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Statistical Convergence of Double Sequences in Neutrosophic Normed Spaces

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Abstract. In this paper, we define and study the notion of statistically convergent and statistically Cauchy double sequences in neutrosophic normed spaces. Moreover, we give the double statistically Cauchy sequence in neutrosophic normed space and present the double statistically completeness in connection with a neutrosophic normed space.

Keywords: Nuetrosophic normed spaces, statistical double convergence.

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1. **Introduction**

The concept of fuzzy set was originally introduced by Zadeh [20] in 1965. The fuzzy theory has become an area of active research for the last fifty years. It has a wide range of applications in the field of science and engineering, population dynamics [1], chaos control [5], computer programming [8], nonlinear dynamical systems [9], fuzzy physics [13] and more. Taking into account the concept of fuzzy set, Smarandache [16] introduced the notion of Neutrosophic set (NS) which is a new version of the idea of the classical set. The first world publication related to the concept of neutrosophy was published in 1998 and included in the literature [17].

On the other hand, Kaleva and Seikkala [10] defined the fuzzy metric spaces (FMS) as a distance between two points to be a non-negative fuzzy number. after that, In [6] some basic properties of FMS were studied and the Baire Category Theorem for FMS was proved. Furthermore, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [7]. Consequently, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decisionmaking and more. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [15] introduced IF metric space (IFMS), that is a generalization of FMS. Then, Park used George and Veeramani's [6] work for applying t-norm and t-conorm to FMS meanwhile defining IFMS and studying its basic properties. Moreover, Bera and Mahapatra introduced the neutrosophic soft linear spaces (NSLS) [3]. Later, neutrosophic soft normed linear spaces(NSNLS) was defined by Bera and Mahapatra [2]. Besides, In [2], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were defined and studied. Recently, Kirisci and Simsek [11] in 2020, introduced and studied the notion of statistical convergence in a neutrosophic normed spaces. Besides, they showed some interesting results.

In this paper, we extend the notion of statistical convergence on neutrosophic normed spaces by using double sequences. Moreover, we prove some of its properties and characterizations. This paper is organized as follows: In the second section, we procure some well-known notions and definitions which are useful for the developing of this paper. In the third part, we define and study the notion of statistical convergence of double sequences on neutrosophic normed spaces (NNS). And the fourth section, we put a a conclusion in which we discuss about the results showed in section 3 and some future studies.

2. Preliminaries

The notion of statistical convergence was defined by Fast [7] and Steinhaus [18] independently and later this notion was studied by various authors.

Let K be a subset of $\mathbb{N}$, then the asymptotic density of K, denoted by $d(K)$ is defined as follows:

$$d(K) = \lim_{n} \frac{1}{n} |k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set. A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, the set $d(\epsilon) = \{k \leq n : |x_k - L| > \epsilon\}$ has asymptotic density zero. Then, taking into account that notion, Mursaleen and Edely [14] defined the notion of statistical convergence of double sequences. Let $K \subset \mathbb{N} \times \mathbb{N}$ be two-dimensional set of positive integers and let $K(m, n)$ be the numbers of (j, k) in K such that $j \leq n$ and $k \leq n$. Then, the two-dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of the set $K \subset \mathbb{N} \times \mathbb{N}$ is defined as:

$$d_2(K) = \lim_{m,n} \inf \frac{K(m,n)}{mn}$$

In case that the sequence $(K(m, n)/mn)$ has a limit in Pringsheim's sense then we say that K has a double density and is defined as:

$$\lim_{m,n} \frac{K(m,n)}{mn} = \delta_2(K).$$

Statistical convergence for double sequence $x = (x_{kj})$ of real which was defined by [14] as: A real double sequence $x = (x_{kj})$ is said to be statistically convergent to the number L if for each $\epsilon > 0$, the set $\{(j,k), j \leq m, k \leq n : |x_{kj} - L| \geq \epsilon\}$, has a double natural density zero. In this case, we write S_2-$\lim x_{jk} = L$.

