{(i)}_{\Im_1}(\hbar)F_{\Im_2}(\hbar))^2})} \right\}$$

for $\hbar \in X$ and i runs from 1 to m.

Example 3.26. For $\Im_1$ and $\Im_2$ given in Example 3.12, we have

$$\Im_1 \otimes \Im_2 = \begin{pmatrix} (0.57, 0.36, 0.37) & (0.37, 0.44, 1.00) & (0.00, 0.55, 0.72) \\ (0.09, 0.08, 1.00) & (0.00, 0.65, 0.15) & (0.49, 0.41, 0.8) \\ (0.00, 0.00, 0.86) & (0.19, 0.09, 1.00) & (0.83, 0.00, 0.48) \end{pmatrix}$$

Definition 3.27. If $\Im_1 = \Im_2$ in Definition 3.25, then we express $\Im_1 \otimes \Im_1$ by $\Im_1^2$. Thus,

$$\begin{aligned} \Im^2 &= \left\{ \frac{\hbar}{((T^{(i)}_{\Im}(\hbar))^2, (I^{(i)}_{\Im}(\hbar))^2, \sqrt{2(F^{(i)}_{\Im}(\hbar))^2 - (F^{(i)}_{\Im}(\hbar))^4})} : \hbar \in X; i = 1, 2, \cdots, m \right\} \\ &= \left\{ \frac{\hbar}{((T^{(i)}_{\Im}(\hbar))^2, (I^{(i)}_{\Im}(\hbar))^2, \sqrt{1 - (1 - ((F^{(i)}_{\Im}(\hbar))^2)^2)})} : \hbar \in X; i = 1, 2, \cdots, m \right\} \end{aligned}$$

The set $\Im^2$ is called as *concentration* of $\Im$, written as $con(\Im)$. If $k \in [0, \infty)$, in general, then

$$\Im^k = \left\{ \frac{\hbar}{((T^{(i)}_{\Im}(\hbar))^k, (I^{(i)}_{\Im}(\hbar))^k, \sqrt{1 - (1 - ((F^{(i)}_{\Im}(\hbar))^2)^k)})} : \hbar \in X; i = 1, 2, \cdots, m \right\}$$

The set

$$\Im^{1/2} = \left\{ \frac{\hbar}{(\sqrt{T^{(i)}_{\Im}(\hbar)}, \sqrt{I^{(i)}_{\Im}(\hbar)}, \sqrt{1 - \sqrt{(1 - (F^{(i)}_{\Im}(\hbar))^2)}})} : \hbar \in X; i = 1, 2, \cdots, m \right\}$$

is called as *dilation* of $\Im$, denoted as $dil(\Im)$.

Example 3.28. For PmFNS $\Im_1$ given in Example 3.12, we have

$$con(\Im) = \begin{pmatrix} (0.32, 0.37, 0.27) & (0.55, 0.37, 0.00) & (0.00, 0.30, 0.31) \\ (0.01, 0.77, 1.00) & (0.24, 0.98, 0.14) & (0.85, 0.45, 0.93) \\ (0.00, 0.13, 0.40) & (0.49, 0.04, 1.00) & (1.00, 0.00, 0.62) \end{pmatrix}$$

and

$$dil(\Im) = \begin{pmatrix} (0.75, 0.78, 0.13) & (0.86, 0.78, 0.00) & (0.00, 0.74, 0.16) \\ (0.33, 0.94, 1.00) & (0.70, 0.99, 0.07) & (0.96, 0.67, 0.63) \\ (0.00, 0.60, 0.21) & (0.84, 0.45, 1.00) & (1.00, 0.00, 0.33) \end{pmatrix}$$

Definition 3.29. The *Cartesian product* of two PmFNSs $\Im_1$ and $\Im_2$ over X is characterized as

$$\Im_1 \times \Im_2 = \left\{ \frac{(\hbar_1, \hbar_2)}{\left(T^{(i)}_{\Im_1}(\hbar)T^{(i)}_{\Im_2}(\hbar), I^{(i)}_{\Im_1}(\hbar)I^{(i)}_{\Im_2}(\hbar), F^{(i)}_{\Im_1}(\hbar)F^{(i)}_{\Im_2}(\hbar)\right)} : \hbar_1, \hbar_2 \in X; i = 1, 2, \cdots, m \right\}$$

Example 3.30. For $\Im_1$ and $\Im_2$ given in Example 3.12, we have

$$\Im_1 \times \Im_2 = \begin{pmatrix} (0.57, 0.36, 0.06) & (0.37, 0.44, 0.00) & (0.00, 0.55, 0.15) \\ (0.44, 0.05, 0.10) & (0.00, 0.40, 0.00) & (0.00, 0.34, 0.00) \\ (0.34, 0.00, 0.16) & (0.21, 0.26, 0.00) & (0.00, 0.22, 0.03) \\ (0.11, 0.52, 0.32) & (0.25, 0.71, 0.10) & (0.30, 0.67, 0.56) \\ (0.09, 0.08, 0.50) & (0.00, 0.65, 0.01) & (0.50, 0.41, 0.00) \\ (0.07, 0.00, 0.85) & (0.14, 0.43, 0.09) & (0.76, 0.27, 0.11) \\ (0.00, 0.21, 0.09) & (0.35, 0.14, 1.00) & (0.33, 0.00, 0.32) \\ (0.00, 0.03, 0.15) & (0.00, 0.13, 0.11) & (0.54, 0.00, 0.00) \\ (0.00, 0.00, 0.25) & (0.20, 0.09, 0.90) & (0.83, 0.00, 0.06) \end{pmatrix}$$

3.1. *Superiority of the proposed work*

The superiority of our suggested work is exhibited in Table 3, which is self explanatory. The same applies for the corresponding topology.

TABLE 3. Concise comparison of PmFNS set with some prevailing structures

Set	Membership function	Indeterminacy	Non-membership function	Multiple membership function
Fuzzy set [18]	✓	×	×	×
Intuitionistic fuzzy set [2]	✓	×	✓	×
Pythagorean fuzzy set [15,16]	✓	×	✓	×
m-polar fuzzy set [4]	✓	×	×	✓
Pythagorean m-polar fuzzy set [8]	✓	×	✓	✓
PmFNS (proposed)	✓	✓	✓	✓

4. **Pythagorean m-polar fuzzy neutrosophic topology**

In this section, we present Pythagorean m-polar fuzzy neutrosophic topology on Pythagorean m-polar fuzzy neutrosophic set and elongate numerous characteristics of crisp topology towards Pythagorean m-polar fuzzy neutrosophic topology. Separation axioms in PmFNSs are also discussed.

Definition 4.1. Let PmFNS($\underset{\sim}{X}$) be the collection of all PmFN-subsets of the absolute PmFNS $\underset{\sim}{X}_A$. For $\underset{\sim}{S}$, $\underset{\sim}{T} \subseteq \underset{\sim}{A}$, a subcollection $\underset{\sim}{\daleth}_{pn}$ of PmFNS($\underset{\sim}{X}$) is known as *Pythagorean m-polar fuzzy neutrosophic topology* (PmFNT) on $\underset{\sim}{X}$ if the following needs are satisfied:

(i) $\underset{\sim}{\emptyset}$, $\underset{\sim}{X}_A \in \underset{\sim}{\daleth}_{pn}$,

(ii) $\jmath_S, \jmath_T \Subset \daleth_{pn}$ then $\jmath_S \Cap \jmath_T \Subset \daleth_{pn}$,

(iii) $\jmath_i \Subset \daleth_{pn}, \forall i \Subset \underline{\mathrm{I}}$, then $\cup_{i \in \underline{\mathrm{I}}} \jmath_i \Subset \daleth_{pn}$.

The doublet ($\underline{\mathrm{X}}$, $\daleth_{pn}$) or simply $\daleth_{pn}$, where $\underline{\mathrm{X}}$ is a non-empty PmFNS and $\daleth_{pn}$ is a Pythagorean m-polar fuzzy neutrosophic topology on $\underline{\mathrm{X}}$, is known as *Pythagorean m-polar fuzzy neutrosophic topological space* (PmFNTS).

Example 4.2. Let $\underline{\mathrm{X}} = \{\hbar_1, \hbar_2\}$ be a universal P3FNS with $\underline{\mathrm{S}}$ and $\underline{\mathrm{T}}$ be as shown in table 4 and table 5 below:

TABLE 4. P3FNS $\underline{\mathrm{S}}$

$\underline{\mathrm{S}}$			
$\hbar_1$	(0.401, 0.210, 0.216)	(0.221, 0.100, 0.363)	(0.632, 0.029, 0.216)
$\hbar_2$	(0.626, 0.111, 0.162)	(0.432, 0.000, 0.163)	(0.221, 0.012, 0.108)

TABLE 5. P3FNS $\underline{\mathrm{T}}$

$\underline{\mathrm{T}}$			
$\hbar_1$	(0.126, 0.621, 0.623)	(0.063, 0.920, 0.706)	(0.276, 0.636, 0.591)
$\hbar_2$	(0.168, 0.702, 0.668)	(0.165, 0.761, 0.726)	(0.149, 0.712, 0.561)

Then $\daleth_{pn5} = \{\underline{\emptyset}, \underline{\mathrm{S}}, \underline{\mathrm{T}}, \underline{\mathrm{X}}_A\}$ is a P3FNT on $\underline{\mathrm{X}}$.

Definition 4.3. The members of $\daleth_{pn}$ are called *Pythagorean m-polar fuzzy neutrosophic open sets* (PmFN-open sets). The complements of Pythagorean m-polar fuzzy neutrosophic open sets are called *Pythagorean m-polar fuzzy neutrosophic closed sets* (PmFN-closed sets) and PmFN-open set as well as PmFN-closed set is called *Pythagorean m-polar fuzzy neutrosophic clopen sets* (PmFN-clopen sets).

Example 4.4. For the P3FNTS $\daleth_{pn5}$ given in Example 4.2, we have $\underline{\emptyset}, \underline{\mathrm{S}}, \underline{\mathrm{T}}, \underline{\mathrm{X}}_A$ are P3FN-open sets because they are members of $\daleth_{pn5}, (\underline{\mathrm{X}}_A)^c = \underline{\emptyset} \Subset \daleth_{pn}$ is a P3FN-closed set and $\underline{\emptyset}, \underline{\mathrm{X}}_A$ are P3FN$-$clopen sets as $\underline{\emptyset}^c = \underline{\mathrm{X}}_A - \underline{\emptyset} = \underline{\mathrm{X}}_A$ and $\underline{\mathrm{X}}_A^c = \underline{\mathrm{X}}_A - \underline{\mathrm{X}}_A = \underline{\emptyset}$

Example 4.5. Consider the P3FNSs $\underline{\mathrm{X}}, \underline{\mathrm{S}}$ and $\underline{\mathrm{T}}$ given in Example 4.2 and

TABLE 6. P3FNS $\underline{\mathrm{U}}$

$\underline{\mathrm{U}}$			
$\hbar_1$	(0.221, 0.561, 0.524)	(0.172, 0.603, 0.367)	(0.307, 0.633, 0.336)
$\hbar_2$	(0.267, 0.623, 0.201)	(0.380, 0.529, 0.419)	(0.162, 0.560, 0.333)

We have,

$$\begin{aligned} \daleth_{pn1} &= \{\underline{\emptyset}, \underline{X}_A\} \\ \daleth_{pn2} &= \{\underline{\emptyset}, \underline{S}, \underline{X}_A\} \\ \daleth_{pn3} &= \{\underline{\emptyset}, \underline{T}, \underline{X}_A\} \\ \daleth_{pn4} &= \{\underline{\emptyset}, \underline{U}, \underline{X}_A\} \\ \daleth_{pn5} &= \{\underline{\emptyset}, \underline{S}, \underline{T}, \underline{X}_A\} \\ \daleth_{pn6} &= \{\underline{\emptyset}, \underline{T}, \underline{U}, \underline{X}_A\} \\ \daleth_{pn7} &= \{\underline{\emptyset}, \underline{S}, \underline{U}, \underline{X}_A\} \\ \daleth_{pn8} &= \{\underline{\emptyset}, \underline{S}, \underline{T}, \underline{U}, \underline{X}_A\} \end{aligned}$$

are Pythagorean 3-polar fuzzy neutrosophic topologies over $\underline{X}$. Here, both $\underline{\emptyset}$ & $\underline{X}_A$ are P3FN-open set as well as P3FN-closed set so it is a P3FN-clopen set.

