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On Refined Neutrosophic Hyperrings

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Abstract. This paper presents the refinement of a type of neutrosophic hyperring in which $+'$ and $\cdot'$ are hyperopraetions and studied some of its properties. Several interesting results and examples are presented.
Keywords: .

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1. Introduction

In a general sense the triple $(R, +, \cdot)$ is an hyperring if the hyperoperations $+$ and $\cdot$ are such that $(R, +)$ is a hypergroup, $(R, \cdot)$ is semihypergroup and $\cdot$ is distributive with respect to $+$. These structures are essentially rings with approximately modified axioms. Different notions of hyperrings have been investigated by researchers in the field of algebraic hyperstructures. For example, Krasner in [20] introduced a type of hyperring in which $''+''$ is an hyperoperation and $''\cdot''$ is a binary operation. This type of hyperring is referred to as a Krasner hyperring. In [24] a type of hyperring called multiplicatve hyperring was introduced by Rota. In this hyperring $''+''$ is considered as an ordinary addition and $''\cdot''$ as an hyperoperation. The type of hyperring in which $''+''$ and $''\cdot''$ were hyperoperations was studied by De Salvo in [14]. These classes of hyperrings were further studied by Barghi [12] , Asokkumar and Velrajan [9–11] .

In 1995, Smarandache generalized fuzzy logic/set and intuitionistic fuzzy logic/set by introducing a new branch of philosophy called Neutrosophy, which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I) and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]-0,1+[$ as can be seen in [22,23] . Ever since the introduction of this theory, several neutrosophic structures have been introduced, some of which includes; neutrosophic group, neutrosophic rings, neutrosophic modules, neutrosophic hypergroups, neutrosophic hyperrings, neutrosophic loops and many more. Smarandache in [22] introduced the concept of refined neutrosophic logic and neutrosophic set which is basically the splitting of the components $< T, I, F >$ into subcomponents of the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdots, I_r; F_1, F_2, \cdots, F_s >$. This concept inspired the work of Agboola in [5] where he introduced refined neutrosophic algebraic structures. A lot of results have been published on the refinement of some of the known neutrosophic algebraic structures/hyperstructures ever since the work of Agboola. A comprehensive review of refined neutrosophic structures/hyperstrutures, can be found in [1,2,8,15–19] .

In this paper, the refinement of neutrosophic hyperring is studied and several interesting results and examples are presented.

2. Preliminaries

In this section, we will give some definitions, examples and results that will be used in the sequel.

Definition 2.1. [13] Let H be a non-empty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 2.2. [13] Let H be a non-empty set and let $+$ be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

(1) $x + y = y + x$, for all $x, y \in H$,
(2) $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,
(3) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,
(4) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
(5) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$. A nonempty subset A of H is called a subcanonical hypergroup if A is a canonical hypergroup under the same hyperaddition as that of H that is, for every $a, b \in A$, $a - b \in A$. If in addition $a + A - a \subseteq A$ for all $a \in H$, A is said to be normal.

Definition 2.3. A hyperring is a triple $(R, +, \cdot)$ satisfying the following axioms:

(1) $(R, +)$ is a canonical hypergroup.
(2) $(R, \cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, that is, 0 is a bilaterally absorbing element,
(3) For all $x, y, z \in R$,
   (a) $x \cdot (y + z) = x \cdot y + x \cdot z$ and

(b) $(x+y)\cdot z = x\cdot z + y\cdot z$. That is, the hyperoperation . is distributive over the hyperoperation $+$.

Definition 2.4. Let $(R, +, \cdot)$ be a hyperring and let A be a nonempty subset of R. A is said to be a subhyperring of R if $(A, +, \cdot)$ is itself a hyperring.

Definition 2.5. Let A be a subhyperring of a hyperring R. Then,

(1) A is called a left hyperideal of R if $r \cdot a \subseteq A$ for all $r \in R, a \in A$.
(2) A is called a right hyperideal of R if $a \cdot r \subseteq A$ for all $r \in R, a \in A$. A is called a hyperideal of R if A is both left and right hyperideal of R.

Definition 2.6. Let A be a hyperideal of a hyperring R. A is said to be normal in R if $r + A - r \subseteq A$ for all $r \in R$.

It will be assumed that I splits into two sub-indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. With the properties that:

$$\begin{aligned} I_1I_1 &= I_1^2 &= I_1, \\ I_2I_2 &= I_2^2 &= I_2 \text{ and} \\ I_1I_2 &= I_2I_1 &= I_1. \end{aligned}$$

Definition 2.7. [4] If $* : X(I_1, I_2) \times X(I_1, I_2) \mapsto X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2), *)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

Definition 2.8. [4] Let $(X(I_1, I_2), +, \cdot)$ be any refined neutrosophic algebraic structure where $+$ and . are ordinary addition and multiplication respectively.
For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$

$$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2).$$

Definition 2.9. [4] If $+$ and . are ordinary addition and multiplication, I_k with $k = 1, 2$ have the following properties:

(1) $I_k + I_k + \cdots + I_k = nI_k$.
(2) $I_k + (-I_k) = 0$.
(3) $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
(4) $0 \cdot I_k = 0$.
(5) I_k^{-1} is undefined and therefore does not exist.

Definition 2.10. [4] Let $(G, *)$ be any group. The couple $(G(I_1, I_2), *)$ is called a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non -commutative refined neutrosophic group.

Definition 2.11. [4] If $(X(I_1, I_2), *)$ and $(Y(I_1, I_2), *')$ are two refined neutrosophic algebraic structures, the mapping

$$\phi : (X(I_1, I_2), *) \longrightarrow (Y(I_1, I_2), *')$$

is called a neutrosophic homomorphism if the following conditions hold:

(1) $\phi((a, bI_1, cI_2) * (d, eI_1, fI_2)) = \phi((a, bI_1, cI_2)) *' \phi((d, eI_1, fI_2))$.
(2) $\phi(I_k) = I_k$ for all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$ and $k = 1, 2$.

Example 2.12. [4] Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2)(0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 2.13. [4] Let $(G(I_1, I_2), *)$ and and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. Let $\phi : G(I_1, I_2) \times H(I_1, I_2) \rightarrow G(I_1, I_2)$ be a mapping defined by $\phi(x, y) = x$ and let $\psi : G(I_1, I_2) \times H(I_1, I_2) \rightarrow H(I_1, I_2)$ be a mapping defined by $\psi(x, y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

Definition 2.14. [6] Let $(H, +)$ be any canonical hypergroup and let I be an indeterminate.
Let $H(I) =< H \cup I >= \{(a, bI) : a, b \in H\}$ be a set generated by H and I. The hyperstructure $(H(I), +)$ is called a neutrosophic canonical hypergroup . For all $(a, bI), (c, dI) \in H(I)$ with $b \neq 0$ or $d \neq 0$, we define $(a, bI) + (c, dI) = \{(x, yI) : x \in a + c,\ y \in a + d \cup b + c \cup b + d\}$. An element $I \in H(I)$ is represented by $(0, I)$ in $H(I)$ and any element $x \in H$ is represented by $(x, 0)$ in $H(I)$. For any nonempty subset $A(I)$ of $H(I)$, we define $-A(I) = \{-(a, bI) = (-a, -bI) : a, b \in H\}$.

Definition 2.15. [6] Let $(H(I), +)$ be a neutrosophic canonical hypergroup .

