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Solvability of System of Neutrosophic Soft Linear Equations

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Abstract. This article exposes a system of Neutrosophic Soft Linear Equations (NSLE) of the form $A \otimes x = b$ and is said to be solvable if $A \otimes x(A; b) = b$ holds, otherwise unsolvable. We derive conditions under which the above system is solvable and further using Chebychev Approximation we find a prinicipal solution if the given systen is not solvable.

Keywords: Neutrosophic Soft Set (NSS), Neutrosophic Soft Matrix(NSM), Neutrosophic Soft Eigenvector(NSEv), System of Neutrosophic Soft Linear Equation(NSLE), Chebychev distance.

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1. Introduction

In human judgment, the importance of relations is almost self-evident. But the problem is mainly to pass from a vague and customary concept to a precisely formulated one. The theory of fuzzy sets is a step in such a direction and we believe that a straightforward study of fuzzy relations deserves to be developed for a better interpretation and explanation of real-world problems. The system of fuzzy relation equations is an important topic in fuzzy set theory. Sanchez [29] first introduced fuzzy relation equations with sup-inf composition in complete Brouwerian lattices. Since then, many authors investigated the methods for solving fuzzy relation equations with different composite operators over various special Brouwerian lattices. Among them, for finite fuzzy relation equations with sup-inf composition, Higashi et,al. [10] showed that the solution set can be determined by minimal solutions and the greatest solution in the linear lattice [0,1]. The solvability and unique solvability of linear systems in the max-min algebra which is one of the most important fuzzy algebra, and the related question of the strong regularity of max-min matrices was considered in [5, 6]. Cechlarova [7] studied the unique solvability of linear systems of equation over the max-min fuzzy algebra on the unit real interval. In 2010 Sriram and Murugadas discussed the relation between row space, column space and regularity of Intuitionistic Fuzzy Matrix(IFM) etc.(see [25,26,31–35]). Pradhan and Pal [27] introduced the concepts that the Intuitionistic Fuzzy Relation Equation of the form $A \otimes x = b$ is consistent when the coefficient IFM A is regular.

But all these theories have their inherent difficulties as pointed out by Molodtsove [24]. The reason for these difficulties is, possibly, the inadequacy of the parameterization tools of the theories. The fuzzy soft set representation of the intuitionistic fuzzy soft set has been studied by Maji et.al, [23]. Likewise, Rajarajeswari et.al [28], proposed new definitions for intuitionistic fuzzy soft matrices and its sort.

The notion of Neutrosophic Set (NS) was introduced by Smarandache [30]. Deli [8] defined Neutrosophic parameterized Neutrosophic soft sets (npn-soft sets) which is the combination of NS and a soft set. Deli and Broumi [9] redefined the notion of NS in a new way and put forward the concept of NSM and different types of matrices in neutrosophic soft theory. They have introduced some new operations and properties on these matrices. For recent development of NS in decision making theory see the work done by Abdel Basset et.al, [1–3] and N . Nabeeh et.al, [18–20]. The minimal solution of NSM was done by Kavitha et.al, [12] based on the notion of NSM given by Sumathi and Arokiarani [4]. As the time goes some works on NSM were done by Kavitha et.al, [13–15,17]. The Monotone interval fuzzy neutrosophic soft eigenproblem and Monotone fuzzy neutrosophic soft eigenspace structures in max-min algebra were investigated by Murugadas et.al, [21,22]. Also, two kinds of fuzzy neutrosophic soft matrices are presented by Uma et.al, [36].

In this paper, we will concentrate on the solvability of the system of NSLEs be solvable of the form $A \otimes x(A; b) = b$. We derived the maximum solution for a system of NSLEs and we define that particular solution $x(A; b)$ as principal solution. In the concluding section-5, we have tried to give an algorithm for coefficient NSM A of an unsolvable system, $A \otimes x = b$ to get a principal solution.

2. **Preliminaries**

In this section, some elementary aspects that are necessary for this paper are introduced.

Definition 2.1. [30] A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x,\ T_A(x),\ I_A(x),\ F_A(x)\rangle, x \in X\}$, where $T, I, F : X \to\ ]^-0, 1^+[$ and
$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$. (1)]

From philosophical point of view the NS set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application especially in Scientific and Engineering problems it is difficult to use NS with value from real standard or non-standard subset of $]^{-}0,1^{+}[$. Hence we consider the NS which takes the value from the subset of $[0,1]$. Therefore we can rewrite equation (1) as $0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3$. In short an element $\widetilde{a}$ in the NS A, can be written as $\widetilde{a}=\langle a^T, a^I, a^F\rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \leq a^T+a^I+a^F \leq 3$.

Definition 2.2. [4] A NS A on the universe of discourse X is defined as $A = \{x, \langle T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$, where $T, I, F : X \rightarrow [0,1]$ and $0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3$.

Definition 2.3. [24] Let U be the initial universe set and E be a set of parameter. Consider a non-empty set $A, A \subset E$. Let $P(U)$ denotes the set of all NSs of U. The collection (F,A) is termed to be the NSS over U, where F is a mapping given by $F : A \rightarrow P(U)$. Here after we simply consider A as NSS over U instead of (F,A).

Definition 2.4. [4] Let $U=\{c_1,c_2,...,c_m\}$ be the universal set and E be the set of parameters given by $E=\{e_1,e_2,...,e_m\}$. Let $A \subset E$. A pair (F,A) be a NSS over U. Then the subset of $U \times E$ is defined by $R_A=\{(u,e);\ e \in A,\ u \in F_A(e)\}$
which is called a relation form of $(F_A,\ E)$. The membership function, indeterminacy membership function and non membership function are written by
$T_{R_A} : U \times E \rightarrow [0,1],\ I_{R_A} : U \times E \rightarrow [0,1]$ and $F_{R_A} : U \times E \rightarrow [0,1]$ where $T_{R_A}(u,e) \in [0,1],\ I_{R_A}(u,e) \in [0,1]$ and $F_{R_A}(u,e) \in [0,1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$.
If $[(T_{ij},\ I_{ij},\ F_{ij})]=[T_{ij}(u_i,e_j),\ I_{ij}(u_i,e_j)\ ,F_{ij}(u_i,e_j)]$ we define a matrix

$$[\langle T_{ij}\ ,I_{ij},\ F_{ij}\rangle]_{m\times n}=\begin{bmatrix}\langle T_{11},\ I_{11},\ F_{11}\rangle & \cdots & \langle T_{1n},\ I_{1n},\ F_{1n}\rangle \\ \langle T_{21},\ I_{21},\ F_{21}\rangle & \cdots & \langle T_{2n},\ I_{2n},\ F_{2n}\rangle \\ \vdots & \vdots & \vdots \\ \langle T_{m1},\ I_{m1},\ F_{m1}\rangle & \cdots & \langle T_{mn},\ I_{mn},\ F_{mn}\rangle\end{bmatrix}.$$

Which is called an $m \times n$ $FNSM$ of the NSS (F_A,E) over U.

