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Construction of New Similarity Measures and Entropy for Interval-Valued Neutrosophic Sets with Applications

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Abstract. The concepts of similarity measures and entropy have practical applications in computational intelligence, machine learning, image processing, neural networks, medical diagnosis, and decision analysis. An interval-valued neutrosophic set (IVNS) is strong model for modeling and handling uncertainties by using independent intervals of truthness, indeterminacy, and untruth. We introduce new similarity measures, entropy and inclusion relation for interval-valued neutrosophic sets (IVNSs). We introduce new inclusion relation named as type-f for ordering of interval neutrosophic sets. Additionally, a robust multi-attribute decision-making (MADM) method is developed by making use of proposed measures of similarity for IVNSs. A practical application for ranking of alternatives with newly developed MADM approach is illustrated by a numerical example for the car selection. The validity and superiority of new similarity measures with existing approaches is also given with the help of a comparison analysis.

Keywords: Similarity measure; entropy;interval-valued neutrosophic set; multi-attribute decision-making.

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1. **Introduction**

Zadeh [23] advanced his significant idea of fuzzy sets in 1965 to deal with various styles of uncertainties. From that time, it has been used prevalently in so many areas. Theory of fuzzy set is a more developed version of crisp set theory. By using fuzzy numbers or linguistic numbers which have numerical representation of inaccurate information, new mathematical methods have been developed for modeling the uncertain structure of today's problems. There is not a single model in fuzzy set theory, it means that many options can be reached considering the features of the system to be modeled by using various extensions of fuzzy sets. Since the problems encountered in life and human thoughts are too complex to be limited, fuzzy numbers have been inadequate at the decision-making stage such as problems involving vague or incomplete information. In this way, a fuzzy set's extension have been proposed by Atanassov [1] as intuitionistic fuzzy sets (IFS) in 1986 that include the degrees of membership and non-membership. It means that an IFS $A = \{< a, \mu_A(a), \lambda_A(a) >: a \in X\}$ has been established by two mappings $\mu_A(a), \lambda_A(a) : X \to [0, 1]$ named as membership function and non-membership function, respectively, with the restriction $0 \leq \mu_A(a) + \lambda_A(a) \leq 1, a \in X$. Later on IFSs extended towards interval-valued intuitionistic fuzzy sets (for brief: IVFSs) by Atanassov and G.Gargov [2], Turksen [15], and Gorzalczany [4]. IVIFSs have been used by these authors in the fields of signal processing, approximate inference, and controller, etc. Smarandache [12] initiated the notion of neutrosophic sets which consider indeterminate/uncertain information in today's problems and incorporated not only membership and non-membership grades, but also indeterminacy grades assigned each component of the discourse universe with is limitation that the sum of three independent grades chosen in the interval $[0, 3]$. Later on, Wang *et al.* [17,18] defined the notion of single valued neutrosophic set (SVNS) and interval neutrosophic set (INS). Besides, in [10] the definitions of fuzzy neutrosophic soft (FNS) σ-algebra, FNS-measure and FNS-outer measure are established considering the concepts of soft sets and neutrosophic sets. Additionally, illustrative examples are given in [10]. Saqlain *et al.* [11] suggested an algorithm involving neutrosophic soft set for decision making problems.

Ye [19] has created neutrosophic linguistic variables as well as any new assemblage operators for interval-valued neutrosophic linguistic data. A new MADM application is also suggested by [19]. Recently, Jun [20] proposed new similarity methods for neutrosophic sets of interval values by using and Hamming distances an developed an application of these measures in MADM problems. Additionally, Simsek and Kirisci [14], and Kirisci [6] defined the neutrosophic contraction mapping and established a fixed point theorem in neutrosophic metric spaces. Similarity measures have been successfully used in various fields, for instance; pattern recognition, image processing, medical diagnosis, decision-making, etc. Majumdar and Samanta [8] suggested a membership degree-based similarity measure between SVNSs. The cosine similarity measure and weighted cosine similarity measure of IVFSs with risk preference were described by Ye [22].

The remainder of this paper is structured as follows: Firstly, fundamental definitions are given about neutrosophic set theory such as interval-valued neutrosophic set, inclusion relations. After, type-f inclusion relation for INS is defined. In section 3, we propose the idea of similarity measures and entropy for interval-valued neutrosophic sets. Section 4 provides the numerical example to indicate how the calculation, correction and suitability of similarity measures were done. Finally in Section 5, a comparative study is given and some conclusions are outlined.