(1) A nonempty subset $A(I)$ of $H(I)$ is called a neutrosophic subcanonical hypergroup of $H(I)$ if $(A(I), +)$ is itself a neutrosophic canonical hypergroup . It is essential that $A(I)$ must contain a proper subset which is a subcanonical hypergroup of H.
If $A(I)$ does not contain a proper subset which is a subcanonical hypergroup of H, then it is called a pseudo neutrosophic subcanonical hypergroup of $H(I)$.
(2) If $A(I)$ is a neutrosophic subcanonical hypergroup (pseudo neutrosophic subcanonical hypergroup), $A(I)$ is said to be normal in $H(I)$ if for all $(a, bI) \in H(I)$, $(a, bI)+A(I)-(a, bI) \subseteq A(I)$.

Definition 2.16. [6] Let $(R, +, \cdot)$ be any hyperring and let I be an indeterminate. The hyperstructure $(R(I), +, \cdot)$ generated by R and I, that is, $R(I) =< R \cup I >$, is called a neutrosophic hyperring. For all $(a, bI), (c, dI) \in R(I)$ with $b \neq 0$ or $d \neq 0$, we define

$$(a, bI) \cdot (c, dI) = \{(x, yI) : x \in a \cdot c, y \in a \cdot d \cup b \cdot c \cup b \cdot d\}.$$

Definition 2.17. [6] Let $(R(I), +, \cdot)$ be a neutrosophic hyperring and let $A(I)$ be a nonempty subset of $R(I)$. $A(I)$ is called a neutrosophic subhyperring of $R(I)$ if $(A(I), +, \cdot)$ is itself a neutrosophic hyperring . It is essential that $A(I)$ must contain a proper subset which is a hyperring. Otherwise, $A(I)$ is called a pseudo neutrosophic subhyperring of $R(I)$.

Definition 2.18. [6] Let $(R(I), +, \cdot)$ be a neutrosophic hyperring and let $A(I)$ be a neutrosophic subhyperring of $R(I)$.

(1) $A(I)$ is called a left neutrosophic hyperideal if $(r, sI) \cdot (a, bI) \subseteq A(I)$ for all $(r, sI) \in R(I)$ and $(a, bI) \in A(I)$.
(2) $A(I)$ is called a right neutrosophic hyperideal if $(a, bI) \cdot (r, sI) \subseteq A(I)$ for all $(r, sI) \in R(I)$ and $(a, bI) \in A(I)$.
(3) $A(I)$ is called a neutrosophic hyperideal if $A(I)$ is both a left and right neutrosophic hyperideal .

A neutrosophic hyperideal $A(I)$ of $R(I)$ is said to be normal in $R(I)$ if for all $(r, sI) \in R(I)$

$$(r, sI) + A(I) - (r, sI) \subseteq A(I).$$

Definition 2.19. [6] Let $(R_1(I), +, \cdot)$ and $(R_2(I), +, \cdot)$ be two neutrosophic hyperring and let $\phi : R_1(I) \longrightarrow R_2(I)$ be a mapping from $R_1(I)$ into $R_2(I)$.

(1) ϕ is called a homomorphism if :
 (a) ϕ is a hyperring homomorphism,
 (b) $\phi((0, I)) = (0, I)$.
(2) ϕ is called a good or strong homomorphism if:
 (a) ϕ is a good or strong hyperring homomorphism,
 (b) $\phi((0, I)) = (0, I)$.
(3) ϕ is called an isomorphism (strong isomorphism) if ϕ is a bijective homomorphism (strong homomorphism).

3. Formulation of a refined neutrosophic hyperrings

In this section, we study and present the development of refined neutrosophic hyperring $(R(I_1, I_2), +, \cdot)$ generated by R, I_1 and I_2 where the operations $''+''$ and $''\cdot''$ are hyperoperations. i.e.,

$$+, \cdot : R(I_1, I_2) \times R(I_1, I_2) \longrightarrow 2^{R(I_1, I_2)}.$$

For all $(a, bI_1, cI_2), (d, eI_1, fI_2) \in R(I_1, I_2)$ with $a, b, c, d, e, f \in R$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = \{(p, qI_1, rI_2) : p \in a + d, q \in (b + e), r \in (c + f)\},$$

$$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = \{(p, qI_1, rI_2) : p \in ad,\ q \in ae + bd + be + bf + ce, r \in af + cd + cf\}.$$

Definition 3.1. A refined neutrosophic hyperring is a tripple $(R(I_1, I_2), +, \cdot)$ satisfying the following axioms:

(1) $(R(I_1, I_2), +)$ is a refined neutrosophic canonical hypergroup .
(2) $(R(I_1, I_2), \cdot)$ is a refined neutrosophic semihypergroup.
(3) For all $(a, bI_1, cI_2), (d, eI_1, fI_2), (g, hI_1, jI_2) \in R(I_1, I_2)$,
   (a) $(a, bI_1, cI_2)\cdot((d, eI_1, fI_2)+(g, hI_1, jI_2)) = (a, bI_1, cI_2)\cdot(d, eI_1, fI_2)+(a, bI_1, cI_2)\cdot(g, hI_1, jI_2)$ and
   (b) $((d, eI_1, fI_2) + (g, hI_1, jI_2)) \cdot (a, bI_1, cI_2) = (d, eI_1, fI_2) \cdot (a, bI_1, cI_2) + (g, hI_1, jI_2) \cdot (a, bI_1, cI_2)$.

Definition 3.2. Let $(R(I_1, I_2), +, \cdot)$ be a refined neutrosophic hyperring. A non-empty subset $M(I_1, I_2)$ of $R(I_1, I_2)$ is called a refined neutrosophic subhyperring of $R(I_1, I_2)$ if $(M(I_1, I_2), +, \cdot)$ is itself a neutrosophic hyperring. It is essential that $M(I_1, I_2)$ must contain a proper subset which is a hyperring. Otherwise, $M(I_1, I_2)$ is called a refined pseudo neutrosophic subhyperring of $R(I_1, I_2)$.

Definition 3.3. Let $R(I_1, I_2)$ be a refined neutrosophic hyperring. The refined neutrosophic subhyperring $M(I_1, I_2)$ is said to be normal in $R(I_1, I_2)$ if and only if $(a, bI_1, cI_2) + M(I_1, I_2) - (a, bI_1, cI_2) \subseteq M(I_1, I_2)$ for all $(a, bI_1, cI_2) \in R(I_1, I_2)$.

Definition 3.4. Let $(R(I_1, I_2)), +, \cdot)$ be a refined neutrosophic hyperring and let $M(I_1, I_2)$ be a refined neutrosophic subhyperring of $R(I_1, I_2)$. $(M(I_1, I_2), +, \cdot)$ is a left(right) refined neutrosophic hyperideal of $R(I_1, I_2)$ if $x \cdot m \in M(I_1, I_2)[m \cdot x \in M(I_1, I_2)]$ for all $x = (a, bI_1, cI_2) \in R(I_1, I_2)$ and $m = (p, qI_1, sI_2) \in M(I_1, I_2)$. $M(I_1, I_2)$ is a refined neutrosophic hyperideal if $M(I_1, I_2)$ is both left and right refined neutrosophic hyperideal.

Remark 3.5. It should be noted that a refined neutrosophic hyperideal $H(I_1, I_2)$ of a refined neutrosophic hyperring $R(I_1, I_2)$ is normal in $R(I_1, I_2)$ only if hyperideal H is normal in hyperring R.

Proposition 3.6. *Let $(R(I_1, I_2), +, \cdot)$ be any refined neutrosophic hyperring. $(R(I_1, I_2), +, \cdot)$ is a hyperring.*

Proof. (1) That $(R(I_1, I_2), +)$ is a canonical hypergroup follows from Proposition 2.3 in [19].