Definition 2.5. [36] Let $A=(\langle a_{ij}^T,\ a_{ij}^I,\ a_{ij}^F\rangle),\ B=\langle(b_{ij}^T,\ b_{ij}^I,\ b_{ij}^F\rangle) \in \mathcal{N}_{(m,n)}$, NSM of order $m \times n$) and $\mathcal{N}_{(n)}$-denotes a square NSM of order n. The component wise addition and component wise multiplication is defined as
$A \oplus B=(sup\{a_{ij}^T,\ b_{ij}^T\},\ sup\{a_{ij}^I,\ b_{ij}^I\},\ inf\{a_{ij}^F,\ b_{ij}^F\})$
$A \otimes B=(inf\{a_{ij}^T,\ b_{ij}^T\},\ inf\{a_{ij}^I,\ b_{ij}^I\},\ sup\{a_{ij}^F,\ b_{ij}^F\})$

Definition 2.6. Let $A \in \mathcal{N}_{(m,n)},\ B \in \mathcal{N}_{(n,p}$, the composition of A and B is defined as

$$A \circ B = \left(\sum_{k=1}^{n} (a_{ik}^T \wedge\ b_{kj}^T),\ \sum_{k=1}^{n} (a_{ik}^I \wedge\ b_{kj}^I),\ \prod_{k=1}^{n} (a_{ik}^F \vee\ b_{kj}^F) \right)$$

equivalently we can write the same as

$$= \left(\bigvee_{k=1}^{n} (a_{ik}^T \wedge b_{kj}^T),\ \bigvee_{k=1}^{n} (a_{ik}^I \wedge b_{kj}^I),\ \bigwedge_{k=1}^{n} (a_{ik}^F \vee b_{kj}^F) \right).$$

The product $A \circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. Then A and B are said to be conformable for multiplication. We shall use AB instead of $A \circ B$.

Where $\sum (a_{ik}^T \wedge\ b_{kj}^T)$ means max-min operation and

$\prod_{k=1}^{n} (a_{ik}^F \vee\ b_{kj}^F)$ means min-max operation.

Definition 2.7. [16] Let V_n will denote the set of all n-tuples $(\langle v_1^T, v_1^I, v_1^F \rangle, ..., \langle v_n^T, v_n^I, v_n^F \rangle)$ over $[0, 1]^3$

An element of V_n is called a Neutrosophic Soft vector (NSV) of dimension n.

Definition 2.8. [16] If $A \in \mathcal{N}_{(m,n)}$ and $X \in \mathcal{N}_{(n,m)}$ satisfies the relation $AXA = A$ then X is called a generalized inverse(g-inverse) of A which is denoted by A^-. The g-inverse of an NSM is not necessarily unique. We denote the set of all g-inverses of A by $A\{1\}$.

Definition 2.9. [16] Let $A = \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \in \mathcal{N}_{(m,n)}$. Then the element $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle$ is called the (i, j) entry of A. Let $A_{i*}(A_{*j})$ denote the i^{th} row (column) of A. The row space $\mathcal{R}(A)$ of A is the subspace of V_n generated by rows $\{A_{i*}\}$ of A. The column space $\mathcal{C}(A)$ of A is the space of V_m generated by the columns $\{A_{*j}\}$ of A.

Definition 2.10. [16] For NSM $A, X \in \mathcal{N}_{(m \times n)}$, are said to be a Moore-Penrose of A, if $AXA = A, XAX = X, (AX)^t = AX$ and $(XA)^t = XA$.

3. **Results**

Definition 3.1. (Linear combination of NSVs)

Let $S = \{\langle a_1^T, a_1^I, a_1^F \rangle, \langle a_2^T, a_2^I, a_2^F \rangle, ..., \langle a_p^T, a_p^I, a_p^F \rangle\}$ be a set of NSV of dimension n. The linear combination of elements of the set S is a finite sum $\sum_{i=1}^{p} \langle c_i^T, c_i^I, c_i^F \rangle \langle a_i^T, a_i^I, a_i^F \rangle$ where $\langle a_i^T, a_i^I, a_i^F \rangle \in S$ and $\langle c_i^T, c_i^I, c_i^F \rangle \in [0, 1]^3$. The set of all linear combinations of the elements of S is calld the span of S, denoted by $\langle S \rangle$.

Here we illustrate the above concept.

Example 3.2. Let $S = \{\langle a_1^T, a_1^I, a_1^F \rangle, \langle a_2^T, a_2^I, a_2^F \rangle, \langle a_3^T, a_3^I, a_3^F \rangle\}$ be a subset of V_3, where $\langle a_1^T, a_1^I, a_1^F \rangle = (\langle 0.8, 0.7, 0.2 \rangle, \langle 0.6, 0.5, 0.4 \rangle, \langle 0.4, 0.3, 0.6 \rangle)$,

$\langle a_2^T, a_2^I, a_2^F\rangle = (\langle 0.5, 0.4, 0.6\rangle, \langle 0.5, 0.4, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle)$,
and $\langle a_3^T, a_3^I, a_3^F\rangle = (\langle 0.7, 0.6, 0.3\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.9, 0.8, 0.1\rangle)$. Then

$$\begin{aligned}\langle S\rangle = \{&\langle c_1^T, c_1^I, c_1^F\rangle(\langle 0.8, 0.7, 0.2\rangle, \langle 0.6, 0.5, 0.2\rangle, \langle 0.4, 0.3, 0.6\rangle)\\ &+\langle c_2^T, c_2^I, c_2^F\rangle(\langle 0.5, 0.4, 0.6\rangle, \langle 0.5, 0.4, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle\\ &+\langle c_3^T, c_3^I, c_3^F\rangle(\langle 0.7, 0.6, 0.3\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.9, 0.8, 0.1\rangle\}.\end{aligned}$$

Definition 3.3 (Dependenece of NSVs)**.** A set S of NSVs is independent if and only if each element of S can be expressed as a linear combination of other elements of S, that is, no element $s \in S$ is a linear combination of $S \setminus \{s\}$. If a vector α can be expressed by some other vectors, then the vector α is called dependent otherwise it is called independent. These terminologies are similar to classical vectors.

An independent and dependent set of vectors are illustrated below.