2. **Preliminaries**

In this section, we review some basic ideas of NSs, IVNSs, and distance measures.
Any variable of is called an interval number and is represented by u. Find the following: $\mathbb{E} = \{u = [u_\ell, u_r] : u_\ell, u_r \in \mathbb{R}, u_\ell \leq u_r\}$. For $u, v \in \mathbb{E}$, we have $u = v \Leftrightarrow u_\ell = v_\ell, u_r = v_r$. Define the fundamental functions of addition $+ : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, multiplication of scalars $\cdot : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$ and product $\cdot : \mathbb{R} \times \mathbb{E} \to \mathbb{E}$, respectively, as follows:

$$+(u, v) = u + v = [u_\ell + v_r, u_r + v_r],$$

$$\alpha u = \begin{cases} [\alpha u_\ell, \alpha u_r], & \alpha \geq 0 \\ [\alpha u_r, \alpha u_\ell], & \alpha < 0, \end{cases}$$

$$\cdot(u, v) = u \cdot v = [\min R, \max R], R = \{u_\ell v_\ell, u_\ell v_r, u_r v_\ell, u_r v_r\}.$$

Any two random elements (interval number) in E may not always be compared using the start and end points. A second way of comparing the interval numbers is given below:
Let $u = [u_\ell, u_r] \in E$. Then $B(u) = \max\{|a - a'| : a, a' \in u\} = u_r - u_\ell$ is called the length of interval number u. By using the property of $B(u)$, the ordering of two interval numbers u and v can be defined as

$$u \leq v \Leftrightarrow B(u) \leq B(v)$$

A fuzzy set F is a function $F : X \to I$ on the universe X, where $I = [0, 1]$. The set of α levels (α-cut) $[F]^\alpha$, and the support of the set F are given as follows:

$$[F]^\alpha = \{a \in X : F(x) \geq \alpha\}, \alpha \in (0, 1];$$

$$supp[F] = \{a \in X : A(a) > 0\}.$$

Definition 2.1. [13] A NS, N over universe X can be given by

$$N = \Big\{[a, (t_N(a), i_N(a), f_N(a))] : a \in X\Big\}$$

where $t_N(a), i_N(a), f_N(a)$ are standard or non-standard subsets of $]_\ell 0, 1_r[$ which represent truth-function, indeterminacy, and untruth-function of $a \in N$, respectively.

Definition 2.2. [18] A single-valued neutrosophic set (SVNS) on the universe X is defined as

$$A = \Big\{ < x, t_A(a), i_A(a), f_A(a) >: a \in X\Big\}$$

$t_A(a), i_A(a), f_A(a) \in [0, 1]$ indicate the degree of truthness, degree of indeterminacy, and degree of untruth, respectively.

Definition 2.3. [17] Given a set X with generic elements showed by a. A neutrosophic set of interval values $\tilde{N}$ (IVNS $\tilde{N}$) is described by an interval truth-membership function $t_{\widetilde{N}}(a) = [t_{\widetilde{N}\ell}, t_{\widetilde{N}r}]$, an interval indeterminacy-membership function $i_{\widetilde{N}}(a) = [i_{\widetilde{N}\ell}, i_{\widetilde{N}r}]$, and an interval untruth-membership function $f_{\widetilde{N}}(a) = [f_{\widetilde{N}\ell}, f_{\widetilde{N}r}]$ for each $x \in X$ and $t_{\widetilde{N}}(a), i_{\widetilde{N}}(a), f_{\widetilde{N}}(a) \subset [0,1]$. An IVNS $\widetilde{N}$ can be represented as

$$\widetilde{N} = \{[a, \left(t_{\widetilde{N}}(a), i_{\widetilde{N}}(a), f_{\widetilde{N}}(a)\right)] : a \in X\}.$$

Additionally, complement of $\widetilde{N}$ will be given as

$$\widetilde{N}^c = \{[a, \left(t_{\widetilde{N}^c}(a), i_{\widetilde{N}^c}(a), f_{\widetilde{N}^c}(a)\right)] : x \in X\}$$

where $t_{\widetilde{N}^c}(a) = f_{\widetilde{N}^c}(a), i_{\widetilde{N}^c}(a) = [1 - i_{\widetilde{N}r}(a), 1 - i_{\widetilde{N}\ell}(a)]$.