(2) We show that $(R(I_1, I_2), \cdot)$ is a semihypergroup.

$$\begin{aligned}
x\cdot(y\cdot z) &= (a, bI_1, cI_2)\cdot((d, eI_1, fI_2)\cdot(g, hI_1, kI_2))\\
&= (a, bI_1, cI_2)\cdot((dg, (dh+eg+eh+ek+fh)I_1, (dk+fg+fk)I_2)\\
&= (a(dg), (a(dh)+a(eg)+a(eh)+a(ek)+a(fh)+b(dg)+b(dh)+b(eg)+b(eh)\\
&\quad +b(ek)+b(fh)+b(dk)+b(fg)+b(fk)+c(dh)+c(eg)+c(eh)+c(ek)+c(fh))I_1,\\
&\quad (a(dk)+a(fg)+a(fk)+c(dg)+c(dk)+c(fg)+c(fk))I_2)\\
&= (ad)g, ((aI_1, ((ad)k+(af)g+(af)k+(cd)g+(cd)k+(cf)g+(cf)k)I_2)\\
&= ((a, bI_1, cI_2)\cdot(d, eI_1, fI_2))\cdot(g, hI_1, kI_2)\\
&= (x\cdot y)\cdot z.
\end{aligned}$$

Accordingly, $(R(I_1, I_2), \cdot)$ is a semihypergroup. Also, for all $(a, bI_1, cI_2) \in R(I_1, I_2)$,

$$(a, bI_1, cI_2)\cdot(0, 0I_1, 0I_2) = \{(x, yI_1, zI_2) : x \in a\cdot 0,\ y \in a\cdot 0+b\cdot 0+c\cdot 0, z \in a\cdot 0+c\cdot 0\} = \{(0, 0I_1, 0I_2)\}.$$

Similarly, it can be shown that $(0, 0I_1, 0I_2)\cdot(a, bI_1, cI_2) = \{(0, 0I_1, 0I_2)\}$. Hence, $(0, 0I_1, 0I_2)$ is a bilaterally absorbing element.

(3) For the distributivity of $\cdot$ over $+$.

Let $a = (x, yI_1, zI_2), b = (u, vI_1, sI_2), c = (k, mI_1, nI_2)$ be arbitrary elements in $R(I_1, I_2)$ with $x, y, z, u, v, s, k, m, n \in R$.

$$\begin{aligned}
a\cdot(b+c) &= a\cdot\{(h_1, h_2I_1, h_3I_2) : h_1 \in u+k, h_2 \in v+m, h_3 \in s+n\}\\
&= \{(x, yI_1, zI_2)\cdot(h_1, h_2I_1, h_3I_2) : h_1 \in u+k, h_2 \in v+m, h_3 \in s+n\}\\
&= \{(p_1, p_2I_1, p_3I_2) : p_1 \in xh_1, p_2 \in xh_2+yh_1+yh_2+yh_3+zh_2, p_3 \in xh_3+zh_1+zh_3\}\\
&= \{(p_1, p_2I_1, p_3I_2) : p_1 \in xu+xk, p_2 \in xv+xm+yu+yk+yv+ym+ys+yn+\\
&\quad zv+zm, p_3 \in xs+xn+zu+zk+zs+zn\}.
\end{aligned}$$

Now if we take $p_1 = t_1+t_1',\ p_2 = t_2+t_2',\ p_3 = t_3+t_3',$ then we have

$$\begin{aligned}
a\cdot(b+c) &= \{(t_1+t_1', (t_2+t_2')I_1, (t_3+t_3')I_2) : t_1+t_1' \in xu+xk,\\
&\quad t_2+t_2' \in xv+xm+yu+yk+yv+ym+ys+yn+zv+zm,\\
&\quad t_3+t_3' \in xs+xn+zu+zk+zs+zn\}\\
&= \{(t_1, t_2I_1, t_3I_2) : t_1 \in xu,\ t_2 \in xv+yu+yv+ys+zv,\ t_3 \in xs+zu+zs\}+\\
&\quad \{(t_1', t_2'I_1, t_3'I_2) : t_1' \in xk,\ t_2' \in xm+yk+ym+yn+zm,\ t_3' \in xn+zk+zn\}\\
&= (x, yI_1, zI2)\cdot(u, vI_1, sI_2)+(x, yI_1, zI_2)\cdot(k, mI_1, nI_2)\\
&= a\cdot b+a\cdot c.
\end{aligned}$$

Similarly, we can show that $(b+c)\cdot a = b\cdot a+c\cdot a$. Therefore $\cdot$ is distributive over $+$.

Hence $R(I_1, I_2)$ is a hyperring. □

Example 3.7. Let $R(I_1, I_2) = \{a_1 = (s, sI_1, sI_2), a_2 = (s, sI_1, tI_2), a_3 = (s, tI_1, sI_2),$ $a_4 = (s, tI_1, tI_2),\ b_1 = (t, tI_1, tI_2), b_2 = (t, tI_1, sI_2), b_3 = (t, sI_1, tI_2), b_4 = (t, sI_1, sI_2)\}$ be a refined neutrosophic set and let $+$ be the hyperoperation on $R(I_1, I_2)$ defined as in the tables below. Let $a = \{a_1, a_2, a_3, a_4\}$ and $b = \{b_1, b_2, b_3, b_4\}$.

It is clear from Table 1 and 2 that $(R(I_1, I_2), +, \cdot)$ is a refined neutrosophic hyperring.

TABLE 1. Cayley table for the binary operation " + "

$+$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_1	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_2	a_2	$\{a_1, a_2\}$	a_4	$\{a_3, a_4\}$	$\{b_1, b_2\}$	b_1	$\{b_3, b_4\}$	b_3
a_3	a_3	a_4	$\{a_1, a_3\}$	$\{a_2, a_4\}$	$\{b_1, b_3\}$	$\{b_2, b_4\}$	b_1	b_2
a_4	a_4	$\{a_3, a_4\}$	$\{a_2, a_4\}$	a	b	$\{b_1, b_3\}$	$\{b_1, b_2\}$	b_1
b_1	b_1	$\{b_1, b_2\}$	$\{b_1, b_3\}$	b	$R(I_1, I_2)$	$\{a_2, a_4, b_1, b_3\}$	$\{a_3, a_4, b_1, b_2\}$	$\{a_4, b_1\}$
b_2	b_2	b_1	$\{b_2, b_4\}$	$\{b_1, b_3\}$	$\{a_2, a_4, b_1, b_3\}$	$\{a_1, a_3, b_2, b_4\}$	$\{a_4, b_1\}$	$\{a_3, b_2\}$
b_3	b_3	$\{b_3, b_4\}$	b_1	$\{b_1, b_2\}$	$\{a_3, a_4, b_1, b_2\}$	$\{a_4, b_1\}$	$\{a_1, a_2, b_3, b_4\}$	$\{a_2, b_3\}$
b_4	b_4	b_3	b_2	b_1	$\{a_4, b_1\}$	$\{a_3, b_2\}$	$\{a_2, b_3\}$	$\{a_1, b_4\}$

Proposition 3.8. *Let $(R(I_1, I_2), +_1, \cdot_1)$ be a refined neutrosophic hyperring and let $(K, +_2, \cdot_2)$ be a hyperring. Define for all $(x_1, k_1), (x_2, k_2) \in R(I_1, I_2) \times K$ the hyperoperations $''+''$ and $''\cdot''$ by*

$$(x_1, k_1) + (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 +_1 x_2,\ \ k_3 \in k_1 +_2 k_2\}$$

and

$$(x_1, k_1) \cdot (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 \cdot_1 x_2,\ \ k_3 \in k_1 \cdot_2 k_2\}.$$

Then $(R(I_1, I_2) \times K, +, \cdot)$ is a refined neutrosophic hyperring.