Definition 4.6. Let $(\underline{X}, \daleth_{pn1})$ and $(\underline{X}, \daleth_{pn2})$ be two PmFNTSs on $\underline{X}$. $\daleth_{pn2}$ is contained in $\daleth_{pn1}$ i.e $\daleth_{pn2} \subseteq \daleth_{pn1}$ if $\underline{\kappa} \Subset \daleth_{pn1}$ for every $\underline{\kappa} \Subset \daleth_{pn2}$. In such case, $\daleth_{pn2}$ is known as *Pythagorean m−polar fuzzy neutrosophic coarser or weaker* (PmFN-coaser/weaker) than $\daleth_{pn1}$ and $\daleth_{pn1}$ is called *Pythagorean m−polar fuzzy neutrosophic finer or stronger* PmFN-finer/stronger than $\daleth_{pn2}$. $\daleth_{pn1}$ and $\daleth_{pn2}$ in such a case are known as *comparable*. In Example 4.5, $\daleth_{pn2}$ is PmFN-coarser than $\daleth_{pn5}$ and $\daleth_{pn5}$ is PmFN-stronger than $\daleth_{pn2}$. Hence $\daleth_{pn2}$ and $\daleth_{pn5}$ are comparable.

Definition 4.7. The PmFNT $\daleth_{pn(indiscrete)} = \{\underline{\emptyset}, \underline{X}_A\}$ is known as *indiscrete Pythagorean m-polar fuzzy neutrosophic topology* (indiscrete-PmFNT) & $\daleth_{pn(discrete)} = \underline{P}(\underline{X}_A)$ (power set of $\underline{X}_A$) is known as *discrete Pythagorean m-polar fuzzy neutrosophic topology* (discrete-PmFNT) over $\underline{X}$.

Remark 4.8. On $\underline{X}$, the smallest PmFNT is $\daleth_{pn(indiscrete)}$ whereas the largest PmFNT is $\daleth_{pn(discrete)}$.

Definition 4.9. Suppose that $(\underline{X}, \daleth_{pnX})$ be a PmFNTS. A few $\underline{Y} \subseteq \underline{X}$ and PmFN-open sets are $\underline{S}_n^{\star} = \underline{S}_n \Cap \underline{Y}_A$ of PmFNT $\daleth_{pnY}$ on $\underline{Y}$ where $\underline{S}_n$ are PmFN-open sets of $\daleth_{pnX}$ & $\underline{Y}_A$ is absolute PmFNS on $\underline{Y}$ then $\daleth_{pnY}$ is reserved as the *Pythagorean m-polar fuzzy neutrosophic subspace* (PmFN-subspace) of $\daleth_{pnX}$. It can be written as:

$$\daleth_{pnY} = \{\underline{S}_n^{\star} : \underline{S}_n^{\star} = \underline{S}_n \Cap \underline{Y}_A, \underline{S}_n \Subset \daleth_{pnX}\}$$

Example 4.10. Let $\daleth_{pnX} = \{\underline{\emptyset}, \underline{S}, \underline{T}, \underline{X}_A\}$, then $\daleth_{pnX}$ is a P3FNT on $\underline{X}$. P3FNS on $\underline{Y} = \{\underline{S}\} \subseteq \underline{X}$ is

TABLE 7. P3FNS $\underline{Y}_A$

$\underline{Y}_A$			
$\hbar_1$	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)

Since

$$\begin{aligned} \underline{Y}_A \cap \underline{\emptyset} &= \underline{\emptyset} \\ \underline{Y}_A \cap \underline{S} &= \underline{S} \\ \underline{Y}_A \cap \underline{T} &= \underline{T} \\ \underline{Y}_A \cap \underline{X}_A &= \underline{Y}_A \end{aligned}$$

So, $\beth_{pnY} = \{\underline{\emptyset}, \underline{S}, \underline{T}, \underline{Y}_A\}$ is a *Pythagorean* 3*-polar fuzzy neutrosophic subtopology* (P3FN-subtopology) of $\beth_{pnX}$ (i.e $\beth_{pnY} \subseteq \beth_{pnX}$).

Remark 4.11. (1) A PmFN-subtopology i.e. $\beth_{pnZ}$ of a PmFN-subtopology $\beth_{pnY}$ of a PmFNTS $\beth_{pnX}$ is also a PmFN-subtopology of $\beth_{pnX}$.
(2) Every PmFN-subspace of a discrete-PmFNTS is always discrete-PmFNTS. Similarly, every PmFN-subspace of indiscrete-PmFNTS ia also an indiscrete-PmFNTS.

Definition 4.12. Let $(\underline{X}, \beth_{pn})$ be a PmFNTS and $\underline{V} \subseteq PmFNS(\underline{X})$. The *Pythagorean m-polar fuzzy neutrosophic interior* (PmFN-interior) $\underset{\cdot}{\underline{V}}$ of $\underline{V}$ is PmFNS which is the union of all PmFNS-open subsets (i.e that are contained in $\underline{V}$) of $\underline{X}$.

Example 4.13. If

TABLE 8. P3FNS $\underline{V}$

$\underline{V}$			
$\hbar_1$	(0.233, 0.449, 0.496)	(0.276, 0.507, 0.365)	(0.332, 0.501, 0.312)
$\hbar_2$	(0.314, 0.416, 0.308)	(0.389, 0.501, 0.402)	(0.267, 0.517, 0.223)

and $\beth_{pn8} = \{\underline{\emptyset}, \underline{S}, \underline{T}, \underline{U}, \underline{X}_A\}$, then $\underset{\cdot}{\underline{V}} = \underline{T} \cup \underline{U} = \underline{U}$
or

TABLE 9. P3FN-interior $\underset{\cdot}{\underline{V}}$

$\underset{\cdot}{\underline{V}}$			
$\hbar_1$	(0.221, 0.561, 0.524)	(0.172, 0.603, 0.367)	(0.307, 0.633, 0.336)
$\hbar_2$	(0.267, 0.623, 0.401)	(0.380, 0.529, 0.419)	(0.162, 0.560, 0.333)

Definition 4.14. Let $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ be a PmFNTS and $\underset{\sim}{V} \subseteq$ PmFN$(\underset{\sim}{X})$. Then the *Pythagorean m-polar fuzzy neutrosophic closure* (PmFN-closure) $\dot{\underset{\sim}{V}}$ of $\underset{\sim}{V}$ is the PmFNS which is intersection of all PmFN-closed supersets (i.e that contain $\underset{\sim}{V}$) of $\underset{\sim}{V}$.

Example 4.15. Let $\underset{\sim}{\daleth}_{pn8} = \{\underset{\sim}{\emptyset}, \underset{\sim}{S}, \underset{\sim}{T}, \underset{\sim}{U}, \underset{\sim}{X}_A\}$, then first of all we've to find $\underset{\sim}{\emptyset}^c, \underset{\sim}{S}^c, \underset{\sim}{T}^c, \underset{\sim}{U}^c, \underset{\sim}{X}_A^c$.
$\underset{\sim}{\emptyset}^c = \underset{\sim}{X}_A, \underset{\sim}{S}^c = \underset{\sim}{S}_1, \underset{\sim}{T}^c = \underset{\sim}{T}_1, \underset{\sim}{U}^c = \underset{\sim}{U}_1, \underset{\sim}{X}_A^c = \underset{\sim}{\emptyset}$ where

Table 10. P3FNS $\underset{\sim}{S}^c/\underset{\sim}{S}_1$

$\underset{\sim}{S}^c$=$\underset{\sim}{S}_1$			
$\hbar_1$	(0.216, 0.790, 0.401)	(0.363, 0.900, 0.221)	(0.216, 0.971, 0.632)
$\hbar_2$	(0.162, 0.889, 0.626)	(0.163, 1.000, 0.432)	(0.108, 0.988, 0.221)

Table 11. P3FNS $\underset{\sim}{T}^c/\underset{\sim}{T}_1$

$\underset{\sim}{T}^c$=$\underset{\sim}{T}_1$			
$\hbar_1$	(0.623, 0.379, 0.126)	(0.706, 0.080, 0.063)	(0.591, 0.364, 0.276)
$\hbar_2$	(0.368, 0.298, 0.368)	(0.726, 0.239, 0.165)	(0.561, 0.288, 0.149)

and

Table 12. P3FNS $\underset{\sim}{U}^c/\underset{\sim}{U}_1$

$\underset{\sim}{U}^c$=$\underset{\sim}{U}_1$			
$\hbar_1$	(0.524, 0.439, 0.221)	(0.367, 0.397, 0.172)	(0.336, 0.367, 0.307)
$\hbar_2$	(0.401, 0.377, 0.267)	(0.419, 0.471, 0.380)	(0.333, 0.440, 0.162)

As $\underset{\sim}{X}_A$ is the only P3FN-closed supersets of $\underset{\sim}{V}$ i.e $\underset{\sim}{V}$ is contained only in $\underset{\sim}{X}_A$. Thus, $\dot{\underset{\sim}{V}} = \underset{\sim}{X}_A$

Remark 4.16. Largest PmFN-open subset of $\underset{\sim}{V}$ is $\underset{\cdot}{\underset{\sim}{V}}$ whereas the smallest PmFN-closed superset of $\underset{\sim}{V}$ is $\dot{\underset{\sim}{V}}$.

Definition 4.17. Let $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ be a PmFNTS and $\underset{\sim}{V} \subseteq$ PmFN$(\underset{\sim}{X})$. Then the *Pythagorean m-polar fuzzy neutrosophic frontier or boundary* (PmFN-frontier/boundary) $F^{\diamond}(\underset{\sim}{V})$ of $\underset{\sim}{V}$ is defined as:

$$F^{\diamond}(\underset{\sim}{V}) = \dot{\underset{\sim}{V}} \Cap \dot{\underset{\sim}{V}}^c$$

Example 4.18. For the P3FNS $\underset{\sim}{V}$ given in Example 4.13, we have

Table 13. P3FNS $\underline{V}^c$

$\underline{V}^c$			
$\hbar_1$	(0.496, 0.551, 0.233)	(0.365, 0.493, 0.276)	(0.312, 0.499, 0.332)
$\hbar_2$	(0.308, 0.584, 0.314)	(0.402, 0.499, 0.389)	(0.223, 0.483, 0.267)

$$\dot{\underline{V}} = \underline{\emptyset}^c \cap \underline{T}^c \cap \underline{U}^c = \underline{U}^c$$
$$\dot{\underline{V}} = \underline{X} \cap \underline{T}_1 \cap \underline{U}_1 = \underline{U}_1$$
$$\Rightarrow F^{\diamond}(\underline{V}) = \underline{X} \cap \underline{U}_1 = \underline{U}_1$$

Definition 4.19. Let $(\underline{X}, \underline{\mathfrak{I}}_{pn})$ be a PmFNTS and $\underline{V} \subseteq$ PmFN$(\underline{X})$. Then the *Pythagorean m-polar fuzzy neutrosophic exterior* (PmFN-exterior) $E^{\diamond}(\underline{V})$ of $\underline{V}$ is defined as:

$$E^{\diamond}(\underline{V}) = \underset{\cdot}{\underline{V}}^c$$

From Example 4.5 and 4.15, we get $\underset{\cdot}{\underline{V}}^c = \underline{S}^c \cup \underline{\emptyset} = \underline{S}^c = \underline{S}_1$

Example 4.20. For the P3FNSs $\underline{S}, \underline{T}, \underline{U}, \underline{V}$ given in Examples 4.5,4.13, and

Table 14. P3FNS $\underline{W}$

$\underline{W}$			
$\hbar_1$	(0.721, 0.110, 0.116)	(0.662, 0.100, 0.265)	(0.621, 0.010, 0.116)
$\hbar_2$	(0.765, 0.011, 0.062)	(0.571, 0.000, 0.006)	(0.795, 0.002, 0.008)

(i) $\underset{\cdot}{\underline{V}} \subseteq \underline{V} \subseteq \dot{\underline{V}}$ (See Table 8 and Table 9) and as we know

Table 15. P3FNS $\underline{X}$

$\underline{X}$			
$\hbar_1$	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)
$\hbar_2$	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)

(ii) $\underset{\vdots}{\underline{V}} = \underset{\cdot}{\underline{V}}$
$\underset{\vdots}{\underline{V}} = \underline{T} \cup \underline{U} = \underline{U}$
$\underset{\cdot}{\underline{V}} = \underline{U}$
From above equations we get, $\underset{\vdots}{\underline{V}} = \underset{\cdot}{\underline{V}}$

(iii) $\overset{\vdots}{\underline{V}} = \dot{\underline{V}}$
$\overset{\vdots}{\underline{V}} = \underline{X}$ and $\dot{\underline{V}} = \underline{X}$

(iv) $\underset{\vdots}{\underset{\sim}{X}} = \underset{\sim}{X}$

$\underset{\vdots}{\underset{\sim}{X}} = \underset{\sim}{S} \cup \underset{\sim}{T} \cup \underset{\sim}{U} \cup \underset{\sim}{X} = \underset{\sim}{X}$

(v) $\underset{\vdots}{\underset{\sim}{\emptyset}} = \underset{\sim}{\emptyset}$

As $\underset{\sim}{\emptyset}$ is superset of itself only.