Inclusion relation is a fundamental to give definitions of union and intersection operations on any sets. In literature, there are two suggestions of the inclusion relation of neutrosophic sets. First inclusion definition for neutrosophic sets is introduced by Smarandache (see [25], [26]), it's referred to it as a inclusion relationship of type-1 and represented by $\subseteq_1$; second is the type-2 inclusion relation, which is demonstrated by $\subseteq 2$. Now, we give definitions of these inclusion relations as in the following, respectively:

Definition 2.4. [12] A single valued neutrosophic set N is included in the other single valued neutrosophic set M, it means that $N \subseteq_1 M \Leftrightarrow t_N(a) \leq t_M(a), i_N(a) \geq i_M(a), f_N(a) \geq f_M(a)$ for any $a \in X$.

Definition 2.5. [18] SVNS, N is included in the other SVNS, M, it means that $N \subseteq_2 M \Leftrightarrow t_N(a) \leq t_M(a), i_N(a) \leq i_M(a), f_N(a) \geq f_M(a)$ for any $a \in X$.

Smarandache [17] proposes an original description relation for the interval neutrosophic set as follows:

Definition 2.6. An interval neutrosophic set $\widetilde{N}$ is included in the other interval neutrosophic set $\widetilde{M}$, it means that $\widetilde{N} \subseteq_1 \widetilde{M} \Leftrightarrow t_{\widetilde{N}\ell}(a) \leq t_{\widetilde{M}\ell}(a), t_{\widetilde{N}r}(a) \leq t_{\widetilde{M}r}(a), i_{\widetilde{N}\ell}(a) \geq i_{\widetilde{M}\ell}(a), i_{\widetilde{N}r}(a) \geq i_{\widetilde{M}r}(a), f_{\widetilde{N}\ell}(a) \geq f_{\widetilde{M}\ell}(a), f_{\widetilde{N}r}(a) \geq f_{\widetilde{M}r}(a)$ for any $a \in X$.

Now, we give new definition named type-f inclusion relation:

Definition 2.7. Let $u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}])$ and $v = ([v_{1\ell}, v_{1r}], [v_{2\ell}, v_{2r}], [v_{3\ell}, v_{3r}])$ be the interval neutrosophic values. We can say $u \leq_f v$ if and only if any conditions is satisfied given as in the following:

(1) $B^t(u) \leq B^t(v)$ and $B^f(u) \geq B^f(v)$
(2) $B^t(u) = B^t(v)$ and $B^f(u) > B^f(v)$
(3) $B^t(u) = B^t(v)$ and $B^f(u) = B^f(v)$ and $B^i(u) \geq B^i(v)$.

By this way, the inclusion relation $\widetilde{N} \subseteq_f \widetilde{M}$ between interval neutrosophic sets $\widetilde{N}$ and $\widetilde{M}$ is satisfied if and only if one of the following three conditions exist:

(1) $B^t(\widetilde{N}) \leq B^t(\widetilde{M})$ and $B^f(\widetilde{N}) \geq B^f(\widetilde{M})$
(2) $B^t(\widetilde{N}) = B^t(\widetilde{M})$ and $B^f(\widetilde{N}) > B^f(\widetilde{M})$
(3) $B^t(\widetilde{N}) = B^t(\widetilde{M})$ and $B^f(\widetilde{N}) = B^f(\widetilde{M})$ and $B^i(\widetilde{N}) \geq B^i(\widetilde{M})$.

3. Similarity and Entropy of Interval Neutrosophic Sets

In this section, firstly, we give definition of similarity measure between interval neutrosophic values by means of [20].

Definition 3.1. (See [20]) Letting $S : \mathfrak{D} \times \mathfrak{D} \to [0,1]$ is similarity between interval neutrosophic values u and v if S has the following properties;

(1) $0 \leq S(u,v) \leq 1$;
(2) $S(u,v) = 1 \Leftrightarrow u = v$;
(3) $S(u,v) = S(v,u)$
(4) If $u \leq v \leq z,$ then $S(u,z) \leq S(u,v), S(u,z) \leq S(v,z)$ for all $u,v,z \in \mathfrak{D}$
$= \{u : u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}])\}$.

Now, we introduce new similarity by considering $\subseteq_f$ as given below.