Proof. (1) That $(R(I_1, I_2) \times K, +)$ is a canonical hypergroup follows from the proof of Proposition 2.6 in [19] .

(2) We shall show that $(R(I_1, I_2) \times K, \cdot)$ is a refined neutrosophic semihypergroup.
Let $(r_1, k_1), (r_2, k_2), (r_3, k_3) \in R(I_1, I_2) \times K$ where $r = (a, bI_1, cI_2)$.
$(r_1, k_1) \cdot ((r_2, k_2) \cdot (r_3, k_3)) =$
$((a_1, b_1I_1, c_1I_2), k_1) \cdot [((a_2, b_2I_1, c_2I_2), k_2) \cdot ((a_3, b_3I_1, c_3I_2), k_3)]$
$= ((a_1, b_1I_1, c_1I_2), k_1) \cdot \{((p, qI_1, sI_2), k_4) :\ p \in a_2 \cdot_1 a_3,$
$q \in a_2 \cdot_1 b_3 +_1 b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3, s \in a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3,\ k_4 \in k_2 \cdot_2 k_3\}$
$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 p,$

TABLE 2. Cayley table for the binary operation $''\cdot''$

$\cdot$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	a	a	$\{a_1, a_2\}$	$\{a_1, a_2\}$
a_3	a_1	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$	$\{a_1, a_3\}$
a_4	a_1	a	$\{a_1, a_3\}$	a	a	a	a	a
b_1	a_1	a	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$R(I_1, I_2)$	$R(I_1, I_2)$	$R(I_1, I_2)$
b_2	a_1	a	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$
b_3	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$R(I_1, I_2)$	$\{a_1, a_2, b_3, b_4\}$	$\{a_1, a_2, b_3, b_4\}$
b_4	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	a	$R(I_1, I_2)$	$\{a_1, a_3, b_2, b_4\}$	$\{a_1, a_2, b_3, b_4\}$	$\{a_1, b_4\}$

$y \in a_1 \cdot_1 q +_1 b_1 \cdot_1 p +_1 b_1 \cdot_1 q +_1 b_1 \cdot_1 s +_1 c_1 \cdot_1 q, z \in a_1 \cdot_1 s +_1 c_1 \cdot_1 p +_1 c_1 \cdot_1 s,\ k_5 \in k_1 \cdot_2 k_4\}$

$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 (a_2 \cdot_1 a_3),$

$y \in a_1 \cdot_1 (a_2 \cdot_1 b_3 +_1 b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 a_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 b_3 +_1$
$b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3) +_1 b_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 b_3 +_1$
$b_2 \cdot_1 a_3 +_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3),$

$z \in a_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 a_3) +_1 c_1 \cdot_1 (a_2 \cdot_1 c_3 +_1 c_2 \cdot_1 a_3 +_1 c_2 \cdot_1 c_3),$
$k_5 \in k_1 \cdot_2 (k_2 \cdot_2 k_3)\}$

$= \{((x, yI_1, zI_2), k_5) : x \in a_1 \cdot_1 a_2 \cdot_1 a_3,$

$y \in a_1 \cdot_1 a_2 \cdot_1 b_3 +_1 a_1 \cdot_1 b_2 \cdot_1 a_3 +_1 a_1 \cdot_1 b_2 \cdot_1 b_3 +_1 a_1 \cdot_1 b_2 \cdot_1 c_3 +_1 a_1 \cdot_1 c_2 \cdot_1 b_3 +_1 b_1 \cdot_1 a_2 \cdot_1 a_3 +_1$
$b_1 \cdot_1 a_2 \cdot_1 b_3 +_1 b_1 \cdot_1 b_2 \cdot_1 a_3 +_1 b_1 \cdot_1 b_2 \cdot_1 b_3 +_1 b_1 \cdot_1 b_2 \cdot_1 c_3 +_1 b_1 \cdot_1 c_2 \cdot_1 b_3 +_1 b_1 \cdot_1 a_2 \cdot_1 c_3 +_1 b_1 \cdot_1$
$c_2 \cdot_1 a_3 +_1 b_1 \cdot_1 c_2 \cdot_1 c_3 +_1 c_1 \cdot_1 a_2 \cdot_1 b_3 +_1 c_1 \cdot_1 b_2 \cdot_1 a_3 +_1 c_1 \cdot_1 b_2 \cdot_1 b_3 +_1 b_2 \cdot_1 c_3 +_1 c_2 \cdot_1 b_3,$

$z \in a_1 \cdot_1 a_2 \cdot_1 c_3 +_1 a_1 \cdot_1 c_2 \cdot_1 a_3 +_1 a_1 \cdot_1 c_2 \cdot_1 c_3 +_1 c_1 \cdot_1 a_2 \cdot a_3 +_1 c_1 \cdot_1 a_2 \cdot_1 c_3 +_1 c_1 \cdot_1 c_2 \cdot_1 a_3 +_1 c_1 \cdot_1 c_2 \cdot_1 c_3, k_5 \in$
$k_1 \cdot_2 k_2 \cdot_2 k_3\}$

$= \{((x, yI_1, zI_2), k_5) : x \in (a_1 \cdot_1 a_2) \cdot_1 a_3,$

$y \in (a_1 \cdot_1 a_2) \cdot_1 b_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1 a_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1$
$a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1 b_3 +_1 (a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2) \cdot_1$

$c_3 +_1 (a_1c_2 +_1 c_1a_2 +_1 c_1c_2) \cdot_1 b_3,$

$z \in (a_1 \cdot_1 a_2) \cdot_1 c_3 +_1 (a_1 \cdot_1 c_2 +_1 c_1 \cdot_1 a_2 +_1 c_1 \cdot_1 c_2) \cdot_1 a_3 +_1 (a_1 \cdot_1 c_2 +_1 c_1 \cdot_1 a_2 +_1 c_1 \cdot_1 c_2) \cdot_1 c_3, k_5 \in$

$(k_1 \cdot_2 k_2) \cdot_2 k_3\}$

$= \{((m, nI_1, hI_2), k) : m \in a_1 \cdot_1 a_2,\ n \in a_1 \cdot_1 b_2 +_1 b_1 \cdot_1 a_2 +_1 b_1 \cdot_1 b_2 +_1 b_1 \cdot_1 c_2 +_1 c_1 \cdot_1 b_2, h \in$

$a_1c_2 +_1 c_1a_2 +_1 c_1c_2, k \in k_1 \cdot_2 k_2\} \cdot ((a_3, b_3I_1, c_3I_2), k_3)$

$= [((a_1, b_1I_1, c_1I_2), k_1) \cdot ((a_2, b_2I_1, c_2I_2), k_2)] \cdot ((a_3, b_3I_1, c_3I_2))$

$= ((r_1, k_1) \cdot (r_2, k_2)) \cdot (r_3, k_3).$

Accordingly, $(R(I_1, I_2) \times K, \cdot)$ is a refined neutrosophic semihypergroup.

