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Soft Subring Theory Under Interval-valued Neutrosophic Environment

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Abstract. The primary goal of this article is to establish and investigate the idea of interval-valued neutrosophic soft subring. Again, we have introduced function under interval-valued neutrosophic soft environment and investigated some of its homomorphic attributes. Additionally, we have established product of two interval-valued neutrosophic soft subrings and analyzed some of its fundamental attributes. Furthermore, we have presented the notion of interval-valued neutrosophic normal soft subring and investigated some of its algebraic properties and homomorphic attributes.

Keywords: Neutrosophic set; Interval-valued neutrosophic soft set; Interval-valued neutrosophic soft subring; Interval-valued neutrosophic normal soft subring

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ABBREVIATIONS

TN indicates "T-norm".

SN indicates "S-norm".

IVTN indicates "Interval-valued T-norm".

IVSN indicates "Interval-valued S-norm".

CS indicates "Crisp set".

US indicates "Universal set".

FS indicates "Fuzzy set".

IFS indicates "Intuitionistic fuzzy set".

NS indicates "Neutrosophic set".

PS indicates "Plithogenic set".

SS indicates "Soft set".

IVFS indicates "Interval-valued fuzzy set".

IVIFS indicates "Interval-valued intuitionistic fuzzy set".

IVNS indicates "Interval-valued neutrosophic set".

NSSR indicates "Neutrosophic soft subring".

NNSSR indicates "Neutrosophic normal soft subring".

IVNSR indicates "Interval-valued neutrosophic subring".

IVNSSR indicates "Interval-valued neutrosophic soft subring".

IVNNSSR indicates "Interval-valued neutrosophic normal soft subring".

DMP indicates "Decision making problem".

$\boldsymbol{\phi(F)}$ indicates "Power set of F".

$\boldsymbol{K}$ indicates "The set $[0, 1]$".

1. Introduction

Uncertainty plays a huge part in different economical, sociological, biological, as well as other scientific fields. It is not always possible to tackle ambiguous data using CS theory. To cope with its limitations Zadeh introduced the groundbreaking concept of FS [1] theory. Which was further generalized by Atanassov as IFS [2] theory. Later on, Smarandache extended these notions by introducing NS [3] theory, which became more reasonable for managing indeterminate situations. From the beginning, NS theory became very popular among various researchers. Nowadays, it is heavily utilized in numerous research domains. PS [4] theory is another innovative concept introduced by Smarandache, which is more general than all the previously mentioned notions. In NS and PS theory some of Smarandache's remarkable contributions are the notions of neutrosophic robotics [5], neutrosophic psychology [6], neutrosophic measure [7], neutrosophic calculus [8], neutrosophic statistics [9], neutrosophic probability [10], neutrosophic triplet group [11], plithogenic logic, probability [12], plithogenic subgroup [13], plithogenic aggregation operators [14], plithogenic hypersoft set [15], plithogenic fuzzy whole hypersoft set [16], plithogenic hypersoft subgroup [17], etc. Moreover, NS and PS theory has several contributions in various other scientific fields, for instance, in selection of suppliers [18], professional selection [19], fog and mobile-edge computing [20], fractional programming [21], linear programming [22], shortest path problem [23–30], supply chain problem [31], DMP [32–37], healthcare [38, 39], etc.

Interval-valued versions of FS [40], IFS [41], and NS [42] are further generalizations of their previously discussed counterparts. Since the beginning, various researchers have carried out this concepts and explored them in different research domains. For instance, nowadays in logic [42], abstract algebra [43–46], graph theory [47, 48], DMPs [49–51], etc., these concepts are widely used.

Another set theory of utmost importance is SS [52] theory. It was introduced by Molodtsov to deal with uncertainty more conveniently and easily. At present, it is extensively used in different scientific areas, like in DMPs [53–57], abstract algebra [58–61], stock treading [62], etc. Furthermore, to achieve higher uncertainty handling potentials researchers have implemented SS theory in different interval-valued environments. The following Table 1 comprises some momentous aspects of different interval-valued soft notions.

TABLE 1. Significance of different interval-valued soft notions in various fields.

Author & references	**Year**	**Contributions in various fields**
Yang et al. [63]	2009	Introduced soft IVFS and defined complement, "and" and "or" operations on them.
Jiang et al. [64]	2010	Proposed soft IVIFS and defined complement, "and", "or", union, intersection, necessity, and possibility operations on them.
Feng et al. [65]	2010	Introduced soft reduct fuzzy sets of soft IVFS and utilizing soft versions of reduct fuzzy sets and level sets, proposed flexible strategy for DMP.
Broumi et al. [66]	2014	Presented generalized soft IVNS, analyzed some set operations and further, applied it in DMP.
Mukherje et al. [67]	2014	Proposed relation on soft IVIFSs and presented a solution to a DMP.
Broumi et al. [68]	2014	Proposed relation on soft IVNSs and studied reflexivity, symmetry, transitivity of it.
Mukherje and Sarkar [69]	2015	Defined Euclidean and Hamming distances between two soft IVNSs and presented similarity measures according to distances within them.
Deli [70]	2017	Defined soft IVNS and introduced some operations. Further, implemented this in DMP.
Garg and Arora [71]	2018	Solved DMP with soft IVIFS information.

Group theory and ring theory are essential parts of abstract algebra, which have various applications in different research domains. But these were initially introduced under the crisp environment, which has certain limitations. From the year 1971, various mathematicians started implementing uncertainty theories to generalize these notions. Some noteworthy contributions in the field of group theory under uncertainty can be found on [72–76]. In ring theory under uncertainty, the following articles [77–80] are some important developments. Again, several researchers introduced these notions under soft environments. For instance, researchers have introduced the concepts of ring theory under soft fuzzy [81], soft intuitionistic fuzzy [82], and soft neutrosophic [83] environments. Also, some more articles which can be helpful to different researchers are [84–91], etc. Now, by mixing interval-valued environment with soft neutrosophic environment, we can introduce a more general version of NSSR, which will be called IVNSSR. Also, their homomorphic attributes can be studied. Again, their product and normal versions can be introduced and studied. Based on these perceptions, the followings are our primary objectives for this article:

- Introducing the concept of IVNSSR and a analyzing its homomorphic attributes.
- Introducing the product of IVNSSRs.
- Introducing subring of a IVNSSR.
- Introducing the concept of IVNNSSR and a analyzing its homomorphic properties.

The arrangement our article is: in Section 2, some desk researches of IVTN, IVSN, NS, IVNS, IVNSS, NSR, NSSR, etc., are discussed. In Section 3, the concept of IVNSSR has been introduced and some fundamental theories are provided. Also, their product and normal versions are defined and some theories are given to understand their different algebraic characteristics. Lastly, in Section 4, mentioning some future scopes, the concluding segment is given.

2. Literature Review

Definition 2.1. [92] A function $T : K \to K$ is known as a TN iff $\forall g, n, z \in K$, the followings can be concluded

(i) $T(g, 1) = g$
(ii) $T(g, n) = T(n, g)$
(iii) $T(g, n) \leq T(z, n)$ if $g \leq z$
(iv) $T(g, T(n, z)) = T(T(g, n), z)$

Definition 2.2. [93] A function $\bar{T} : \phi(K) \times \phi(K) \to \phi(K)$ defined as $\bar{T}(\bar{g}, \bar{n}) = [T(g^-, n^-), T(g^+, n^+)]$ (T is a TN) is known as an IVTN.

Definition 2.3. [92] A function $S : K \to K$ is known as SN iff $\forall g, n, z \in K$, the followings can be concluded

(i) $S(g, 0) = g$
(ii) $S(g, n) = S(n, g)$
(iii) $S(g, n) \leq S(z, n)$ if $g \leq z$
(iv) $S(g, S(n, z)) = S(S(g, n), z)$

Definition 2.4. [93] The function $\bar{S} : \phi(K) \times \phi(K) \to \phi(K)$ defined as $\bar{S}(\bar{g}, \bar{n}) = [S(g^-, n^-), S(g^+, n^+)]$ (S is a SN) is called an IVSN.

Definition 2.5. [3] A NS σ of a CS Q is denoted as $\sigma = \big\{\big(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)\big) : g \in Q\big\}$. Here $\forall g \in Q$, $t_\sigma(g), i_\sigma(g),$ and $f_\sigma(g)$ are known as degree of truth, indeterminacy, and falsity which satisfy the inequality ${}^{-}0 \leq t_\sigma(g) + i_\sigma(g) + f_\sigma(g) \leq 3^{+}$.

The set of all NSs of Q will be expressed as NS(Q).

Definition 2.6. [52] Let Q be a US and A be a set of parameters. Also, let $L \subseteq A$. Then the ordered pair (f, L) is called a SS over Q, where $f : L \to \phi(Q)$ is a function.