Example 3.4. Let $S = \{\langle a_1^T, a_1^I, a_1^F\rangle, \langle a_2^T, a_2^I, a_2^F\rangle, \langle a_3^T, a_3^I, a_3^F\rangle\}$ be a subset of V_3, where
$\langle a_1^T, a_1^I, a_1^F\rangle = (\langle 0.8, 0.7, 0.2\rangle, \langle 0.6, 0.5, 0.4\rangle, \langle 0.4, 0.3, 0.6\rangle)$,
$\langle a_2^T, a_2^I, a_2^F\rangle = (\langle 0.5, 0.4, 0.6\rangle, \langle 0.5, 0.4, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle)$, and
$\langle a_3^T, a_3^I, a_3^F\rangle = (\langle 0.7, 0.6, 0.3\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.9, 0.8, 0.1\rangle)$.

Here the set S is an independent set.
If not then $\langle a_1^T, a_1^I, a_1^F\rangle = \langle\alpha^T, \alpha^I, \alpha^F\rangle\langle a_2^T, a_2^I, a_2^F\rangle + \langle\beta^T, \beta^I, \beta^F\rangle\langle a_3^T, a_3^I, a_3^F\rangle$
for $\langle\alpha^T, \alpha^I, \alpha^F\rangle, \langle\beta^T, \beta^I, \beta^F\rangle \in \mathcal{N}$. So

$$\begin{aligned}\langle a_1^T, a_1^I, a_1^F\rangle &= \langle\alpha^T, \alpha^I, \alpha^F\rangle(\langle 0.5, 0.4, 0.6\rangle, \langle 0.5, 0.4, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle)\\ &\quad +\langle\beta^T, \beta^I, \beta^F\rangle(\langle 0.7, 0.6, 0.3\rangle, \langle 0.7, 0.6, 0.3\rangle, \langle 0.9, 0.8, 0.1\rangle\\ &= (\langle\max\{\min(0.5, \alpha^T), \min(0.7, \beta^T)\}, \max\{\min(0.4, \alpha^I), \min(0.6, \beta^I)\},\\ &\quad \min\{\max(0.6, \alpha^F), \max(0.3, \beta^F)\}\}),\\ &\quad (\langle\max\{\min(0.5, \alpha^T), \min(0.7, \beta^T)\}, \max\{\min(0.4, \alpha^I), \min(0.6, \beta^I)\},\\ &\quad \min\{\max(0.6, \alpha^F), \max(0.3, \beta^F)\}\}),\\ &(\langle\max\{\min(0.4, \alpha^T), \min(0.9, \beta^T)\}, \max\{\min(0.3, \alpha^I), \min(0.8, \beta^I)\},\\ &\quad \min\{\max(0.6, \alpha^F), \max(0.1, \beta^F)\}\}).\end{aligned}$$

It is not possible to find any $\langle\alpha^T, \alpha^I, \alpha^F\rangle, \langle\beta^T, \beta^I, \beta^F\rangle \in \mathcal{N}$ such that the corresponding coefficients on both sides will be equal. That is,
$\langle a_1^T, a_1^I, a_1^F\rangle \neq \langle\alpha^T, \alpha^I, \alpha^F\rangle\langle a_2^T, a_2^I, a_2^F\rangle + \langle\beta^T, \beta^I, \beta^F\rangle\langle a_3^T, a_3^I, a_3^F\rangle$. Similarly,
$\langle a_2^T, a_2^I, a_2^F\rangle \neq \langle\alpha^T, \alpha^I, \alpha^F\rangle\langle a_1^T, a_1^I, a_1^F\rangle + \langle\beta^T, \beta^I, \beta^F\rangle\langle a_3^T, a_3^I, a_3^F\rangle$ and
$\langle a_3^T, a_3^I, a_3^F\rangle \neq \langle\alpha^T, \alpha^I, \alpha^F\rangle\langle a_2^T, a_2^I, a_2^F\rangle + \langle\beta^T, \beta^I, \beta^F\rangle\langle a_1^T, a_1^I, a_1^F\rangle$. So the set S is independent.

Let $S = \{\langle a_1^T, a_1^I, a_1^F\rangle, \langle a_2^T, a_2^I, a_2^F\rangle\}$ be a subset of V_3,
where $\langle a_1^T, a_1^I, a_1^F\rangle = (\langle 0.7, 0.6, 0.3\rangle, \langle 0.5, 0.4, 0.5\rangle, \langle 0.6, 0.5, 0.4\rangle)$ and
$\langle a_2^T, a_2^I, a_2^F\rangle = (\langle 0.8, 0.7, 0.2\rangle, \langle 0.5, 0.4, 0.5\rangle, \langle 0.6, 0.5, 0.4\rangle)$.
Here $\langle a_1^T, a_1^I, a_1^F\rangle = \langle c^T, c^I, c^F\rangle(\langle a_2^T, a_2^I, a_2^F\rangle)$ for $\langle c^T, c^I, c^F\rangle = \langle 0.7, 0.6, 0.3\rangle$. So S is a dependent set.

Definition 3.5 (Basis)**.** Let W be an Neutrosophic Soft Subspace of V_n and S be a subset of W such that the elements of S are independent. If every element of W can be expressed uniquely as a linear combination of the elements of S, then S is called a basis of neutrosophic soft subspace W.

Definition 3.6 (Standard basis)**.** A basis B of an Neutrosophic Soft Vector Space (NSVS) W is a standard basis if and only if whenever
$\langle b_i^T, b_i^I, b_i^F\rangle = \sum\limits_{j=1}^{n}\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle\langle b_j^T, b_j^I, b_j^F\rangle$ for $\langle b_i^T, b_i^I, b_i^F\rangle, \langle b_j^T, b_j^I, b_j^F\rangle \in \mathcal{N}$
and $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \in [1, 0]$ then $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle\langle b_i^T, b_i^I, b_i^F\rangle = \langle b_i^T, b_i^I, b_i^F\rangle$.