Also, for all $((a, bI_1, cI_2), k) \in R(I_1, I_2) \times K$,

$$\begin{aligned}((a, bI_1, cI_2), k) \cdot ((0, 0I_1, 0I_2), 0) &= \{((x, yI_1, zI_2), k_1) : x \in a \cdot_1 0, y \in a \cdot_1 0 +_1 b \cdot_1 0 +_1 c \cdot_1 0,\\ &\quad z \in a \cdot_1 0 +_1 c \cdot 0,\ k_1 \in k \cdot_2 0\}\\ &= \{((0, 0I_1, 0I_2), 0)\}.\end{aligned}$$

Similarly, it can be shown that $((0, 0I_1, 0I_2), 0) \cdot ((a, bI_1, cI_2), k) = \{((0, 0I_1, 0I_2), 0)\}$.

Hence, $((0, 0I_1, 0I_2), 0)$ is a bilaterally absorbing element.

(3) For the distributivity of $\cdot$ over $+$.

Let $a = ((x, yI_1, zI_2), t_1),\ b = ((u, vI_1, sI_2), t_2),\ c = ((k, mI_1, nI_2), t_3)$ be arbitrary elements in $R(I_1, I_2) \times K$ with $x, y, z, u, v, s, k, m, n \in R$ and $t_1, t_2, t_3 \in K$.

$$\begin{aligned}a \cdot (b + c) &= a \cdot \{((h_1, h_2I_1, h_3I_2), t_4) : h_1 \in u +_1 k, h_2 \in v +_1 m, h_3 \in s +_1 n, t_4 \in t_2 +_2 t_3\}\\ &= \{((x, yI_1, zI_2), t_1) \cdot ((h_1, h_2I_1, h_3I_2), t_4) : h_1 \in u +_1 k, h_2 \in v +_1 m, h_3 \in s +_1 n,\\ &\quad t_4 \in t_2 +_2 t_3\}\\ &= \{((p_1, p_2I_1, p_3I_2), t_5) : p_1 \in x \cdot_1 h_1, p_2 \in x \cdot_1 h_2 +_1 y \cdot_1 h_1 +_1 y \cdot_1 h_2 +_1 y \cdot_1 h_3\\ &\quad +_1 z \cdot_1 h_2, p_3 \in x \cdot_1 h_3 +_1 z \cdot_1 h_1 +_1 z \cdot_1 h_3, t_5 \in t_1 \cdot_2 t_4\}\\ &= \{((p_1, p_2I_1, p_3I_2), t_5) : p_1 \in x \cdot_1 u +_1 x \cdot_1 k,\\ &\quad p_2 \in x \cdot_1 v +_1 x \cdot_1 m +_1 y \cdot_1 u +_1 y \cdot_1 k +_1 y \cdot_1 v +_1 y \cdot_1 m +_1 y \cdot_1 s +_1 y \cdot_1 n +_1\\ &\quad z \cdot_1 v +_1 z \cdot_1 m,\ p_3 \in x \cdot_1 s +_1 x \cdot_1 n +_1 z \cdot_1 u +_1 z \cdot_1 k +_1 z \cdot_1 s +_1 z \cdot_1 n,\\ &\quad t_5 \in t_1 \cdot_2 t_2 +_2 t_1 \cdot_2 t_3\}.\end{aligned}$$

Now if we take $p_1 = g_1 +_1 g'_1,\ p_2 = g_2 +_1 g'_2,\ p_3 = g_3 +_1 g'_3,\ t_5 = h_1 +_2 h'_1$ then we have

$$\begin{aligned}a \cdot (b + c) &= \{((g_1 +_1 g'_1, (g_2 +_1 g'_2)I_1, (g_3 +_1 g'_3)I_2), (h_1 +_2 h'_1)) : g_1 +_1 g'_1 \in x \cdot_1 u +_1 x \cdot_1 k,\\ &\quad g_2 +_1 g'_2 \in x \cdot_1 v +_1 x \cdot_1 m +_1 y \cdot_1 u +_1 y \cdot_1 k +_1 y \cdot_1 v +_1 y \cdot_1 m +_1 y \cdot_1 s +_1 y \cdot_1 n +_1\\ &\quad z \cdot_1 v +_1 z \cdot_1 m,\ g_3 +_1 g'_3 \in x \cdot_1 s +_1 x \cdot_1 n +_1 z \cdot_1 u +_1 z \cdot_1 k +_1 z \cdot_1 s +_1 z \cdot_1 n,\\ &\quad h_1 +_2 h'_1 \in t_1 \cdot_2 t_2 +_2 t_1 \cdot_2 t_3\}\end{aligned}$$

$$\begin{aligned}&= \{((g_1, g_2I_1, g_3I_2), h_1) : g_1 \in x \cdot_1 u, g_2 \in x \cdot_1 v +_1 y \cdot_1 u +_1 y \cdot_1 v +_1 y \cdot_1 s +_1 z \cdot_1 v,\\ &\quad g_3 \in x \cdot_1 s +_1 z \cdot_1 u +_1 z \cdot_1 s, h_1 \in t_1 \cdot_2 t_2\} +\\ &\quad \{((g'_1, g'_2I_1, g'_3I_2), h'_1) : g'_1 \in x \cdot_1 k, g'_2 \in x \cdot_1 m +_1 y \cdot_1 k +_1 y \cdot_1 m +_1 y \cdot_1 n +_1 z \cdot_1 m,\\ &\quad g'_3 \in x \cdot_1 n +_1 z \cdot_1 k +_1 z \cdot_1 n, h'_1 \in t_1 \cdot_2 t_3\}\\ &= a \cdot b + a \cdot c.\end{aligned}$$

Similarly, we can show that $(b + c) \cdot a = b \cdot a + c \cdot a$.

Therefore $\cdot$ is distributive over $+$. Hence $(R(I_1, I_2), \times K, +, \cdot)$ is a refined neutrosophic Hyperring. □

Proposition 3.9. *Let* $(R(I_1, I_2), +_1, \cdot_1)$ *and* $(K(I_1, I_2), +_2, \cdot_2)$ *be any two refined neutrosophic hyperring. Define for all* $(x_1, k_1), (x_2, k_2) \in R(I_1, I_2) \times K(I_1, I_2)$ *the hyperoperations* $''+''$ *and* $''\cdot''$ *by*

$$(x_1, k_1) + (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 +_1 x_2,\ k_3 \in k_1 +_2 k_2\}$$

and

$$(x_1, k_1) \cdot (x_2, k_2) = \{(x_3, k_3) : x_3 \in x_1 \cdot_1 x_2,\ k_3 \in k_1 \cdot_2 k_2\}.$$

Then $(R(I_1, I_2) \times K(I_1, I_2), +, \cdot)$ *is a refined neutrosophic hyperring.*

Proof. The proof is similar to the proof of Proposition 3.8. □

Lemma 3.10. *Let* $R(I_1, I_2)$ *be a refined neutrosophic hyperring. A non-empty subset* $M(I_1, I_2)$ *of* $R(I_1, I_2)$ *is a left(right) refined neutrosophic hyperideal if and only if for* $m_1 = (p_1, q_1I_1, s_1I_1), m_2 = (p_2, q_2I_1, s_2I_1) \in M(I_1, I_2)$ *and* $x = (a, bI_1, cI_2) \in R(I_1, I_2)$

(1) $m_1 - m_2 \subseteq M(I_1, I_2)$,

(2) $x \cdot m_1 \in M(I_1, I_2)$ $[m_1 \cdot x \in M(I_1, I_2)]$.

Definition 3.11. Let $H(I_1, I_2)$ and $J(I_1, I_2)$ be any two nonempty subsets of a refined neutrosophic hyperring $R(I_1, I_2)$.

(1) The sum $H(I_1, I_2) + J(I_1, I_2) = \{(x, yI_1, zI_2) : x \in x_1 + x_2,\ y \in y_1 + y_2,\ z \in z_1 + z_2\}$.
For some $x_1, y_1, z_1 \in H,\ x_2, y_2, z_2 \in J$.