Definition 2.7. [94] Let Q be a US and A be a set of parameters. Also, let $M \subseteq A$. Then a NSS over Q is denoted as (f, M) where $f : M \to \mathrm{NS}(Q)$ is a function.

The following Definition 2.7 is a redefined version of NSS, which we have adopted in this article.

Definition 2.8. [56] Let Q be a US and A be a set of parameters. Then a NSS δ of Q is denoted as $\delta = \big\{\big(r, l_\delta(r)\big) : r \in A\big\}$ where $l_\delta : A \to \mathrm{NS}(Q)$ is a function which is also known as an approximate function of NSS δ and $l_\delta(r) = \big\{\big(g, t_{l_{\delta(r)}}(g), i_{l_{\delta(r)}}(g), f_{l_{\delta(r)}}(g)\big) : g \in Q\big\}$. Here, $\forall g \in Q$, $t_{l_{\delta(r)}}(g)$, $i_{l_{\delta(r)}}(g)$, and $f_{l_{\delta(r)}}(g) \in [0, 1]$ and they satisfy the inequality $3 \geq t_{l_{\delta(r)}}(g) + i_{l_{\delta(r)}}(g) + f_{l_{\delta(r)}}(g) \geq 0$.

The set of all NSSs of a set Q will be expressed as NSS(Q).

Definition 2.9. [42] An IVNS of Q is defined as the mapping $\bar{\sigma} : Q \to \phi(K) \times \phi(K) \times \phi(K)$, where $\bar{\sigma}(g) = \big\{\big(g, \bar{t}_{\bar{\sigma}}(g), \bar{i}_{\bar{\sigma}}(g), \bar{f}_{\bar{\sigma}}(g)\big) : g \in Q\big\}$, where $\forall g \in Q$, $\bar{t}_{\bar{\sigma}}(g)$, $\bar{i}_{\bar{\sigma}}(g)$, and $\bar{f}_{\bar{\sigma}}(g) \subseteq [0, 1]$.

The set of all IVNSs of a set Q will be expressed as IVNS(Q).

Definition 2.10. [70] Let Q be a US and A be a set of parameters. Then a IVNSS Ψ of Q is denoted as $\Psi = \big\{\big(r, l_\Psi(r)\big) : r \in A\big\}$, where $l_\Psi : A \to \mathrm{IVNS}(Q)$ is a function which is also known as an approximate function of IVNSS Ψ and $l_\Psi(r) = \big\{\big(g, \bar{t}_{l_{\Psi(r)}}(g), \bar{i}_{l_{\Psi(r)}}(g), \bar{f}_{l_{\Psi(r)}}(g)\big) : g \in Q\big\}$. Here, $\forall g \in Q$, $\bar{t}_{l_{\Psi(r)}}(g)$, $\bar{i}_{l_{\Psi(r)}}(g)$, and $\bar{f}_{l_{\Psi(r)}}(g) \subseteq [0, 1]$.

The set of all IVNSSs of a set Q will be expressed as IVNSS(Q).

Definition 2.11. [70] $\Psi_1 = \big\{\big(r, l_{\Psi_1}(r)\big) : r \in A\big\}$ and $\Psi_2 = \big\{\big(r, l_{\Psi_2}(r)\big) : r \in A\big\}$ be two IVNSSs of Q. Then $\Psi = \Psi_1 \cup \Psi_2 = \big\{\big(r, l_\Psi(r)\big) : r \in A\big\}$ is defined as

$$\bar{t}_{l_{\Psi(r)}} = \big[\max\big\{\bar{t}^{-}_{l_{\Psi_1}(r)}, \bar{t}^{-}_{l_{\Psi_2}(r)}\big\}, \max\big\{\bar{t}^{+}_{l_{\Psi_1}(r)}, \bar{t}^{+}_{l_{\Psi_2}(r)}\big\}\big]$$
$$\bar{t}_{l_{\Psi(r)}} = \big[\min\big\{\bar{i}^{-}_{l_{\Psi_1}(r)}, \bar{i}^{-}_{l_{\Psi_2}(r)}\big\}, \min\big\{\bar{i}^{+}_{l_{\Psi_1}(r)}, \bar{i}^{+}_{l_{\Psi_2}(r)}\big\}\big]$$
$$\bar{t}_{l_{\Psi(r)}} = \big[\min\big\{\bar{f}^{-}_{l_{\Psi_1}(r)}, \bar{f}^{-}_{l_{\Psi_2}(r)}\big\}, \min\big\{\bar{f}^{+}_{l_{\Psi_1}(r)}, \bar{f}^{+}_{l_{\Psi_2}(r)}\big\}\big]$$

Definition 2.12. [70] $\Psi_1 = \big\{\big(r, l_{\Psi_1}(r)\big) : r \in A\big\}$ and $\Psi_2 = \big\{\big(r, l_{\Psi_2}(r)\big) : r \in A\big\}$ be two IVNSSs of Q. Then $\Psi = \Psi_1 \cap \Psi_2 = \big\{\big(r, l_{\Psi}(r)\big) : r \in A\big\}$ is defined as

$$\bar{t}_{l_\Psi(r)} = \big[\min\big\{\bar{t}^-_{l_{\Psi_1}(r)}, \bar{t}^-_{l_{\Psi_2}(r)}\big\}, \min\big\{\bar{t}^+_{l_{\Psi_1}(r)}, \bar{t}^+_{l_{\Psi_2}(r)}\big\}\big]$$
$$\bar{t}_{l_\Psi(r)} = \big[\max\big\{\bar{i}^-_{l_{\Psi_1}(r)}, \bar{i}^-_{l_{\Psi_2}(r)}\big\}, \max\big\{\bar{i}^+_{l_{\Psi_1}(r)}, \bar{i}^+_{l_{\Psi_2}(r)}\big\}\big]$$
$$\bar{t}_{l_\Psi(r)} = \big[\max\big\{\bar{f}^-_{l_{\Psi_1}(r)}, \bar{f}^-_{l_{\Psi_2}(r)}\big\}, \max\big\{\bar{f}^+_{l_{\Psi_1}(r)}, \bar{f}^+_{l_{\Psi_2}(r)}\big\}\big]$$

2.1. *Neutrosophic subring*

Definition 2.13. [80] Let $(Q, +, \cdot)$ be a crisp ring. A NS $\sigma = \big\{\big(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)\big) : g \in Q\big\}$ is called a NSR of F, iff $\forall g, n \in Q$,

(i) $t_\sigma(g+n) \geq T\big(t_\sigma(g), t_\sigma(n)\big),\ i_\sigma(g+n) \geq I\big(i_\sigma(g), i_\sigma(n)\big),\ f_\sigma(g+n) \leq F\big(f_\sigma(g), f_\sigma(n)\big)$
(ii) $t_\sigma(-g) \geq t_\sigma(g),\ i_\sigma(-g) \geq i_\sigma(g),\ f_\sigma(-g) \leq f_\sigma(g)$
(iii) $t_\sigma(g \cdot n) \geq T\big(t_\sigma(g), t_\sigma(n)\big),\ i_\sigma(g \cdot n) \geq I\big(i_\sigma(g), i_\sigma(n)\big),\ f_\sigma(g \cdot n) \leq S\big(f_\sigma(g), f_\sigma(n)\big)$.

Here, T and I are two TNs and S is a SN.

The set of all NSR of a crisp ring $(Q, +, \cdot)$ will be expressed as NSR(Q).

Proposition 2.1. *[80] A NS $\sigma = \big\{\big(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)\big) : g \in Q\big\}$ is called a NSR of Q, iff $\forall g, n \in Q$,*

(i) $t_\sigma(g-n) \geq T\big(t_\sigma(g), t_\sigma(n)\big),\ i_\sigma(g-n) \geq I\big(i_\sigma(g), i_\sigma(n)\big),\ f_\sigma(g-n) \leq F\big(f_\sigma(g), f_\sigma(n)\big)$
(ii) $t_\sigma(g \cdot n) \geq T\big(t_\sigma(g), t_\sigma(n)\big),\ i_\sigma(g \cdot n) \geq I\big(i_\sigma(g), i_\sigma(n)\big),\ f_\sigma(g \cdot n) \leq S\big(f_\sigma(g), f_\sigma(n)\big)$.

Here, T and I are two TNs and S is a SN.

Proposition 2.2. *[80] Let $\sigma_1, \sigma_2 \in$ NSR(Q). Then $\sigma_1 \cap \sigma_2 \in$ NSR(Q).*

Theorem 2.3. *[80] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If σ is a NSR of Q then $h(\sigma)$ is a NSR of Y.*

Theorem 2.4. *[80] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If σ' is a NSR of Y then $h^{-1}(\sigma')$ is a NSR of Q.*

Definition 2.14. [80] Let $\sigma = \big\{\big(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)\big) : g \in Q\big\}$ be a NSR of Q. Then $\forall s \in [0, 1]$ the s-level sets of Q are defined as

(i) $(t_\sigma)_s = \{g \in Q : t_\sigma(g) \geq s\}$,
(ii) $(i_\sigma)_s = \{g \in Q : i_\sigma(g) \geq s\}$, and
(iii) $(f_\sigma)^s = \{g \in Q : f_\sigma(g) \leq s\}$.

