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Rough Semigroups in Connection with Single Valued Neutrosophic $(\in, \in)$-Ideals

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Abstract. The scheme of rough sets is an effective procedure that handle ambiguous, inexact or uncertain information configuration. Rough set theory for algebraic structures like semigroups is a formal approximation space consisting of a universal set and an equivalence relation. This article achieves a new utilization of rough sets in the theory of semigroups via single valued neutrosophic (SVN) subsemigroups/ideals. The conceptions of an SVN $(\in, \in)$-subsemigroup and an SVN $(\in, \in)$-ideal in semigroups are introduced, and its properties are investigated. Special congruence relations induced by an SVN $(\in, \in)$-ideal are introduced in semigroups. Using these notions, the lower and upper approximations, so called the $\mathcal{R}_q$-lower approximation and the $\mathcal{R}_q$-upper approximation for $q \in \{T, I, F\}$ based on an SVN $(\in, \in)$-ideal in semigroups are presented, and related characteristics are discussed. The notions of lower and upper subsemigroups/ideals, so called the $\mathcal{R}_q$-lower subsemigroup/ideal and the $\mathcal{R}_q$-upper subsemigroup/ideal for $q \in \{T, I, F\}$, are defined, and then the relationships between subsemigroups/ideals and $\mathcal{R}_q$-lower (upper) subsemigroups/ideals are considered.

Keywords: single valued neutrosophic $(\in, \in)$-subsemigroup/ideal; $\mathcal{R}_q$-lower subsemigroup/ideal; $\mathcal{R}_q$-upper subsemigroup/ideal.

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1. Introduction

Rough sets were originally suggested by Pawlak (see [1]), as an official approximation of the classical set in terms of a couple of sets that specify the upper and lower approximations of the crisp set. The approach of rough set is adequate for rule induction from sets of imperfect information. This approach helps in set apart between three patterns of missing attribute values; those are lost value, attribute-concept value and "do not care" conditions. Rough set can be seen as being used in a variety of fields (see [2–9]).

In 1965, Zadeh fetched up the idea of fuzzy set to handle imprecise information (see [10]). He used a single value to represent the degree of membership of the fuzzy set defined in a universe. There is a difficulty that not all problems with imprecise information are expressed in the class of single point membership value. To defeat such difficulties, an interval valued fuzzy set is adopted by Turksen (see [11]). As an extended notion of fuzzy sets, Atanassov attained a new scope called intuitionistic fuzziness sets (see [12]). In intuitionistic fuzzy sets, the membership (resp. nonmembership) function represents truth (resp. false) part. Smarandache used indeterminacy membership function as an independent component to introduce neutrosophic sets, which are a widen of intuitionistic fuzzy sets, by using three independent components: truth, indeterminacy and falsehood (see [13–15]). Wang et al. formed the idea of SVN sets which is an instance of neutrosophic sets which can be utilized in various disciplines of real-life issues, etc. (see [16]). It is already well known that neutrosophic sets are being applied in almost every field of study.

In this article, we state a SVN $(\in, \in)$-subsemigroup and a SVN $(\in, \in)$-ideal in semigroups, and investigate their properties. We define some special congruence relations $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ induced by a SVN $(\in, \in)$-ideal, and discuss a few properties in semigroups. Using these notions, we introduce the lower and upper approximations, so called the $\mathcal{R}_q$-lower approximation and the $\mathcal{R}_q$-upper approximation for $q \in \{T, I, F\}$, based on a SVN $(\in, \in)$-ideal in semigroups, and investigate related properties. Using the notion of $\mathcal{R}_q$-lower approximation and $\mathcal{R}_q$-upper approximation, we define lower and upper subsemigroups/ideals, so called the $\mathcal{R}_q$-lower subsemigroup/ideal and the $\mathcal{R}_q$-upper subsemigroup/ideal for $q \in \{T, I, F\}$, are defined, and then we provide the relationships between subsemigroups/ideals and $\mathcal{R}_q$-lower (upper) subsemigroups/ideals.

2. Preliminaries

This segment lists the basic well-known contents that are relevant to the current paper.

Definition 2.1. A set $S \neq \phi$ together with a binary operation "$\cdot$" such that $(w \cdot z) \cdot \hbar = w \cdot (z \cdot \hbar)$ for all $w, z, \hbar \in S$ is called a *semigroup*.

We use wz instead of $w \cdot z$ in what follows. Given two subsets G and H of a semigroup S, we define:

$$GH := \{wz | w \in G,\ z \in H\}.$$

Definition 2.2. A subset $N \neq \phi$ of a semigroup S is a *subsemigroup* of S if $NN \subseteq N$, and a *left ideal* (resp., *right ideal*) of S if $SN \subseteq N$ (resp., $NS \subseteq N$). We say that N is an *ideal* of S if it is both a left and a right ideal of S.

Definition 2.3 ([16])**.** Let $S \neq \phi$. An SVN set in S is defined as:

$$\Psi_{\text{TIF}} := \{\langle w; \Psi_T(w), \Psi_I(w), \Psi_F(w)\rangle | w \in S\} \tag{1}$$

where $\Psi_T, \Psi_I, \Psi_F : S \to [0,1]$ are functions.

For the sake of clarity, the SVN set in (1) will be symbolized by $\Psi_{\text{TIF}} := (\Psi_T, \Psi_I, \Psi_F)$.

Given an SVN set $\Psi_{\text{TIF}} := (\Psi_T, \Psi_I, \Psi_F)$ in S, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we describe:

$$\begin{aligned} T_\in(\Psi_{\text{TIF}}; \alpha) &:= \{w \in S | \Psi_T(w) \geq \alpha\}, \\ I_\in(\Psi_{\text{TIF}}; \beta) &:= \{w \in S | \Psi_I(w) \geq \beta\}, \\ F_\in(\Psi_{\text{TIF}}; \gamma) &:= \{w \in S | \Psi_F(w) \leq \gamma\}, \end{aligned}$$

which are called SVN $\in$-subsets.