(2) The product

$$H(I_1, I_2)J(I_1, I_2) = \{(x, yI_1, zI_2) : (x, yI_1, zI_2) \in \sum_{i=1}^{n}(a_i, b_iI_1, c_iI_2) \cdot (d_i, e_iI_1, f_iI_1),\ n \in \mathbb{Z}^+\}.$$

Proposition 3.12. *Let* $R(I_1, I_2)$ *be a refined neutrosophic hyperring. Let* $H(I_1, I_2)$ *and* $J(I_1, I_2)$ *be refined neutrosophic hyperideals of* $R(I_1, I_2)$ *then :*

(1) $H(I_1, I_2) + J(I_1, I_2)$ *is a refined neutrosophic hyperideal.*

(2) $H(I_1, I_2)J(I_1, I_2)$ *is a refined neutrosophic hyperideal.*

Proof. (1) Let $x = (a, bI_1, cI_2), y = (d, eI_1, fI_2) \in H(I_1, I_2) + J(I_1, I_2)$ and let $r = (g, hI_1, kI_2) \in R(I_1, I_2)$.

$$\begin{aligned}
(i)\ x - y &= (a, bI_1, cI_2) - (d, eI_1, fI_2) = (a, bI_1, cI_2) + (-d, -eI_1, -fI_2) \\
&= \{(p, qI_1, rI_2) : p \in a + (-d), q \in b + (-e),\ r \in c + (-f)\} \\
&= \{(p_1 + p_2, (q_1 + q_2)I_1, (r_1 + r_2)I_2) : p_1 + p_2 \in (a_1 + a_2) + (-d_1 + (-d_2)), \\
&\quad q_1 + q_2 \in (b_1 + b_2) + (-e_1 + (-e_2)),\ r_1 + r_2 \in (c_1 + c_2) + (-f_1 + (-f_2))\} \\
&= \{(p_1, q_1I_1, r_1I_2) : p_1 \in a_1 + (-d_1),\ q_1 \in b_1 + (-e_1),\ r_1 \in c_1 + (-f_1)\} + \\
&\quad \{(p_2, q_2I_1, r_2I_2) : p_2 \in a_2 + (-d_2),\ q_2 \in b_2 + (-e_2),\ r_2 \in c_2 + (-f_2)\} \\
&= \{(p_1, q_1I_1, r_1I_2) : p_1 \in a_1 - d_1,\ q_1 \in b_1 - e_1, r_1 \in c_1 - f_1\} + \\
&\quad \{(p_2, q_2I_1, r_2I_2) : p_2 \in a_2 - d_2,\ q_2 \in b_2 - e_2, r_2 \in c_2 - f_2\} \\
&= (x_1 - y_1) + (x_2 - y_2) \\
&\subseteq H(I_1, I_2) + J(I_1, I_2).
\end{aligned}$$

$$\begin{aligned}(ii)\ r\cdot x &= (g,hI_1,kI_2)\cdot(a,bI_1,cI_2)\\ &= \{(u,vI_1,mI_2): u\in ga,\ v\in gb+ha+hb+hc+kb,\ m\in gc+ka+kc\}\\ &= \{(u_1+u_2,(v_1+v_2)I_1,(m_1+m_2)I_2): u_1+u_2\in g(a_1+a_2),\\ &\quad v_1+v_2\in g(b_1+b_2)+h(a_1+a_2)+h(b_1+b_2)+h(c_1+c_2)+k(b_1+b_2),\\ &\quad m_1+m_2\in g(c_1+c_2)+k(a_1+a_2)+k(c_1+c_2)\}\\ &= \{(u_1,v_1I_1,m_1I_2): u_1\in ga_1, v_1\in gb_1+ha_1+hb_1+hc_1+kb_1, m\in gc_1+ka_1+kc_1\}+\\ &\quad \{(u_2,v_2I_1,m_2I_2): u_2\in ga_2, v_2\in gb_2+ha_2+hb_2+hc_2+kb_2, m_2\in gc_2+ka_2+kc_2\}\\ &= r\cdot x_1+r\cdot x_2\\ &\subseteq H(I_1,I_2)+J(I_1,I_2).\end{aligned}$$

$$\begin{aligned}(iii)\ x\cdot r &= (a,bI_1,cI_2)\cdot(g,hI_1,kI_2)\\ &= \{(u,vI_1,mI_2): u\in ag,\ v\in ah+bg+bh+bk+ch,\ m\in ak+cg+ck\}\\ &= \{(u_1+u_2,(v_1+v_2)I_1,(m_1+m_2)I_2): u_1+u_2\in (a_1+a_2)g,\\ &\quad v_1+v_2\in (a_1+a_2)h+(b_1+b_2)g+(b_1+b_2)h+(b_1+b_2)k+(c_1+c_2)h,\\ &\quad m_1+m_2\in (a_1+a_2)k+(c_1+c_2)g+(c_1+c_2)k\}\\ &= \{(u_1,v_1I_1,m_1I_2): u_1\in a_1g, v_1\in a_1h+b_1g+b_1h+b_1k+c_1h, m_1\in a_1k+c_1g+c_1k\}+\\ &\quad \{(u_2,v_2I_1,m_2I_2): u_2\in a_2g, v_2\in a_2h+b_2g+b_2h+b_2k+c_2h, m_2\in a_2k+c_2g+c_2k\}\\ &= x_1\cdot r+x_2\cdot r\\ &\subseteq H(I_1,I_2)+J(I_1,I_2).\end{aligned}$$

(2) Let $x=(a,bI_1,cI_2), y=(d,eI_1,fI_2)\in H(I_1,I_2)J(I_1,I_2)$ and let $r=(g,hI_1,kI_2)\in R(I_1,I_2)$.

Here

$$(a,bI_1,cI_2)\in\sum_{i=1}^{n}(a_i,b_iI,c_iI)\cdot(a_i',b_i'I_1,c_i'I_2) \text{ and } (d,eI_1,fI_2)\in\sum_{i=1}^{n}(d_i,e_iI_1,f_iI_2)\cdot(d_i',e_i'I_1,f_i'I_2).$$

For $(a_i,b_iI_1,c_iI_2),(d_i,e_iI_1,f_iI_2)\in H(I_1,I_2)$, $(a_i',b_i'I_1,c_i'I_2),(d_i',e_i'I_1,f_i'I_2)\in J(I_1,I_2)$, $a_i,b_i,c_i,d_i,e_i,f_i\in H$ and $a_i',b_i',c_i',d_i',e_i',f_i'\in J$.