Proposition 2.5. *[80] A NS $\sigma = \big\{\big(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)\big) : g \in Q\big\}$ of a crisp ring $(Q, +, \cdot)$ is a NSR of Q iff $\forall s \in [0, 1]$ the s-level sets of Q, i.e. $(t_\sigma)_s$, $(i_\sigma)_s$, and $(f_\sigma)^s$ are crisp rings of Q.*

2.2. *Neutrosophic soft subring*

Definition 2.15. [83] Let $(Q, +, \cdot)$ be a crisp ring and A be a set of parameters. Then a NSS $\delta = \{(r, l_\delta(r)) : r \in A\}$ with $l_\delta : A \to \mathrm{NS}(Q)$ is called a NSSR if $\forall r \in A$, $l_\delta(r) \in \mathrm{NSR}(Q)$.

The set of all NSSR of a crisp ring $(Q, +, \cdot)$ will be expressed as NSSR(Q).

Proposition 2.6. *[83] A NSS* $\delta = \Big\{\Big(r, \big\{\big(g, t_{l_\delta(r)}(g), i_{l_\delta(r)}(g), f_{l_\delta(r)}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$ *over a crisp ring $(Q, +, \cdot)$ is called a NSSR iff the following conditions hold:*

(i) $t_{l_\delta(r)}(g - n) \geq T\big(t_{l_\delta(r)}(g), t_{l_\delta(r)}(n)\big)$, $i_{l_\delta(r)}(g - n) \geq I\big(i_{l_\delta(r)}(g), i_{l_\delta(r)}(n)\big)$, $f_{l_\delta(r)}(g - n) \leq F\big(f_{l_\delta(r)}(g), f_{l_\delta(r)}(n)\big)$ *and*

(ii) $t_{l_\delta(r)}(g \cdot n) \geq T\big(t_{l_\delta(r)}(g), t_{l_\delta(r)}(n)\big)$, $i_{l_\delta(r)}(g \cdot n) \geq I\big(i_{l_\delta(r)}(g), i_{l_\delta(r)}(n)\big)$, $f_{l_\delta(r)}(g \cdot n) \leq S\big(f_{l_\delta(r)}(g), f_{l_\delta(r)}(n)\big)$.

Proposition 2.7. *[83] Let $\delta_1, \delta_2 \in NSSR(Q)$. Then $\delta_1 \cap \delta_2 \in NSSR(Q)$.*

Theorem 2.8. *[83] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be an isomorphism. If δ is a NSSR of Q then $h(\delta)$ is a NSSR of Y.*

Theorem 2.9. *[83] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If δ' is a NSSR of Y then $h^{-1}(\delta')$ is a NSSR of Q.*

Theorem 2.10. *$\delta_1 \in NSSR(Q)$ and $\delta_2 \in NSSR(Y)$, then their cartesian product $\delta_1 \times \delta_2 \in NSSR(Q \times Y)$.*

Definition 2.16. [83] A NSSR $\delta = \{(r, l_\delta(r)) : r \in A\}$ of a crisp ring $(Q, +, \cdot)$ is known as a NNSSR of Q iff $t_{l_\delta(r)}(g \cdot n) = t_{l_\delta(r)}(n \cdot g)$, $i_{l_\delta(r)}(g \cdot n) = i_{l_\delta(r)}(n \cdot g)$, and $f_{l_\delta(r)}(g \cdot n) = f_{l_\delta(r)}(n \cdot g)$.

The set of all NNSSR of Q will be expressed as NNSSR(Q).

Proposition 2.11. *[83] Let $\delta_1, \delta_2 \in NNSSR(Q)$. Then $\delta_1 \cap \delta_2 \in NNSSR(Q)$.*

Theorem 2.12. *[83] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be an isomorphism. If δ is a NNSSR of Q then $h(\delta)$ is a NNSSR of Y.*

Theorem 2.13. *[83] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a ring homomorphism. If δ' is a NNSSR of Y then $h^{-1}(\delta')$ is a NNSSR of Q.*

3. Proposed notion of interval-valued neutrosophic soft subring

Definition 3.1. Let $(Q, +, \cdot)$ be a crisp ring and A be a set of parameters. An IVNSS $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$ is called an IVNSSR of $(Q, +, \cdot)$ if $\forall g, n \in Q$, and $\forall r \in A$, the followings can be concluded:

(i) $\begin{cases} \bar{t}_{l_\Psi(r)}(g+n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big), \\ \bar{i}_{l_\Psi(r)}(g+n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \\ \bar{f}_{l_\Psi(r)}(g+n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) \end{cases}$

(ii) $\begin{cases} \bar{t}_{l_\Psi(r)}(-r) \geq t_{l_\Psi(r)}(g), \\ \bar{i}_{l_\Psi(r)}(-r) \leq i_{l_\Psi(r)}(g), \\ \bar{f}_{l_\Psi(r)}(-r) \leq f_{l_\Psi(r)}(g) \end{cases}$

(iii) $\begin{cases} \bar{t}_{l_\Psi(r)}(g \cdot n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big), \\ \bar{i}_{l_\Psi(r)}(g \cdot n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \\ \bar{f}_{l_\Psi(r)}(g \cdot n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big), \end{cases}$

The set of all IVNSSR of a crisp ring $(Q, +, \cdot)$ will be expressed as IVNSSR(Q).

Example 3.2. Let $(\mathbb{Z}, +, \cdot)$ be the ring and $\mathbb{N}$ be a set of parameters. Also, let $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in \mathbb{Z}\big\}\Big) : e \in \mathbb{N}\Big\}$ be an IVNSS of $\mathbb{Z}$, where $l_\Psi : \mathbb{N} \to$ IVNS(Q) and $\forall g \in \mathbb{Z}$, $\forall r \in \mathbb{N}$ corresponding memberships are

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \Big[\dfrac{1}{r+1}, \dfrac{1}{r}\Big] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{1}{2r+2}, \dfrac{1}{2r}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{r-1}{r}, \dfrac{r}{r+1}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\forall r \in \mathbb{N}$, $\Psi \in$ IVNSSR$(\mathbb{Z})$.

Example 3.3. Let $(\mathbb{Z}_4, +, \cdot)$ be the ring of integers modulo 4 and $A = \{r_1, r_2, r_3\}$ be a set of parameters. Also, let $\Psi = \Big\{\Big(r, \big\{\big(r, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in \mathbb{Z}_4\big\}\Big) : r \in A\Big\}$ be an IVNSS of $\mathbb{Z}_4$, where $l_\Psi : A \to$ IVNS(Q). Again, let the membership values of the elements belonging to Ψ are specified in Table 2, 3, and 4.

Table 2. Membership values of elements with respect to parameter r_1

$\Psi(r_1)$	$\bar{t}_{l_\Psi(r_1)}$	$\bar{i}_{l_\Psi(r_1)}$	$\bar{f}_{l_\Psi(r_1)}$
$\bar{0}$	$[0.64, 0.66]$	$[0.33, 0.35]$	$[0.13, 0.14]$
$\bar{1}$	$[0.7, 0.72]$	$[0.21, 0.23]$	$[0.77, 0.79]$
$\bar{2}$	$[0.74, 0.76]$	$[0.24, 0.26]$	$[0.51, 0.53]$
$\bar{3}$	$[0.66, 0.68]$	$[0.31, 0.33]$	$[0.28, 0.3]$

TABLE 3. Membership values of elements with respect to parameter r_2

$\Psi(r_2)$	$\bar{t}_{l_{\Psi}(r_2)}$	$\bar{i}_{l_{\Psi}(r_2)}$	$\bar{f}_{l_{\Psi}(r_2)}$
$\bar{0}$	$[0.68, 0.7]$	$[0.3, 0.32]$	$[0.31, 0.33]$
$\bar{1}$	$[0.61, 0.63]$	$[0.31, 0.33]$	$[0.41, 0.43]$
$\bar{2}$	$[0.57, 0.59]$	$[0.4, 0.42]$	$[0.65, 0.67]$
$\bar{3}$	$[0.7, 0.72]$	$[0.26, 0.28]$	$[0.52, 0.54]$

TABLE 4. Membership values of elements with respect to parameter r_3

$\Psi(r_3)$	$\bar{t}_{l_{\Psi}(r_3)}$	$\bar{i}_{l_{\Psi}(r_3)}$	$\bar{f}_{l_{\Psi}(r_3)}$
$\bar{0}$	$[0.71, 0.73]$	$[0.2, 0.23]$	$[0.15, 0.17]$
$\bar{1}$	$[0.83, 0.85]$	$[0.15, 0.17]$	$[0.24, 0.26]$
$\bar{2}$	$[0.68, 0.7]$	$[0.3, 0.32]$	$[0.38, 0.4]$
$\bar{3}$	$[0.78, 0.8]$	$[0.18, 0.2]$	$[0.4, 0.43]$

Here, considering the Łukasiewicz TN ($T(g,n) = \max\{0, g+n-1\}$) and bounded sum SNs ($S(g,n) = \min\{g+n, 1\}$), $\forall r \in A$, $\Psi \in$ IVNSSR($\mathbb{Z}_4$).