Definition 2.4 ([17])**.** An SVN set Ψ_{TIF} in a semigroup S is an SVN $(\in, \in)$-subsemigroup of S if it satisfies:

$$\begin{aligned} &w \in T_\in(\Psi_{\text{TIF}}; \alpha_w),\ z \in T_\in(\Psi_{\text{TIF}}; \alpha_z) \ \Rightarrow\ wz \in T_\in(\Psi_{\text{TIF}}; \min\{\alpha_w, \alpha_z\}), \\ &w \in I_\in(\Psi_{\text{TIF}}; \beta_w),\ z \in I_\in(\Psi_{\text{TIF}}; \beta_z) \ \Rightarrow\ wz \in I_\in(\Psi_{\text{TIF}}; \min\{\beta_w, \beta_z\}), \\ &w \in F_\in(\Psi_{\text{TIF}}; \gamma_w),\ z \in F_\in(\Psi_{\text{TIF}}; \gamma_z) \ \Rightarrow\ wz \in F_\in(\Psi_{\text{TIF}}; \max\{\gamma_w, \gamma_z\}). \end{aligned} \tag{2}$$

Lemma 2.5 ([17])**.** *An* SVN *set* Ψ_{TIF} *in a semigroup* S *is an* SVN $(\in, \in)$*-subsemigroup of* S *if and only if it satisfies:*

$$(\forall w, z \in S) \begin{pmatrix} \Psi_T(wz) \geq \min\{\Psi_T(w), \Psi_T(z)\} \\ \Psi_I(wz) \geq \min\{\Psi_I(w), \Psi_I(z)\} \\ \Psi_F(wz) \leq \max\{\Psi_F(w), \Psi_F(z)\} \end{pmatrix}. \tag{3}$$

3. **Rough semigroups based on single valued neutrosophic $(\in, \in)$-ideals**

Here, let S be a semigroup unless otherwise stated.

Definition 3.1. An SVN set Ψ_{TIF} in S is a left SVN $(\in, \in)$-ideal of S if it is an SVN $(\in, \in)$-subsemigroup of S satisfying the following condition:

$$(\forall w, z \in S) \begin{pmatrix} z \in T_\in(\Psi_{\text{TIF}}; \alpha) \ \Rightarrow\ wz \in T_\in(\Psi_{\text{TIF}}; \alpha) \\ z \in I_\in(\Psi_{\text{TIF}}; \beta) \ \Rightarrow\ wz \in I_\in(\Psi_{\text{TIF}}; \beta) \\ z \in F_\in(\Psi_{\text{TIF}}; \gamma) \ \Rightarrow\ wz \in F_\in(\Psi_{\text{TIF}}; \gamma) \end{pmatrix}. \tag{4}$$

Definition 3.2. An SVN set $\Psi_{\rm TIF}$ in S is a right SVN $(\in,\in)$-ideal of S if it is an SVN $(\in,\in)$-subsemigroup of S satisfying the following condition:

$$(\forall w,z\in S)\left(\begin{array}{l} z\in T_{\in}(\Psi_{\rm TIF};\alpha) \ \Rightarrow \ zw\in T_{\in}(\Psi_{\rm TIF};\alpha) \\ z\in I_{\in}(\Psi_{\rm TIF};\beta) \ \Rightarrow \ zw\in I_{\in}(\Psi_{\rm TIF};\beta) \\ z\in F_{\in}(\Psi_{\rm TIF};\gamma) \ \Rightarrow \ zw\in F_{\in}(\Psi_{\rm TIF};\gamma) \end{array}\right). \tag{5}$$

If $\Psi_{\rm TIF}$ is a left and a right SVN $(\in,\in)$-ideal of S, we say that $\Psi_{\rm TIF}$ is an SVN $(\in,\in)$-ideal of S.

Example 3.3. Consider a semigroup $S=\{\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4\}$ with the "$\cdot$" operation given by Table 1.

TABLE 1. Table for "$\cdot$" operation

$\cdot$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_2	ς_2	ς_4
ς_2	ς_2	ς_2	ς_2	ς_4
ς_3	ς_2	ς_2	ς_2	ς_4
ς_4	ς_4	ς_4	ς_4	ς_4

Let $\Psi_{\rm TIF}$ be an SVN set in S which is shown as:

$$\begin{aligned}\Psi_{\rm TIF}=\{&\langle\varsigma_1,(0.33,0.27,0.68)\rangle,\langle\varsigma_2,(0.55,0.47,0.57)\rangle,\\ &\langle\varsigma_3,(0.11,0.17,0.89)\rangle,\langle\varsigma_4,(0.88,0.77,0.36)\rangle\}.\end{aligned}$$

It is routine to show that $\Psi_{\rm TIF}$ is an SVN $(\in,\in)$-ideal of S.

Theorem 3.4. *An* SVN *set* $\Psi_{\rm TIF}$ *in* S *is a left (resp. right)* SVN $(\in,\in)$*-ideal of* S $\Leftrightarrow$ *it satisfies* (3) *and*

$$(\forall w,z\in S)\left(\begin{array}{l} \Psi_T(wz)\geq\Psi_T(z) \text{ (resp. } \Psi_T(w)) \\ \Psi_I(wz)\geq\Psi_I(z) \text{ (resp. } \Psi_I(w)) \\ \Psi_F(wz)\leq\Psi_F(z) \text{ (resp. } \Psi_F(w)) \end{array}\right). \tag{6}$$

Proof. Let $\Psi_{\rm TIF}$ be a left SVN $(\in,\in)$-ideal of S. Obviously, the condition (3) is true by Lemma 2.5. If $\exists\ w,z\in S$ such that $\Psi_T(wz)<\Psi_T(z)$, then $z\in T_{\in}(\Psi_{\rm TIF};\Psi_T(z))$ but $wz\notin T_{\in}(\Psi_{\rm TIF};\Psi_T(z))$, a contradiction. So $\Psi_T(wz)\geq\Psi_T(z)$ $\forall w,z\in S$. Assume that $\Psi_I(ab)<\Psi_I(b)$ for some $a,b\in S$ and take $\beta:=\frac{1}{2}(\Psi_I(ab)+\Psi_I(b))$. Then, $b\in I_{\in}(\Psi_{\rm TIF};\beta)$ and $ab\notin I_{\in}(\Psi_{\rm TIF};\beta)$, which is a contradiction. Hence, $\Psi_I(wz)\geq\Psi_I(z)$ for all $w,z\in S$. If $\Psi_F(wz)>\Psi_F(z)$ for some $w,z\in S$, then $\exists\ \gamma\in[0,1)$ such that $\Psi_F(wz)\geq\gamma>\Psi_F(z)$. Then, $z\in F_{\in}(\Psi_{\rm TIF};\gamma)$ and $wz\notin F_{\in}(\Psi_{\rm TIF};\gamma)$, which induces a contradiction. Therefore, $\Psi_F(wz)\leq$

$\Psi_F(z)\ \forall w, z \in S$. Similarly, if Ψ_{TIF} is a right SVN $(\in, \in)$-ideal of S, then $\Psi_T(wz) \geq \Psi_T(w)$, $\Psi_I(wz) \geq \Psi_I(w)$ and $\Psi_F(wz) \leq \Psi_F(w)$ for all $w, z \in S$.