So we have

$$a\in\sum_{i=1}^{n}a_ia_i',\ b\in\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i'),\ c\in\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i')$$

and

$$d\in\sum_{i=1}^{n}d_id_i',\ e\in\sum_{i=1}^{n}(d_ie_i'+e_id_i'+e_ie_i'+e_if_i'+f_ie_i'),\ f\in\sum_{i=1}^{n}(d_if_i'+f_id_i'+f_if_i').$$

$$\begin{aligned}(i)\ x-y &= (a,bI_1,cI_2)-(d,eI_1,fI_2)=(a,bI_1,cI_2)+(-d,-eI_1,-fI_2)\\ &= \{(u,vI_1,mI_2): u\in a-d,\ v\in b-e,\ m\in c-f\}\\ &= \{(u,vI_1,mI_2): u\in \textstyle\sum_{i=1}^{n}a_ia_i'-\sum_{i=1}^{n}d_id_i',\\ &\quad v\in\textstyle\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i')-\sum_{i=1}^{n}(d_ie_i'+e_id_i'+e_ie_i'+e_if_i'+f_ie_i'),\\ &\quad m\in\textstyle\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i')-\sum_{i=1}^{n}(d_if_i'+f_id_i'+f_if_i')\}\\ &= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^{n}(a_ia_i'+(-d_id_i')),\\ &\quad v\in\textstyle\sum_{i=1}^{n}(a_ib_i'+b_ia_i'+b_ib_i'+b_ic_i'+c_ib_i'+(-d_ie_i')+(-e_id_i')+(-e_ie_i')+(-e_if_i')\\ &\quad +(-f_ie_i')),\ m\in\textstyle\sum_{i=1}^{n}(a_ic_i'+c_ia_i'+c_ic_i'+(-d_if_i')+(-f_id_i')+(-f_if_i'))\}\\ &\subseteq H(I_1,I_2)J(I_1,I_2).\end{aligned}$$

$$\begin{aligned}(ii)\ r\cdot x &= (g,hI_1,kI_2)\cdot(a,bI_1,cI_2)\\ &= \{(u,vI_1,mI_2): u\in ga,\ v\in gb+ha+hb+hc+kb,\ m\in gc+ka+kc\}\\ &= \{(u,vI_1,mI_2): u\in g\textstyle\sum_{i=1}^n a_ia'_i,\\ &\quad v\in g\textstyle\sum_{i=1}^n(a_ib'_i+b_ia'_i+b_ib'_i+b_ic'_i+c_ib'_i)+h\sum_{i=1}^n a_ia'_i+\\ &\quad h\textstyle\sum_{i=1}^n(a_ib'_i+b_ia'_i+b_ib'_i+b_ic'_i+c_ib'_i)+h\sum_{i=1}^n(a_ic'_i+c_ia'_i+c_ic'_i)+\\ &\quad k\textstyle\sum_{i=1}^n(a_ib'_i+b_ia'_i+b_ib'_i+b_ic'_i+c_ib'_i),\\ &\quad m\in g\textstyle\sum_{i=1}^n(a_ic'_i+c_ia'_i+c_ic'_i)+k\sum_{i=1}^n a_ia'_i+k\sum_{i=1}^n(a_ic'_i+c_ia'_i+c_ic'_i)\}\\ &= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^n ga_ia'_i,\\ &\quad v\in\textstyle\sum_{i=1}^n(ga_ib'_i+gb_ia'_i+gb_ib'_i+gb_ic'_i+gc_ib'_i+ha_ia'_i+ha_ib'_i+hb_ia'_i+hb_ib'_i+\\ &\quad hb_ic'_i+hc_ib'_i+ha_ic'_i+hc_ia'_i+hc_ic'_i+ka_ib'_i+kb_ia'_i+kb_ib'_i+kb_ic'_i+kc_ib'_i),\\ &\quad m\in\textstyle\sum_{i=1}^n(ga_ic'_i+gc_ia'_i+gc_ic'_i+ka_ia'_i+ka_ic'_i+kc_ia'_i+kc_ic'_i)\}\\ &\subseteq H(I_1,I_2)J(I_1,I_2).\end{aligned}$$

$$\begin{aligned}(ii)\ x\cdot r &= (a,bI_1,cI_2)\cdot(g,hI_1,kI_2)\\ &= \{(u,vI_1,mI_2): u\in ag,\ v\in ah+bg+bh+bk+ch,\ m\in ak+cg+ck\}\\ &= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^n a_ia'_ig,\\ &\quad v\in\textstyle\sum_{i=1}^n a_ia'_ih+\sum_{i=1}^n(a_ib'_i+b_ia'_i+b_ib'_i+b_ic'_i+c_ib'_i)g+\\ &\quad \textstyle\sum_{i=1}^n(a_ib'_i+b_ia'_i+b_ib'_i+b_ic'_i+c_ib'_i)h+\sum_{i=1}^n(a_ib'_i+b_ia'_i+b_ib'_i+b_ic'_i+c_ib'_i)k+\\ &\quad \textstyle\sum_{i=1}^n(a_ic'_i+c_ia'_i+c_ic'_i)h,\\ &\quad m\in\textstyle\sum_{i=1}^n a_ia'_ik+\sum_{i=1}^n(a_ic'_i+c_ia'_i+c_ic'_i)g+\sum_{i=1}^n(a_ic'_i+c_ia'_i+c_ic'_i)k\}\\ &= \{(u,vI_1,mI_2): u\in\textstyle\sum_{i=1}^n a_ia'_ig,\\ &\quad v\in\textstyle\sum_{i=1}^n(a_ia'_ih+a_ib'_ig+b_ia'_ig+b_ib'_ig+b_ic'_ig+c_ib'_ig+a_ib'_ih+b_ia'_ih+b_ib'_ih+\\ &\quad b_ic'_ih+c_ib'_ih+a_ib'_ik+b_ia'_ik+b_ib'_ik+b_ic'_ik+c_ib'_ik+a_ic'_ih+c_ia'_ih+c_ic'_ih),\\ &\quad m\in\textstyle\sum_{i=1}^n(a_ia'_ik+a_ic'_ig+c_ia'_ig+c_ic'_ig+a_ic'_ik+c_ia'_ik+c_ic'_ik)\}\\ &\subseteq H(I_1,I_2)J(I_1,I_2).\end{aligned}$$

Hence $H(I_1,I_2)J(I_1,I_2)$ is a refined neutrosophic hyperideal of $R(I_1,I_2)$. □

Proposition 3.13. *Let $R(I_1,I_2)$ be a refined neutrosophic hyperrings and $J_i(I_1,I_2)_{i\in\Lambda}$ be a family of refined neutrosophic hyperideals of $R(I_1,I_2)$, then $\bigcap_{i\in\Lambda}J_i(I_1,I_2)$ is a refined neutrosophic hyperideal of $R(I_1,I_2)$.*

Proof. The proof is the same as the proof in classical case. □

Proposition 3.14. *Let $H(I_1,I_2)$ and $J(I_1,I_2)$ be a refined neutrosophic hyperideals of a refined neutrosophic hyperring $R(I_1,I_2)$ such that $J(I_1,I_2)$ is normal in $R(I_1,I_2)$. Then*

(1) *$H(I_1,I_2)\cap J(I_1,I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1,I_2)$.*
(2) *$J(I_1,I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1,I_2)+J(I_1,I_2)$.*

Proof. (1) That $H(I_1,I_2)\cap J(I_1,I_2)$ is a refined neutrosophic hyperideal of $H(I_1,I_2)$ can be easily established. So, it remains to show that $H(I_1,I_2)\cap J(I_1,I_2)$ is normal in $H(I_1,I_2)$.

Let $x = (a, bI_1, cI_2) \in H(I_1, I_2) \cap J(I_1, I_2)$, $h = (u, vI_1, tI_2) \in H(I_1, I_2)$ with $a, b, c \in H \cap J$ and $u, v, t \in H$. Then

$$\begin{aligned}
h + H(I_1, I_2) \cap J(I_1, I_2) - h &= h + x - h \ for \ x \in H(I_1, I_2) \cap J(I_1, I_2) \\
&= (u, vI_1, tI_2) + (a, bI_1, cI_2) - (u, vI_1, tI_2) \\
&= \{(p, qI_1, rI_2) : p \in u + a - u,\ q \in v + b - v,\ r \in t + c - t\} \\
&= \{(p, qI_1, rI_2) : p \in u + (H \cap J) - u,\ q \in v + (H \cap J) - v, \\
&\quad r \in t + (H \cap J) - t\} \\
&= \{(p, qI_1, rI_2) : p \in u + (H \cap J) - u \subseteq H \cap J, \\
&\quad q \in v + (H \cap J) - v \subseteq H \cap J,\ r \in t + (H \cap J) - t \subseteq H \cap J\} \\
&= \{(p, qI_1, rI_2) : p \in H \cap J,\ q \in H \cap J,\ r \in H \cap J\} \\
&\subseteq H(I_1, I_2) \cap J(I_1, I_2).
\end{aligned}$$

Accordingly, $H(I_1, I_2) \cap J(I_1, I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1, I_2)$.