On the other hand, Triangular norms (t-norms) (TN) were initiated by Menger [13]. In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers for distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations(intersections and unions).

Definition 2.1. ([13]) Give an operation $\circ : [0,1] \times [0,1] \to [0,1]$. If the operation $\circ$ is satisfying the following conditions:

(1) $s \circ 1 = s$,
(2) If $s \leq u$ and $t \leq v$, then $s \circ t \leq u \circ v$,
(3) $\circ$ is continuous,
(4) $\circ$ is continuous and associative.

Then, it is called that the operation $\circ$ is continuous TN, for $s, t, u, v \in [0,1]$.

Definition 2.2. ([13]) Give an operation $\bullet : [0,1] \times [0,1] \to [0,1]$. If the operation $\bullet$ is satisfying the following conditions:

(1) $s \bullet 0 = s$,
(2) If $s \leq u$ and $t \leq v$, then $s \bullet t \leq u \bullet v$,
(3) $\bullet$ is continuous,
(4) $\bullet$ is continuous and associative.

Then, it is called that the operation $\bullet$ is continuous TC, for $s, t, u, v \in [0,1]$.

Remark 2.3. [11]) From the above definitions, we can see that if we take $0 < \epsilon_1$, $\epsilon_2 < 1$ for $\epsilon_1 > \epsilon_2$, then there exist $0 < \epsilon_3$, $\epsilon_4 < 0,1$ such that $\epsilon_1 \circ \epsilon_3 \geq \epsilon_2$, $\epsilon_1 \geq \epsilon_4 \bullet \epsilon_2$. Moreover, if we take $\epsilon_5 \in (0,1)$, then there exist $\epsilon_6, \epsilon_7 \in (0,1)$ such that $\epsilon_6 \circ \epsilon_6 \geq \epsilon_5$ and $\epsilon_7 \bullet \epsilon_7 \leq \epsilon_5$.

Definition 2.4. ([12]) Take F be an arbitrary set, $\mathbb{N} = \{< u, Q(u), W(u), E(u) >: u \in F\}$ be a NS (neutrosophic set) such that $\mathbb{N} : F \times F \times R^+ \to [0,1]$. Let $\circ$ and $\bullet$ show the continuous TN and continuous TC, respectively. If the following conditions are satisfied, then the four-tuple $(F, \mathbb{N}, \circ, \bullet)$ is called neutrosophic metric space (NMS):

(1) $0 \leq Q(u,v,\lambda) \leq 1$, $0 \leq W(u,v,\lambda) \leq 1$, $0 \leq E(u,v,\lambda) \leq 1$ for all $\lambda \in R^+$,
(2) $Q(u,v,\lambda) + W(u,v,\lambda) + E(u,v,\lambda) \leq 3$, for$\lambda \in R^+$,

(3) $Q(u, v, \lambda) = 1$, for $\lambda > 0$ if and only if $u = v$,

(4) $Q(u, v, \lambda) = Q(v, u, \lambda)$, for $\lambda > 0$,

(5) $Q(u, v, \lambda) \circ Q(v, y, \mu) \leq Q(u, y, \lambda + \mu)$, for all $\mu, \lambda > 0$,

(6) $Q(u, v, .) : [0, \infty) \to [0, 1]$ is continuous,

(7) $\lim_{\lambda \to \infty} Q(u, v, \lambda) = 1$, for all $\lambda > 0$,

(8) $W(u, v, \lambda) = 0$, for $\lambda > 0$ if and only if $u = v$,

(9) $W(u, v, \lambda) = W(v, u, \lambda)$, for $\lambda > 0$,

(10) $W(u, v, \lambda) \bullet W(v, y, \mu) \geq W(u, y, \lambda + \mu)$, for all $\mu, \lambda > 0$,