Definition 3.2. Let $u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}])$ and $v = ([v_{1\ell}, v_{1r}], [v_{2\ell}, v_{2r}], [v_{3\ell}, v_{3r}])$. Then the similarity measure of u and v is defined by

$$S(u,v) = 1 - \frac{\max\{|u_{2r} - v_{2r}|, |u_{2\ell} - v_{2\ell}|\}}{2} \tag{1}$$

in the case $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}]$ and $[u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}]$ and

$$S(u,v) = \frac{1}{2} - \frac{1}{4}\{\max\{|u_{1r} - v_{1r}|, |u_{1\ell} - v_{1\ell}|\} + \max\{|u_{1r} - v_{1r}|, |u_{1\ell} - v_{1\ell}|\}\} \tag{2}$$

otherwise.

Theorem 3.3. *The values $S(u,v)$ defined by (1) and (2) are similarity measure between u and v.*

Proof. Let $u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}]), v = ([v_{1\ell}, v_{1r}], [v_{2\ell}, v_{2r}], [v_{3\ell}, v_{3r}]) \in \mathfrak{D}$. If we choose $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}]$ and $[u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}]$; then $0.5 \leq S(u,v) \leq 1$. In otherwise, $0 \leq S(u,v) \leq 0.5$.

(1) It is clear that $0 \leq S(u,v) \leq 1$,
(2) $S(u,v) = 1 \Leftrightarrow u = v$,
(3) $S(u,v) = S(v,u)$ is clearly satisfied,
(4) Let $u,v,z \in \mathfrak{D}$ and $u \leq v \leq z$, then the following cases hold:

(a) $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$.

(b) $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$.

(c) $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$. From here, $[u_{1\ell}, u_{1r}] < [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] \geq [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$.

(d) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$.

(e) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$.

(f) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] > [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$.

(g) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}]$ and $[v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] < [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}] \geq [z_{3\ell}, z_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}]$.

(h) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}] > [z_{3\ell}, z_{3r}]$.

(i) $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}]$ and $[v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$. From here, $[u_{1\ell}, u_{1r}] = [v_{1\ell}, v_{1r}] = [z_{1\ell}, z_{1r}], [u_{3\ell}, u_{3r}] = [v_{3\ell}, v_{3r}] = [z_{3\ell}, z_{3r}], [u_{2\ell}, u_{2r}] \geq [v_{2\ell}, v_{2r}] \geq [z_{2\ell}, z_{2r}]$.

Finally, in all cases it is deduced that $S(u, z) \leq S(u, v), S(u, z) \leq S(v, z)$.

Consequently, it is deduced that $S(u, v)$ is similarity on u and v. □

Fuzziness is significant topic in neutrosophic sets and there exist soo many ways to measure this fuzziness. Here, firstly we give definition of entropy for interval neutrosophic value then construct original entropy for neutrosophic value u.

Definition 3.4. [7] If E has the following properties, $E : \mathfrak{D} \rightarrow [0, 1]$ is an entropy of interval neutrosophic value:

(1) $E(u) = 0 \Leftrightarrow [u_{1\ell}, u_{1r}] = [0,0]$ or $[1,1]$ and $[u_{3\ell}, u_{3r}] = [0,0]$ or $[1,1]$;
(2) $E(u) = 1 \Leftrightarrow [u_{1\ell}, u_{1r}] = [u_{2\ell}, u_{2r}] = [u_{3\ell}, u_{3r}] = [0.5, 0.5]$;
(3) $E(u) = E(u^c)$;
(4) Let $u, v \in \mathfrak{D}$ then $v^c = ([v_{3\ell}, v_{3r}], [1 - v_{2r}, 1 - v_{2\ell}], [v_{1\ell}, v_{1r}])$ and $E(u) \leq E(v)$ if $u \leq_f v$ when $v \leq_f v^c$ or $v \leq_f u$ when $v^c \leq_f v$.