(vi) $\underset{\sim}{V} \subseteq \underset{\sim}{W} \Rightarrow \underset{\vdots}{\underset{\sim}{V}} \subseteq \underset{\vdots}{\underset{\sim}{W}}$ and $\dot{\underset{\sim}{V}} \subseteq \dot{\underset{\sim}{W}}$

We know that, $\underset{\vdots}{\underset{\sim}{V}} = \underset{\sim}{U}$ and $\underset{\vdots}{\underset{\sim}{W}} = \underset{\sim}{S} \cup \underset{\sim}{T} \cup \underset{\sim}{U} = \underset{\sim}{S} \Rightarrow \underset{\vdots}{\underset{\sim}{V}} \subseteq \underset{\vdots}{\underset{\sim}{W}} (\because \underset{\sim}{U} \subseteq \underset{\sim}{S})$

Now, $\dot{\underset{\sim}{V}} = \underset{\sim}{X}$ also $\dot{\underset{\sim}{W}} = \underset{\sim}{X} \Rightarrow \dot{\underset{\sim}{V}} \subseteq \dot{\underset{\sim}{W}} (\because \underset{\sim}{X} \subseteq \underset{\sim}{X})$

(vii) $\underset{\vdots}{(V \underset{\sim}{\cap} W)} = \underset{\vdots}{\underset{\sim}{V}} \underset{\sim}{\cap} \underset{\vdots}{\underset{\sim}{W}}$

Table 16. P3FNS $\underset{\sim}{V} \underset{\sim}{\cap} \underset{\sim}{W}$

$\underset{\sim}{V} \underset{\sim}{\cap} \underset{\sim}{W}$			
$\hbar_1$	(0.233, 0.449, 0.496)	(0.276, 0.507, 0.365)	(0.332, 0.501, 0.312)
$\hbar_2$	(0.314, 0.416, 0.308)	(0.389, 0.501, 0.402)	(0.267, 0.517, 0.223)

$\underset{\vdots}{(V \underset{\sim}{\cap} W)} = \underset{\sim}{T} \cup \underset{\sim}{U} = \underset{\sim}{U}$ and $\underset{\vdots}{\underset{\sim}{V}} \underset{\sim}{\cap} \underset{\vdots}{\underset{\sim}{W}} = \underset{\sim}{S} \cap \underset{\sim}{U} = \underset{\sim}{U}$

From above equations, we get the result, $\underset{\vdots}{(V \underset{\sim}{\cap} W)} = \underset{\vdots}{\underset{\sim}{V}} \underset{\sim}{\cap} \underset{\vdots}{\underset{\sim}{W}}$

Proposition 4.21. *Let* $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ *be a PmFNTS and* $\underset{\sim}{Q} \subseteq \underset{\sim}{X}$*, then*

(i) $(\underset{\vdots}{\underset{\sim}{Q}})^c = \dot{\underset{\sim}{Q}}^c$

(ii) $(\dot{\underset{\sim}{Q}})^c = \underset{\vdots}{\underset{\sim}{Q}}^c$

Proof. (i) $\underset{\sim}{Q} = \left\{ \frac{\hbar_1}{(T^{(i)}(\hbar_1), I^{(i)}(\hbar_1), F^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i = 1, 2, \cdots, m \right\}$

Let PmFN-open sets contained in $\underset{\sim}{Q}$ be indexed by the collection

$$\left\{ \frac{\hbar_1}{(T_j^{(i)}(\hbar_1), I_j^{(i)}(\hbar_1), F_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i = 1, 2, \cdots, m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}.$$

By definition,

$$\underset{\vdots}{\underset{\sim}{Q}} = \left\{ \frac{\hbar_1}{(\max\ T_j^{(i)}(\hbar_1), \min\ I_j^{(i)}(\hbar_1), \min\ F_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i = 1, 2, \cdots, m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}$$

and

$$(\underset{\vdots}{\underset{\sim}{Q}})^c = \left\{ \frac{\hbar_1}{(\min\ F_j^{(i)}(\hbar_1), 1 - \min\ I_j^{(i)}(\hbar_1), \max\ T_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i = 1, 2, \cdots, m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}$$

$$\because \underset{\sim}{Q}^c = \left\{ \frac{\hbar_1}{(F^{(i)}(\hbar_1), 1 - I^{(i)}(\hbar_1), T^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i = 1, 2, \cdots, m \right\}$$

Also, $T_j^{(i)}(\hbar_1) \leq T^{(i)}(\hbar_1), 1 - (I_j^{(i)}(\hbar_1)) \geq 1 - (I^{(i)}(\hbar_1)), F_j^{(i)}(\hbar_1) \geq F^{(i)}(\hbar_1), \forall$ values of i & $j \underset{\sim}{\in} \underset{\sim}{J}$, so it develops that

$$\left\{ \frac{\hbar_1}{(F_j^{(i)}(\hbar_1), 1 - I_j^{(i)}(\hbar_1), T_j^{(i)}(\hbar_1))} : \hbar_1 \underset{\sim}{\in} \underset{\sim}{X}, i = 1, 2, \cdots, m; j \underset{\sim}{\in} \underset{\sim}{J} \right\}$$

is the entire of PmFN-closed sets contained contained $\underset{\sim}{Q}^c$ i.e.

$$\dot{\underset{\sim}{Q}}^c = \left\{ \frac{\hbar_1}{(\min\ F_j^{(i)}(\hbar_1), 1-(\min\ I_j^{(i)}(\hbar_1)), \max\ T_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{X}, i = 1,2,\cdots,m; j \underline{\in} \underline{J} \right\}$$

which completes the proof.

(ii) $\underset{\sim}{Q} = \left\{ \frac{\hbar_1}{(T^{(i)}(\hbar_1), I^{(i)}(\hbar_1), F^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{X}, i = 1,2,\cdots,m \right\}$

Let PmFN-closed supersets of $\underset{\sim}{Q}$ be indexed by the collection

$$\left\{ \frac{\hbar_1}{(T_j^{(i)}(\hbar_1), I_j^{(i)}(\hbar_1), F_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{X}, i = 1,2,\cdots,m; j \underline{\in} \underline{J} \right\}$$

By definition,

$$\dot{\underset{\sim}{Q}} = \left\{ \frac{\hbar_1}{(min\ T_j^{(i)}(\hbar_1), max\ I_j^{(i)}(\hbar_1), max\ F_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{X}, i = 1,2,\cdots,m; j \underline{\in} \underline{J} \right\}$$

and

$$(\dot{\underset{\sim}{Q}})^c = \left\{ \frac{\hbar_1}{(max\ F_j^{(i)}(\hbar_1), 1-(max\ I_j^{(i)}(\hbar_1)), min\ T_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{X}, i = 1,2,\cdots,m; j \underline{\in} \underline{J} \right\}$$

Now,

$$\because \underset{\sim}{Q}^c = \left\{ \frac{\hbar_1}{(F^{(i)}(\hbar_1), 1-I^{(i)}(\hbar_1), T^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{X}, i = 1,2,\cdots,m \right\}$$

and $T_j^{(i)}(\hbar_1) \leq T^{(i)}(\hbar_1), 1-(I_j^{(i)}(\hbar_1)) \geq 1-(I^{(i)}(\hbar_1)), F_j^{(i)}(\hbar_1) \geq F^{(i)}(\hbar_1), \forall$ values of i and $j \underline{\in} \underline{J}$ so it follows that

$$\underset{\dot{\sim}}{Q}^c = \left\{ \frac{\hbar_1}{(max\ F_j^{(i)}(\hbar_1), 1-(max\ I_j^{(i)}(\hbar_1)), min\ T_j^{(i)}(\hbar_1))} : \hbar_1 \underline{\in} \underline{X}, i = 1,2,\cdots,m; j \underline{\in} \underline{J} \right\}$$

which completes the proof.

0.1cm□

Proposition 4.22.

(i) $\underset{\dot{\sim}}{Q} \neq \underset{\sim}{Q} - \dot{\underset{\sim}{Q}}^c$

(ii) $E^\diamond(\underset{\sim}{Q})^c = \underset{\dot{\sim}}{Q}$

(iii) $E^\diamond(\underset{\sim}{Q}) = \underset{\dot{\sim}}{Q}^c$

(iv) $E^\diamond(\underset{\sim}{Q}) \underline{\cup} F^\diamond(\underset{\sim}{Q}) \underline{\cup} \underset{\dot{\sim}}{Q} \neq \underset{\sim}{X}_A$

(v) $F^\diamond(\underset{\sim}{Q}) = F^\diamond(\underset{\sim}{Q}^c)$

(vi) $\underset{\dot{\sim}}{Q} \underline{\cap} F^\diamond(\underset{\sim}{Q}) \neq \underline{\emptyset}$

(vii) $\dot{\underset{\sim}{Q}} \neq \underset{\sim}{Q} \underline{\cup} F^\diamond(\underset{\sim}{Q})$

(viii) $\dot{\underset{\sim}{Q}} \neq \underset{\dot{\sim}}{Q} \underline{\cup} F^\diamond(\underset{\sim}{Q})$

Proof. Follows directly from definition. 0.1cm□

Proposition 4.23. *Let $(\underset{\sim}{X}, \beth_{pn})$ be a PmFNTS and $\underset{\sim}{Q} \subseteq \underset{\sim}{X}$, then $F^{\diamond}(\underset{\sim}{Q})$=$F^{\diamond}(\underset{\sim}{Q}^c)$*

Proof. By definition; $F^{\diamond}(\underset{\sim}{Q}) = \dot{\underset{\sim}{Q}} \sqcap \dot{\underset{\sim}{Q}}^{c} = \dot{\underset{\sim}{Q}}^{c} \sqcap \dot{\underset{\sim}{Q}} = \dot{\underset{\sim}{Q}}^{c} \sqcap (\dot{\underset{\sim}{Q}}^{c})^{c} = F^{\diamond}(\underset{\sim}{Q}^{c})$ 0.1cm□

Remark 4.24. The intersection of two or more PmFNTSs is always a PmFNTS but it is not necessary that their union is also a PmFNTS.