(11) $W(u, v, .) : [0, \infty) \to [0, 1]$ is continuous,

(12) $\lim_{\lambda \to \infty} W(u, v, \lambda) = 1$, for all $\lambda > 0$,

(13) $E(u, v, \lambda) = 0$, for $\lambda > 0$ if and only if $u = v$,

(14) $E(u, v, \lambda) = E(v, u, \lambda)$, for $\lambda > 0$,

(15) $E(u, v, \lambda) \bullet E(v, y, \mu) \geq E(u, y, \lambda + \mu)$, for all $\mu, \lambda > 0$,

(16) $W(u, v, .) : [0, \infty) \to [0, 1]$ is continuous,

(17) $\lim_{\lambda \to \infty} E(u, v, \lambda) = 1$, for all $\lambda > 0$,

(18) if $\lambda \geq 0$, then $Q(u, v, \lambda) = 0$, $W(u, v, \lambda) = 1$ and $E(u, v, \lambda) = 1$.

For all $u, v, y \in F$. Then, $N = (Q, W, E)$ is called Neutrosophic metric (NM) on F.

The notion of neutrosophic normed space (NNS) was defined by [11], as well as, the definition statistical convergence with respect to NNS was given.

Definition 2.5. ([11])Take F as a vector space $N = \{< u, G(u), B(u), Y(u) >: u \in F\}$ be a normed space (NS) such that $N : F \times R^+ \to [0, 1]$. Let $\circ$ and $\bullet$ show the continuous TN and continuous TC, respectively. If the following contritions are satisfied, then the four-tuple $V = (F, N, \circ, \bullet)$ is called NNS, for all $u, v \in F$, $\lambda, \mu > 0$ and for each $\sigma \neq 0$:

(1) $0 \leq G(u, \lambda) \leq 1$, $0 \leq B(u, \lambda) \leq 1$, $0 \leq Y(u, \lambda) \leq 1$ for all $\lambda \in R^+$,

(2) $G(u, \lambda) + B(u, \lambda) + Y(u, \lambda) \leq 3$, for$\lambda \in R^+$,

(3) $G(u, \lambda) = 1$, for $\lambda > 0$ if and only if $u = 0$,

(4) $G(\sigma u, \lambda) = G(u, \frac{\lambda}{|\sigma|})$,

(5) $G(u, \mu) \circ G(v, \lambda) \leq G(u + v, \lambda + \mu)$,

(6) $G(u, .)$ is continuous non-decreasing function,

(7) $\lim_{\lambda \to \infty} G(u, \lambda) = 1$,

(8) $B(u, \lambda) = 0$, for $\lambda > 0$ if and only if $u = 0$,

(9) $B(\sigma u, \lambda) = B(u, \frac{\lambda}{|\sigma|})$,

(10) $B(u, \mu) \bullet B(v, \lambda) \geq B(u + v, \lambda + \mu)$,

(11) $B(u, .)$ is continuous non-increasing function,

(12) $\lim_{\lambda \to \infty} B(u, \lambda) = 0$,

(13) $Y(u, \lambda) = 0$, for $\lambda > 0$ if and only if $u = 0$,
(14) $Y(\sigma u, \lambda) = Y(u, \frac{\lambda}{|\sigma|})$,
(15) $Y(u, \mu) \bullet Y(v, \lambda) \geq Y(u + v, \lambda + \mu)$,
(16) $Y(u, .)$ is continuous non-increasing function,
(17) $\lim_{\lambda\to\infty} Y(u, \lambda) = 0$,
(18) if $\lambda \leq 0$, then $G(u, \lambda) = 0$, $B(u, \lambda) = 1$ and $Y(u, \lambda) = 1$.

Then, $N = (G, B, Y)$ is called neutrosophic norm (NN).