Example 3.7. Let $S = \{\langle a_1^T, a_1^I, a_1^F\rangle, \langle a_2^T, a_2^I, a_2^F\rangle, \langle a_3^T, a_3^I, a_3^F\rangle\}$ be a subset of V_3 given by
$a_1 = (\langle 0.5, 0.4, 0.5\rangle, \langle 0.5, 0.4, 0.5\rangle, \langle 0.5, 0.4, 0.5\rangle)$ and
$a_2 = (\langle 0.5, 0.4, 0.5\rangle, \langle 0.6, 0.5, 0.4\rangle, \langle 0.8, 0.7, 0.2\rangle)$ and
$a_3 = (\langle 0.4, 0.3, 0.6\rangle, \langle 0.4, 0.3, 0.6\rangle, \langle 0.8, 0.7, 0.2\rangle)$.
Then S is independent set, since
$\langle a_1^T, a_1^I, a_1^F\rangle \neq \langle c_1^T, c_1^I, c_1^F\rangle(\langle a_2^T, a_2^I, a_2^F\rangle) + \langle c_2^T, c_2^I, c_2^F\rangle(\langle a_3^T, a_3^I, a_3^F\rangle)$,
$\langle a_2^T, a_2^I, a_2^F\rangle \neq \langle c_3^T, 3_3^I, c_3^F\rangle\langle a_1^T, a_1^I, a_1^F\rangle + \langle c_4^T, c_4^I, c_4^F\rangle\langle a_3^T, a_3^I, a_3^F\rangle$ and
$\langle a_3^T, a_3^I, a_3^F\rangle \neq \langle c_5^T, c_5^I, c_5^F\rangle(\langle a_1^T, a_1^I, a_1^F\rangle) + \langle c_6^T, c_6^I, c_6^F\rangle(\langle a_2^T, a_2^I, a_2^F\rangle)$.
So $\{\langle a_1^T, a_1^I, a_1^F\rangle, \langle a_2^T, a_2^I, a_2^F\rangle, \langle a_3^T, a_3^I, a_3^F\rangle\}$ is a basis for $\langle S\rangle$.
Now this is a standard basis . For, $\langle a_1^T, a_1^I, a_1^F\rangle = \langle c_{11}^T, c_{11}^I, c_{11}^F\rangle(\langle a_1^T, a_1^I, a_1^F\rangle) + \langle c_{12}^T, c_{12}^I, c_{12}^F\rangle(\langle a_2^T, a_2^I, a_2^F\rangle) + \langle c_{13}^T, c_{13}^I, c_{13}^F\rangle(\langle a_3^T, a_3^I, a_3^F\rangle)$ holds if $\langle c_{11}^T, c_{11}^I, c_{11}^F\rangle = \langle 0.8, 0.7, 0.2\rangle$, $\langle c_{12}^T, c_{12}^I, c_{12}^F\rangle = \langle 0.5, 0.4, 0.5\rangle$ and $\langle c_{13}^T, c_{13}^I, c_{13}^F\rangle = \langle 0.6, 0.5, 0.4\rangle$.
Also $\langle a_1^T, a_1^I, a_1^F\rangle = \langle c_{11}^T, c_{11}^I, c_{11}^F\rangle(\langle a_1^T, a_1^I, a_1^F\rangle)$ for $\langle c_{11}^T, c_{11}^I, c_{11}^F\rangle = \langle 0.8, 0.7, 0.2\rangle$.
Similarly for $\langle a_2^T, a_2^I, a_2^F\rangle$ and $\langle a_3^T, a_3^I, a_3^F\rangle$.

4. **Solvability**

In this section, we are going to study the system of NSLEs of the form,

$$A \otimes x = b \ (1)$$

that is

$$\langle \max_j \min(a_{ij}^T, x_j^T), \max_j \min(a_{ij}^I, x_j^I), \min_j \max(a_{ij}^F, x_j^F)\rangle = \langle b_i^T, b_i^I, b_i^F\rangle \ (2)$$

where the NSM $A \in \mathcal{N}_{(m\times n)}$ and the NSV $b \in \mathcal{N}_{(m)}$ are given and the NSV $x \in \mathcal{N}_{(n)}$ is unknown.

The solution set of the system defined in (1) for a given NSM A and an NSV b will be denoted by $S(A,b) = \{x \in \mathcal{N}_n | A \otimes x = b\}$.
Now our aim is to find whether the system (1) is solvable, that is, whether the solution set $S(A,b)$ is non-empty.

Lemma 4.1. Let us consider the system of NSLE $A \otimes x = b$.
If $\max_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) < \langle b^T_j, b^I_j, b^F_j\rangle$ for some k, then $S(A,b) = \phi$, that is the sysem is not solvable.
Proof: If $\max_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) < \langle b^T_j, b^I_j, b^F_j\rangle$ for some j, then
$\min_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) \leq \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \leq \max_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) < (\langle b^T_j, b^I_j, b^F_j\rangle)$
Hence, $\langle \max_j \min(a^T_{ij}, x^T_i), \max_j \min(a^I_{ij}, x^I_i), \min_j \max(a^F_{ij}, x^F_i)\rangle < (\langle b^T_j, b^I_j, b^F_j\rangle)$ for some j, and by equation (2) no values $\langle x^T_i, x^I_i, x^F_i\rangle$ exists that satisfy the equation (1). Therefor $S(A,b) = \phi$.

Remark 4.2. Let us consider the condition of the Lemma 4.1 be
$\max_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) > (\langle b^T_j, b^I_j, b^F_j\rangle)$ for some j. Then according to the proof of the Lemma 4.1, $\min_j(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle, \langle x^T_i, x^I_i, x^F_i\rangle) \geq \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \geq \max(\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle) > (\langle b^T_j, b^I_j, b^F_j\rangle)$ implies the only possibility is, $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle$ are same for all i. Then two case may arises,
Case-1: If $\langle b^T_j, b^I_j, b^F_j\rangle$ are equal for all j. Then the system reduce to one equation. So that the system is solvable.
Case-2: If $\langle b^T_j, b^I_j, b^F_j\rangle$ are different for some j. Then the equation of the system will be such that, all have the same left side with some different right side. Hence the system is not solvable.

Example 4.3. Let us consider the system of NSLEs $A \otimes x = b$ where,

$$A = \begin{bmatrix} \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.6\ 0.5\ 0.4\rangle \\ \langle 0.8\ 0.7\ 0.2\rangle & \langle 0.4\ 0.3\ 0.6\rangle \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} \langle 0.4\ 0.3\ 0.6\rangle \\ \langle 1, 1, 0\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle \end{bmatrix}.$$

Here for $j = 2$,
$\max\{\langle 0.3\ 0.2\ 0.7\rangle, \langle 0.6\ 0.5\ 0.4\rangle, \langle 0.4\ 0.3\ 0.6\rangle\} = \langle 0.6\ 0.5\ 0.4\rangle < \langle 1, 1, 0\rangle$. Hence by Lemma 4.1, the system of NSLEs $A \otimes x = b$ is not solvable.

The following theorem deduce the fact its solvability of a system of NSLEs of the form (1) depends upon the characteristics of the coefficient NSM A.

Theorem 4.4. The system of NSLEs of the form (1) has a solution if the non-zero rows of the coefficient NSM A forms a standard basis for the row space of itself.