Proposition 3.1. *An IVNSS* $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$ *of a crisp ring* $(Q, +, \cdot)$ *is an IVNSSR iff the following conditions hold (considering idempotent IVTN and IVSNs):*

(i) $\bar{t}_{l_{\Psi}(r)}(g-n) \geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(n)\big)$, $\bar{i}_{l_{\Psi}(r)}(g-n) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(n)\big)$, $\bar{f}_{l_{\Psi}(r)}(g-n) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(n)\big)$ *and*

(ii) $\bar{t}_{l_{\Psi}(r)}(g \cdot n) \geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(n)\big)$, $\bar{i}_{l_{\Psi}(r)}(g \cdot n) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(n)\big)$, $\bar{f}_{l_{\Psi}(r)}(g \cdot n) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(n)\big)$.

Proof. Let $\Psi \in$ IVNSSR(Q). Then

$$\begin{aligned}\bar{t}_{l_{\Psi}(r)}(g-n) &\geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(-n)\big) \text{ [by Definition 3.1]}\\ &\geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(n)\big) \text{ [by Definition 3.1]}\end{aligned}$$

Similary, we will have

$$\begin{aligned}\bar{i}_{l_{\Psi}(r)}(g-n) &\leq \bar{I}\big(\bar{i}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(n)\big), \text{ and}\\ \bar{f}_{l_{\Psi}(r)}(g-n) &\leq \bar{F}\big(\bar{f}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(n)\big),\end{aligned}$$

Again, (ii) follows immediately from condition (iii) of Definition 3.1.

Conversely, let conditions (i) and (ii) of Proposition 3.1 hold. Assuming θ_Q as the additive neutral member of $(Q, +, \cdot)$, we have

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(\theta_Q) &= \bar{t}_{l_\Psi(r)}(g-g)\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(g)\big)\\ &= \bar{t}_{l_\Psi(r)}(g) \end{aligned} \tag{3.1}$$

Similaly,

$$\bar{i}_{l_\Psi(r)}(\theta_Q) \leq \bar{i}_{l_\Psi(r)}(g) \tag{3.2}$$

$$\bar{f}_{l_\Psi(r)}(\theta_Q) \leq \bar{f}_{l_\Psi(r)}(g) \tag{3.3}$$

Now,

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(-g) &= \bar{t}_{l_\Psi(r)}(\theta_Q - g)\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(\theta_Q), \bar{t}_{l_\Psi(r)}(g)\big)\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(g)\big) \text{ [by 3.1]}\\ &= \bar{t}_{l_\Psi(r)}(g) \text{ [since } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.4}$$

Similarly,

$$\bar{i}_{l_\Psi(r)}(-g) \leq \bar{i}_{l_\Psi(r)}(g) \text{ [since } \bar{I} \text{ is idempotent]} \tag{3.5}$$

$$\bar{f}_{l_\Psi(r)}(-g) \leq \bar{f}_{l_\Psi(r)}(g) \text{ [since } \bar{F} \text{ is idempotent]} \tag{3.6}$$

Hence,

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(g+n) &= \bar{t}_{l_\Psi(r)}(g-(-n))\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(-n)\big)\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \text{ [by 3.4]} \end{aligned} \tag{3.7}$$

Similarly,

$$\bar{i}_{l_\Psi(r)}(g+n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big) \text{ [by 3.5]} \tag{3.8}$$

$$\bar{f}_{l_\Psi(r)}(g+n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) \text{ [by 3.6]} \tag{3.9}$$

Hence, Equations 3.7, 3.8, and 3.9 prove part (i) of Proposition 3.1. Again, part (ii) of Proposition 3.1 is similar to condition (iii) of Definition 3.1. So, $\Psi \in$ IVNSSR(Q). $\square$

Theorem 3.2. *Let $(Q, +, \cdot)$ be a crisp ring. If $\Psi_1, \Psi_2 \in$ IVNSSR(Q), then $\Psi_1 \cap \Psi_2 \in$ IVNSSR(Q) (considering idempotent IVTN and IVSNs).*

Proof. Let $\Psi = \Psi_1 \cap \Psi_2$. Now, $\forall g, n \in Q$ and $\forall r \in A$

$$\begin{aligned}
\bar{t}_{l_{\Psi}(r)}(g+n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g+n), \bar{t}_{l_{\Psi_2}(r)}(g+n)\big) \\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_1}(r)}(n)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2}(r)}(g), \bar{t}_{l_{\Psi_2}(r)}(n)\big)\Big) \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_1}(r)}(n)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2}(r)}(n), \bar{t}_{l_{\Psi_2}(r)}(g)\big)\Big) \text{ [as } \bar{T} \text{ is commutative]} \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Psi_1}(g), \bar{t}_{l_{\Psi_2}(r)}(g)\big), \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(n), \bar{t}_{l_{\Psi_2}(r)}(n)\big)\Big) \text{ [as } \bar{T} \text{ is associative]} \\
&= \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(n)\big) \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
\bar{t}_{l_{\Psi}(r)}(-g) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(-g), \bar{t}_{l_{\Psi_2}(r)}(-g)\big) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_2}(r)}(g)\big) \text{ [by Definition 3.1]} \\
&= \bar{t}_{l_{\Psi}(r)}(g) \qquad (3.11)
\end{aligned}$$

Similarly, we can show

$$\bar{i}_{l_{\Psi}(r)}(g+n) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(n)\big) \qquad (3.12)$$

$$\bar{f}_{l_{\Psi}(r)}(g+n) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(n)\big) \qquad (3.13)$$

and

$$\bar{i}_{l_{\Psi}(r)}(-g) \leq \bar{i}_{l_{\Psi}(r)}(g) \qquad (3.14)$$

$$\bar{f}_{l_{\Psi}(r)}(-g) \leq \bar{f}_{l_{\Psi}(r)}(g) \qquad (3.15)$$

Also, we can show that

$$\bar{t}_{l_{\Psi}(r)}(g \cdot n) \geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(n)\big), \qquad (3.16)$$

$$\bar{i}_{l_{\Psi}(r)}(g \cdot n) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(n)\big), \text{ and} \qquad (3.17)$$

$$\bar{f}_{l_{\Psi}(r)}(g \cdot n) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(n)\big) \qquad (3.18)$$

So, from Equations 3.10–3.18 $\Psi \in$ IVNSSR(Q). □

Remark 3.3. *In general, if $\Psi_1, \Psi_2 \in IVNSSR(Q)$, then $\Psi_1 \cup \Psi_2$ may not always be an IVNSSR of $(Q, +, \cdot)$.*

The following Example 3.4 will prove Remark 3.3.

Example 3.4. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers and $\mathbb{N}$ be a set of parameters. Again, let $\Psi_1 = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_{\Psi_1}(r)}(g), \bar{i}_{l_{\Psi_1}(r)}(g), \bar{f}_{l_{\Psi_1}(r)}(g)\big) : g \in \mathbb{Z}\big\}\Big) : r \in \mathbb{N}\Big\}$ and $\Psi_2 =$