Example 4.25. Let $\underset{\sim}{X} = \{\hbar_1, \hbar_2\}$ be a universal non-empty P3FNS and let

Table 17. P3FNS $\underset{\sim}{Q}_1$

$\underset{\sim}{Q}_1$			
$\hbar_1$	(0.211, 0.301, 0.451)	(0.251, 0.321, 0.420)	(0.021, 0.567, 0.481)
$\hbar_2$	(0.100, 0.500, 0.256)	(0.257, 0.421, 0.000)	(0.424, 0.567, 0.291)

Table 18. P3FNS $\underset{\sim}{Q}_2$

$\underset{\sim}{Q}_2$			
$\hbar_1$	(0.312, 0.217, 0.111)	(0.171, 0.367, 0.582)	(0.361, 0.272, 0.391)
$\hbar_2$	(0.111, 0.421, 0.156)	(0.167, 0.568, 0.721)	(0.321, 0.666, 0.382)

be P3FNSs over $\underset{\sim}{X}$, then $\beth_{pno1} = \{\underset{\sim}{\emptyset}, \underset{\sim}{Q}_1, \underset{\sim}{X}_A\}$ and $\beth_{pno2} = \{\underset{\sim}{\emptyset}, \underset{\sim}{Q}_2, \underset{\sim}{X}_A\}$ are two P3FNTs over $\underset{\sim}{X}$. However, $\beth_{pno1} \sqcup \beth_{pno2} = \{\underset{\sim}{\emptyset}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2, \underset{\sim}{X}_A\}$ fails to be P3FNT on $\underset{\sim}{X}$ and intersection of P3FNT over $\underset{\sim}{X}$, $\beth_{pno1} \sqcap \beth_{pno2} = \{\underset{\sim}{\emptyset}, \underset{\sim}{X}_A\}$ is also a P3FNT.

Theorem 4.26. *Let $(\underset{\sim}{X}, \beth_{pn})$ be a PmFNTS then the following conditions are satisfied:*

(1) $\underset{\sim}{\emptyset}$, $\underset{\sim}{X}_A$ *are PmFN−open sets.*
(2) *Union of any number of PmFN−open sets is PmFN−open set.*
(3) *Intersection of any number of PmFN−closed sets is PmFN−closed set.*
(4) *The intersection of any two PmFN−open sets (and hence of any finite number of PmFN−open sets) is PmFN−open set.*
(5) *The union of any two PmFN−closed sets (and hence of any finite number of PmFN−closed sets) is PmFN−closed set.*
(6) $\underset{\sim}{\emptyset}$, $\underset{\sim}{X}_A$ *are PmFN−closed set.*

Proof. (1) The proof is obvious.

(2) Let $\{< \hbar, (\daleth_1^{(i)}(\hbar), \ddot{\underset{\sim}{I}}_1^{(i)}(\hbar), \underset{\sim}{F}_1^{(i)}(\hbar)) >: \hbar \in \underset{\sim}{X}\}$ be a collection of PmFN-open sets.
Also, $\underset{\sim}{Y} = \bigcup_{\hbar \in \underset{\sim}{X}} \{< \hbar, (\daleth_1^{(i)}(\hbar), \ddot{\underset{\sim}{I}}_1^{(i)}(\hbar), \underset{\sim}{F}_1^{(i)}(\hbar)) >\}$

Let $\hbar^{\dagger} \Subset \text{Y}$ implies that $\hbar^{\dagger} \Subset \{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\}$ for some $\hbar \Subset \text{X}$ and $\text{B}(\text{y}, \text{r}) \subseteq \{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\} \subseteq \bigcup_{\hbar \Subset \text{X}} \{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\} = \text{Y}$

$\Rightarrow$ Y is PmFN-open set.

(3) Let $\{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >: \hbar \Subset \text{X}\}$ be any number of PmFN-closed sets. We shall show that $\bigcap_{\hbar \Subset \text{X}} \{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\}$, is P$m$FN-closed set, by proving that its complement is PmFN-open set.

By De Morgan's law,

$$[\bigcap_{\hbar \Subset \text{X}} \{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\}]^c = \bigcup_{\hbar \Subset \text{X}} \{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\}^c$$

Since each $< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >$ is PmFN-closed set, each $\{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\}^c$ is a PmFN-open set (by definition of PmFN-closed set).

So, $\bigcup_{\hbar \Subset \text{X}} \{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\}^c$ is PmFN-open set.

Hence $\bigcap_{\hbar \Subset \text{X}} \{< \hbar, (\beth_1^{(i)}(\hbar), \ddot{\text{I}}_1^{(i)}(\hbar), \digamma_1^{(i)}(\hbar)) >\}$ is PmFN-closed set.

(4) and (5) may be established in the similar way.

(6) The complement of X_A is the PmFN-open set $\emptyset$ and the complement of $\emptyset$ is the PmFN-open set X_A. So, X_A and $\emptyset$ are PmFN-closed sets.

0.1cm□

Definition 4.27. Let $(\text{X}, \beth_{pn})$ be a PmFNTS and let $\hbar$ be a PmFN-point of X. $\aleph^{\dagger} \subseteq \text{X}$ is called a *neighborhood* of $\hbar$ iff there exists a PmFN-open set $\text{Ł}^{\dagger}$ s.t. $\hbar \Subset \text{Ł}^{\dagger}$ and $\text{Ł}^{\dagger} \Subset \aleph^{\dagger}$ (or, for short, $\hbar \Subset \text{Ł}^{\dagger} \subseteq \aleph^{\dagger}$). In other words, $\aleph^{\dagger}$ is a neighborhood of $\hbar$, iff it contains some PmFN-open set to which $\hbar$ belongs.

Example 4.28. Let $\mathbb{X} = \{\text{e}, \text{f}, \text{g}\}$ be a universal P3FNS and $\beth_{pn} = \{\emptyset, \mathbb{D}_1, \mathbb{D}_2, \mathbb{X}_A\}$ where,

TABLE 19. P3FNS $\mathbb{D}_1$

$\mathbb{D}_1$			
e	(0.672, 0.421, 0.221)	(0.567, 0.420, 0.111)	(0.242, 0.121, 0.199)
f	(0.211, 0.467, 0.520)	(0.562, 0.721, 0.221)	(0.444, 0.333, 0.111)
g	(0.167, 0.437, 0.561)	(0.466, 0.167, 0.321)	(0.252, 0.467, 0.490)

and,

Table 20. P3FNS $\mathbb{D}_2$

$\mathbb{D}_2$			
$\underline{f}$	(0.115, 0.226, 0.421)	(0.462, 0.621, 0.221)	(0.555, 0.222, 0.001)
$\underline{g}$	(0.267, 0.337, 0.461)	(0.366, 0.017, 0.421)	(0.452, 0.376, 0.241)

$\mathbb{X}_A$ is the only P3FN−open set of

Table 21. P3FNS $\hbar_1^*$

$\hbar_1^*$			
$\underline{e}$	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)

So, $\mathbb{X}_A$ is the only neighborhood of $\hbar_1^*$.
The P3FN−point

Table 22. P3FNS $\hbar_2^*$

$\hbar_2^*$			
$\underline{f}$	(0.111, 0.562, 0.621)	(0.461, 0.921, 0.178)	(0.321, 0.642, 0.316)

has three neighborhoods, namely, $\mathbb{D}_1, \mathbb{D}_2$ and $\mathbb{X}_A$.
Similarly, the P3FN−point

Table 23. P3FNS $\hbar_3^*$

$\hbar_3^*$			
$\underline{e}$	(0.462, 0.562, 0.398)	(0.367, 0.572, 0.192)	(0.120, 0.499, 0.400)

has two neighborhoods $\mathbb{D}_1$ and $\mathbb{X}_A$.

Remark 4.29. In an indiscrete−PmFNTS, each PmFN-point has a single neighborhood which is the ground PmFNS itself.

The following example illustrate the PmFN-point that a neighborhood of a PmFN-point may not be PmFN-open set.

Example 4.30. Let $\mathbb{X} = \{\underline{e}, \underline{f}, \underline{g}\}$ be an universal non-empty P3FNS and $\beth_{pn} = \{\underline{\emptyset}, \mathbb{D}_4, \mathbb{X}_A\}$ where,

TABLE 24. P3FNS $\mathbb{D}_4$

$\mathbb{D}_4$			
$\underline{f}$	(0.315, 0.226, 0.421)	(0.162, 0.621, 0.221)	(0.555, 0.222, 0.001)
$\underline{g}$	(0.267, 0.337, 0.461)	(0.366, 0.017, 0.421)	(0.452, 0.376, 0.241)

clearly the P3FNS

TABLE 25. P3FNS $\mathbb{D}_3$

$\mathbb{D}_3$			
$\underline{e}$	(0.672, 0.421, 0.221)	(0.567, 0.420, 0.111)	(0.242, 0.121, 0.199)
$\underline{f}$	(1.000, 0.000, 0.000)	(0.715, 0.421, 0.226)	(1.000, 0.000, 0.000)
$\underline{g}$	(0.452, 0.421, 0.324)	(1.000, 0.210, 0.000)	(0.667, 0.210, 0.140)

is a neighborhood of $\mathbb{D}_4$, but it is not P3FN-open set because it is not an element of $\beth_{pn}$.

The following theorem enables us to recognize PmFN-open sets by knowing all the neighborhoods of a point and conversely. Thus, knowledge about PmFN-open sets enables us to determine the neighborhood of a point and conversely.

Theorem 4.31. *If $(\underline{X}, \beth_{pn})$ is a PmFNTS, then a PmFN-subset $\mathbb{A}$ of $\underline{X}$ is PmFN-open set, iff $\mathbb{A}$ is a neighborhood of each of its PmFN-points.*

Proof. Assume that $\mathbb{A}$ is PmFN-open set. We shall show that $\mathbb{A}$ is a neighborhood of each of its PmFN-points. Let $\varkappa$ be any PmFN-point of $\mathbb{A}$, then $\mathbb{A}$ itself can play the role of the PmFN-open set, whose existence qualifies $\mathbb{A}$ to be a neighborhood of $\varkappa$. Symbolically, $\varkappa \Subset \mathbb{A} \subseteq \mathbb{A}$ where $\mathbb{A}$ is PmFN-open set. It follows that $\mathbb{A}$ is neighborhood of each of its PmFN-points.
Conversely, if $\mathbb{A}$ is a neighborhood of every PmFN-point belonging to it, then for each $\varkappa \Subset \mathbb{A}$ there exists a PmFN-open set χ such that $\varkappa \Subset \chi \subseteq \mathbb{A}$. Then

$$\mathbb{A} = \cup\{< \hbar, (\daleth^{(i)}_{A1}(\hbar), \ddot{\mathrm{I}}^{(i)}_{A1}(\hbar), F^{(i)}_{A1}(\hbar)) >: \hbar \Subset \mathbb{A}\}$$
$$\subseteq \cup\{< \hbar, (\daleth^{(i)}_{A2}(\hbar), \ddot{\mathrm{I}}^{(i)}_{A2}(\hbar), F^{(i)}_{A2}(\hbar)) >: \hbar \Subset \mathbb{A}\} \subseteq \mathbb{A}$$

The simultaneous validity of

$$\mathbb{A} \subseteq \cup\{< \hbar, (\daleth^{(i)}_{A2}(\hbar), \ddot{\mathrm{I}}^{(i)}_{A2}(\hbar), F^{(i)}_{A2}(\hbar)) >: \hbar \Subset \mathbb{A}\}$$

and

$$\cup\{< \hbar, (\daleth^{(i)}_{A2}(\hbar), \ddot{\mathrm{I}}^{(i)}_{A2}(\hbar), F^{(i)}_{A2}(\hbar)) >: \hbar \Subset \mathbb{A}\} \subseteq \mathbb{A}$$
$$\Rightarrow \mathbb{A} = \cup\{< \hbar, (\daleth^{(i)}_{A2}(\hbar), \ddot{\mathrm{I}}^{(i)}_{A2}(\hbar), F^{(i)}_{A2}(\hbar)) >: \hbar \Subset \mathbb{A}\}$$

Since the union of PmFN-open sets is also PmFN-open set, it follows that $\mathbb{A}$ is PmFN-open set. 0.1cm□

The most important properties of neighborhoods in a PmFNTS are established in the following:

Definition 4.32. Let $\varkappa$ be a PmFN-point in a PmFNTS $(\underline{\mathrm{X}}, \daleth_{pn})$. Then the set of all neighborhoods of $\varkappa$ is called the *neighborhood system* of the PmFN-point $\varkappa$ and is denoted by $\aleph^{\dagger}(\varkappa)$.

Definition 4.33. Let $(\underline{\mathrm{X}}, \daleth_{pn})$ be a PmFNTS and $\mathbb{A}$ is a PmFN$-$subset of $\underline{\mathrm{X}}$. A point $\varkappa \Subset \underline{\mathrm{X}}$ is known as *Pythagorean m-polar fuzzy neutrosophic limit point* (PmFN-limit point) or *Pythagorean m-polar fuzzy neutrosophic cluster point* or *Pythagorean m$-$polar fuzzy neutrosophic accumulation point* $\mathbb{A}$ if every PmFN-open set,containing $\varkappa$ contains a PmFN-point of $\mathbb{A}$ different from $\varkappa$.