Definition 3.5. Let $u = ([u_{1\ell}, u_{1r}], [u_{2\ell}, u_{2r}], [u_{3\ell}, u_{3r}])$. Then the entropy for u is defined by

$$E(u) = \begin{cases} 1 - \frac{|u_{2r}+u_{2\ell}-1|}{2}, & [u_{1\ell}, u_{1r}] = [u_{3\ell}, u_{3r}] = [\frac{1}{2}, \frac{1}{2}] \\ \frac{1}{2} - \frac{1}{2}\left\{\max\{|u_{1\ell} - u_{3\ell}|, |u_{1r} - u_{3r}|\}\right\}, & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 3.6. *$E(u)$ introduced as (3) isn entropy for u.*

Proof. If $[u_{1\ell}, u_{1r}] = [u_{3\ell}, u_{3r}] = [\frac{1}{2}, \frac{1}{2}]$, then it is easy to see that $\frac{1}{2} \leq E(u) \leq 1$. In other case, $0 \leq E(u) \leq \frac{1}{2}$.

(1) $E(u) = 0 \Leftrightarrow \frac{1}{2} - \frac{1}{2}\left\{\max\{|u_{1\ell} - u_{3\ell}|, |u_{1r} - u_{3r}|\}\right\} = 0$
$\Leftrightarrow 1 = \max\{|u_{1\ell} - u_{3\ell}|, |u_{1r} - u_{3r}|\}$
$\Leftrightarrow [u_{1\ell}, u_{1r}] = [0,0]$ or $[1,1]$, $[u_{3\ell}, u_{3r}] = [1,1]$ or $[0,0]$.
(2) $E(u) = 1 \Leftrightarrow [u_{1\ell}, u_{1r}] = [u_{2\ell}, u_{2r}] = [u_{3\ell}, u_{3r}] = [\frac{1}{2}, \frac{1}{2}]$.
(3) $E(u) = E(u^c) =$ is clearly satisfied.
(4) Let $u, v \in \mathfrak{D}$ and $v^c = ([v_{3\ell}, v_{3r}], [1 - v_{2r}, 1 - v_{2\ell}], [v_{1\ell}, v_{1r}])$. If $u \leq_f v$ when $v \leq_f v^c$ or $v \leq_f u$ when $v^c \leq_f v$ then $E(u) \leq E(v)$.

This completes the proof. □

3.1. *Definition of Similarity and Entropy of INSs*

In [21], similarity and entropy measure definitions of interval neutrosophic values expanded to interval neutrosophic sets. Now, we introduce this definition as follows, respectively.

Definition 3.7. (See [21]) Let M, N be two interval neutrosophic sets. Then, S is called similarity measure between M and N, if the following properties are satisfied:

(1) $0 \leq S(M, N) \leq 1$;
(2) $S(M, N) = 1 \Leftrightarrow M = N$;
(3) $S(M, N) = S(N, M)$
(4) If $M \subseteq N \subseteq P$, then $S(M, P) \leq S(M, N)$, $S(M, P) \leq S(B, P)$ for all $M, N, P \in$ INSs.

Definition 3.8. (See [21]) Let M be an interval neutrosophic set, then we give the definition E as interval neutrosophic sets' entropy if E contains the following assertions:

(1) $E(M) = 0 \Leftrightarrow B^t(M) = [0,0]$ or $[1,1]$, $B^f(M) = [0,0]$ or $[1,1]$;
(2) $E(M) = 1 \Leftrightarrow B^t(M) = B^i(M) = B^f(M) = [0.5, 0.5]$;

(3) $E(M) = E(M^c)$;

(4) Let M, N are two INSs, $E(M) \leq E(N)$ if $M \subseteq_f N$ when $N \subseteq_f N^c$, or $N \subseteq_f M$ when $N^c \subseteq_f N$.

In addition to above definitions, [21] gain the literature similarity and entropy concepts of two neutrosophic sets. It means that similarity measure of interval neutrosophic values is carried on the interval neutrosophic sets as showed in the following definition.

Definition 3.9. [21]

$$S(M,N) = \frac{1}{n}\sum_{i=1}^{n} s(M(x_i), N(x_i))$$

where $X = \{x_1, x_2, ..., x_n\}$ is a NS and $s : \mathfrak{D} \times \mathfrak{D} \to [0,1]$ is similarity of INS for $M, N \subseteq X$. And

$$E(M) = \frac{1}{n}\sum_{i=1}^{n} e(M(x_i)), e : \mathfrak{D} \to [0,1].$$

4. Multi-attribute Decision-making

Ye [20] employs a multi-attribute decision-making process for single valued neutrosophic sets. First we discuss the Hamming distance, Euclidean distance, and measure of similarities for INSs developed by Ye [20].