Conversely, suppose that Ψ_{TIF} satisfies $\Psi_T(wz) \geq \Psi_T(w)$, $\Psi_I(wz) \geq \Psi_I(w)$ and $\Psi_F(wz) \leq \Psi_F(w)\ \forall w, z \in S$. Let $w \in T_\in(\Psi_{\text{TIF}}; \alpha) \cap I_\in(\Psi_{\text{TIF}}; \beta) \cap F_\in(\Psi_{\text{TIF}}; \gamma)$. Then,

$$\Psi_T(wz) \geq \Psi_T(w) \geq \alpha,$$

$$\Psi_I(wz) \geq \Psi_I(w) \geq \beta$$

and

$$\Psi_F(wz) \leq \Psi_F(w) \leq \gamma,$$

which imply that $wz \in T_\in(\Psi_{\text{TIF}}; \alpha) \cap I_\in(\Psi_{\text{TIF}}; \beta) \cap F_\in(\Psi_{\text{TIF}}; \gamma)$. Hence, Ψ_{TIF} is a right SVN $(\in, \in)$-ideal of S. Similarly, if Ψ_{TIF} satisfies $\Psi_T(wz) \geq \Psi_T(z)$, $\Psi_I(wz) \geq \Psi_I(z)$ and $\Psi_F(wz) \leq \Psi_F(z)$ for all $w, z \in S$, then Ψ_{TIF} is a left SVN $(\in, \in)$-ideal of S. □

Let Δ be the diagonal relation on S and let χ_Δ be the characteristic function of Δ in $S \times S$. Given an SVNS Ψ_{TIF} in S, consider the following relations on S:

$$\begin{aligned} \mathcal{R}_{(T,\alpha)} &:= \{(w, z) \in S \times S \mid \max\{\chi_\Delta(w, z), \min\{\Psi_T(w), \Psi_T(z)\}\} \geq \alpha\} \\ \mathcal{R}_{(I,\beta)} &:= \{(w, z) \in S \times S \mid \max\{\chi_\Delta(w, z), \min\{\Psi_I(w), \Psi_I(z)\}\} \geq \beta\} \\ \mathcal{R}_{(F,\gamma)} &:= \{(w, z) \in S \times S \mid \min\{f_\Delta(w, z), \max\{\Psi_F(w), \Psi_F(z)\}\} \leq \gamma\} \end{aligned} \tag{7}$$

where $\alpha, \beta \in (0, 1]$, $\gamma \in [0, 1)$ and

$$f_\Delta : S \times S \to [0, 1],\ (w, z) \mapsto 1 - \chi_\Delta(w, z).$$

It is simple to demonstrate that $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are equivalence relations on S. Let Ψ_{TIF} be an SVN $(\in, \in)$-ideal of S. Let $a, w, z \in S$ be such that $(w, z) \in \mathcal{R}_{(T,\alpha)}$. If $aw = az$, then $\chi_\Delta(aw, az) = 1$ and so

$$\max\{\chi_\Delta(aw, az), \min\{\Psi_T(aw), \Psi_T(az)\}\} = 1 \geq \alpha.$$

Thus $(aw, az) \in \mathcal{R}_{(T,\alpha)}$. Similarly, we can verify that

$$\max\{\chi_\Delta(aw, az), \min\{\Psi_I(aw), \Psi_I(az)\}\} = 1 \geq \beta,$$

that is, $(aw, az) \in \mathcal{R}_{(I,\beta)}$. If $aw = az$, then $f_\Delta(w, z) = 1 - (w, z) = 0$ and so

$$\min\{f_\Delta(w, z), \max\{\Psi_F(w), \Psi_F(z)\}\} = 0 \leq \gamma,$$

i.e., $(aw, az) \in \mathcal{R}_{(F,\gamma)}$. Suppose that $aw \neq az$. Then, $\chi_\Delta(aw, az) = 0$ and $w \neq z$. Since Ψ_{TIF} is a left SVN $(\in, \in)$-ideal of S, it follows that

$$\begin{aligned}\max\{\chi_\Delta(aw, az), \min\{\Psi_T(aw), \Psi_T(az)\}\} &= \min\{\Psi_T(aw), \Psi_T(az)\}\\ &\geq \min\{\Psi_T(w), \Psi_T(z)\}\\ &\geq \alpha,\end{aligned}$$

$$\begin{aligned}\max\{\chi_\Delta(aw, az), \min\{\Psi_I(aw), \Psi_I(az)\}\} &= \min\{\Psi_I(aw), \Psi_I(az)\}\\ &\geq \min\{\Psi_I(w), \Psi_I(z)\}\\ &\geq \beta\end{aligned}$$

and

$$\begin{aligned}\min\{f_\Delta(ax, ay), \max\{\Psi_F(aw), \Psi_F(az)\}\} &= \max\{\Psi_F(aw), \Psi_F(az)\}\\ &\leq \max\{\Psi_F(w), \Psi_F(z)\}\\ &\leq \gamma.\end{aligned}$$

Thus $(aw, az) \in \mathcal{R}_{(T,\alpha)}$, $(aw, az) \in \mathcal{R}_{(I,\beta)}$ and $(aw, az) \in \mathcal{R}_{(F,\gamma)}$. Similarly, we can verify that $(wa, za) \in \mathcal{R}_{(T,\alpha)}$, $(wa, za) \in \mathcal{R}_{(I,\beta)}$ and $(wa, za) \in \mathcal{R}_{(F,\gamma)}$. Therefore, $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are congruence relations on S.

We summarize the result as a lemma.

Lemma 3.5. *If Ψ_{TIF} is an* SVN $(\in, \in)$*-ideal of S, then $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are congruence relations on S.*

Given $w \in S$, let $[w]_{(T,\alpha)}$ (resp., $[w]_{(I,\beta)}$ and $[w]_{(F,\gamma)}$) denote the equivalence class of x which is called T*-equivalence class* (resp. I*-equivalence class* and F*-equivalence class*) of x.

Lemma 3.6. *If Ψ_{TIF} is an* SVN $(\in, \in)$*-ideal of S, then $[w]_{(T,\alpha)}[z]_{(T,\alpha)} \subseteq [wz]_{(T,\alpha)}$, $[w]_{(I,\beta)}[z]_{(I,\beta)} \subseteq [wz]_{(I,\beta)}$ and $[w]_{(F,\gamma)}[z]_{(F,\gamma)} \subseteq [wz]_{(F,\gamma)}$ for every $\alpha, \beta, \gamma \in [0, 1]$.*