Example 4.34. Let $(\underline{\mathrm{X}}, \daleth_{pn})$ is a P3FNTS, $\mathbb{X} = \{\underline{\mathrm{e}}, \underline{\mathrm{f}}, \underline{\mathrm{g}}\}$ be an universal non-empty P3FNS and

TABLE 26. P3FNS $\mathbb{C}$

$\mathbb{C}$			
$\underline{\mathrm{e}}$	(0.000, 1.000, 1.000)	(0.000, 1.000, 1.000)	(0.000, 1.000, 1.000)
$\underline{\mathrm{f}}$	(0.511, 0.062, 0.211)	(0.312, 0.270, 0.137)	(0.921, 0.266, 0.152)
$\underline{\mathrm{g}}$	(0.232, 0.101, 0.431)	(0.466, 0.352, 0.121)	(0.368, 0.572, 0.400)

TABLE 27. P3FNS $\hbar_4^*$

$\hbar_4^*$			
$\underline{\mathrm{e}}$	(0.417, 0.312, 0.356)	(0.312, 0.270, 0.137)	(0.012, 0.374, 0.436)
$\underline{\mathrm{f}}$	(0.412, 0.117, 0.362)	(0.333, 0.672, 0.491)	(0.068, 0.772, 0.221)

and,

TABLE 28. P3FNS $\hbar_5^*$

$\hbar_5^*$			
$\underline{\mathrm{e}}$	(0.324, 0.467, 0.576)	(0.247, 0.657, 0.421)	(0.001, 0.476, 0.891)

then,

Table 29. P3FNS $\hbar_4^* - \hbar_5^*$

$\hbar_4^* - \hbar_5^*$			
$\underline{e}$	(0.417, 0.312, 0.356)	(0.312, 0.270, 0.247)	(0.012, 0.374, 0.436)
$\underline{f}$	(0.000, 0.117, 1.000)	(0.000, 0.672, 1.000)	(0.000, 0.772, 1.000)

Table 30. P3FNS $(\hbar_4^* - \hbar_5^*) \underline{\cap} \mathbb{C}$

$(\hbar_4^* - \hbar_5^*) \underline{\cap} \mathbb{C}$			
$\underline{e}$	(0.000, 1.000, 1.000)	(0.000, 1.000, 1.000)	(0.000, 1.000, 1.000)
$\underline{f}$	(0.000, 0.117, 1.000)	(0.000, 0.672, 1.000)	(0.000, 0.772, 1.000)

As $(\hbar_4^* - \hbar_5^*) \underline{\cap} \mathbb{C} \neq \underline{\emptyset}$. So, $\hbar_5^*$ is the P3FN-limit point of $\mathbb{C}$.

Definition 4.35. Let $(\underline{X}, \underline{\daleth}_{pn})$ be a PmFNTS then *Pythagorean m-polar fuzzy neutrosophic basis* (PmFN-basis) $\underline{\mathbb{B}}^{\circledcirc} \subseteq \underline{\daleth}_{pn}$ for $\underline{\daleth}_{pn}$ if for each $\underline{¥} \underline{\in} \underline{\daleth}_{pn}, \exists \underline{\ } \underline{\in} \mathbb{B}$ such that $\underline{¥} = \underline{\cup}\underline{\ }$.

4.1. *Separation Axioms in Pythagorean m-Polar Fuzzy Neutrosophic Sets*

Definition 4.36. A PmFNTS $(\underline{X}, \underline{\daleth}_{pn})$ is known as a *Pythagorean m-polar fuzzy neutrosophic T_0 space* (PmFNT_0S) if for every pair of distinct PmFN-points $ð_1, ð_2 \exists$ at any rate 1 PmFN-open set $\underline{ð}$ including precisely one of the PmNF-points.

Example 4.37. Each discrete PmFNTS is a PmFNT_0S for $\exists$ a PmFN-open set $\{ð_1\}$ that clearly contains $ð_1$ but not $ð_2$.

Remark 4.38. Each PmFN-subspace of a PmFNT_0S is PmFNT_0S means property of being a PmFNT_0S of any PmFNTS $(\underline{X}, \underline{\daleth}_{pn})$ is innate.

Definition 4.39. A PmFNTS $(\underline{X}, \underline{\daleth}_{pn})$ is *Pythagorean m-polar fuzzy neutrosophic T_1 space* (PmFNT_1S), *Pythagorean m-polar fuzzy Tychonoff space* or *Pythagorean m-polar fuzzy accessible space* if for any two unique PmFN-points $ð_1, ð_2$ of $(\underline{X}, \underline{\daleth}_{pn}), \exists$ two PmFN-open sets $\underline{ð}$ and Υ s.t. $ð_1 \underline{\in} \underline{ð}, ð_2 \underline{\notin} \underline{ð}$ and $ð_2 \underline{\in} \Upsilon, ð_1 \underline{\notin} \Upsilon$.

Example 4.40. Every discrete PmFNTS is a PmFNT_1S if $ð_1$ and $ð_2$ are two distinct PmFN-points then there are PmFN-open points $\{ð_1\}$ and $\{ð_2\}$ in $(\underline{X}, \underline{\daleth}_{pn})$ s.t. $ð_1 \underline{\in} \{ð_1\}$ whereas $ð_2 \underline{\notin} \{ð_1\}$.

Theorem 4.41. *The following assertions about a PmFNTS $(\underline{X}, \underline{\daleth}_{pn})$ are equivalent:*

(1) *$(\underline{X}, \underline{\daleth}_{pn})$ is a PmFNT_1S.*

(2) *Every PmFN singleton subset of* $\underline{X}$ *is PmFN-closed.*

(3) *Every PmFN-subset* $\eth$ *of* $\underline{X}$ *is the intersection of all its PmFN-open supersets.*

Proof. The proof is obvious. 0.1cm□

Remark 4.42. Every subspace of a PmFNT_1S is PmFNT_1S means property of being a PmFNT_1S of any PmFNTS $(\underline{X}, \underline{\beth}_{pn})$ is innate.

Definition 4.43. A PmFNTS $(\underline{X}, \underline{\beth}_{pn})$ is called a *Pythagorean m-polar fuzzy neutrosophic* T_2 *space* (PmFNT_2S), *Pythagorean m-polar fuzzy neutrosophic Hausdorff space* or *Pythagorean m-polar fuzzy neutrosophic separated space* if for any two unique PmFN-points $\eth_1$ & $\eth_2$ of $(\underline{X}, \underline{\beth}_{pn}), \exists$ two PmFN-open sets $\underline{\eth}$ & Υ in such a way $\eth_1 \underline{\in} \underline{\eth}, \eth_2 \underline{\in} \Upsilon$ and $\underline{\eth} \cap \Upsilon = \emptyset$.

Example 4.44. Consider the discrete PmFNTS $(\underline{X}, \underline{\beth}_{pn})$. If $\eth_1$ and $\eth_2$ are two distinct PmFN-points in $\underline{X}$, then clearly $\{\eth_1\}$ and $\{\eth_2\}$ are disjoint PmFN-open sets such that $\eth_1 \underline{\in} \{\eth_1\}$ and $\eth_2 \underline{\in} \{\eth_2\}$. Thus, $(\underline{X}, \underline{\beth}_{pn})$ is a PmFN$\beth_2^{\triangleright}$S.

Theorem 4.45. *A PmFNTS* $(\underline{X}, \underline{\beth}_{pn})$ *is a PmFN*T_2*S iff for any two distinct PmFN-points* $\eth_1$ *and* $\eth_2$*, there are PmFN-closed sets* $\underline{\eth}$ *and* Υ *such that* $\eth_1 \underline{\in} \underline{\eth}, \eth_2 \underline{\notin} \underline{\eth}, \eth_1 \underline{\notin} \Upsilon, \eth_2 \underline{\in} \Upsilon$ *and* $\underline{\eth} \cup \Upsilon = \underline{X}_A$.

Proof. Assume that $(\underline{X}, \underline{\beth}_{pn})$ is a PmFNT_2S and let $\eth_1$ and $\eth_2$ be two distinct PmFN-points of $(\underline{X}, \underline{\beth}_{pn})$. Then, by definition, there must exists two PmFN-open sets $\underline{\eth}$ and Υ such that $\eth_1 \underline{\in} \underline{\eth}, \eth_2 \underline{\notin} \underline{\eth}$ and $\eth_1 \underline{\notin} \Upsilon, \eth_2 \underline{\in} \Upsilon$ and $\underline{\eth} \cap \Upsilon = \underline{\emptyset}$. But then, $\underline{\eth}^c \cup \Upsilon^c = \underline{X}_A$ and $\eth_1 \underline{\notin} \underline{\eth}^c, \eth_2 \underline{\in} \underline{\eth}^c, \eth_1 \underline{\in} \Upsilon^c, \eth_2 \underline{\notin} \Upsilon^c$.
Conversely, assume that for any two distinct PmFN-points $\eth_1, \eth_2 \underline{\in} (\underline{X}, \underline{\beth}_{pn})$, there are P$m$FN-closed sets $\underline{\eth}$ and Υ such that $\eth_1 \underline{\in} \underline{\eth}, \eth_2 \underline{\notin} \underline{\eth}, \eth_1 \underline{\notin} \Upsilon, \eth_2 \underline{\in} \Upsilon$ and $\underline{\eth} \cup \Upsilon = \underline{X}_A$. Then $\underline{\eth}^c$ and Υ^c are PmFN-open sets such that $\eth_1 \underline{\notin} \underline{\eth}^c, \eth_2 \underline{\in} \underline{\eth}^c, \eth_1 \underline{\in} \Upsilon^c, \eth_2 \underline{\notin} \Upsilon^c$ and $\underline{\eth}^c \cap \Upsilon^c = \underline{X}_A{}^c = \underline{\emptyset}$. So, $(\underline{X}, \underline{\beth}_{pn})$ is a PmFNT_2S. 0.1cm□

Remark 4.46. Each PmFN-subspace of a PmFNT_2S is also a PmFNT_2S means property of being a PmFNT_2S of any PmFNTS $(\underline{X}, \underline{\beth}_{pn})$ is innate.

Definition 4.47. A PmFNTS $(\underline{X}, \underline{\beth}_{pn})$ is called a *Pythagorean m-polar fuzzy neutrosophic regular space* (PmFN-regular space) if unspecified PmFN-closed set $\underline{\eth}$ & any PmFN-point $\eth_1 \underline{\notin} \underline{\eth}$ and here PmFN-open sets Υ & $\underline{\Upsilon}^*$ such that $\eth_1 \in \Upsilon, \underline{\eth} \subseteq \underline{\Upsilon}^*$ and $\Upsilon \cap \underline{\Upsilon}^* = \underline{\emptyset}$.

Definition 4.48. A PmFNTS $(\underline{X}, \underline{\beth}_{pn})$ is called *Pythagorean m-polar fuzzy neutrosophic* T_3 *space* (PmFNT_3S) if it is a PmFN regular T_1 space.

Definition 4.49. A PmFNTS $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ is called *Pythagorean m-polar fuzzy neutrosophic normal space* if unspecified two PmFN-closed disjoint subsets $\underset{\sim}{\eth}$ & Υ of $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ and here PmFN-open sets $\underset{\sim}{\Upsilon}^*$ and $\underset{\sim}{\Upsilon}^\bullet$ such that $\underset{\sim}{\eth} \subseteq \underset{\sim}{\Upsilon}^*, \Upsilon \subseteq \underset{\sim}{\Upsilon}^\bullet$ and $\underset{\sim}{\Upsilon}^* \Cap \underset{\sim}{\Upsilon}^\bullet = \underset{\sim}{\emptyset}$. A P$m$FN-normal T_1 space is called a *Pythagorean m-polar fuzzy neutrosophic T_4 space* (PmFNT_4S).

Remark 4.50. We have the following chain for different PmFNTSs studied above:

$$T_e \supseteq T_{e+1}$$

for $0 \leq e \leq 3$. The reverse chain, however, may not hold. The forthcoming Example 4.51 supports our claim.