Proof: As the non-zero rows of the NSM A forms a standard basis for the row space of A, then the NSM A be regular. That is there exists a g-inverse A^- of A such that $A \otimes A^- \otimes A = A$. Now, $A \otimes x = b$ gives $A \otimes A^- \otimes A \otimes x = b$.

That implies, $A \otimes A^- \otimes b = b$. Which shows, $(A^- \otimes b)$ is a solution of the given sytem. Therefore the system of NSLE is solvable.

Example 4.5. Let us consider the system of NSLEs $A \otimes x = b$. with

$$A = \begin{bmatrix} \langle 0.7\ 0.6\ 0.3 \rangle & \langle 0.6\ 0.5\ 0.4 \rangle & \langle 0.5\ 0.4\ 0.5 \rangle \\ \langle 0.5\ 0.4\ 0.5 \rangle & \langle 0.6\ 0.5\ 0.4 \rangle & \langle 0.8\ 0.7\ 0.2 \rangle \end{bmatrix}$$

$$X = [\langle x_1^T, x_1^I, x_1^F \rangle, \langle x_2^T, x_2^I, x_2^F \rangle, \langle x_3^T, x_3^I, x_3^F \rangle]^T \text{ and}$$

$$b = \begin{bmatrix} \langle 0.6\ 0.5\ 0.4 \rangle \\ \langle 0.5, 0.4, 0.5 \rangle \end{bmatrix}.$$

Here the non-zero rows of the NSM S are linearly independent and form s standard basis . So
A is regular and one of its $g-$ inverse is

$$A^- = \begin{bmatrix} \langle 0.8\ 0.7\ 0.2 \rangle & \langle 0.5\ 0.4\ 0.5 \rangle \\ \langle 0.5\ 0.4\ 0.5 \rangle & \langle 0.5\ 0.4\ 05 \rangle \\ \langle 0.5\ 0.4\ 0.5 \rangle & \langle 0.8\ 0.7\ 0.2 \rangle \end{bmatrix}$$

$$x = A^- b = \begin{bmatrix} \langle 0.6\ 0.5\ 0.4 \rangle \\ \langle 0.5\ 0.4\ 0.3 \rangle \\ \langle 0.5\ 0.4\ 0.5 \rangle \end{bmatrix}$$

This is one of the solution of the above system of NSLEs.

The assertion of the $g-$inverse of a NSM A is not unique. So the solution of a system of NSLEs may have many solution. Among these solutions the maximum solution is defined as follows.

Definition 4.6. Any arbitrary element $\bar{x}$ of $S(A, b)$ is called a maximum solution of the system $A \otimes x = b$ if for all $x \in S(A, b), x \geq \bar{x}$ implies $x = \bar{x}$.

The following theorem demonstrate how to find the maximum solution of the system of NSLEs.

Theorem 4.7. If for a system of NSLEs $A \otimes x = b$ has a solution denoted by $\bar{x}(A,b)$ and is defined by

$$\bar{x} = \langle \bar{x}^T, \bar{x}^I, \bar{x}^F \rangle = \begin{cases} \langle 1,1,0 \rangle \ if \ \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \leq \langle b_j^T, b_j^I, b_j^F \rangle \ \forall i \\ \min\{\langle b_j^T, b_j^I, b_j^F \rangle\} \ if \ \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle > \langle b_j^T, b_j^I, b_j^F \rangle, \end{cases}$$

is the maximum solution.

Proof: As the system of NSLEs $A \otimes x = b$ has a solution, so it is consistent, then $\bar{x}$ is a solution of the system. If $\bar{x}$ is not a solution, then $A \otimes x \neq b$ and therefore
$\max_j \min(a_{ij}^T, x_j^T), \max_j \min(a_{ij}^I, x_j^I) \min_j \max(a_{ij}^F, x_j^F) \neq (\langle b_{j0}^T, b_{j0}^I, b_{j0}^F \rangle)$ for at least one j_0. The above definition of $\bar{x}$,
since $\langle \bar{x}_i^T, \bar{x}_i^I, \bar{x}_i^F \rangle \leq \langle b_j^T, b_j^I, b_j^F \rangle$ for each j, so
$\langle \bar{x}_i^T, \bar{x}_i^I, \bar{x}_i^F \rangle \leq \langle b_{j0}^T, b_{j0}^I, b_{j0}^F \rangle$. By our assumption, $\max_j(\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle < \langle b_{j0}^T, b_{j0}^I, b_{j0}^F \rangle)$ for some j_0 and by Lemma 4.1 it follows that $S(A,b) = \phi$, which is a contradiction. Hence $\bar{x}$ is a solution of the system $A \otimes x = b$.
Now let us prove that $\bar{x}$ is a maximum solution. If possible let us assume that $y = \langle y^T, y^I, y^F \rangle$ be a solution of the system such that $y > \bar{x}$, that is
$\langle y_{i0}^T, y_{i0}^I, y_{i0}^F \rangle > \langle \bar{x}_{i0}^T, \bar{x}_{i0}^I, \bar{x}_{i0}^F \rangle$ for at least one i_0.
Therefore by definition of $\bar{x}$, we have $\langle y_{i0}^T, y_{i0}^I, y_{i0}^F \rangle > \min(\langle b_j^T, b_j^I, b_j^F \rangle)$ when $\langle a_{i_0 j}^T, a_{i_0 j}^I, a_{i_0 j}^F \rangle > \langle b_j^T, b_j^I, b_j^F \rangle$ for some j. Again, since $S(A,b) \neq \emptyset$, by Lemma 4.1,
$\max_i(\langle a_{ij_0}^T, a_{ij_0}^I, a_{ij_0}^F \rangle > \langle b_{j0}^T, b_{j0}^I, b_{j0}^F \rangle)$ for each j_0.
Hence, $\langle b_{j0}^T, b_{j0}^I, b_{j0}^F \rangle \neq \langle \max_i \min(a_{ij_0}^T, y_i^T), \max_i \min(a_{ij_0}^I, y_i^I), \min_i \max(a_{ij_0}^F, y_i^F)$, which contradicts our assumption $y \in S(A,b)$.

Therefore, $\bar{x}$ is the maximum solution of the system of NSLEs $A \otimes x = b$.