(1) The Hamming Distance:
$d_1(A,B) = \frac{1}{6}\sum_{i=1}^{n}(|t_{A\ell}(u_i) - t_{B\ell}(u_i)| + |t_{Ar}(u_i) - t_{Br}(u_i)| + |i_{A\ell}(u_i) - i_{B\ell}(u_i)| + |i_{Ar}(u_i) - i_{Br}(u_i)| + |f_{A\ell}(u_i) - f_{B\ell}(u_i)| + |f_{Ar}(u_i) - f_{Br}(u_i)|)$.

(2) The Euclidean Distance:
$d_2(A,B) = \frac{1}{6}\sum_{i=1}^{n}(t_{A\ell}(u_i) - t_{B\ell}(u_i))^2 + (t_{Ar}(u_i) - t_{Br}(u_i))^2 + (i_{A\ell}(u_i) - i_{B\ell}(u_i))^2 + (i_{Ar}(u_i) - i_{Br}(u_i))^2 + (f_{A\ell}(u_i) - f_{B\ell}(u_i))^2 + (f_{Ar}(u_i) - f_{Br}(u_i))^2$.

(3) Similarity Measure:
$S_1(A,B) = 1 - \frac{1}{6}\sum_{i=1}^{n}(|t_{A\ell}(u_i) - t_{B\ell}(u_i)| + |t_{Ar}(u_i) - t_{Br}(u_i)| + |i_{A\ell}(u_i) - i_{B\ell}(u_i)| + |i_{Ar}(u_i) - i_{Br}(u_i)| + |f_{A\ell}(u_i) - f_{B\ell}(u_i)| + |f_{Ar}(u_i) - f_{Br}(u_i)|)$.

(4) Similarity Measure:
$S_2(A,B) = 1 - \frac{1}{6}\sum_{i=1}^{n}(t_{A\ell}(u_i) - t_{B\ell}(u_i))^2 + (t_{Ar}(u_i) - t_{Br}(u_i))^2 + (i_{A\ell}(u_i) - i_{B\ell}(u_i))^2 + (i_{Ar}(u_i) - i_{Br}(u_i))^2 + (f_{A\ell}(u_i) - f_{B\ell}(u_i))^2 + (f_{Ar}(u_i) - f_{Br}(u_i))^2$.

Next we give a numerical example for MADM and a comparison analysis of Ye's methods, Wang's method with our proposed MADM method.

4.1. *Numerical Example*

Let $\{M_1, M_2, M_3, M_4\}$ be the set of cars (alternatives), and $\{P_1, P_2, P_3\}$ be the set criterion for the selection of a suitable car, where P_1 is fuel compatibility and performance, M_2 is resale value and affordability, M_3 is safety and ride.
Let us consider the following interval neutrosophic set

$$M = (([0.7, 0.8], [0.1, 0.2], [0.1, 0.3]), ([0.8, 0.9], [0, 0.1], [0.2, 0.4]), ([0.6, 0.8], [0.1, 0.2], [0.3, 0.6]))$$

as a model option for the selection of a best car under given criterion.

The alternatives are evaluated under the given criterion and the interval neutrosophic decision matrix is computed and it is given in Table 1, where the columns represent the criteria and the rows represent the alternatives.
First we calculate similarity measure values by using proposed similarity measures under interval neutrosophic set as given below:

$$S_z(M_1, M) = 0.408, S_z(M_2, M) = 0.425, S_z(M_1, M) = 0.366, S_z(M_4, M) = 0.4.$$

Hence,

$$S(M_2, M) \succ S(M_1, M) \succ S(M_4, M) \succ S(M_3, M)$$

That is,

$$M_2 \succ M_1 \succ M_4 \succ M_3$$

Here M_2 is the best choice for selecting a car.
Now we consider the similarity measure values by means of [20] as given below.

$$\left(\left[\max_i \left(t_{\ell M_i}(x_i)\right), \max_i \left(t_{r M_i}(x_i)\right)\right], \left[\min_i \left(i_{\ell M_i}(x_i)\right), \min_i \left(i_{r M_i}(x_i)\right)\right], \left[\min_i \left(f_{\ell M_i}(x_i)\right), \min \left(f_{r M_i}(x_i)\right)\right]\right).$$

Secondly, we use Ye's method and obtain the following results:

$$S_1(M_1, M) = 0.55, S_1(M_2, M) = 0.7, S_1(M_3, M) = 0.4, S_1(M_4, M) = 0.6$$

$$S_2(M_1, M) = 0.881, S_2(M_2, M) = 0.92, S_2(M_3, M) = 0.823, S_2(M_4, M) = 0.886.$$

Hence, it is deduced that

$$S_2(M_2, M) \succ S_2(M_4, M) \succ S_2(M_1, M) \succ S_2(M_3, M)$$

That is,

$$M_2 \succ M_4 \succ M_1 \succ M_3$$

Thus and so, M_2 is the most suitable alternative.