Example 2.6. ([11]) Let $(F, \|.\|)$ be a NS. Give the operations $\circ$ and $\bullet$ as TN $u \circ v = uv$; TC $u \bullet v = u + v - uv$. For $\lambda > \|u\|$,

$$G(u, \lambda) = \frac{\lambda}{\lambda + \|u\|},\ B(u, \lambda) = \frac{\|u\|}{\lambda + \|u\|},\ Y(u, \lambda) = \frac{\|u\|}{\lambda},$$

for all $u, v \in F$ and $\lambda > 0$. If we take $\lambda \leq \|u\|$, then $G(u, \lambda) = 0$, $B(u, \lambda) = 1$ and $Y(u, \lambda) = 1$. Then, $(F, N, \circ, \bullet)$ is NNS such that $N : F \times R^+ \to [0, 1]$.

Definition 2.7. ([11]) Let V be a NNS and (x_k) be a sequence in V such that $0 < \epsilon < 1$ and $\lambda > 0$. Then, (x_k) converges to x if and only if there exists $n_0 \in \mathbb{N}$ such that $G(x_k - x, \lambda) > 1 - \epsilon$, $B(x_k - x, \lambda) < \epsilon$ and $Y(x_k - x, \lambda) < \epsilon$. That is $\lim_{k\to\infty} G(x_k - x, \lambda) = 1$, $\lim_{k\to\infty} B(x_k - x, \lambda) = 0$ and $\lim_{k\to\infty} Y(x_k - x, \lambda) = 0$ as $\lambda > 0$. In this case, the sequence (x_k) is said to be a convergent sequence in V. The convergent in NNS is denoted by $N\text{-}\lim x_k = L$.

Definition 2.8. ([11]) Let V be a NNS, the sequence (x_k) in V where $0 < \epsilon < 1$ and $\lambda > 0$. Then, the sequence (x_k) is Cauchy in a NNS V if there is a $n_0 \in \mathbb{N}$ such that $G(x_k - x_q, \lambda) > 1 - \epsilon$, $B(x_k - x_q, \lambda) < \epsilon$ and $Y(x_k - x_q, \lambda) < \epsilon$ for $k, q \geq n_0$.

Definition 2.9. ([11]) Let V be a NNS. For $\lambda > 0$, $u \in F$ and $0 < \epsilon < 1$,

$$O(u, \epsilon, \lambda) = \{vinF : G(u - v, \lambda) > 1 - \epsilon,\ B(u - v, \lambda) < \epsilon,\ Y(u - v, \lambda) < \epsilon\}$$

is called open ball (OB)with center u and radius ϵ.

Definition 2.10. ([11]) The set $A \subset F$ is called neutrosophic-bounded (NB)in NNS V, if there exist $\lambda > 0$, and $\epsilon \in (0, 1)$ such that $G(u, \lambda) > 1 - \epsilon$, $B(u, \lambda) < \epsilon$ and $Y(u, \lambda) < \epsilon$ for each $u \in A$.

3. **Statistical convergence of double sequences on NNS**

In this section, we define and study the notion of statistical double convergence in a Neutrosophic normed space

Definition 3.1. Let V be a NNS and (x_{kj}) be a double sequence in V such that $0 < \epsilon < 1$ and $\lambda > 0$. Then, (x_{kj}) converges to x if and only if there exists $n_0 \in \mathbb{N}$ such that $G(x_{kj} - x, \lambda) > 1 - \epsilon$, $B(x_{kj} - x, \lambda) < \epsilon$ and $Y(x_{kj} - x, \lambda) < \epsilon$. That is $\lim_{k,j\to\infty} G(x_{kj} - x, \lambda) = 1$,

$\lim_{k,j\to\infty} B(x_{kj}-x,\lambda)=0$ and $\lim_{k,j\to\infty} Y(x_{kj}-x,\lambda)=0$ as $\lambda>0$. In this case, the double sequence (x_{kj}) is said to be a double convergent sequence in V. The double convergent in NNS is denoted by $N_2\text{-}\lim x_{kj}=L$.