$\left\{\left(r, \left\{\left(g, \bar{t}_{l_{\Psi_2}(r)}(g), \bar{i}_{l_{\Psi_2}(r)}(g), \bar{f}_{l_{\Psi_2}(r)}(g)\right) : g \in \mathbb{Z}\right\}\right) : r \in \mathbb{N} \setminus \{1\}\right\}$ be two IVNSSs of $\mathbb{Z}$, where $l_{\Psi_1} : \mathbb{N} \to \text{IVNSS}(Q)$ be defined as

$$\bar{t}_{l_{\Psi_1}(r)}(g) = \begin{cases} \left[\dfrac{1}{r+1}, \dfrac{1}{r}\right] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_{\Psi_1}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \left[\dfrac{1}{2r+2}, \dfrac{1}{2r}\right] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_{\Psi_1}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \left[\dfrac{r-1}{r}, \dfrac{r}{r+1}\right] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

and $l_{\Psi_2} : \mathbb{N} \setminus \{1\} \to \text{IVNSS}(Q)$ be defined as

$$\bar{t}_{l_{\Psi_2}(r)}(g) = \begin{cases} \left[\dfrac{1}{r}, \dfrac{1}{r-1}\right] & \text{if } g \in 3\mathbb{Z} \\ [0,0] & \text{if } g \in 3\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_{\Psi_2}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 3\mathbb{Z} \\ \left[\dfrac{1}{2r}, \dfrac{1}{2r-2}\right] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_{\Psi_2}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 3\mathbb{Z} \\ \left[\dfrac{r-2}{r-1}, \dfrac{r-1}{r}\right] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\Psi_1, \Psi_2 \in \text{IVNSSR}(\mathbb{Z})$. Let $\Psi = \Psi_1 \cup \Psi_2$. Now considering $r = 3$ we will have

$$\bar{t}_{l_{\Psi_1}(3)}(g) = \begin{cases} \left[\dfrac{1}{4}, \dfrac{1}{3}\right] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases} \quad \text{and}$$

$$\bar{t}_{l_{\Psi_2}(3)}(g) = \begin{cases} \left[\dfrac{1}{3}, \dfrac{1}{2}\right] & \text{if } g \in 3\mathbb{Z} \\ [0,0] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}$$

Now, taking $g = 10$ and $n = 15$, we will have

$$\begin{aligned} \bar{t}_{l_{\Psi}(3)}(g+n) &= \bar{t}_{l_{\Psi}(3)}(10+15) \\ &= \bar{t}_{l_{\Psi}(3)}(25) \\ &= \max\{\bar{t}_{l_{\Psi_1}(3)}(25), \bar{t}_{l_{\Psi_2}(3)}(25)\} \\ &= \max\{[0,0],[0,0]\} \\ &= [0,0] \end{aligned}$$

Again, if $\Psi \in$ IVNSSR(Q), then $\forall g, n \in Q$, $\bar{t}_{l_\Psi(3)}(g+n) \geq \min\{\bar{t}_{l_\Psi(3)}(g), \bar{t}_{l_\Psi(3)}(n)\}$. But, here for $g = 10$ and $n = 15$, $\min\{\bar{t}_{l_\Psi(3)}(10), \bar{t}_{l_\Psi(3)}(15)\} = \min\left\{\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right\} = \left[\frac{1}{4}, \frac{1}{3}\right] \nleq [0,0] = \bar{t}_{l_\Psi(3)}(10+15)$. So, $\Psi \notin$ IVNSSR(Q).

Corollary 3.4. *If $\Psi_1, \Psi_2 \in IVNSSR(Q)$, then $\Psi_1 \cup \Psi_2 \in IVNSSR(Q)$ iff one is a subset of other.*

Definition 3.5. let $\Psi = \left\{\left(r, \left\{\left(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\right) : g \in \mathbb{Z}_4\right\}\right) : r \in A\right\}$ be an IVNSS of a crisp ring $(Q, +, \cdot)$. Also, let $[g_1, n_1]$, $[g_2, n_2]$, and $[g_3, n_3] \in \phi(K)$. Then the CS $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is called a level set of IVNSSR Ψ, where for any $g \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ the following inequalities will hold: $\bar{t}_{l_\Psi(r)}(g) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(g) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(g) \leq [g_3, n_3]$.

Theorem 3.5. *Let $(Q, +, \cdot)$ be a crisp ring. Then $\Psi \in IVNSSR(Q)$ iff $\forall [g_1, n_1], [g_2, n_2], [g_3, n_3] \in \phi(K)$ with $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is a crisp subring of $(Q, +, \cdot)$ (considering idempotent IVTN and IVSNs).*

Proof. Since, $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\theta_Q \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$, i.e., $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is non-empty. Now, let $\Psi \in$ IVNSSR(Q) and $g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$. To show that, $(g - n)$ and $g \cdot n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$. Here,

$$\begin{aligned} \bar{t}_{l_\Psi(r)}(g-n) &\geq \bar{T}\left(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\right) \text{ [by Proposition 3.1]} \\ &\geq \bar{T}\left([g_1, n_1], [g_1, n_1]\right) \left[\text{as } g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}\right] \\ &\geq [g_1, n_1] \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.19}$$

Again,

$$\begin{aligned} \bar{t}_{l_\Psi(r)}(g \cdot n) &\geq \bar{T}\left(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\right) \text{ [by Proposition 3.1]} \\ &\geq \bar{T}\left([g_1, n_1], [g_1, n_1]\right) \left[\text{as } g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}\right] \\ &\geq [g_1, n_1] \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.20}$$

Similarly, as $\bar{I}$ and $\bar{F}$ are idempotent, we can prove that

$$\bar{i}_{l_\Psi(r)}(g-n) \leq [g_2, n_2], \tag{3.21}$$

$$\bar{i}_{l_\Psi(r)}(g \cdot n) \leq [g_2, n_2], \tag{3.22}$$

$$\bar{f}_{l_\Psi(r)}(g-n) \leq [g_3, n_3], \text{ and} \tag{3.23}$$

$$\bar{f}_{l_\Psi(r)}(g \cdot n) \leq [g_3, n_3]. \tag{3.24}$$

So, from Equations 3.19–3.24 $(g-n)$ and $g\cdot n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$, i.e., $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is a crisp subring of $(Q,+,\cdot)$.

Conversely, let $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is a crisp subring of $(Q,+,\cdot)$. To show that, $\Psi \in$ IVNSSR(Q).

Let $g,n \in Q$, then there exists $[g_1,n_1] \in \phi(K)$ such that $\bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) = [g_1,n_1]$. Wherefrom $\bar{t}_{l_\Psi(r)}(g) \geq [g_1,n_1]$ and $\bar{t}_{l_\Psi(r)}(n) \geq [g_1,n_1]$. Also, let there exist $[g_2,n_2],[g_3,n_3] \in \phi(K)$ such that $\bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big) = [g_2,n_2]$ and $\bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) = [g_3,n_3]$. Then $g,n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$.

Now, as $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is a crisp subring, $g-n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ and $g\cdot n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$.

Hence,

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(g-n) &\geq [k_1,s_1]\\ &= \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \text{ and} \end{aligned} \tag{3.25}$$

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(g\cdot n) &\geq [k_1,s_1]\\ &= \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \end{aligned} \tag{3.26}$$

Similarly, we can prove that

$$\begin{aligned}\bar{i}_{l_\Psi(r)}(g-n) &\leq [k_2,s_2]\\ &= \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \end{aligned} \tag{3.27}$$

$$\begin{aligned}\bar{i}_{l_\Psi(r)}(g\cdot n) &\leq [k_2,s_2]\\ &= \bar{T}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \end{aligned} \tag{3.28}$$

$$\begin{aligned}\bar{f}_{l_\Psi(r)}(g-n) &\leq [k_3,s_3]\\ &= \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big), \text{ and} \end{aligned} \tag{3.29}$$

$$\begin{aligned}\bar{f}_{l_\Psi(r)}(g\cdot n) &\leq [k_3,s_3]\\ &= \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) \end{aligned} \tag{3.30}$$

Hence, from Equations 3.25–3.30 $\Psi \in$ IVNSSR(Q). □

Definition 3.6. Let Ψ and Ψ' be two IVNSSs of two CSs Q and Y, respectively. Also, let $h : Q \to Y$ be a function. Then

(i) image of Ψ under h will be
$h(\Psi) = \Big\{\Big(r, \big\{\big(n, \bar{t}_{h(l_\Psi(r))}(n), \bar{i}_{h(l_\Psi(r))}(n), \bar{f}_{h(l_\Psi(r))}(n)\big) : n \in Y\big\}\Big) : r \in A\Big\}$,
where $\bar{t}_{h(l_\Psi(r))}(n) = \underset{s\in h^{-1}(n)}{\vee} \bar{t}_{l_\Psi(r)}(s)$, $\bar{i}_{h(l_\Psi(r))}(n) = \underset{s\in h^{-1}(n)}{\wedge} \bar{i}_{l_\Psi(r)}(s)$, and $\bar{f}_{h(l_\Psi(r))}(v) =$

$\bigwedge\limits_{s\in h^{-1}(n)} \bar{f}_{l_\Psi(r)}(s)$. Wherefrom, if h is injective then $\bar{t}_{h(l_\Psi(r))}(n) = \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big)$, $\bar{i}_{h(l_\Psi(r))}(n) = \bar{i}_{l_\Psi(r)}\big(h^{-1}(n)\big)$, $\bar{f}_{h(l_\Psi(r))}(n) = \bar{f}_{l_\Psi(r)}\big(h^{-1}(n)\big)$.