Lastly, we calculate similarity measure by taking into account Yang *et al.* [21] method and determine the best option also as below:

$$S(M_1, M) = 0.420, S(M_2, M) = 0.445, S(M_1, M) = 0.375, S(M_4, M) = 0.416.$$

We have following results,

$$S(M_2, M) \succ S(M_1, M) \succ S(M_4, M) \succ S(M_3, M)$$

That is,

$$M_2 \succ M_1 \succ M_4 \succ M_3$$

Finally, M_2 is the best option for car selection. As shown in Table 2, the suggested MADM method is compared to established MADM methods. It can be noted in the comparison Table 2, the selected alternative given by any one proposed method acknowledges the authenticity and the efficacy of the existing methodology.

5. Conclusion

The concept of interval neutrosophic set (INS) is a strong model for MADM. We introduced new similarity measures, entropy, and inclusion relation named as type-f for interval neutrosophic sets (INSs). Then we developed robust MADM method for car selection by using proposed similarity measures for INSs. Meanwhile, a practical application for ranking of alternatives with newly developed MADM approach is illustrated by a numerical example. We computed similarity measures by our proposed method and compared the results with existing methods of Ye [20] and Yang *et al.* [21]. The validity and superiority of new similarity measures with existing approaches is also given with the help of a comparison analysis. Finally, it is deduced that proposed similarity measure and inclusion relations are more efficient, impressive and suitable.

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Appendix

Table 1. The interval neutrosophic decision matrix

	C_1	C_2	C_3
M_1	([0.3,0.5],[0.2,0.4],[0.3,0.5])	([0.4,0.6],[0.1,0.2],[0.2,0.5])	([0.6,0.8],[0.2,0.4],[0.3,0.6])
M_2	([0.7,0.8],[0.1,0.2],[0.2,0.3])	([0.8,0.9],[0,0.1],[0.3,0.4])	([0.4,0.5],[0.3,0.5],[0.7,0.8])
M_3	([0.4,0.5],[0.1,0.3],[0.4,0.5])	([0.5,0.6],[0.3,0.4],[0.2,0.4])	([0.3,0.5],[0.1,0.2],[0.7,0.9])
M_4	([0.7,0.8],[0.2,0.4],[0.1,0.3])	([0.4,0.5],[0,0.2],[0.4,0.5])	([0.5,0.6],[0.1,0.2],[0.7,0.8])

Table 2. Comparison analysis of final ranking with existing methods.

Method	Ranking of alternatives	The optimal alternative
Proposed method	$M_2 \succ M_1 \succ M_4 \succ M_3$	M_2
Ye's method	$M_2 \succ M_4 \succ M_1 \succ M_3$	M_2
Wang's method	$M_2 \succ M_1 \succ M_4 \succ M_3$	M_2

References

[1] Atanassov, K.T.; Intuitionistic Fuzzy Sets, pp. 87-96, Fuzzy Sets and Systems, 1986.

[2] Atanassov, K.T. & Gargov, G.; Interval Valued Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 31(3), pp. 343-349, 1989.

[3] Chen, L.; Distance Between Interval-valued Fuzzy Sets, The 28th North American Fuzzy Information Processing Society Annual Conference, Cincinnati, Ohio, USA, 2009.

[4] Gorzalczany, B.; Approximate Inference with Interval-valued Fuzzy Sets - an outline, in: Proceedings of the Polish Symposium on Interval and Fuzzy Mathematics, Poznan, Poland, pp.89-95, 1983.

[5] Hong, D.H, Moon, E.L. & Kim, J.D.; A Note On the Core of Fuzzy Numbers, Fuzzy Sets and Systems,v.98, pp.331-335, 1998.