Proof. Let $a \in [w]_{(T,\alpha)}[z]_{(T,\alpha)}$. Then, $a = w'z'$ for some $w' \in [w]_{(T,\alpha)}$ and $z' \in [z]_{(T,\alpha)}$. Thus $\Psi_T(w, w') \geq \alpha$ and $\Psi_T(z, z') \geq \alpha$. Since $\mathcal{R}_{(T,\alpha)}$ is a congruence relation on S, it follows that $\Psi_T(wz, w'z') \geq \alpha$, that is, $a = w'z' \in [wz]_{(T,\alpha)}$. Hence, $[w]_{(T,\alpha)}[z]_{(T,\alpha)} \subseteq [wz]_{(T,\alpha)}$. If $b \in [w]_{(I,\beta)}[z]_{(I,\beta)}$, then $b = w'z'$ for some $w' \in [w]_{(I,\beta)}$ and $z' \in [z]_{(I,\beta)}$. Hence, $\Psi_I(w, w') \geq \beta$ and $\Psi_I(z, z') \geq \beta$ which imply that $\Psi_I(wz, w'z') \geq \beta$, that is, $b = w'z' \in [wz]_{(I,\beta)}$. This shows that $[w]_{(I,\beta)}[z]_{(I,\beta)} \subseteq [wz]_{(I,\beta)}$. Suppose that $c \in [w]_{(F,\gamma)}[z]_{(F,\gamma)}$. Then, $c = ab$ for some $a \in [w]_{(F,\gamma)}$ and $b \in [z]_{(F,\gamma)}$. Thus, $\Psi_F(a, w) \leq \gamma$ and $\Psi_F(b, z) \leq \gamma$, and so $\Psi_F(ab, wz) \leq \gamma$ since $\mathcal{R}_{(F,\gamma)}$ is a congruence relation on S. Therefore, $c = ab \in [wz]_{(F,\gamma)}$, which proves $[w]_{(F,\gamma)}[z]_{(F,\gamma)} \subseteq [wz]_{(F,\gamma)}$. □

The following example illustrates Lemma 3.6.

Example 3.7. Consider the SVN $(\in, \in)$-ideal Ψ_{TIF} of S in Example 3.3. If we take $(\alpha, \beta, \gamma) = (0.44, 0.37, 0.63)$, then

$$\mathcal{R}_{(T,\alpha)} = \{(\varsigma_1, \varsigma_1), (\varsigma_2, \varsigma_2), (\varsigma_3, \varsigma_3), (\varsigma_4, \varsigma_4), (\varsigma_2, \varsigma_4)\},$$

$$\mathcal{R}_{(I,\beta)} = \{(\varsigma_1, \varsigma_1), (\varsigma_2, \varsigma_2), (\varsigma_3, \varsigma_3), (\varsigma_4, \varsigma_4), (\varsigma_2, \varsigma_4)\}$$

and

$$\mathcal{R}_{(F,\gamma)} = \{(\varsigma_1, \varsigma_1), (\varsigma_2, \varsigma_2), (\varsigma_3, \varsigma_3), (\varsigma_4, \varsigma_4), (\varsigma_2, \varsigma_4)\}.$$

Hence, $[\varsigma_1]_{(T,\alpha)} = \{\varsigma_1\}$, $[\varsigma_2]_{(T,\alpha)} = \{\varsigma_2, \varsigma_4\}$, $[\varsigma_3]_{(T,\alpha)} = \{\varsigma_3\}$, and $[\varsigma_4]_{(T,\alpha)} = \{\varsigma_2, \varsigma_4\}$. It follows that $[\varsigma_1]_{(T,\alpha)}[\varsigma_3]_{(T,\alpha)} = \{\varsigma_2\} \subseteq \{\varsigma_2, \varsigma_4\} = [\varsigma_2]_{(T,\alpha)} = [\varsigma_1\varsigma_3]_{(T,\alpha)}$. In the same way, we can check $[w]_{(I,\beta)}[z]_{(I,\beta)} \subseteq [wz]_{(I,\beta)}$ and $[w]_{(F,\gamma)}[z]_{(F,\gamma)} \subseteq [wz]_{(F,\gamma)}$ for $w, z \in S$.

Definition 3.8. The congruence relation $\mathcal{R}_{(T,\alpha)}$ (resp., $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$) on S is said to be *complete* if $[w]_{(T,\alpha)}[z]_{(T,\alpha)} = [wz]_{(T,\alpha)}$ (resp., $[w]_{(I,\beta)}[z]_{(I,\beta)} = [wz]_{(I,\beta)}$ and $[w]_{(F,\gamma)}[z]_{(F,\gamma)} = [wz]_{(F,\gamma)}$) for all $w, z \in S$.

Example 3.9. Consider a semigroup $S = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ with the "·" operation given by Table 2.

TABLE 2. Table for "·" operation

$\cdot$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_2	ς_3	ς_4
ς_2	ς_2	ς_2	ς_3	ς_4
ς_3	ς_3	ς_3	ς_3	ς_4
ς_4	ς_4	ς_4	ς_4	ς_3

Let Ψ_{TIF} be an SVNS in S which is shown as:

$$\Psi_{\mathrm{TIF}} = \{\langle \varsigma_1, (0.11, 0.27, 0.68)\rangle, \langle \varsigma_2, (0.44, 0.47, 0.57)\rangle,$$
$$\langle \varsigma_3, (0.77, 0.67, 0.29)\rangle, \langle \varsigma_4, (0.77, 0.67, 0.29)\rangle\}.$$

Then, Ψ_{TIF} is an SVN $(\in, \in)$-ideal of S. It is routine to verify that $[w]_{(T,\alpha)}[z]_{(T,\alpha)} = [wz]_{(T,\alpha)}$, $[w]_{(I,\beta)}[z]_{(I,\beta)} = [wz]_{(I,\beta)}$ and $[w]_{(F,\gamma)}[z]_{(F,\gamma)} = [wz]_{(F,\gamma)}$ for all $w, z \in S$ where $(\alpha, \beta, \gamma) = (0.77, 0.67, 0.29)$. Therefore, $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are complete congruence relations on S for $(\alpha, \beta, \gamma) = (0.77, 0.67, 0.29)$.

Definition 3.10. Let Ψ_{TIF} be an SVN $(\in, \in)$-ideal of S and let N be a nonempty subset of S. Given $q \in \{T, I, F\}$, the $\mathcal{R}_q$-*lower approximation* and $\mathcal{R}_q$-*upper approximation* of X are defined to be the sets

$$\begin{aligned}
\underline{\mathcal{R}}_T(N;\alpha) &:= \{w \in S | [w]_{(T,\alpha)} \subseteq N\} \\
\underline{\mathcal{R}}_I(N;\beta) &:= \{w \in S | [w]_{(I,\beta)} \subseteq N\} \\
\underline{\mathcal{R}}_F(N;\gamma) &:= \{w \in S | [w]_{(F,\gamma)} \subseteq N\}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathcal{R}}_T(N;\alpha) &:= \{w \in S | [w]_{(T,\alpha)} \cap N \neq \emptyset\} \\
\overline{\mathcal{R}}_I(N;\beta) &:= \{w \in S | [w]_{(I,\beta)} \cap N \neq \emptyset\} \\
\overline{\mathcal{R}}_F(N;\gamma) &:= \{w \in S | [w]_{(F,\gamma)} \cap N \neq \emptyset\},
\end{aligned}$$

respectively.

By routine calculations, we have the next proposition.