(i) preimage of Ψ' under h will be
$h^{-1}(\Psi') = \Big\{\Big(r, \big\{\big(g, \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$,
where $\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{t}_{l_{\Psi'}(r)}\big(h(g)\big)$, $\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{i}_{l_{\Psi'}(r)}\big(h(g)\big)$, $\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{f}_{l_{\Psi'}(r)}\big(h(g)\big)$.

Theorem 3.6. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be an isomorphism. If Ψ is an IVNSSR of Q then $h(\Psi)$ is an IVNSSR of Y.*

Proof. Let $n_1 = h(g_1)$ and $n_2 = h(g_2)$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Now,

$$\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 - n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 - n_2)\big) \text{ [as } h \text{ is injective]}\\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) - h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]}\\
&= \bar{t}_{l_\Psi(r)}(g_1 - g_2)\\
&\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g_1), \bar{t}_{l_\Psi(r)}(g_2)\big)\\
&= \bar{T}\Big(\bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1)\big), \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2)\big)\Big)\\
&= \bar{T}\big(\bar{t}_{h(l_\Psi(r))}(n_1), \bar{t}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.31)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 \cdot n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 \cdot n_2)\big) \text{ [as } h \text{ is injective]}\\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) \cdot h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]}\\
&= \bar{t}_{l_\Psi(r)}(g_1 \cdot g_2)\\
&\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g_1), \bar{t}_{l_\Psi(r)}(g_2)\big)\\
&= \bar{T}\Big(\bar{t}_{l_\Psi(r)}\Big(h^{-1}(n_1)\big), \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2)\big)\Big)\\
&= \bar{T}\big(\bar{t}_{h(l_\Psi(r))}(n_1), \bar{t}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.32)
\end{aligned}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n_1 - n_2) \leq \bar{I}\big(\bar{i}_{h(l_\Psi(r))}(n_1), \bar{i}_{h(l_\Psi(r))}(n_2)\big), \qquad (3.33)$$

$$\bar{i}_{h(l_\Psi(r))}(n_1 \cdot n_2) \leq \bar{I}\big(\bar{i}_{h(l_\Psi(r))}(n_1), \bar{i}_{h(l_\Psi(r))}(n_2)\big), \qquad (3.34)$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 - n_2) \leq \bar{F}\big(\bar{f}_{h(l_\Psi(r))}(n_1), \bar{f}_{h(l_\Psi(r))}(n_2)\big), \text{ and} \qquad (3.35)$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 \cdot n_2) \leq \bar{F}\big(\bar{f}_{h(l_\Psi(r))}(n_1), \bar{f}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.36)$$

So, from Equations 3.31–3.36 $h(\Psi)$ is an IVNSSR of Y. □

Theorem 3.7. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If Ψ' is an IVNSSR of Y then $h^{-1}(\Psi')$ is an IVNSSR of Q. (Note that, h^{-1} may not be an inverse function but $h^{-1}(\Psi')$ is an inverse image of Ψ').*

Proof. Let $n_1 = h(g_1)$ and $n_2 = h(g_2)$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Now,

$$\begin{aligned}
\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) &= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1 - g_2)\big) \\
&= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1) - h(g_2)\big) \text{ [as } h \text{ is a homomorphism]} \\
&= \bar{t}_{l_{\Psi'}(r)}(n_1 - n_2) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi'}(r)}(n_1), \bar{t}_{l_{\Psi'}(r)}(n_2)\big) \\
&= \bar{T}\Big(\bar{t}_{l_{\Psi'}(r)}\big(h(g_1)\big), \bar{t}_{l_{\Psi'}(r)}\big(h(g_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.37)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) &= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1 \cdot g_2)\big) \\
&= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1) \cdot h(g_2)\big) \text{ [as } h \text{ is a homomorphism]} \\
&= \bar{t}_{l_{\Psi'}(r)}(n_1 \cdot n_2) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi'}(r)}(n_1), \bar{t}_{l_{\Psi'}(r)}(n_2)\big) \\
&= \bar{T}\Big(\bar{t}_{l_{\Psi'}(r)}\big(h(g_1)\big), \bar{t}_{l_{\Psi'}(r)}\big(h(g_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.38)
\end{aligned}$$

Similarly,

$$\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) \leq \bar{I}\big(\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.39)$$

$$\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) \leq \bar{I}\big(\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.40)$$

$$\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) \leq \bar{F}\big(\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.41)$$

$$\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) \leq \bar{F}\big(\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.42)$$

So, from Equations 3.37–3.42 $h^{-1}(\Psi')$ is an IVNSSR of Q. □

Definition 3.7. Let $(Q, +, \cdot)$ be a crisp ring and $\Psi \in \text{IVNSSR}(Q)$. Again, let $\bar{\alpha} = [\alpha_1, \alpha_2], \bar{\nu} = [\nu_1, \nu_2], \bar{\chi} = [\chi_1, \chi_2] \in \phi(K)$. Then

(i) Ψ is called a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$identity IVNSSR over Q, if $\forall g\in Q$

$$\bar{t}_{l_\Psi(r)}(g)=\begin{cases}\bar{\alpha} & \text{if } g=\theta_Q\\ [0,0] & \text{if } g\neq\theta_Q\end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g)=\begin{cases}\bar{\nu} & \text{if } g=\theta_Q\\ [1,1] & \text{if } g\neq\theta_Q\end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g)=\begin{cases}\bar{\chi} & \text{if } g=\theta_Q\\ [1,1] & \text{if } g\neq\theta_Q\end{cases},$$

where θ_Q is the additive zero element of Q.

(ii) Ψ is called a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$absolute IVNSSR over Q, if $\forall g\in Q$, $\bar{t}_{l_\Psi(r)}(g)=\bar{\alpha}$, $\bar{i}_{l_\Psi(r)}(g)=\bar{\nu}$, and $\bar{f}_{l_\Psi(r)}(g)=\bar{\chi}$.

Theorem 3.8. *Let $(Q,+,\cdot)$ and $(Y,+,\cdot)$ be two crisp rings and $\Psi\in$IVNSSR (Q). Again, let $h:Q\to Y$ be a homomorphism. Then*

(i) *$h(\Psi)$ will be a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$identity IVNSSR over Y, if $\forall g\in Q$*

$$\bar{t}_{l_\Psi(r)}(g)=\begin{cases}\bar{\alpha} & \textit{if } g\in Ker(h)\\ [0,0] & \textit{otherwise}\end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g)=\begin{cases}\bar{\nu} & \textit{if } g\in Ker(h)\\ [1,1] & \textit{otherwise}\end{cases}, \textit{ and}$$

$$\bar{f}_{l_\Psi(r)}(g)=\begin{cases}\bar{\chi} & \textit{if } g\in Ker(h)\\ [1,1] & \textit{otherwise}\end{cases},$$

(ii) *$h(\Psi)$ will be a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$absolute IVNSSR over Y, if Ψ is a $(\bar{\alpha},\bar{\nu},\bar{\chi})-$absolute IVNSSR over Q.*

Proof. (i) Clearly, by Theorem 3.6 $h(\Psi)\in$ IVNSSR(Y). Let $g\in Ker(h)$, then $h(g)=\theta_Y$. So,

$$\begin{aligned}\bar{t}_{h(l_\Psi(r))}(\theta_Y)&=\bar{t}_{l_\Psi(r)}\big(h^{-1}(\theta_Y)\big)\\&=\bar{t}_{l_\Psi(r)}(g)\\&=\bar{\alpha}\end{aligned}\tag{3.43}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(\theta_Y)=\bar{\nu},\ \text{and}\tag{3.44}$$

$$\bar{f}_{h(l_\Psi(r))}(\theta_Y)=\bar{\chi}\tag{3.45}$$

Again, let $g \in Q \setminus Ker(h)$ and $h(g) = n$. Then

$$\begin{aligned}\bar{t}_{h(l_\Psi(r))}(n) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big)\\ &= \bar{t}_{l_\Psi(r)}(g)\\ &= [0,0] \qquad (3.46)\end{aligned}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n) = [1,1] \text{ and} \qquad (3.47)$$

$$\bar{f}_{h(l_\Psi(r))}(n) = [1,1] \qquad (3.48)$$

So, from the Equations 3.43–3.48 $h(\Psi)$ is a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$identity IVNSSR over Y.