Example 4.8. Given

$$A = \begin{bmatrix} \langle 0.7 \ 0.6 \ 0.3 \rangle & \langle 0.6 \ 0.5 \ 0.4 \rangle & \langle 0.5 \ 0.4 \ 0.5 \rangle \\ \langle 0.5 \ 0.4 \ 0.5 \rangle & \langle 0.6 \ 0.5 \ 0.4 \rangle & \langle 0.8 \ 0.7 \ 0.2 \rangle \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} \langle 0.5 \ 0.4 \ 0.5 \rangle \\ \langle 0.6, 0.5, 0.4 \rangle \end{bmatrix}.$$

From the definition of maximum solution,
$x_1 = \langle 0.5 \ 0.4 \ 0.5 \rangle, x_2 = \langle 0.6 \ 0.5 \ 0.4 \rangle$,
$x_3 = \langle 0.5 \ 0.4 \ 0.5 \rangle$. So $\bar{x} = [\langle 0.5 \ 0.4 \ 0.5 \rangle, \langle 0.6 \ 0.5 \ 0.4 \rangle, \langle 0.5 \ 0.4 \ 0.5 \rangle]^T$. Thus, $S(A,b) \neq \phi$ and $A \otimes \bar{x} = b$ hold. Hence $x = [\langle 0.5 \ 0.4 \ 0.5 \rangle, \langle 0.6 \ 0.5 \ 0.4 \rangle, \langle 0.5 \ 0.4 \ 0.5 \rangle]^t = \bar{x}$ is the maximum solution.

Now we consider the definition 2.10 of Moore-Penrose Inverse.

Theorem 4.9. Let us consider a system of NSLEs (1). The system must have a solution, that is, must be consistent if the coefficient NSM A is a symmetric and idempotent of order n.

Proof: Since A is symmetric and idempotent square NSM, that is A itself is a Moore-Penrose inverse. That is, $A = A^{+}$. So in the case the solution will be
$x = A^{+}b = Ab$.

Example 4.10. Consider the system of NSLEs $A \otimes x = b$ where,

$$A = \begin{bmatrix} \langle 0.8\ 0.7\ 0.2\rangle & \langle 0.6\ 0.5\ 0.4\rangle \\ \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.7\ 0.6\ 0.3\rangle \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} \langle 0.8\ 0.7\ 0.2\rangle \\ \langle 0.6, 0.5, 0.4\rangle \end{bmatrix}.$$

Here, $A^T = A$ and $A^2 = A$, that is, the NSM A is symmetric and idempotent. So the Moore-Penrose inverse A^{+} of A is itself A. Then the solution will be
$x = A^{+}b = Ab = [\langle 0.8\ 0.7\ 0.2\rangle, \langle 0.6, 0.5, 0.4\rangle]^t$.

5. Chebychev Approximation

In this section, we describe an algorithm by which we approach the right hand side of the system of NSLEs $A \otimes x = b$ by successively changing the original NSM $A \in \mathcal{N}_{m\times n}$ to a NSM $D \in \mathcal{N}_{m\times n}$ such that $D \otimes x = b$ is solvable.

Let us consider the solution or tolerable solution $x^{'}(A; b)$ of the system of NSLEs

$$A \otimes x = b\ as\ x^{'}(A; b) = \begin{cases} \langle 1, 1, 0\rangle\ if\ \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \leq \langle b^T_i, b^I_i, b^F_i\rangle\ \forall i \\ \min\{\langle b^T_i, b^I_i, b^F_i\rangle\}\ if\ \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle\ (3) \end{cases}$$

Now if we define that the system (1) is solvable if and only if (3) is its solution,
that is $A \otimes x^{'}(A, b) = b$ holds, but in general $A \otimes x^{'}(A; b) \leq b$ holds always. So our aim is, by changing the NSM A and retain the right hand side of the system same to make the system solvable.

First we have to define some importent Definitions.

Definition 5.1. The Chebychev distance of two NSMs $A, B \in \mathcal{N}_{(m\times n)}$ is denoted by $\rho(A, B)$ and is defined by
$\rho(A, B) = \langle \max\limits_{i,j} |a^T_{ij} - b^T_{ij}|, \max\limits_{i,j} |a^I_{ij} - b^I_{ij}|, \min\limits_{ij} |a^F_{i,j} - b^F_{ij}|\rangle$.
The Chebychev distance of a NSM $A \in \mathcal{N}_{(m\times n)}$ and the set $S \in \mathcal{N}_{(m\times n)}$ is defined by $\rho(A, S) = \inf\limits_{B\in S} \rho(A, B)$.

Definition 5.2. We say that a NSM $B \in \mathcal{N}_{(m\times n)}$ is closer to a NSV $v \in \mathcal{N}_{(m)}$ than a NSM $A \in \mathcal{N}_{(m\times n)}$ if
$\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \leq \langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle \leq \langle v^T_i, v^I_i, v^F_i\rangle$ or $\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \geq \langle b^T_{ij}, b^I_{ij}, b^F_{ij}\rangle \geq \langle v^T_i, v^I_i, v^F_i\rangle$ for all indices $i \in M$ and $j \in N$ and we denote by $A \to B \leftarrow v$.

Lemma 5.3. Let us consider two NSMs $A, C \in \mathcal{N}_{(m\times n)}$ and the NSV $b \in \mathcal{N}_{(m)}$ such that $A \to C \leftarrow b$. Then $x^{'}(C; b) \geq x^{'}(A; b)$.

Proof: From the definition of the solution of the system of NSLEs of the form $A \otimes x = b$ we have,

$$x^{'}(C; b) = \begin{cases} \langle 1, 1, 0\rangle \ if \ \langle c^T_{ij}, c^I_{ij}, c^F_{ij}\rangle \leq \langle b^T_i, b^I_i, b^F_i\rangle \ \forall i \\ \min\{\langle b^T_i, b^I_i, b^F_i\rangle\} \ if \ \langle c^T_{ij}, c^I_{ij}, c^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle \end{cases}$$

and

$$x^{'}(A; b) = \begin{cases} \langle 1, 1, 0\rangle \ if \ \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \leq \langle b^T_i, b^I_i, b^F_i\rangle \ \forall i \\ \min\{\langle b^T_i, b^I_i, b^F_i\rangle\} \ if \ \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle. \end{cases}$$

Now, as $A \to C \leftarrow b$, we have
$\{i; \langle c^T_{ij}, c^I_{ij}, c^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle\} \subseteq \{i; \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle > \langle b^T_i, b^I_i, b^F_i\rangle\}$ for each $j \in N$. So $x^{'}(C; b) \geq x^{'}(A; b)$.

Lemma 5.4. Let A and C be two NSMs of order $(m \times n)$ and $b \in \mathcal{N}_{(m)}$ be a NSV with $A \to C \leftarrow b$. If $A \otimes x = b$ is solvable then $C \otimes x = b$ is solvable.