Example 4.51. Let $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ be a PmFNTS, where $\underset{\sim}{X} = \{\hbar_1, \hbar_2\}, \underset{\sim}{\daleth}_{pn} = \{\underset{\sim}{\emptyset}, \underset{\sim}{B}, \underset{\sim}{X}_A\}$. Then

TABLE 31. P3FNS $\underset{\sim}{B}$

$\underset{\sim}{B}$			
$\hbar_1$	(0.000, 0.423, 0.801)	(0.167, 0.210, 0.562)	(0.472, 0.421, 0.301)
$\hbar_2$	(0.162, 0.423, 0.004)	(0.000, 0.409, 0.210)	(0.100, 0.432, 0.720)

is a P3FNT_0S but it is not a P3FNT_1S.

Theorem 4.52. *Each PmFNT_4S is a PmFN regular means each PmFN normal T_1 space is PmFN regular.*

Proof. Let $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ be a PmFNT_4S. Let $\eth_1$ be a PmFN-point in $\underset{\sim}{X}$. Then, by Theorem 4.41, $\{\eth_1\}$ is a closed PmFNS in $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$. Suppose that $\underset{\sim}{\eth}$ be a PmFN-closed set not containing $\eth_1$. Since $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ is PmFN normal, there are PmFN-open set namely $\Upsilon, \underset{\sim}{\Upsilon}^*$ such that $\{\eth_1\} \subseteq \Upsilon, \underset{\sim}{\eth} \subseteq \underset{\sim}{\Upsilon}^*$ and $\Upsilon \Cap \underset{\sim}{\Upsilon}^* = \underset{\sim}{\emptyset}$. But then, $\{\eth_1\} \Subset \Upsilon, \underset{\sim}{\eth} \subseteq \underset{\sim}{\Upsilon}^*$ and $\Upsilon \Cap \underset{\sim}{\Upsilon}^* \underset{\sim}{\emptyset}$. So, $(\underset{\sim}{X}, \underset{\sim}{\daleth}_{pn})$ is a Pythagorean m-polar fuzzy neutrosophic regular topological space. 0.1cm□

5. Intelligent Decision Making using PmFNS TOPSIS

In this section, we present an application of PmFNS in decision making.

Case Study:

A desert is a desolate region of land with hardly any rainfall and, as a result, unhealthy living conditions for flora and fauna. The absence of habitat reveals the ground's vulnerable surface to geomorphic activities. Around 33% of the world's land surface is sandy or semi-arid. The piece of land that attains fewer than 25 cm of rainfall per annum is considered a desert. Deserts are part of a broader class of regions named dry lands. Pakistan has five significant deserts comprising Cholistan, Katpana, Thar, Thal and Kharan deserts.

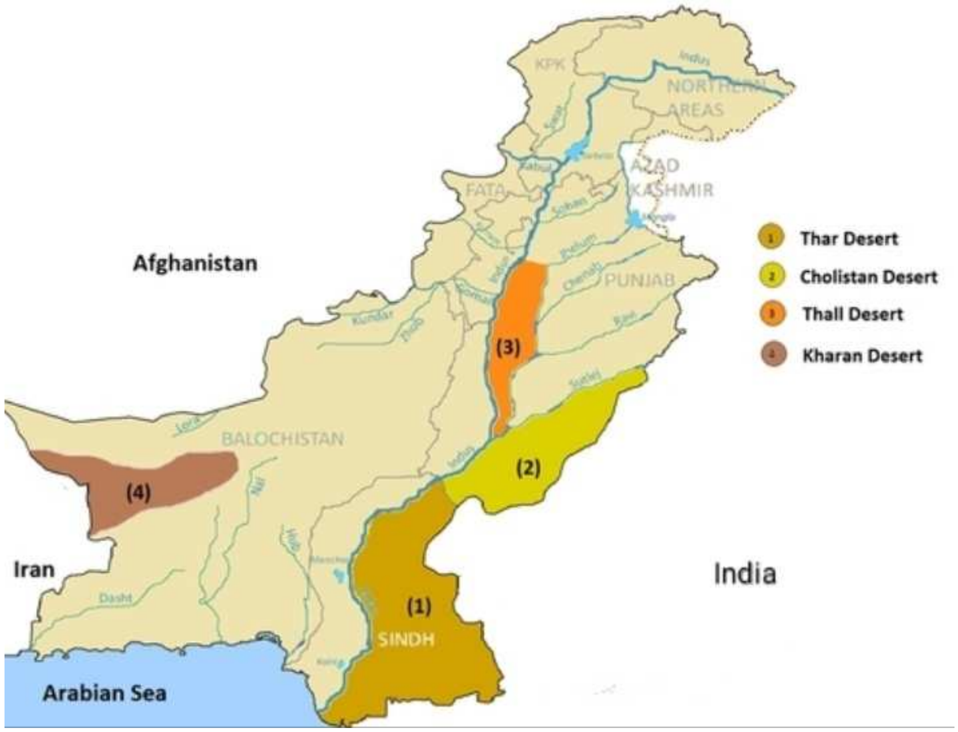

FIGURE 1. Deserts of Pakistan

About 85% of the Thar desert, also called the Great Indian Desert, is situated inside India, with the excess 15% in Pakistan. It covers around 170,000 km^2, and the leftover 30,000 km^2 of the desert is inside Pakistan. Thar desert is the world's seventeenth biggest desert, and the world's ninth biggest subtropical desert. During different periods of predominant breeze is the dry northeast storm. May and June are the most sweltering a long time of the year, with mercury ascending to 50^0 C. In January, considered to be the coldest month there, the average minimum temperature drops down to 10^0 C, and frost is frequent. Dust storms and dust-raising winds, often blow with a speed of 140 to 150 km per hour, are frequent in the months of May and June. The amount of annual rainfall in the desert is generally low, ranging from about less in the west to about 20 inches (500 mm) in the east or 4 inches (100 mm), mostly decreasing from July to September.

The desert of Kharan is situated in Balochistan. It makes a nature limit among Pakistan, Iran and Afghanistan. It is situated in Kharan region. The Kharan desert is a sandiest desert in Pakistan. It is particular from the remainder of the province's landscape because of its sandy nature and all the more even ter. The desert was utilized for atomic testing by the Pakistan military, making it the most renowned of the five deserts. In altitude these central deserts slope from about 1,000 m in the north to about 250 m on in the southwest. Maximum, average and minimum temperatures of kharan desert are 42^0 C, 38^0 C and 26^0 C respectively. Average annual rainfall throughout these deserts is well under 100 mm. The desert includes areas of inland drainage and dry lakes.

The Cold Desert, otherwise called the Katpana Desert or Biama Nakpo, is a high-elevation desert situated close Skardu, northern Gilgit-Baltistan area of Pakistan controlled Jammu and Kashmir. The desert contains costs of huge sand rises that are once in a while shrouded in snow during winter. Situated at an elevation of 2,226 m (7,303 feet) above ocean level, the Katpana Desert is one of the most noteworthy deserts in the world. The desert actually extends from the Khaplu Valley to Nubra in Ladakh, yet the biggest desert area is found in Skardu and Shigar Valley. The part most visited is situated close Skardu Airport. Temperatures range from a maximum of 27^0 C and a minimum (in October) 8^0 C which can drop further to beneath -17^0 C in December and January. The temperature infrequently drops as low as -25^0 C.

The Thal Desert is situated in Bhakkar area of Pakistan between the Indus and Jhelum rivers. A huge canal-building venture is in progress to flood the land. Water system will make a large portion of the desert appropriate for cultivating. In the north of the Thal Desert there are salt reaches, in the east the Jhelum and Chenab streams and toward the west the Indus waterway. The maximum temperature is 34^0 C and minimum temperature is 25^0 C in Thal desert. The average annual temperature for Thal is 29^0 C. It is dry for 207 days a year with an average humidity of 36%. The average annual rainfall varies from 385 mm in the north-east to 170 mm in the south. Approximately three-fourth of annual rainfall is received during monsoon.

Cholistan Desert is locally known as Rohi. It abuts the Thar Desert, stretching out over to Sindh and into India. Cholistan desert hosts an yearly Jeep rally, known as Cholistan Desert Jeep Rally which is the greatest engine game in Pakistan. Cholistan's atmosphere is described as a bone-dry and semi-dry Tropical desert, with exceptionally low yearly dampness. The mean temperature in Cholistan is 28.33^0 C, with most smoking month being July with a mean temperature of 38.5^0 C. Summer temperatures can outperform 46^0 C and now and then ascents more than 50^0 C during times of dry season. Winter temperatures infrequently dip to 0^0 C. Normal precipitation in Cholistan is up to 180mm, with July and August being the wettest months, despite the fact that dry seasons are normal. Water is gathered occasionally in an arrangement of normal pools called Toba, or man made pools called Kund. Earth water is found at a profundity of 30-40 meters, yet is commonly bitter, and unacceptable for most plant development.

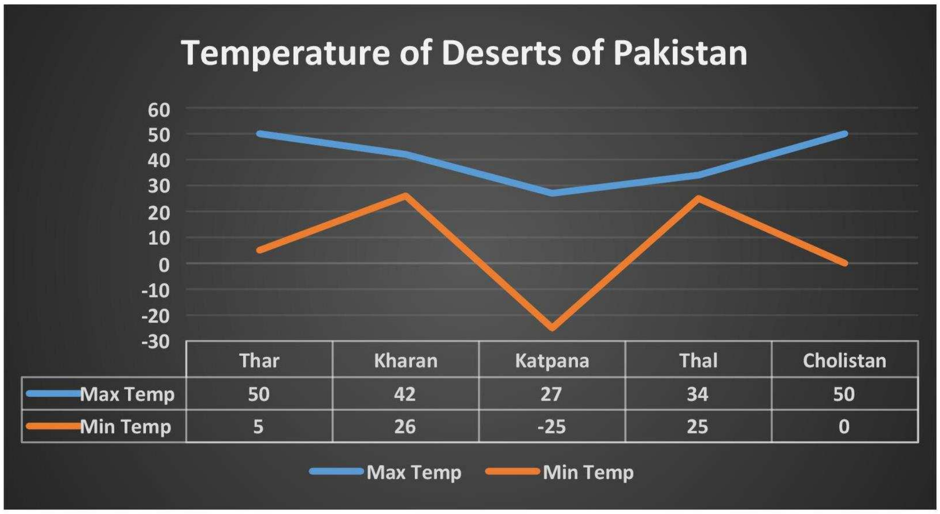

Figure 2. Temperature of Deserts of Pakistan

These deserts contains an extremely dry part, the Marusthali area in the west, and a semi desert locale in the east with less sand hills and somewhat more precipitation. For the most vital problem and the main hinderance, in the way of progress. Government considers that issue of lack of water in desert has to solve as early as possible.

The basic need-water, has greatly affected the lives of residents of desert. It can be said that water has not only changed their social life style but also economy has affected badly. Inadequate sanitary conditions have invited many diseases which can be said epidemic like cholera, typhoid etc. These disaster ruin the human race as well as their cattle.

Cultivation also wiped away due to scarcity of water. Indirectly water is the primary source of food also people face the horrible face of famine. Specially children, represent the reflection of poor humanity. Their body, without any health, you may say their skeletons cry for help or for water.

Scarcity of water has also a deep impact on the psyche of residents of desert. Their temperament, attitudes and behaviors indirectly affected by this vital problem. Tolerance, courtesy, desire, for progress, achievements, dreams and all ways leading to bright future are cover in mist. They cannot see or even have the eagerness for better living style. Their struggle only moves around the availability of water. So it is the need of the time that all the possible steps should be taken at all levels for the sake of humanity.

A city named Nagarparkar in Thar is consist upon 1 lac population people use under ground clean and clear water for the necessities of life but it is very hard to get it in summer.

In summary the level of underground water decreases at the lowest level and to get water becomes impossible by hand pump. For the last many years no proper planning has been made to provide water. In city water is brought far from areas. In this age of dearness to getting

water is difficult. The fare of a cane is 20 to 25 rupees. The people are compelled to drink that kind of water which is jot acceptable to the animals of Lahore. Animals and human drink water from the same place there is no distinguish of camel, goat and the king of all races.
It is a hot issue, so a commission has established in which all the concerning problem experts were included. This commission visited the desert and collected all eye bared witness.
First of all they prepare a report in which they point out the problems facing towards water supply.