Proposition 3.11. *Let* Ψ_{TIF} *be an* SVN $(\in, \in)$*-ideal of* S*. For any nonempty subsets* G *and* H *of* S*, the following assertions are valid.*

$$\begin{aligned}
&\underline{\mathcal{R}}_T(G;\alpha) \subseteq G \subseteq \overline{\mathcal{R}}_T(G;\alpha), \\
&\underline{\mathcal{R}}_I(G;\beta) \subseteq G \subseteq \overline{\mathcal{R}}_I(G;\beta), \\
&\underline{\mathcal{R}}_F(G;\gamma) \subseteq G \subseteq \overline{\mathcal{R}}_F(G;\gamma),
\end{aligned} \tag{8}$$

$$\begin{aligned}
\underline{\mathcal{R}}_T(G \cap H;\alpha) &= \underline{\mathcal{R}}_T(G;\alpha) \cap \underline{\mathcal{R}}_T(H;\alpha), \\
\underline{\mathcal{R}}_I(G \cap H;\beta) &= \underline{\mathcal{R}}_I(G;\beta) \cap \underline{\mathcal{R}}_I(H;\beta), \\
\underline{\mathcal{R}}_F(G \cap H;\gamma) &= \underline{\mathcal{R}}_F(G;\gamma) \cap \underline{\mathcal{R}}_F(H;\gamma),
\end{aligned} \tag{9}$$

$$\begin{aligned}
\overline{\mathcal{R}}_T(G \cap H;\alpha) &\subseteq \overline{\mathcal{R}}_T(G;\alpha) \cap \overline{\mathcal{R}}_T(H;\alpha), \\
\overline{\mathcal{R}}_I(G \cap H;\beta) &\subseteq \overline{\mathcal{R}}_I(G;\beta) \cap \overline{\mathcal{R}}_I(H;\beta), \\
\overline{\mathcal{R}}_F(G \cap H;\gamma) &\subseteq \overline{\mathcal{R}}_F(G;\gamma) \cap \overline{\mathcal{R}}_F(H;\gamma),
\end{aligned} \tag{10}$$

$$G \subseteq H \;\Rightarrow \begin{pmatrix} \underline{\mathcal{R}}_T(G;\alpha) \subseteq \underline{\mathcal{R}}_T(H;\alpha), \\ \underline{\mathcal{R}}_I(G;\beta) \subseteq \underline{\mathcal{R}}_I(H;\beta), \\ \underline{\mathcal{R}}_F(G;\gamma) \subseteq \underline{\mathcal{R}}_F(H;\gamma), \end{pmatrix}, \tag{11}$$

$$G \subseteq H \;\Rightarrow \begin{pmatrix} \overline{\mathcal{R}}_T(G;\alpha) \subseteq \overline{\mathcal{R}}_T(H;\alpha), \\ \overline{\mathcal{R}}_I(G;\beta) \subseteq \overline{\mathcal{R}}_I(H;\beta), \\ \overline{\mathcal{R}}_F(G;\gamma) \subseteq \overline{\mathcal{R}}_F(H;\gamma), \end{pmatrix}, \tag{12}$$

$$\begin{aligned}&\underline{\mathcal{R}}_T(G;\alpha)\cup\underline{\mathcal{R}}_T(H;\alpha)\subseteq\underline{\mathcal{R}}_T(G\cup H;\alpha),\\&\underline{\mathcal{R}}_I(G;\beta)\cup\underline{\mathcal{R}}_I(H;\beta)\subseteq\underline{\mathcal{R}}_I(G\cup H;\beta),\\&\underline{\mathcal{R}}_F(G;\gamma)\cup\underline{\mathcal{R}}_F(H;\gamma)\subseteq\underline{\mathcal{R}}_F(G\cup H;\gamma),\end{aligned}\tag{13}$$

$$\begin{aligned}&\overline{\mathcal{R}}_T(G\cup H;\alpha)=\overline{\mathcal{R}}_T(G;\alpha)\cup\overline{\mathcal{R}}_T(H;\alpha),\\&\overline{\mathcal{R}}_I(G\cup H;\beta)=\overline{\mathcal{R}}_I(G;\beta)\cup\overline{\mathcal{R}}_I(H;\beta),\\&\overline{\mathcal{R}}_F(G\cup H;\gamma)=\overline{\mathcal{R}}_F(G;\gamma)\cup\overline{\mathcal{R}}_F(H;\gamma),\end{aligned}\tag{14}$$

$$\begin{aligned}&\underline{\mathcal{R}}_T(\underline{\mathcal{R}}_T(G;\alpha);\alpha)=\underline{\mathcal{R}}_T(G;\alpha),\\&\underline{\mathcal{R}}_I(\underline{\mathcal{R}}_I(G;\beta);\beta)=\underline{\mathcal{R}}_I(G;\beta),\\&\underline{\mathcal{R}}_F(\underline{\mathcal{R}}_F(G;\gamma);\gamma)=\underline{\mathcal{R}}_F(G;\gamma),\end{aligned}\tag{15}$$

$$\begin{aligned}&\overline{\mathcal{R}}_T(\overline{\mathcal{R}}_T(G;\alpha);\alpha)=\overline{\mathcal{R}}_T(G;\alpha),\\&\overline{\mathcal{R}}_I(\overline{\mathcal{R}}_I(G;\beta);\beta)=\overline{\mathcal{R}}_I(G;\beta),\\&\overline{\mathcal{R}}_F(\overline{\mathcal{R}}_F(G;\gamma);\gamma)=\overline{\mathcal{R}}_F(G;\gamma),\end{aligned}\tag{16}$$

$$\begin{aligned}&\underline{\mathcal{R}}_T(\overline{\mathcal{R}}_T(G;\alpha);\alpha)=\underline{\mathcal{R}}_T(G;\alpha),\\&\underline{\mathcal{R}}_I(\overline{\mathcal{R}}_I(G;\beta);\beta)=\underline{\mathcal{R}}_I(G;\beta),\\&\underline{\mathcal{R}}_F(\overline{\mathcal{R}}_F(G;\gamma);\gamma)=\underline{\mathcal{R}}_F(G;\gamma),\end{aligned}\tag{17}$$

$$\begin{aligned}&\overline{\mathcal{R}}_T(\underline{\mathcal{R}}_T(G;\alpha);\alpha)=\overline{\mathcal{R}}_T(G;\alpha),\\&\overline{\mathcal{R}}_I(\underline{\mathcal{R}}_I(G;\beta);\beta)=\overline{\mathcal{R}}_I(G;\beta),\\&\overline{\mathcal{R}}_F(\underline{\mathcal{R}}_F(G;\gamma);\gamma)=\overline{\mathcal{R}}_F(G;\gamma).\end{aligned}\tag{18}$$