(2) That $J(I_1, I_2)$ is a refined neutrosophic hyperideal of $H(I_1, I_2) + J(I_1, I_2)$ can be easily established. So, it remains to show that $J(I_1, I_2)$ is normal in $H(I_1, I_2) + J(I_1, I_2)$. Let $x = (a, bI_1, cI_2) \in J(I_1, I_2)$, $h = (u, vI_1, tI_2) = (u_1 + u_2, (v_1 + v_2)I_1, (t_1 + t_2)I_2) \in H(I_1, I_2) + J(I_1, I_2)$ with $a, b, c, u_2, v_2, t_2 \in J$ and $u_1, v_1, t_2 \in H$. Then

$$\begin{aligned}
h + J(I_1, I_2) - h &= h + x - h \ for \ x \in J(I_1, I_2) \\
&= (u, vI_1, tI_2) + (a, bI_1, cI_2) - (u, vI_1, tI_2) \\
&= ((u_1 + u_2), (v_1 + v_2)I_1, (t_1 + t_2)I_2) + (a, bI_1, cI_2) \\
&\quad -((u_1 + u_2), (v_1 + v_2)I_1, (t_1 + t_2)I_2) \\
&= \{(p, qI_1, rI_2) : p \in (u_1 + u_2) + a - (u_1 + u_2),\ q \in (v_1 + v_2) + b - (v_1 + v_2), \\
&\quad r \in (t_1 + t_2) + c - (t_1 + t_2) \\
&= \{(p, qI_1, rI_2) : p \in (u_1 + u_2) + J - (u_1 + u_2),\ q \in (v_1 + v_2) + J - (v_1 + v_2), \\
&\quad r \in (t_1 + t_2) + J - (t_1 + t_2)\} \\
&= \{(p, qI_1, rI_2) : p \in u_1 + (u_2 + J - u_2) - u_1,\ q \in v_1 + (v_2 + J - v_2) - v_1, \\
&\quad r \in t_1 + (t_2 + J - t_2) - t_1\} \\
&\subseteq \{(p, qI_1, rI_2) : p \in u_1 + J - u_1,\ q \in v_1 + J - v_1, r \in t_1 + J - t_1\} \\
&= \{(p, qI_1, rI_2) : p \in u_1 + J - u_1 \subseteq J,\ q \in v_1 + J - v_1 \subseteq J,\ r \in t_1 + J - t_1 \subseteq J\} \\
&= \{(p, qI_1, rI_2) : p \in J,\ q \in J,\ r \in J\} \\
&\subseteq J(I_1, I_2).
\end{aligned}$$

Accordingly, $J(I_1, I_2)$ is a normal refined neutrosophic hyperideal of $H(I_1, I_2) + J(I_1, I_2)$. □

Let $R(I_1, I_2)$ be a refined neutrosophic hyperring, and let $H(I_1, I_2)$ be a refined neutrosophic hyperideal of $R(I_1, I_2)$. Since $H(I_1, I_2)$ is a refined neutrosophic subcanonical hypergroup of $R(I_1, I_2)$, if $(R/H, +)$ is a canonical hypergroup then

$$R(I_1, I_2)/H(I_1, I_2) = \{\bar{(}x, yI_1, zI_2) : (x, yI_1, zI_2) \in R(I_1, I_2)\}$$

is a refined neutrosophic canonical hypergroup under the hyperaddition $+'$ defined for $r_1 + H(I_1, I_2), r_2 + H(I_1, I_2) \in R(I_1, I_2)/H(I_1, I_2)$ with $r_1 = (x_1, y_1 I_1, z_i I_2), r_2 = (x_2, y_2 I_1, z_2 I_2)$, by

$$r_1 + H(I_1, I_2) +' r_2 + H(I_1, I_2) = (r_1 +' r_2) + H(I_1, I_2).$$

Define on $R(I_1, I_2)/H(I_1, I_2)$ a hypermultiplication $\cdot'$ by

$$r_1 + H(I_1, I_2) \cdot' r_2 + H(I_1, I_2) = (r_1 r_2) + H(I_1, I_2).$$

It can be shown that $(R(I_1, I_2)/H(I_1, I_2), +', \cdot')$ is a refined neutrosophic hyperring if $(R/H, +, \cdot)$ is a hyperring.

Definition 3.15. Let $(R(I_1, I_2), +_1, \cdot_1)$ and $(P(I_1, I_2), +_2, \cdot_2)$ be any two refined neutrosophic hypergrings and let

$$\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$$

be a mapping from $R(I_1, I_2)$ into $P(I_1, I_2)$.

(1) ϕ is called a refined neutrosophic hyperring homomorphism if:
   (a) ϕ is hyperring homomorphism,
   (b) $\phi(I_k) = I_k$ for $\quad k = 1, 2$.
(2) ϕ is called a good refined neutrosophic hyperring homomorphism if:
   (a) ϕ is good hyperring homomorphism,
   (b) $\phi(I_k) = I_k$ for $\quad k = 1, 2$.
(3) ϕ is called a refined neutrosophic hyperring isomorphism if ϕ is a refined neutrosophic hyperring homomorphism and ϕ^{-1} is also a refined neutrosophic hyperring homomorphism.

Definition 3.16. Let $\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$ be a refined neutrosophic hyperring homomorphism from a refined neutrosophic hyperring $R(I_1, I_2)$ into a refined neutrosophic hyperring $P(I_1, I_2)$.

(1) The $Ker\phi = \{(u, vI_1, wI_2) \in R(I_1, I_2) : \phi((u, vI_1, wI_2)) = (0, 0I_1, 0I_2)\}$.
(2) The $Im\phi = \{\phi((u, vI_1, wI_2)) : (u, vI_1, wI_2) \in R(I_1, I_2)\}$.

Proposition 3.17. *Let $\phi : R(I_1, I_2) \longrightarrow P(I_1, I_2)$ be a refined neutrosophic homomorphism.*

(1) *The kernel of ϕ is not a neutrosophic subhyperring of $R(I_1, I_2)$.*
(2) *The kernel of ϕ is not a neutrosophic hyper ideal of $R(I_1, I_2)$.*
(3) *The image of ϕ is a neutrosophic subhyperring of $P(I_1, I_2)$.*

Proof. (1) It follows easily from 1 of definition 3.16.
(2) It follows from the Proof of 1.
(3) The proof is similar to the proof in classical case.

It can be shown that $ker\phi$ is just a subhyperrings of $R(I_1, I_2)$. □

4. Conclusions

This paper studied the refinement of a type of neutrosophic hyperrings in which $''+''$ and $''\cdot''$ are hyperoperations and presented their basic properties. It was established that every refined neutrosophic hyperring is a hyperring. It was also shown that the kernel of a refined neutrosophic hyperring homomorphism is not a refined neutrosophic hyperideal but the image is a refined neutrosophic subhyperring.

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