Poor decision making: The commission strongly condemned that decision making policies are not harmonized to the circumstances.

Economically costs: In Thar with boring a place of water is served 8 to 9 villages approximately water is available to 7 km distance. Government do not take solid steps only visits are arranged and due to lack of budgets, no attention is given for this reason people are deprived of water. It has also observed that which projects had passes in past they were very costly. Government could not afford them.

Environmental and social problem: Desert environment needs something special which can appropriate to its hottest environment and social settlement.

Encouragement of local persons: A reason which is also very important is that people do not have much facilities that they can bore or drill the land and can make it easily to get water because they are illiterate and cannot drive correct solution by correct strategy. It is also necessary to take help from the local persons and encouraged them to solve this problem with the help of government.

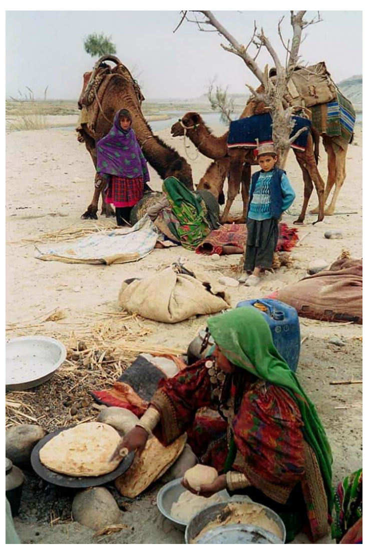

FIGURE 3. Environmental and social problem

For all these issues, they suggested some positive and skilled opinions.

(i) Government should take solid decisions. And the motto of these decisions should be welfare and progress because if the start is good then the end will be best.

(ii) Those projects should be of low cost and much beneficial.

(iii) It should be keep in mind that the trust of local persons is very necessary for their welfare because the negativity of being ignore has been kept its place in their minds.

(iv) Government should start small projects as they would be called tribal units or tribal beneficiary projects.

FIGURE 4. Lack of water

We clarify the procedure bit by bit as follows:

Algorithm:

Stage 1: Firstly analyze the issue to see that what we have and actually what we have need to do: Suppose that $R = \{\sigma_i : i = 1, 2, \cdots, n\}$ is the finite aggregate of alternatives under consideration and $G = \{g_j : j = 1, 2, \cdots, m\}$ is the family of captains. So the $(i, j)^{th}$ entry of the PmFNS matrix represents weight given by j^{th} Captains to i^{th} options.

Stage 2: Develop weighted parameter matrix P as

$$P = [w_{ij}]_{n\times m} = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1m} \\ w_{21} & w_{22} & \cdots & w_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & \cdots & w_{im} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nm} \end{pmatrix}$$

where w_{ij} is the fuzzy weight given by the Captains g_j to the options σ_i by thinking about the phonetic entitle are given (for example) in Table 32.

Table 32. Phonetic terms for benefits of projects

Phonetic Terms	Fuzzy Weights
Not fruitful (NF)	[0.00, 0.25]
Fruitful (F)	(0.25, 0.50]
More or less fruitful (MF)	(0.50, 0.75]
Extremely fruitful (EF)	(0.75, 1.00]

Stage 3: Develop normalized weighted matrix

$$N = [\hat{w}_{ij}]_{n\times m} = \begin{pmatrix} \hat{w}_{11} & \hat{w}_{12} & \cdots & \hat{w}_{1m} \\ \hat{w}_{21} & \hat{w}_{22} & \cdots & \hat{w}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{w}_{i1} & \hat{w}_{i2} & \cdots & \hat{w}_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{w}_{n1} & \hat{w}_{n2} & \cdots & \hat{w}_{nm} \end{pmatrix}$$

where $\hat{w}_{ij} = \frac{w_{ij}}{\sqrt{\sum_{i=1}^{n} w_{ij}^2}}$ and obtaining the weighted vector $W = (w_j : j = 1, 2, \cdots, m)$, where $w_j = \frac{\sum_{i=1}^{n} \hat{w}_{ij}}{n\sum_{k=1}^{m} \hat{w}_{ik}}$

Step 4: Develop PmFNS decision matrix $G_i = [\varsigma^i_{jk}]_{n\times m}$, where $\varsigma^i_{jk} = (\tau^i_{jk}, \upsilon^i_{jk}, \omega^i_{jk})$. Then obtain the mean proportional matrix

$$X = \sqrt[n]{G_1 G_2 \cdots G_n} = [\dot{\varsigma}_{jk}]_{n\times m} = \left[\left(\sqrt[n]{\prod_{i=1}^{n} \tau^i_{jk}}, \sqrt[n]{\prod_{i=1}^{n} \upsilon^i_{jk}}, \sqrt[n]{\prod_{i=1}^{n} \omega^i_{jk}}\right)\right]_{n\times m}$$

Stage 5: Compute weighted PmFNS decision matrix $Y = [\ddot{\varsigma}_{jk}]_{n\times m}$, where $\ddot{\varsigma}_{jk} = w_k \times \dot{\varsigma}_{jk} = (\tau_{jk}, \upsilon_{jk}, \omega_{jk})$.

Stage 6: Get PmFNSV-PIS (PmFNS- valued positive ideal solution) and PmFNSV-NIS (PmFNS- valued negative ideal solution), by using

$$\begin{aligned} PmFNS - PIS &= \{\ddot{\varsigma}_1^+, \ddot{\varsigma}_2^+, \cdots, \ddot{\varsigma}_m^+\} \\ &= \{(\max_k \tau_{jk}, \min_k \upsilon_{jk}, \min_k \omega_{jk}) : k = 1, 2, \cdots, m\} \\ &= \{(\tau_k^+, \upsilon_k^+, \omega_k^+) : k = 1, 2, \cdots, m\} \end{aligned}$$

and

$$\begin{aligned} PmFNS - NIS &= \{\ddot{\varsigma}_1^-, \ddot{\varsigma}_2^-, \cdots, \ddot{\varsigma}_m^-\} \\ &= \{(\min_k \tau_{jk}, \max_k \upsilon_{jk}, \max_k \omega_{jk}) : k = 1, 2, \cdots, m\} \\ &= \{(\tau_k^-, \upsilon_k^-, \omega_k^-) : k = 1, 2, \cdots, m\} \end{aligned}$$

respectively.

Stage 7: Find PmFNS-Euclidean separations of every other option from PmFNS-PIS and PmFNS-NIS respectively, by making use of

$$g_j^+ = \sqrt{\sum_{k=1}^{m} (\tau_{jk} - \tau_k^+)^2 + (\upsilon_{jk} - \upsilon_k^+)^2 + (\omega_{jk} - \omega_k^+)^2}$$

$$g_j^- = \sqrt{\sum_{k=1}^{m} (\tau_{jk} - \tau_k^-)^2 + (\upsilon_{jk} - \upsilon_k^-)^2 + (\omega_{jk} - \omega_k^-)^2}$$

for $j = 1, 2, \cdots, n$.

Step 8: Compute the relative closeness using

$$C_j^* = \frac{g_j^-}{g_j^+ + g_j^-}$$

Stage 9: So as to get the inclination request of the other options, rank the options in descending (or ascending) order.

The procedural steps of above Algorithm are portrayed in Figure 5:

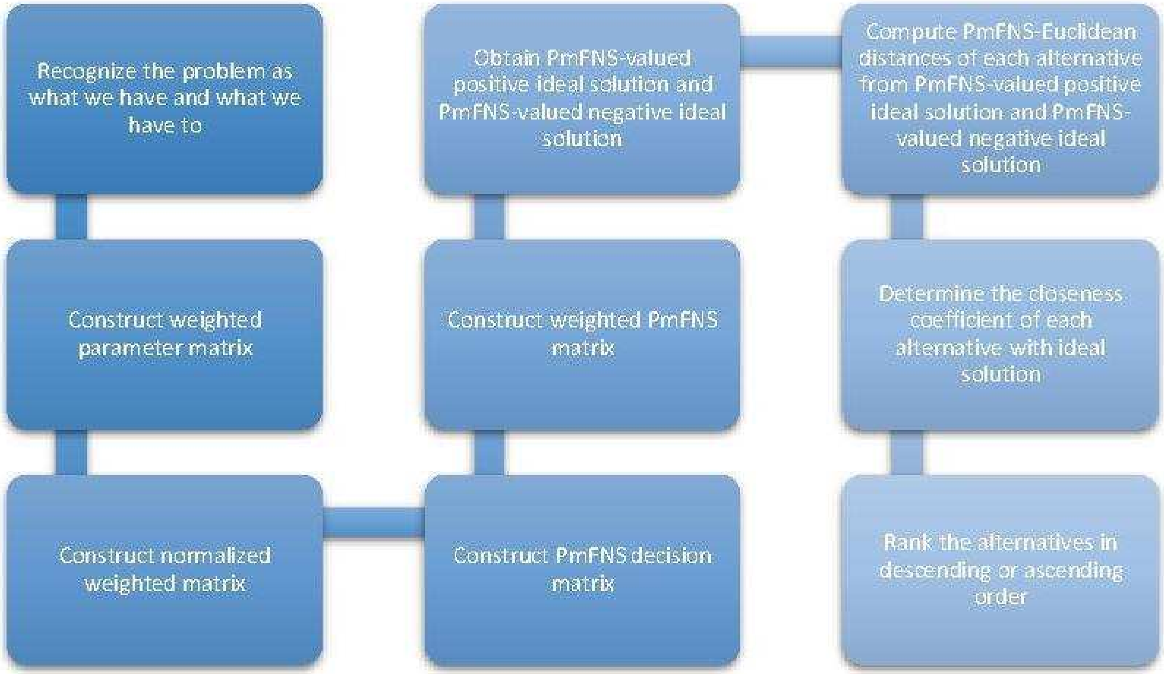

FIGURE 5. Flow chart of Algorithm

Example 5.1. Assume that experts wishes to determine the most vital problems and the main hinderance facing by desert. The experts establish a committee of four members.

Stage 1: Analyze the problem: Assume that $\mathrm{R} = \{\sigma_i : i = 1, 2, \cdots, 4\}$ is the set of choices viable and $\mathrm{G} = \{g_j : j = 1, 2, 3, 4\}$ is the family of experts, where

$$\begin{aligned}\sigma_1 &= \text{Poor decision making,}\\ \sigma_2 &= \text{Economic costs,}\\ \sigma_3 &= \text{Environmental and social problem, and}\\ \sigma_4 &= \text{Encouragement of local persons.}\end{aligned}$$

Stage 2: The weighted parameter matrix, by selecting phonetic terms from Table 32, is

$$\begin{aligned} P &= [w_{ij}]_{4\times 4} \\ &= \begin{pmatrix} \text{F} & \text{NF} & \text{MF} & \text{EF} \\ \text{NF} & \text{F} & \text{EF} & \text{MF} \\ \text{MF} & \text{NF} & \text{F} & \text{EF} \\ \text{EF} & \text{MF} & \text{NF} & \text{F} \end{pmatrix} \\ &= \begin{pmatrix} 0.50 & 0.25 & 0.75 & 1.00 \\ 0.25 & 0.50 & 1.00 & 0.75 \\ 0.75 & 0.25 & 0.50 & 1.00 \\ 1.00 & 0.75 & 0.25 & 0.50 \end{pmatrix} \end{aligned}$$

Where w_{ij} is the weight given by the decision maker g_j to the choices σ_i.