Proposition 3.12. *Let Ψ_{TIF} be an* SVN $(\in, \in)$*-ideal of S. For any nonempty subsets G and H of S, we have the following assertion.*

$$\begin{aligned}&\overline{\mathcal{R}}_T(G;\alpha)\overline{\mathcal{R}}_T(H;\alpha)\subseteq\overline{\mathcal{R}}_T(GH;\alpha),\\&\overline{\mathcal{R}}_I(G;\beta)\overline{\mathcal{R}}_I(H;\beta)\subseteq\overline{\mathcal{R}}_I(GH;\beta),\\&\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(H;\gamma)\subseteq\overline{\mathcal{R}}_F(GH;\gamma).\end{aligned}\tag{19}$$

Proof. Let $w\in\overline{\mathcal{R}}_T(G;\alpha)\overline{\mathcal{R}}_T(H;\alpha)$. Then, $w=ab$ for some $a\in\overline{\mathcal{R}}_T(G;\alpha)$ and $b\in\overline{\mathcal{R}}_T(H;\alpha)$. It follows that $\exists\ w_a,w_b\in S$ such that $w_a\in[a]_{(T,\alpha)}\cap G$ and $w_b\in[b]_{(T,\alpha)}\cap H$. Since $\mathcal{R}_{(T,\alpha)}$ is a congruence relations on S, we have $w_aw_b\in[ab]_{(T,\alpha)}\cap GH$, and so $w=ab\in\overline{\mathcal{R}}_T(GH;\alpha)$. Similarly, we get $\overline{\mathcal{R}}_I(G;\beta)\overline{\mathcal{R}}_I(H;\beta)\subseteq\overline{\mathcal{R}}_I(GH;\beta)$. If $w\in\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(H;\gamma)$, then $\exists\ a\in\overline{\mathcal{R}}_F(G;\gamma)$ and $b\in\overline{\mathcal{R}}_F(H;\gamma)$ such that $w=ab$. Hence, $[a]_{(F,\gamma)}\cap G\neq\emptyset$ and $[b]_{(F,\gamma)}\cap H\neq\emptyset$, which imply that $\exists\ w_a\in[a]_{(F,\gamma)}\cap G$ and $w_b\in[b]_{(F,\gamma)}\cap H$. Since $\mathcal{R}_{(F,\gamma)}$ is a congruence relations on S, it follows that $w_aw_b\in[ab]_{(F,\gamma)}\cap GH$. Therefore, $w=ab\in\overline{\mathcal{R}}_F(GH;\gamma)$, and so $\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(H;\gamma)\subseteq\overline{\mathcal{R}}_F(GH;\gamma)$. □

In Proposition 3.12, the reverse inclusion relationship does not hold as seen in the next example.

Example 3.13. Consider the SVN $(\in, \in)$-ideal Ψ_{TIF} of S in Example 3.3. If we take $(\alpha,\beta,\gamma) = (0.44, 0.37, 0.63)$, then $\overline{\mathcal{R}}_T(\{\varsigma_1\};\alpha)\overline{\mathcal{R}}_T(\{\varsigma_3\};\alpha) = \{\varsigma_1\}\{\varsigma_3\} = \{\varsigma_2\}$, $\overline{\mathcal{R}}_I(\{\varsigma_1\};\beta)\overline{\mathcal{R}}_I(\{\varsigma_3\};\beta) = \{\varsigma_1\}\{\varsigma_3\} = \{\varsigma_2\}$, and $\overline{\mathcal{R}}_F(\{\varsigma_1\};\gamma)\overline{\mathcal{R}}_F(\{\varsigma_3\};\gamma) = \{\varsigma_1\}\{\varsigma_3\} = \{\varsigma_2\}$. Also $\overline{\mathcal{R}}_T(\{\varsigma_1\}\{\varsigma_3\};\alpha) = \{\varsigma_2,\varsigma_4\}$, $\overline{\mathcal{R}}_I(\{\varsigma_1\}\{\varsigma_3\};\beta) = \{\varsigma_2,\varsigma_4\}$ and $\overline{\mathcal{R}}_F(\{\varsigma_1\}\{\varsigma_3\};\gamma) = \{\varsigma_2,\varsigma_4\}$. Therefore, $\overline{\mathcal{R}}_T(\{\varsigma_1\}\{\varsigma_3\};\alpha) \nsubseteq \overline{\mathcal{R}}_T(\{\varsigma_1\};\alpha)\overline{\mathcal{R}}_T(\{\varsigma_3\};\alpha)$, $\overline{\mathcal{R}}_I(\{\varsigma_1\}\{\varsigma_3\};\beta) \nsubseteq \overline{\mathcal{R}}_I(\{\varsigma_1\};\beta)\overline{\mathcal{R}}_I(\{\varsigma_3\};\beta)$, and $\overline{\mathcal{R}}_F(\{\varsigma_1\}\{\varsigma_3\};\gamma) \nsubseteq \overline{\mathcal{R}}_F(\{\varsigma_1\};\gamma)\overline{\mathcal{R}}_F(\{\varsigma_3\};\gamma)$.

Proposition 3.14. *If congruence relations $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ on S are complete, then*

$$\begin{aligned} &\underline{\mathcal{R}}_T(G;\alpha)\underline{\mathcal{R}}_T(H;\alpha) \subseteq \underline{\mathcal{R}}_T(GH;\alpha),\\ &\underline{\mathcal{R}}_I(G;\beta)\underline{\mathcal{R}}_I(H;\beta) \subseteq \underline{\mathcal{R}}_I(GH;\beta),\\ &\underline{\mathcal{R}}_F(G;\gamma)\underline{\mathcal{R}}_F(H;\gamma) \subseteq \underline{\mathcal{R}}_F(GH;\gamma) \end{aligned} \tag{20}$$

for all nonempty subsets G and H of S.

Proof. Let $w \in \underline{\mathcal{R}}_T(G;\alpha)\underline{\mathcal{R}}_T(H;\alpha)$. Then, $w = ab$ for some $a \in \underline{\mathcal{R}}_T(G;\alpha)$ and $b \in \underline{\mathcal{R}}_T(H;\alpha)$. Since $\mathcal{R}_{(T,\alpha)}$is a complete congruence relations on S, we get $[a]_{(T,\alpha)}[b]_{(T,\alpha)} = [ab]_{(T,\alpha)} \subseteq GH$. Hence, $w = ab \in \underline{\mathcal{R}}_T(GH;\alpha)$. Therefore, $\underline{\mathcal{R}}_T(G;\alpha)\underline{\mathcal{R}}_T(H;\alpha) \subseteq \underline{\mathcal{R}}_T(GH;\alpha)$. Similarly, we have $\underline{\mathcal{R}}_I(G;\beta)\underline{\mathcal{R}}_I(H;\beta) \subseteq \underline{\mathcal{R}}_I(GH;\beta)$. If $w \in \underline{\mathcal{R}}_F(G;\gamma)\underline{\mathcal{R}}_F(H;\gamma)$, then $\exists\ a, b \in S$ such that $w = ab$, $a \in \underline{\mathcal{R}}_F(G;\gamma)$ and $b \in \underline{\mathcal{R}}_F(H;\gamma)$. Hence, $[a]_{(F,\gamma)}[b]_{(F,\gamma)} = [ab]_{(F,\gamma)} \subseteq GH$, and so $w = ab \in \underline{\mathcal{R}}_F(GH;\alpha)$. Therefore, $\underline{\mathcal{R}}_F(G;\gamma)\underline{\mathcal{R}}_F(H;\gamma) \subseteq \underline{\mathcal{R}}_F(GH;\gamma)$. □

In Proposition 3.14, if congruence relations $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ on S are not complete, then the inclusion relationship does not hold as seen in the next example.