Proof: From our assumption, solvability of $A \otimes x = b$ means that $A \otimes x^{'}(A, b) = b$. Then i^{th} equation of which gives,

$$\sum_{j=1}^{n} \langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle \otimes x^{'}_j(A; b) = b_i. \ (4)$$

Le us suppose that in (4) the equality has been achieved in term k.
Thus, $\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle \otimes x^{'}(A; b) = b_i$ which is only possible if
$\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle \geq \langle b^T_i, b^I_i, b^F_i\rangle$ as well as $x^{'}_k(X; b) \geq b_i$.

Since, $A \to C \leftarrow b$, we get $\langle a^T_{ik}, a^I_{ik}, a^F_{ik}\rangle \geq \langle c^T_{ik}, c^I_{ik}, c^F_{ik}\rangle \geq \langle b^T_i, b^I_i, b^F_i\rangle$ and Lemma 5.3 gives, $x^{'}_k(C; b) \geq x^{'}_k(A; b) \geq b_i$. This implies, $\langle c^T_{ik}, c^I_{ik}, c^F_{ik}\rangle \otimes x^{'}_k(C; b) \geq b_j$. Again for any NSM $C, C \otimes x^{'}(C; b) \leq b_i$.
Hence the only possibility is, $C \otimes x^{'}(C; b) = b$, that is, $C \otimes b = b$ is solvable.

Lemma 5.5. Let us consider the system of NSLE $A \otimes x = b$ and $x^{'}(A; b)$ be its tolerable solution. If there exists a NSM D such that, $D \otimes x = b$ is solvable with $\rho(A, D) = \delta$, then there exists NSM C such that $A \to C \leftarrow b$ and $\rho(A, C) \leq \delta$ with $C \otimes x = b$ is solvable.

Proof: The NSM C can be chosen in three different way.

Case-1: If $\langle b_i^T, b_i^I, b_i^F\rangle \leq \langle a_i^T, a_i^I, a_i^F\rangle \leq \langle d_i^T, d_i^I, d_i^F\rangle$ or
$\langle b_i^T, b_i^I, b_i^F\rangle \geq \langle a_i^T, a_i^I, a_i^F\rangle \geq \langle d_i^T, d_i^I, d_i^F\rangle$, we set

$$\begin{aligned}c_{ij} = \langle c_{ij}^T, c_{ij}^I, c_{ij}^F\rangle &= \langle \max\{b_i^T, a_{ij}^T - (d_{ij}^T - a_{ij}^T)\}, \max\{b_i^I, a_{ij}^I - (d_{ij}^I - a_{ij}^I)\},\\ &\quad \min\{b_i^F, a_{ij}^F + (a_{ij}^F - d_{ij}^F)\}\rangle\\ &= \langle \max\{b_i^T, (2a_{ij}^T - d_{ij}^T)\}, \max\{b_i^I, (2a_{ij}^I - d_{ij}^I)\},\\ &\quad \min\{b_i^F, (2a_{ij}^F - d_{ij}^F)\}\rangle, \text{ or}\end{aligned}$$

$$\begin{aligned}c_{ij} = \langle c_{ij}^T, c_{ij}^I, c_{ij}^F\rangle &= \langle \min\{b_i^T, a_{ij}^T + (a_{ij}^T - d_{ij}^T)\}, \min\{b_i^I, a_{ij}^I + (a_{ij}^I - d_{ij}^I)\},\\ &\quad \max\{b_i^F, a_{ij}^F - (d_{ij}^F - a_{ij}^F)\}\rangle\\ &= \langle \min\{b_i^T, (2a_{ij}^T - d_{ij}^T)\}, \min\{b_i^I, (2a_{ij}^I - d_{ij}^I)\},\\ &\quad \max\{b_i^F, (2a_{ij}^F - d_{ij}^F)\}\rangle,\end{aligned}$$

respectively.

Case-2: If $\langle a_i^T, a_i^I, a_i^F\rangle \leq \langle d_i^T, d_i^I, d_i^F\rangle \leq \langle b_i^T, b_i^I, b_i^F\rangle$ or

$$\langle a_i^T, a_i^I, a_i^F\rangle \geq \langle d_i^T, d_i^I, d_i^F\rangle \geq \langle b_i^T, b_i^I, b_i^F\rangle,$$

then take $c_{ij} = d_{ij}$

Case-3: If $\langle a_i^T, a_i^I, a_i^F\rangle \leq \langle b_i^T, b_i^I, b_i^F\rangle \leq \langle d_i^T, d_i^I, d_i^F\rangle$ or

$$\langle a_i^T, a_i^I, a_i^F\rangle \geq \langle b_i^T, b_i^I, b_i^F\rangle \geq \langle d_i^T, d_i^I, d_i^F\rangle,$$

then take $c_{ij} = b_{ij}$

Now from the construction of C by the above three cases, it is obviouse that $\rho(A; C) \leq \delta$ and $A \to C \leftarrow b$. More over, $D \to C \leftarrow b$, hence by Lemma 5.4, $C \otimes x = b$ is solvable.

Definition 5.6. For a given NSM $A \in \mathcal{N}_{(m\times n)}$ and the NSV $b \in \mathcal{N}_{(n)}$ we denote the NSM $D \in \mathcal{N}_{(m\times n)}$ by $(A, \Delta \to b)$ such that for each $i \in \{1, 2, 3, ..., m\}$ and $j \in \{1, 2, 3, ..., n\}$,

$$\langle d_{ij}^T, d_{ij}^I, d_{ij}^F\rangle = \begin{cases}\min\{a_{ij}^T + \Delta^T, b_i^T\}, \min\{a_{ij}^I + \Delta^I, b_i^I\}, \max\{a_{ij}^F - \Delta^F, b_i^F\} & if \ a_{ij} < b_i\\ \max\{a_{ij}^T - \Delta^T, b_i^T\}, \max\{a_{ij}^I - \Delta^I, b_i^I\}, \min\{a_{ij}^F + \Delta^F, b_i^F\} & if \ a_{ij} \geq b_i\end{cases}$$

It is obvious that, $A \to (A, \Delta \to b) \leftarrow b$ for any non-negative $\Delta = \langle \Delta^T, \Delta^I, \Delta^F\rangle$. More over as Δ increase, we finally arrive at a NSM D such that $d_{ij} = b_i$ for all $i \in M, j \in N$, which satisfy the condition, $D \otimes x^{'}(D; b) = b$. So computation of the NSM D is an iterative process, which can be described by the following flowchart.