Stage 3: The normalized weighted matrix is

$$\begin{aligned} N &= [\hat{w}_{ij}]_{4\times 4} \\ &= \begin{pmatrix} 0.37 & 0.26 & 0.55 & 0.60 \\ 0.18 & 0.52 & 0.73 & 0.45 \\ 0.55 & 0.26 & 0.37 & 0.60 \\ 0.73 & 0.77 & 0.18 & 0.30 \end{pmatrix} \end{aligned}$$

and thus the weight vector is

$$W = (0.25, 0.24, 0.25, 0.26)$$

Stage 4: Suppose that the four experts give the following PmFNS matrix in which the $(i, j)^{th}$ elements shows the PFN (τ, υ, ω), where choices are showed by row-wise and the PFN assigned by experts are showed by column-wise.

$$G_1 = \begin{pmatrix} (0.61, 0.22, 0.39) & (0.73, 0.52, 0.11) & (0.66, 0.42, 0.33) & (0.36, 0.15, 0.49) \\ (0.38, 0.17, 0.50) & (0.48, 0.29, 0.30) & (0.61, 0.00, 0.18) & (0.46, 0.24, 0.17) \\ (0.54, 0.29, 0.32) & (0.46, 0.35, 0.45) & (0.24, 0.18, 0.59) & (0.78, 0.55, 0.12) \\ (0.08, 0.37, 0.88) & (1.00, 0.00, 0.00) & (0.34, 0.63, 0.35) & (0.69, 0.13, 0.04) \end{pmatrix}$$

$$G_2 = \begin{pmatrix} (0.52, 0.19, 0.22) & (0.39, 0.52, 0.35) & (0.43, 0.61, 0.50) & (0.66, 0.57, 0.14) \\ (0.43, 0.54, 0.29) & (0.48, 0.25, 0.40) & (0.76, 0.10, 0.22) & (0.45, 0.53, 0.41) \\ (0.24, 0.26, 0.30) & (0.37, 0.06, 0.19) & (0.00, 0.48, 0.71) & (0.33, 0.41, 0.28) \\ (0.36, 0.17, 0.29) & (0.62, 0.28, 0.00) & (0.05, 0.18, 0.77) & (0.23, 0.64, 0.59) \end{pmatrix}$$

$$G_3 = \begin{pmatrix} (0.54, 0.58, 0.38) & (1.00, 0.00, 0.00) & (0.52, 0.44, 0.39) & (0.23, 0.10, 0.11) \\ (0.30, 0.59, 0.20) & (0.52, 0.22, 0.33) & (0.13, 0.14, 0.04) & (0.51, 0.06, 0.44) \\ (0.41, 0.28, 0.51) & (0.29, 0.64, 0.39) & (0.78, 0.02, 0.16) & (0.31, 0.13, 0.64) \\ (0.57, 0.55, 0.37) & (0.36, 0.88, 0.14) & (0.40, 0.00, 0.53) & (0.05, 0.27, 0.77) \end{pmatrix}$$

$$G_4 = \begin{pmatrix} (0.37, 0.55, 0.30) & (0.43, 0.58, 0.19) & (0.35, 0.28, 0.44) & (0.59, 0.56, 0.17) \\ (0.35, 0.73, 0.12) & (0.41, 0.27, 0.39) & (0.67, 0.37, 0.21) & (0.64, 0.16, 0.20) \\ (0.00, 0.28, 0.72) & (0.58, 0.06, 0.41) & (0.40, 0.51, 0.31) & (0.35, 0.10, 0.57) \\ (0.47, 0.40, 0.26) & (0.44, 0.51, 0.38) & (0.44, 0.64, 0.26) & (0.28, 0.31, 0.60) \end{pmatrix}$$

Thus, the mean proportional matrix X is

$$\begin{aligned} X &= [\dot{\varsigma}_{jk}]_{4\times 4} \\ &= \begin{pmatrix} (0.50, 0.34, 0.31) & (0.59, 0.00, 0.00) & (0.48, 0.42, 0.41) & (0.42, 0.26, 0.19) \\ (0.36, 0.45, 0.24) & (0.47, 0.26, 0.35) & (0.45, 0.00, 0.14) & (0.51, 0.19, 0.28) \\ (0.00, 0.28, 0.43) & (0.41, 0.17, 0.34) & (0.00, 0.17, 0.38) & (0.41, 0.23, 0.33) \\ (0.30, 0.34, 0.40) & (0.56, 0.00, 0.00) & (0.23, 0.00, 0.44) & (0.22, 0.29, 0.32) \end{pmatrix} \end{aligned}$$

where $\ddot{\varsigma}_{jk} = w_k \times \dot{\varsigma}_{jk}$

Stage 5: The weighted PmFN matrix is

$$\begin{aligned} Y &= [\ddot{\varsigma}_{jk}]_{4\times 4} \\ &= \begin{pmatrix} (0.13, 0.09, 0.08) & (0.14, 0.00, 0.00) & (0.12, 0.11, 0.10) & (0.11, 0.07, 0.05) \\ (0.09, 0.11, 0.06) & (0.11, 0.06, 0.08) & (0.11, 0.00, 0.04) & (0.13, 0.05, 0.07) \\ (0.00, 0.07, 0.11) & (0.10, 0.04, 0.08) & (0.00, 0.04, 0.10) & (0.11, 0.06, 0.09) \\ (0.08, 0.09, 0.10) & (0.13, 0.00, 0.00) & (0.06, 0.00, 0.11) & (0.06, 0.08, 0.08) \end{pmatrix} \end{aligned}$$

Stage 6: Thus, PmFNS-PIS and PmFNS-NIS, are respectively

PmFNSV-PIS $= \{\ddot{\varsigma}_1^+, \cdots, \ddot{\varsigma}_4^+\}$

$= \{(0.13, 0.07, 0.06), (0.14, 0.00, 0.00), (0.12, 0.00, 0.04), (0.13, 0.05, 0.05)\}$

and

PmFNSV-NIS $= \{\ddot{\varsigma}_1^-, \cdots, \ddot{\varsigma}_4^-\}$

$= \{(0.00, 0.11, 0.11), (0.10, 0.06, 0.08), (0.00, 0.11, 0.11), (0.06, 0.08, 0.09)\}$

Stage 7 and 8: The Euclidean separation of every issue from PmFNS-PIS and PmFNS-NIS and corresponding relative coefficients of closeness are given in Table 33:

Table 33. Separation and coefficient of closeness of each issue

Issue ($\ddot{\varsigma}_i$)	g_i^+	g_i^-	C_i^*
$\ddot{\varsigma}_1$	0.13	0.22	0.63
$\ddot{\varsigma}_2$	0.12	0.78	0.87
$\ddot{\varsigma}_3$	0.23	0.10	0.30
$\ddot{\varsigma}_4$	0.14	0.18	0.56

Stage 9: Thus, the preference ranking of the issues is

$$\ddot{\varsigma}_2 \succ \ddot{\varsigma}_1 \succ \ddot{\varsigma}_4 \succ \ddot{\varsigma}_3$$

This ranking is portrayed in Figure 6:

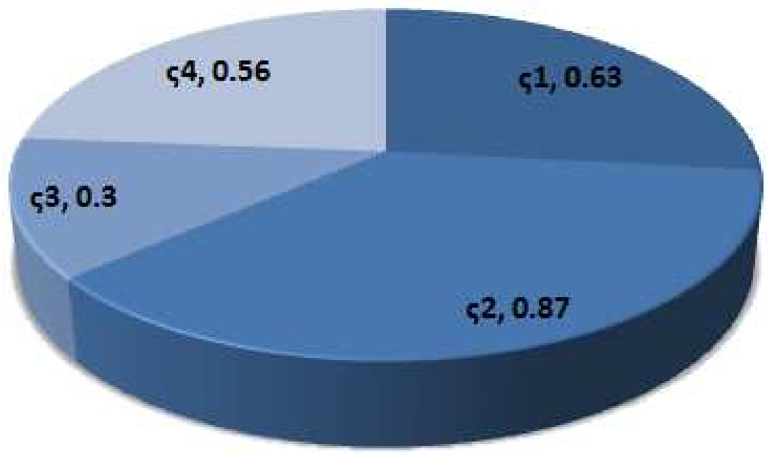

FIGURE 6. Ranking of alternatives

Hence, in view of above ranking, it may be concluded that poor decision making is the core issue.

6. Conclusion

We reviewed fuzzy set theory along with its tabular illustration and examples briefly. We established the axiomatic definitions of Pythagorean m-polar fuzzy neutrosophic set. We presented some fundamental properties of Pythagorean m-polar fuzzy neutrosophic topological space (PmFNTS) by numerous characteristics of crisp topology on the way to the PmFNTS. We defined Pythagorean m-polar fuzzy separation axioms. T_0, T_1, T_2, T_3 and T_4 spaces are modified in the aspects of PmFNS.

We presented example of decision making from real world situations based on TOPSIS accompanied by case study. We presented algorithm and flowcharts of method for comfort. We also showed 3D bar chart with application to make the contrast between different alternatives effectively.

These above mentioned concepts can be used in several real world difficulties such as in economics, business, robotics, medical sciences, water management, electoral systems, transportation problems and much more. We hope that this paper will gives new ideas to the researchers to promote research work in this field.

The notions presented in this article may be extended to define other sorts of topological structures like nano topology and pentapartitioned topology etc.

Conflicts of Interest: The authors declare that there is no conflict of interests.

Authors' Contributions: The authors contributed to each part of this paper equally.

References

1. Arockiarani, I.; Sumathi I.R.; Jency, J.M. Fuzzy neutrosophic soft topological spaces. Int J of Math Arch (2013), **4**(10), 225-238.
2. Atanassov, K.T. Intuitionistic Fuzzy sets. Fuzzy Set Syst (1986), 20, 87-96.
3. Atanassov, K. T. More on Intuitionistic Fuzzy sets. Fuzzy Set Syst (1989), 33, 37-46.
4. Chen, J.; Li, S.; Ma, S.; Wang, X. m-polar fuzzy sets: An extension of bipolar fuzzy sets. The Scientific World Journal (2014), 1-8.
5. Çoker, D. An introduction to intuitionistic fuzzy topological spaces. Fuzzy Set Syst (1997) 88(1), 81-89, http://dx.doi.org/10.1155/2014/416530.
6. Jansi, R.; Mohana, K. Pairwise Pythagorean Neutrosophic P-spaces (with dependent neutrosophic components between T and F). Neutrosohic Sets and Systems 2021, 41, 246-257.
7. Lee, S.J.; Lee, E.P. The Category of Intituionistic Fuzzy Topological Spaces. Bulletin of Korean Mathematical Society, (2000), 37(1), 63-76.
8. Naeem, K.; Riaz, M.; Afzal, D. Pythagorean m-polar fuzzy sets and TOPSIS method for the selection of advertisement mode. J. Intell Fuzzy Syst (2019), 37(6) 8441-8458. DOI: 10.3233/JIFS-191087.
9. Olgun, M.; $\ddot{U}$nver, M.; Yardimci, S. Pythagorean fuzzy topological spaces. Complex and Intelligent Systems (2019) 5(2), https://doi.org/10.1007/s40747-019-0095-2
10. Pao-Ming, P.; Ying-Ming, L. Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. Journal of Mathematical Analysis and Applications (1980), 76, 571-599.
11. Riaz, M.; Naeem, K.; Afzal, D. Pythagorean m-polar fuzzy soft sets with TOPSIS method for MCGDM. Punjab University Journal of Mathematics (2020), 52(3), 21-46.
12. Smarandache, F. Neutrosophy: neutrosophic probability, set, and logic. ProQuest Information and Learning. Ann Arbor, Michigan, USA, 105, (1998), 118-123.
13. Smarandache, F. Neutrosophic set: a generalisation of the intuitionistic fuzzy sets. International Journal of Pure and Applied Mathematics, (2005), 24, 287-297.
14. Smarandache, F. Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets - Revisited. Neutrosophic Sets and Systems (2018), 21, 153-166.
15. Yager, R.R. Pythagorean fuzzy subsets. IFSA World Congress and NAFIPS Annual Meeting, 2013 Joint, Edmonton, Canada, IEEE (2013), 57-61. DOI:10.1109/ifsa-nafips.2013.6608375.
16. Yager, R.R.; Abbasov, A.M. Pythagorean membership grades, complex numbers, and decision making. International Journal of Intelligent Systems (2013), 28(5), 436-452.
17. Yager, R.R. Pythagorean membership grades in multi-criteria decision making. IEEE Transactions on Fuzzy Systems, **22**(4)(2014), 958-965.
18. Zadeh, L.A. Fuzzy sets. Information and Control (1965), 8, 338-353.

Received: Nov 22, 2021. Accepted: Feb 2, 2022