Example 3.15. Consider the SVN $(\in, \in)$-ideal Ψ_{TIF} of S in Example 3.3, and take $(\alpha,\beta,\gamma) = (0.44, 0.37, 0.63)$. Then, $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are not complete. Obviously, $\underline{\mathcal{R}}_T(G;\alpha)\underline{\mathcal{R}}_T(H;\alpha) = \{\varsigma_2\} \nsubseteq \emptyset = \underline{\mathcal{R}}_T(GH;\alpha)$, $\underline{\mathcal{R}}_I(G;\beta)\underline{\mathcal{R}}_I(H;\beta) = \{\varsigma_2\} \nsubseteq \emptyset = \underline{\mathcal{R}}_I(GH;\beta)$, and $\underline{\mathcal{R}}_F(G;\gamma)\underline{\mathcal{R}}_F(H;\gamma) = \{\varsigma_2\} \nsubseteq \emptyset = \underline{\mathcal{R}}_F(GH;\gamma)$ where $G = H = \{\varsigma_2,\varsigma_3\}$.

The results discussed above will contribute to the study of rough subsemigroups and ideals.

Definition 3.16. Let Ψ_{TIF} be an SVN $(\in, \in)$-ideal of S and let X be a nonempty subset of S. Given $q \in \{T, I, F\}$, if $\mathcal{R}_q$-lower approximation (resp., $\mathcal{R}_q$-upper approximation) of X is a subsemigroup of S, then we say that X is a *$\mathcal{R}_q$-lower rough subsemigroup* (resp., *$\mathcal{R}_q$-upper rough subsemigroup*) of S. If $\mathcal{R}_q$-lower approximation (resp., $\mathcal{R}_q$-upper approximation) of X is an ideal of S, then we say that X is a *$\mathcal{R}_q$-lower rough ideal* (resp., *$\mathcal{R}_q$-upper rough ideal*) of S.

Theorem 3.17. *Let* Ψ_{TIF} *be an* SVN $(\in, \in)$*-ideal of* S *and* $(\alpha, \beta, \gamma) \in (0, 1] \times (0, 1] \times [0, 1)$. *If* G *is a subsemigroup (resp., ideal) of* S*, then it is an* $\mathcal{R}_q$*-upper rough subsemigroup (resp.,* $\mathcal{R}_q$*-upper rough ideal) of* S *for* $q \in \{T, I, F\}$.

Proof. Suppose G is a subsemigroup of S, then $GG \subseteq G$, and so

$$\overline{\mathcal{R}}_T(G;\alpha)\overline{\mathcal{R}}_T(G;\alpha) \subseteq \overline{\mathcal{R}}_T(GG;\alpha) \subseteq \overline{\mathcal{R}}_T(G;\alpha),$$
$$\overline{\mathcal{R}}_I(G;\beta)\overline{\mathcal{R}}_I(G;\beta) \subseteq \overline{\mathcal{R}}_I(GG;\beta) \subseteq \overline{\mathcal{R}}_I(G;\beta)$$

and

$$\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(G;\gamma) \subseteq \overline{\mathcal{R}}_F(GG;\gamma) \subseteq \overline{\mathcal{R}}_F(G;\gamma)$$

by (12) and Proposition 3.12. Hence, $\overline{\mathcal{R}}_T(G;\alpha)$, $\overline{\mathcal{R}}_I(G;\beta)$ and $\overline{\mathcal{R}}_F(G;\gamma)$ are subsemigroups of S, and so G is an $\mathcal{R}_q$-upper rough subsemigroup of S for $q \in \{T, I, F\}$. If G is an ideal of S, then $SGS \subseteq G$. Using (12) and Proposition 3.12, we have

$$\overline{\mathcal{R}}_T(S;\alpha)\overline{\mathcal{R}}_T(G;\alpha)\overline{\mathcal{R}}_T(S;\alpha) \subseteq \overline{\mathcal{R}}_T(SGS;\alpha) \subseteq \overline{\mathcal{R}}_T(G;\alpha),$$
$$\overline{\mathcal{R}}_I(S;\beta)\overline{\mathcal{R}}_I(G;\beta)\overline{\mathcal{R}}_I(S;\beta) \subseteq \overline{\mathcal{R}}_I(SGS;\beta) \subseteq \overline{\mathcal{R}}_I(G;\beta)$$

and

$$\overline{\mathcal{R}}_F(S;\gamma)\overline{\mathcal{R}}_F(G;\gamma)\overline{\mathcal{R}}_F(S;\gamma) \subseteq \overline{\mathcal{R}}_F(SGS;\gamma) \subseteq \overline{\mathcal{R}}_F(G;\gamma).$$

This shows that $\overline{\mathcal{R}}_T(G;\alpha)$, $\overline{\mathcal{R}}_I(G;\beta)$ and $\overline{\mathcal{R}}_F(G;\gamma)$ are ideals of S. Therefore, G is an $\mathcal{R}_q$-upper rough ideal of S for $q \in \{T, I, F\}$. □

Next example demonstrates that there is an $\mathcal{R}_q$-upper rough ideal for $q \in \{T, I, F\}$ which is not an ideal.

Example 3.18. Let $S = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ be a semigroup with the "·" operation given by Table 3.