[6] Kirişçi, M.; On a New Metric Space. Conference Proceedings of Science and Technology, v.2, no.1, pp.51-54, 2019.

[7] Liu, C.F. & Lou, Y.S.; A New Method to Construct Entropy of Interval-Valued Neutrosophic Set. Nuetrosophic Sets Syst., v.11, pp.8-11, 2016.

[8] Majumdar, P. & Samanta, S.K.; On Similarity and Entropy of Neutrosophic Sets. Journal of Intelligent and Fuzzy Systems. v.26, no.3, pp.1245-1252, 2013.

[9] Meenakshi, A.R. & Kaliraja, M.; Regular Interval Valued Fuzzy Matrices. Advanced in Fuzzy Mathematics, v.5, pp.7-15, 2010.

[10] Naeem, K., Riaz, M. & Afzal, D.; Fuzzy Neutrosophic Soft σ-algebra and Fuzzy Neutrosophic Soft Measure with Applications, Journal of Intelligent and Fuzzy Systems, vol.39, no.1, pp.277-287, 2020.

[11] Saqlain, M., Naveed Jafar, M. & Riaz, M.; A New Approach of Neutrosophic Soft Set with Generalized Fuzzy TOPSIS in Application of Smart Phone Selection. Neutrosophic Sets and Systems, vol.32, pp.307-316, 2020.

[12] Smarandache, F.; Neutrosophy: Neutrosophic Probability, Set and Logic: Analytic Synthesis & Synthetic Analysis. Rehoboth: American Research Press, 1998.

[13] Smarandache, F., Neutrosophic: Neutrosophic Probability, Set and Logic; American Researcher Press: Rehoboth, DE, USA, 1999.

[14] Şimşek, N. & Kirişçi, M.; Neutrosophic Metric Spaces and Fixed Point Results. Conference Proceedings of Science and Technology, v.2, no.1, pp.64-67, 2019.

[15] Turksen, B.; Interval-Valued Fuzzy Sets Based on Normal Forms. Fuzzy Sets and Systems, v.20, pp. 191-210), 1986.

[16] Wang, G. & Li, X.; The Applications of Interval-Valued Fuzzy Numbers and Interval-Distribution Numbers, Fuzzy Sets and Systems v.98, 1998 pp.331-335.

[17] Wang, H., Smarandache F., Zhang, Y.Q. & Sunderraman, R.; Interval Neutrosophic Sets and Logic: Theory and Applications in Computing. Arizona, Hexis, 2005.

[18] Wang, H., Smarandache F., Zhang, Y.Q. & Sunderraman, R.; Single Valued Neutrosophic Sets. Multispace Multistruct., v.4, 2010, pp.410-413.

[19] Ye, J.; Some Aggregation Operators of Interval Neutrosophic Linguistic Numbers for Multiple-attribute Decision-making. Journal of Intelligent and Fuzzy System, 2014, DOI:10.3233/IFS-141187.

[20] Ye, J.; Similarity Measures Between Interval Neutrosophic Sets and Their Applications in Multicriteria Decision-making. J. Intell. Fuzzy Syst., v.26, 2014, pp.165-172.

[21] Yang, H., Wang, X. & Qin, K. New Similarity and Entropy Measures of Interval Neutrosophic Sets with Applications in Multi-attribute Decision-making. Symmetry, v.11, no.3, 2019, doi:10.3390/sym11030370.

[22] Ye, J. Multicriteria Decision-making Method Based on Cosine Similarity Measures Between Interval-valued Fuzzy Sets with Risk Preference. Economic Computation and Economic Cybernetics Studies and Research, v.50, no.4, 2016, pp.205-215.

[23] Zadeh, L.A.; Fuzzy Sets. 1965, pp.338-353. Inf. Control.

[24] Zararsız, Z.; Similarity Measures of Sequence of Fuzzy Numbers and Fuzzy Risk Analysis. Advances in Mathematical Physics, 12 pages, 2015. Article ID 724647, http://dx.doi.org/10.1155/2015/724647.

[25] Zhang, X.H., Bo, C.X., Smarandache, F. & Dai, J.H.; New Inclusion Relation of Neutrosophic Sets with Applications and Related Lattice Structure. Int. J. Mach. Learn. Cybern. 2018.

[26] Zhang, X.H., Bo, C.X., Smarandache, F. & Park, C.; New Operations of Totally Dependent-neutrosophic

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