(ii) Let $h(g) = n$, for $g \in Q$ and $n \in Y$. Then

$$\begin{aligned}\bar{t}_{h(l_\Psi(r))}(n) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big)\\ &= \bar{t}_{l_\Psi(r)}(g)\\ &= \bar{\alpha} \qquad (3.49)\end{aligned}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n) = \bar{\nu} \text{ and} \qquad (3.50)$$

$$\bar{f}_{h(l_\Psi(r))}(n) = \bar{\chi} \qquad (3.51)$$

So, from the Equations 3.48–3.51 $h(\Psi)$ is a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$absolute IVNSSR over Y. □

3.1. *Product of interval-valued neutrosophic subrings*

Definition 3.8. Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Again, let $\Psi_1 \in$ IVNSSR(Q) and $\Psi_2 \in$ IVNSSR(Y), where $\Psi_1 = \Big\{\Big(r_1, \big\{\big(g, \bar{t}_{l_{\Psi_1}(r_1)}(g), \bar{i}_{l_{\Psi_1}(r_1)}(g), \bar{f}_{l_{\Psi_1}(r_1)}(g)\big) : g \in Q\big\}\Big) : r_1 \in A\Big\}$ and $\Psi_2 = \Big\{\Big(r_2, \big\{\big(v, \bar{t}_{l_{\Psi_2}(r_2)}(n), \bar{i}_{l_{\Psi_2}(r_2)}(n), \bar{f}_{l_{\Psi_2}(r_2)}(n)\big) : n \in Y\big\}\Big) : r_2 \in A\Big\}$. Then cartesian product of Ψ_1 and Ψ_2 will be

$$\begin{aligned}\Psi &= \Psi_1 \times \Psi_2\\ &= \big\{\big((r_1, r_2), l_{\Psi_1 \times \Psi_2}(r_1, r_2)\big) : (r_1, r_2) \in A \times A\big\}\end{aligned}$$

where the approximate function $l_{\Psi_1 \times \Psi_2} : A \times A \to$ IVNS$(Q \times Y)$ is defined as

$$\begin{aligned}\bar{t}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g), \bar{t}_{l_{\Psi_2}(r_2)}(n)\big),\\ \bar{i}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{I}\big(\bar{i}_{l_{\Psi_1}(r_1)}(g), \bar{i}_{l_{\Psi_2}(r_2)}(n)\big), \text{ and}\\ \bar{f}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{F}\big(\bar{f}_{l_{\Psi_1}(r_1)}(g), \bar{f}_{l_{\Psi_2}(r_2)}(n)\end{aligned}$$

Similarly, product of 3 or more IVNSSRs can be defined.

Theorem 3.9. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings with $\Psi_1 \in IVNSSR(Q)$ and $\Psi_2 \in IVNSSR(Y)$. Then $\Psi_1 \times \Psi_2 \in IVNSSR(Q \times Y)$.*

Proof. Let $\Psi = \Psi_1 \times \Psi_2$ and $(g_1, n_1), (g_2, n_2) \in Q \times R$. Then

$$\begin{aligned}
&\bar{t}_{l_{\Psi}(r_1,r_2)}\big((g_1, n_1) - (g_2, n_2)\big) \\
&\quad = \bar{t}_{l_{\Psi_1 \times \Psi_2}(r_1,r_2)}\big((g_1 - g_2, n_1 - n_2)\big) \\
&\quad = \bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g_1 - g_2), \bar{t}_{l_{\Psi_2}(r_2)}(n_1 - n_2)\big) \\
&\quad \geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g_1), \bar{t}_{l_{\Psi_1}(r_1)}(g_2)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2}(r_2)}(n_1), \bar{t}_{l_{\Psi_2}(r_2)}(n_2)\big)\Big) \\
&\quad = \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g_1), \bar{t}_{l_{\Psi_2}(r_2)}(n_1)\big), \bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g_2), \bar{t}_{l_{\Psi_2}(r_2)}(n_2)\big)\Big) \\
&\qquad\qquad \text{[as } \bar{T} \text{ is associative]} \\
&\quad = \bar{T}\big(\bar{t}_{l_{\Psi}(r_1,r_2)}(g_1, n_1), \bar{t}_{l_{\Psi}(r_1,r_2)}(g_2, n_2)\big) \qquad (3.52)
\end{aligned}$$

Again,

$$\begin{aligned}
&\bar{t}_{l_{\Psi}(r_1,r_2)}\big((g_1, n_1) \cdot (g_2, n_2)\big) \\
&\quad = \bar{t}_{l_{\Psi_1 \times \Psi_2}(r_1,r_2)}\big((g_1 \cdot g_2, n_1 \cdot n_2)\big) \\
&\quad = \bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g_1 \cdot g_2), \bar{t}_{l_{\Psi_2}(r_2)}(n_1 \cdot n_2)\big) \\
&\quad \geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g_1), \bar{t}_{l_{\Psi_1}(r_1)}(g_2)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2}(r_2)}(n_1), \bar{t}_{l_{\Psi_2}(r_2)}(n_2)\big)\Big) \\
&\quad = \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g_1), \bar{t}_{l_{\Psi_2}(r_2)}(n_1)\big), \bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g_2), \bar{t}_{l_{\Psi_2}(r_2)}(n_2)\big)\Big) \\
&\qquad\qquad \text{[as } \bar{T} \text{ is associative]} \\
&\quad = \bar{T}\big(\bar{t}_{l_{\Psi}(r_1,r_2)}(g_1, n_1), \bar{t}_{l_{\Psi}(r_1,r_2)}(g_2, n_2)\big) \qquad (3.53)
\end{aligned}$$

Similary,

$$\bar{i}_{l_{\Psi}(r_1,r_2)}\big((g_1, n_1) - (g_2, n_2)\big) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r_1,r_2)}(g_1, n_1), \bar{i}_{l_{\Psi}(r_1,r_2)}(g_2, n_2)\big), \qquad (3.54)$$

$$\bar{i}_{l_{\Psi}(r_1,r_2)}\big((g_1, n_1) \cdot (g_2, n_2)\big) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r_1,r_2)}(g_1, n_1), \bar{i}_{l_{\Psi}(r_1,r_2)}(g_2, n_2)\big), \qquad (3.55)$$

$$\bar{f}_{l_{\Psi}(r_1,r_2)}\big((g_1, n_1) - (g_2, n_2)\big) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r_1,r_2)}(g_1, n_1), \bar{f}_{l_{\Psi}(r_1,r_2)}(g_2, n_2)\big), \text{ and} \qquad (3.56)$$

$$\bar{f}_{l_{\Psi}(r_1,r_2)}\big((g_1, n_1) \cdot (g_2, n_2)\big) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r_1,r_2)}(g_1, n_1), \bar{f}_{l_{\Psi}(r_1,r_2)}(g_2, n_2)\big) \qquad (3.57)$$

So, by Proposition 3.1 and from Equations 3.52–3.57 $\Psi_1 \times \Psi_2 \in$ IVNSSR$(Q \times Y)$. □

Corollary 3.10. *Let $\forall i \in \{1, 2, ..., n\}$, $(Q_i, +, \cdot)$ are crisp rings and $\Psi_i \in IVNSSR(Q_i)$. Then $\Psi_1 \times \Psi_2 \times \cdots \times \Psi_n$ is a IVNSSR of $Q_1 \times Q_2 \times \cdots \times Q_n$, where $n \in \mathbb{N}$.*

3.2. *Subring of a interval-valued neutrosophic soft subgring*

Definition 3.9. Let $(Q, +, \cdot)$ be a crisp ring and $\Psi_1, \Psi_2 \in$ IVNSSR(Q), where $\Psi_1 = \Big\{\big(r, \{\big(g, \bar{t}_{l_{\Psi_1}(r)}(g), \bar{i}_{l_{\Psi_1}(r)}(g), \bar{f}_{l_{\Psi_1}(r)}(g)\big) : g \in Q\}\big) : r \in A\Big\}$ and $\Psi_2 = \Big\{\big(r, \{\big(g, \bar{t}_{l_{\Psi_2}(r)}(g), \bar{i}_{l_{\Psi_2}(r)}(g), \bar{f}_{l_{\Psi_2}(r)}(g)\big) : g \in Q\}\big) : r \in A\Big\}$. Then Ψ_1 is called a subring of Ψ_2 if $\forall g \in Q$, $\bar{t}_{l_{\Psi_1}(r)}(g) \le \bar{t}_{l_{\Psi_2}(r)}(g)$, $\bar{i}_{l_{\Psi_1}(r)}(g) \ge \bar{i}_{l_{\Psi_2}(r)}(g)$, and $\bar{f}_{l_{\Psi_1}(r)}(g) \ge \bar{f}_{l_{\Psi_2}(r)}(g)$.