Algorithm **MATRIX**

begin $k = 0; \Delta_k = \langle 0, 0, 0\rangle;\ A(\Delta_k) = A;$

compute $x^{'}(A; b);$

If $A \otimes x^{'}(A; b) \neq b$ then

repeat $\Delta_{k+1} = \langle \Delta^T_{k+1}, \Delta^I_{k+1}, \Delta^F_{k+1}\rangle$

$= \langle \Delta^T_k + \min\{|A(\delta_k)_{ij} - b^T_i|; A(\delta^T_k)_{ij} \neq b^T_i\},$

$\Delta^I_k + \min\{|A(\delta_k)_{ij} - b^I_i|; A(\delta^I_k)_{ij} \neq b^I_i\},$

$\Delta^F_k + \min\{|A(\delta_k)_{ij} - b^F_i|; A(\delta^F_k)_{ij} \neq b^F_i\}\rangle,$

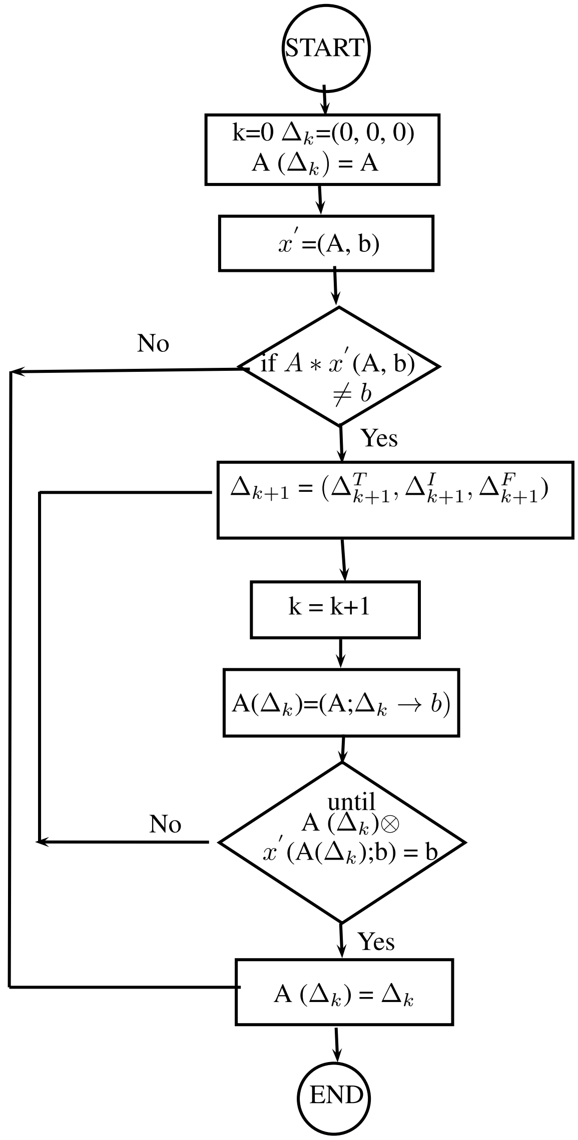
$k = k + 1;$

$A(\Delta_k) = (A; \delta_k \to b)$

until $A(\delta_k) \otimes x^{'}(A(\delta_k); b) = b;$

output: $A(\delta_k); \Delta_k$

end MATRIX.



The following example illustrate the concept of the above flowchart.

Let us consider the system of NSLEs $A \otimes x = b$ where,

Example 5.7. $A = \begin{bmatrix} \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.4\ 0.5\ 0.6\rangle & \langle 0.2\ 0.1\ 0.8\rangle \\ \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.2\ 0.1\ 08\rangle & \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.1\ 0.1\ 0.9\rangle & \langle 0.6\ 0.5\ 0.4\rangle \\ \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.8\ 0.7\ 0.2\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.2\ 0.1\ 0.8\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.3\ 0.2\ 0.7\rangle \end{bmatrix}$

and

$$b = \begin{bmatrix} \langle 0.4\ 0.3\ 0.6\rangle \\ \langle 0.9\ 0.8\ 0.1\rangle \\ \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle \end{bmatrix}.$$

The corresponding tolerable solution will be
$x^{'}(A;b) = [\langle 0.5\ 0.4\ 0.5\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.5\ 0.4\ 0.5\rangle]^t$ but

$A \otimes x^{'}(A;b) \leq b$ so the system is unsolvable.

In the first iteration,

$\Delta_1\langle 0.1, 0.1, 0.9\rangle, A(\Delta_1) =$

$$\begin{bmatrix} \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.4\ 0.5\ 0.6\rangle & \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.3\ 0.2\ 07\rangle & \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.2\ 0.1\ 0.8\rangle & \langle 0.7\ 0.6\ 0.3\rangle \\ \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.7\ 0.6\ 0.3\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.6\ 0.5\ 0.4\rangle & \langle 0.6\ 0.5\ 0.4\rangle \end{bmatrix}$$

and
$x^{'}(A(\Delta_1);b) = [\langle 1\ 1\ 0\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 0.5\ 0.4\ 0.5\rangle, \langle 0.5\ 0.4\ 0.5\rangle]^t$.

Here , $A \otimes x^{'}(A(\Delta_1);b) \leq b$.
In the second iteration,

$\Delta_2 = \langle 0.2\ 0.2\ 0.8\rangle, A(\Delta_2) =$

$$\begin{bmatrix} \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.4\ 0.5\ 0.6\rangle & \langle 0.4\ 0.3\ 0.6\rangle \\ \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.5\ 0.4\ 05\rangle & \langle 0.9\ 0.8\ 0.1\rangle & \langle 0.4\ 0.3\ 0.6\rangle & \langle 0.9\ 0.8\ 0.1\rangle \\ \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.3\ 0.2\ 0.7\rangle & \langle 0.3\ 0.2\ 0.7\rangle \\ \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 0.5\rangle & \langle 0.5\ 0.4\ 0.5\rangle \end{bmatrix}$$

and
$x^{'}(A(\Delta_2);b) = [\langle 1\ 1\ 0\rangle, \langle 0.3\ 0.2\ 0.7\rangle, \langle 1\ 1\ 0\rangle, \langle 1\ 1\ 0\rangle, \langle 1\ 1\ 0\rangle]^t$.

In this case, $A \otimes x^{'}(A(\Delta_2);b) = b$. So $D = A(\Delta_2)$ is the Chebychev best approximation of the coefficient NSM A of the given system and $x^{'}(A(\Delta_2);b)$ is the principal solution.

6. **Conclusion**

In this piece of work, we try to find the conditions under which a system of NSLE is solvable. We have provided necessary examples to describe the theory. Further using the Chebychev approximation discussed the principal solution when the given system (1) has no solution. As a future work we are trying to apply this theory in all operation research problems.

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