TABLE 3. Table for "·" operation

$\cdot$	ς_1	ς_2	ς_3	ς_4
ς_1	ς_1	ς_2	ς_3	ς_4
ς_2	ς_2	ς_2	ς_2	ς_2
ς_3	ς_3	ς_3	ς_3	ς_3
ς_4	ς_4	ς_3	ς_2	ς_1

Let $\Psi_{\rm TIF}$ be an SVNS in S which is shown as :

$$\Psi_{\rm TIF} = \{\langle \varsigma_1, (0.5, 0.6, 0.6)\rangle, \langle \varsigma_2, (0.7, 0.9, 0.2)\rangle, \langle \varsigma_3, (0.7, 0.9, 0.2)\rangle, \langle \varsigma_4, (0.3, 0.4, 0.8)\rangle\}.$$

Clearly, $\Psi_{\rm TIF}$ is an SVN $(\in, \in)$-ideal of S. Consider $(\alpha, \beta, \gamma) \in (0, 1] \times (0, 1] \times [0, 1)$ such that the subsets $\{\varsigma_1\}$, $\{\varsigma_4\}$ and $\{\varsigma_2, \varsigma_3\}$ are the $\mathcal{R}_q$-congruence classes for $q \in \{(T, \alpha), (I, \beta), (F, \gamma)\}$. Then, $\overline{\mathcal{R}}_T(\{\varsigma_2\}; \alpha) = \{\varsigma_2, \varsigma_3\}$, $\overline{\mathcal{R}}_I(\{\varsigma_2\}; \beta) = \{\varsigma_2, \varsigma_3\}$ and $\overline{\mathcal{R}}_F(\{\varsigma_2\}; \gamma) = \{\varsigma_2, \varsigma_3\}$ which are ideals of S. Hence, $\{\varsigma_2\}$ is an $\mathcal{R}_q$-upper rough ideal for $q \in \{T, I, F\}$. But it is not an ideal of S since $S\{\varsigma_2\} = \{\varsigma_2, \varsigma_3\} \nsubseteq \{\varsigma_2\}$.

Theorem 3.19. *Let $\Psi_{\rm TIF}$ be an* SVN $(\in, \in)$*-ideal of S. in which $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are complete congruence relations on S. If G is a subsemigroup (resp., ideal) of S, then it is an $\mathcal{R}_q$-lower rough subsemigroup (resp., $\mathcal{R}_q$-lower rough ideal) of S for $q \in \{T, I, F\}$.*

Proof. If G is a subsemigroup of S, then $GG \subseteq G$ and thus

$$\underline{\mathcal{R}}_T(G; \alpha)\underline{\mathcal{R}}_T(G; \alpha) \subseteq \underline{\mathcal{R}}_T(GG; \alpha) \subseteq \underline{\mathcal{R}}_T(G; \alpha),$$
$$\underline{\mathcal{R}}_I(G; \beta)\underline{\mathcal{R}}_I(G; \beta) \subseteq \underline{\mathcal{R}}_I(GG; \beta) \subseteq \underline{\mathcal{R}}_I(G; \beta),$$
$$\underline{\mathcal{R}}_F(G; \gamma)\underline{\mathcal{R}}_F(G; \gamma) \subseteq \underline{\mathcal{R}}_F(GG; \gamma) \subseteq \underline{\mathcal{R}}_F(G; \gamma)$$

by (11) and (20). Therefore, $\underline{\mathcal{R}}_T(G; \alpha)$, $\underline{\mathcal{R}}_I(G; \alpha)$ and $\underline{\mathcal{R}}_F(G; \alpha)$ are subsemigroups of S, that is, G is an $\mathcal{R}_q$-lower rough subsemigroup of S for $q \in \{T, I, F\}$. If G is an ideal of S, then $SGS \subseteq G$. It follows from (11) and (20) that

$$\underline{\mathcal{R}}_T(S; \alpha)\underline{\mathcal{R}}_T(G; \alpha)\underline{\mathcal{R}}_T(S; \alpha) \subseteq \underline{\mathcal{R}}_T(SGS; \alpha) \subseteq \underline{\mathcal{R}}_T(G; \alpha),$$
$$\underline{\mathcal{R}}_I(S; \beta)\underline{\mathcal{R}}_I(G; \beta)\underline{\mathcal{R}}_I(S; \beta) \subseteq \underline{\mathcal{R}}_I(SGS; \beta) \subseteq \underline{\mathcal{R}}_I(G; \beta),$$
$$\underline{\mathcal{R}}_F(S; \gamma)\underline{\mathcal{R}}_F(G; \gamma)\underline{\mathcal{R}}_F(S; \gamma) \subseteq \underline{\mathcal{R}}_F(SGS; \gamma) \subseteq \underline{\mathcal{R}}_F(G; \gamma).$$

Hence, $\underline{\mathcal{R}}_T(G; \alpha)$, $\underline{\mathcal{R}}_I(G; \alpha)$ and $\underline{\mathcal{R}}_F(G; \alpha)$ are ideals of S, and therefore G is an $\mathcal{R}_q$-lower rough ideal of S for $q \in \{T, I, F\}$. □

The example below demonstrates that there is an $\mathcal{R}_q$-lower rough subsemigroup for $q \in \{T, I, F\}$ which is not a subsemigroup.

Example 3.20. Consider the SVN $(\in, \in)$-ideal $\Psi_{\rm TIF}$ of S in Example 3.9. Then, $\mathcal{R}_{(T,\alpha)}$, $\mathcal{R}_{(I,\beta)}$ and $\mathcal{R}_{(F,\gamma)}$ are complete congruence relations on S for $(\alpha, \beta, \gamma) = (0.77, 0.67, 0.29)$. Also, $\underline{\mathcal{R}}_T(\{\varsigma_1, \varsigma_2, \varsigma_4\}; \alpha) = \{\varsigma_1, \varsigma_2\}$, $\underline{\mathcal{R}}_I(\{\varsigma_1, \varsigma_2, \varsigma_4\}; \beta) = \{\varsigma_1, \varsigma_2\}$ and $\underline{\mathcal{R}}_F(\{\varsigma_1, \varsigma_2, \varsigma_4\}; \gamma) = \{\varsigma_1, \varsigma_2\}$ are subsemigroups of S. Hence, $\{\varsigma_1, \varsigma_2, \varsigma_4\}$ is an $\mathcal{R}_q$-lower rough subsemigroup of S for $q \in \{T, I, F\}$,. but it is not a subsemigroup of S since $\{\varsigma_1, \varsigma_2, \varsigma_4\}\{\varsigma_1, \varsigma_2, \varsigma_4\} = S \nsubseteq \{\varsigma_1, \varsigma_2, \varsigma_4\}$.

4. **Conclusions**

The application of the SVN set gained attention among researchers. This paper found a new link between semigroups and SVNSs by introducing an SVN $(\in, \in)$-subsemigroup and an SVN $(\in, \in)$-ideal in semigroups, and studying their properties. Special congruence relations induced by an SVN $(\in, \in)$-ideal in semigroups have been introduced. We have introduced the lower ($\mathcal{R}_q$-lower approximation) and upper approximations ($\mathcal{R}_q$-upper approximation) for $q \in \{T, I, F\}$ based on an SVN $(\in, \in)$-ideal in semigroups, and have discussed related properties. We also have defined the concepts of lower and upper subsemigroups/ideals, so called the $\mathcal{R}_q$-lower subsemigroup/ideal and the $\mathcal{R}_q$-upper subsemigroup/ideal for $q \in \{T, I, F\}$, and have considered the relationships between subsemigroups/ideals and $\mathcal{R}_q$-lower (upper) subsemigroups/ideals. In future work, various types of rough SVN ideals in semigroups will be defined and discussed. In addition, the idea in this research article can be analyzed according to the works in [18–22], which will be the way for much future work.

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