Theorem 3.11. *Let $(Q, +, \cdot)$ be a crisp ring and $\Psi \in$ IVNSSR(Q). Again, let Ψ_1 and Ψ_2 be two subrings of Ψ. Then $\Psi_1 \cap \Psi_2$ is also a subring of Ψ, considering all the IVTN and IVSNs as idempotent.*

Proof. Here, $\forall g \in Q$

$$\begin{aligned}\bar{t}_{l_{\Psi_1 \cap \Psi_2}(r)}(g) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_2}(r)}(g)\big)\\ &\le \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(g)\big)\\ &= \bar{t}_{l_{\Psi}(r)}(g) \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.58}$$

Similarly, since $\bar{I}$ and $\bar{F}$ are idempotent we have,

$$\bar{i}_{l_{\Psi_1 \cap \Psi_2}(r)}(g) \ge \bar{i}_{l_{\Psi}(r)}(g) \text{ and} \tag{3.59}$$

$$\bar{f}_{l_{\Psi_1 \cap \Psi_2}(r)}(g) \ge \bar{f}_{l_{\Psi}(r)}(g) \tag{3.60}$$

So, from Equations 3.58–3.60 $\Psi_1 \cap \Psi_2$ is a subring of Ψ. □

Theorem 3.12. *Let $(Q, +, \cdot)$ be a crisp ring and $\Psi_1, \Psi_2 \in$ IVNSSR(Q) such that Ψ_1 is a subring of Ψ_2. Let $(Y, +, \cdot)$ is another crisp ring and $h : Q \to Y$ be an isomorphism. Then*

(i) *$h(\Psi_1)$ and $h(\Psi_2)$ are two IVNSSRs over Y and*

(i) *$h(\Psi_1)$ is a subring of $h(\Psi_2)$.*

Proof. (i) can be proved by using Theorem 3.6.
(ii) Let $n = h(g)$, where $g \in Q$ and $n \in Y$. Then

$$\begin{aligned}&\bar{t}_{l_{\Psi_1}(r)}(g) \le \bar{t}_{l_{\Psi_2}(r)}(g) \text{ [as } \Psi_1 \text{ is a subring of } \Psi_2]\\ \Rightarrow &\bar{t}_{l_{\Psi_1}(r)}\big(h^{-1}(n)\big) \le \bar{t}_{l_{\Psi_2}(r)}\big(h^{-1}(n)\big)\\ \Rightarrow &\bar{t}_{h(l_{\Psi_1}(r))}(n) \le \bar{t}_{h(l_{\Psi_2}(r))}(n)\end{aligned} \tag{3.61}$$

Similarly,

$$\bar{i}_{h(l_{\Psi_1}(r))}(n) \ge \bar{i}_{h(l_{\Psi_2}(r))}(n) \text{ and} \tag{3.62}$$

$$\bar{f}_{h(l_{\Psi_1}(r))}(n) \ge \bar{f}_{h(l_{\Psi_2}(r))}(n) \tag{3.63}$$

So, from Equations 3.61–3.63 $h(\Psi_1)$ is a subring of $h(\Psi_2)$. □

3.3. *Interval-valued neutrosophic normal soft subrings*

Definition 3.10. Let $(Q,+,\cdot)$ be a crisp ring and Ψ is an IVNSS of Q, where $\Psi = \Big\{\big(r,\{\big(g,\bar{t}_{l_\Psi(r)}(g),\bar{i}_{l_\Psi(r)}(g),\bar{f}_{l_\Psi(r)}(g)\big) : g \in Q\}\big) : r \in A\Big\}$. Then Ψ is called an IVNNSSR over Q if

(i) Ψ is an IVNSSR of Q and

(ii) $\forall g,n \in Q$, $\bar{t}_{l_\Psi(r)}(g\cdot n) = \bar{t}_{l_\Psi(r)}(n\cdot g)$, $\bar{i}_{l_\Psi(r)}(g\cdot n) = \bar{i}_{l_\Psi(r)}(n\cdot g)$, and $\bar{f}_{l_\Psi(r)}(g\cdot n) = \bar{f}_{l_\Psi(r)}(n\cdot g)$.

The set of all IVNNSSR of $(Q,+,\cdot)$ will be expressed as IVNNSSR(Q).

Example 3.11. Let $(\mathbb{Z},+,\cdot)$ be the ring and $\mathbb{N}$ be the set of parameters. Also, let $\Psi = \Big\{\Big(r,\{\big(g,\bar{t}_{l_\Psi(r)}(g),\bar{i}_{l_\Psi(r)}(g),\bar{f}_{l_\Psi(r)}(g)\big) : g \in \mathbb{Z}\}\Big) : r \in \mathbb{N}\Big\}$ be an IVNSS of $\mathbb{Z}$, where $l_\Psi(r) : \mathbb{N} \to$ IVNSS(Q) and $\forall g \in \mathbb{Z}$, $\forall r \in \mathbb{N}$ corresponding membership values are

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \left[\dfrac{1}{r+1},\dfrac{1}{r-1}\right] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \left[\dfrac{1}{2r+2},\dfrac{1}{2r-2}\right] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \left[\dfrac{r-2}{r-1},\dfrac{r}{r+1}\right] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\forall r \in \mathbb{N}$, $\Psi \in$ IVNNSSR$(\mathbb{Z})$.

Theorem 3.13. *Let $(Q,+,\cdot)$ be a crisp ring. If $\Psi_1,\Psi_2 \in$ IVNNSSR(Q), then $\Psi_1 \cap \Psi_2 \in$ IVNNSSR(Q).*

Proof. As $\Psi_1,\Psi_2 \in$ IVNSSR(Q) by Theorem 3.2 $\Psi_1 \cap \Psi_2 \in$ IVNSSR(Q). Again,

$$\begin{aligned} \bar{t}_{l_{\Psi_1\cap\Psi_2}(r)}(g\cdot n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g\cdot n),\bar{t}_{l_{\Psi_2}(r)}(g\cdot n)\big) \\ &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(n\cdot g),\bar{t}_{l_{\Psi_2}(r)}(n\cdot g)\big) \text{ [as } \Psi_1,\Psi_2 \in \text{IVNNSSR}(Q)] \\ &= \bar{t}_{\Psi_1\cap\Psi_2}(n\cdot g) \end{aligned} \tag{3.64}$$

Similarly,

$$\bar{i}_{l_{\Psi_1\cap\Psi_2}(r)}(g\cdot n) = \bar{i}_{l_{\Psi_1\cap\Psi_2}(r)}(n\cdot g) \tag{3.65}$$

$$\bar{f}_{l_{\Psi_1\cap\Psi_2}(r)}(g\cdot n) = \bar{f}_{l_{\Psi_1\cap\Psi_2}(r)}(n\cdot g) \tag{3.66}$$

Hence, $\Psi_1 \cap \Psi_2 \in$ IVNNSSR(Q). □

Remark 3.14. *In general, if $\Psi_1, \Psi_2 \in IVNNSSR(Q)$, then $\Psi_1 \cup \Psi_2$ may not always be an IVNNSSR of $(Q, +, \cdot)$.*

Remark 3.14 can be shown by Example 3.4.

Theorem 3.15. *Let $(Q, +, \cdot)$ be a crisp ring. Then $\Psi \in IVNNSSR(Q)$ iff $\forall [g_1, n_1], [g_2, n_2], [g_3, n_3] \in \phi(K)$ with $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ is a crisp normal subring of $(Q, +, \cdot)$ (considering idempotent IVTN and IVSNs).*

Proof. This can be proved using Theorem 3.5. □

Theorem 3.16. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a ring isomorphism. If Ψ is an IVNNSSR of Q then $h(\Psi)$ is an IVNNSSR of Y.*

Proof. As Ψ is an IVNSSR of Q, by Theorem 3.6 $h(\Psi)$ is an IVNSSR of Y. Let $h(g_1) = n_1$ and $h(g_2) = n_2$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Then

$$\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 \cdot n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 \cdot n_2)\big) \text{ [as } h \text{ is injective]} \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) \cdot h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_{l_\Psi(r)}(g_1 \cdot g_2) \\
&= \bar{t}_{l_\Psi(r)}(g_2 \cdot g_1) \text{ [as } \Psi \text{ is an IVNNSSR of } Q\text{]} \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2) \cdot h^{-1}(n_1)\big) \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2 \cdot n_1)\big) \\
&= \bar{t}_{h(l_\Psi(r))}(n_2 \cdot n_1) \qquad (3.67)
\end{aligned}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n_1 \cdot n_2) = \bar{i}_{h(l_\Psi(r))}(n_2 \cdot n_1) \text{ and} \qquad (3.68)$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 \cdot n_2) = \bar{f}_{h(l_\Psi(r))}(n_2 \cdot n_1) \qquad (3.69)$$

So, from Equations 3.67–3.69 $h(\Psi)$ is an IVNNSSR of Y. □

4. Conclusions

Interval-valued neutrosophic field is a dynamic research domain. Under soft environment, it becomes more general and productive. For this reason, we have adopted this mixed environment and defined the notions of interval-valued neutrosophic soft subring along with its normal version. Also, we have studied several homomorphic attributes of these newly introduced notions. Again, we have introduced the product of two interval-valued neutrosophic soft subrings. Furthermore, we have given several fundamental theories to understand some of its algebraic characteristics. These newly introduced notions have the potentials to become fruitful research domains. In future, for generalizing this concepts one can introduce them under the hypersoft set environment.

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