Theorem 3.2. *Let V be a NNS and (x_{kj}) be a double sequence in V. Then, the following statements hold:*

(1) *If (x_{kj}) in V is convergent, then the limit point is unique.*
(2) *In V, if $\lim_{k,j\to\infty} x_{kj}=x$ and $\lim_{k,j\to\infty} y_{kj}=y$, then $\lim_{k,j\to\infty} x_{kj}+y_{kj}=x+y$.*
(3) *in V, if $\lim_{k,j\to\infty} x_{kj}=x$ and $\alpha\neq 0$, then $\lim_{k,j\to\infty} \alpha x_{kj}=\alpha x$.*

Proof: Since the proof of this Theorem is straightforward, we omitted it.

Definition 3.3. Let V be a NNS, the double sequence (x_{kj}) in V where $0<\epsilon<1$ and $\lambda>0$. Then, the double sequence (x_{kj}) is Cauchy in a NNS V if there is a $n_0\in\mathbb{N}$ such that $G(x_{kj}-x_{qw},\lambda)>1-\epsilon$, $B(x_{kj}-x_{qw},\lambda)<\epsilon$ and $Y(x_{kj}-x_{qw},\lambda)<\epsilon$ for $k,j,q,w\geq n_0$. A NNS V is called complete if and only if every double Cauchy sequence (x_{kj}) is convergent to x in a NNS V.

Example 3.4. Consider G,B and Y from Example 2.6 . Then, V is a NNS. Besides,

$$\lim_{k,j,q,w\to\infty}\frac{\lambda}{\lambda+\|x_{kj}-x_{qw}\|}=1,\ \lim_{k,j,q,w\to\infty}\frac{\|x_{kj}-x_{qw}\|}{\lambda+\|x_{kj}-x_{qw}\|}=0,\ \lim_{k,j,q,w\to\infty}\frac{\|x_{kj}-x_{qw}\|}{\lambda}=0,$$

That is

$$\lim_{k,j,q,w\to\infty} G(x_{kj}-x_{qw},\lambda)=1,\ \lim_{k,j,q,w\to\infty} B(x_{kj}-x_{qw},\lambda)=0\ ,\ \lim_{k,j,q,w\to\infty} Y(x_{kj}-x_{qw},\lambda)=0.$$

Therefore, we can say that the double sequence (x_{kj}) is a double Cauchy sequence in NNS V.

Remark 3.5. It is clear that every double convergent sequence in V is a double Cauchy sequence. But the inverse of this expression is not be true.

Theorem 3.6. *Let V be a NNS and (x_{kj}) be a double sequence in V. Then, the following statements hold:*

(1) *If for $u,v\in[0,1]$, we choose the continuous TN $u\circ v=\min\{u,v\}$ and the continuous TC $u\bullet v=\max\{u,v\}$, then every double Cauchy sequence is bounded in NNS V.*
(2) *Let the double sequences (x_{nmj}) and (y_{kj}) be double Cauchy and the double sequence (α_{kj}) is scalars in NNS V. Then, the double sequences $(x_{kj}+y_{kj})$ and $(\alpha_{kj}x_{kj})$ are also double Cauchy in NNS V.*
(3) *V is a complete NNS, if every double Cauchy sequence has a double convergent subsequence in NNS V.*

Proof: The proof of this Theorem is followd by the definitions of NNS, $G;B;Y$,double Cauchy sequence in V and completeness.

Definition 3.7. Let V a NNS. A double sequence (x_{kj}) is said to be statistical convergence with respect to neutrosophic normed (DSC-NN), if there exist $L \in F$ such that the set

$$K_{\epsilon_2} = \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda) \geq \epsilon,\ Y(x_{kj} - L, \lambda) \geq \epsilon\}$$

or equivalently

$$K_{\epsilon_2} = \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) > 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda) < \epsilon,\ Y(x_{kj} - L, \lambda) < \epsilon\}.$$

has double neutrosophic density (DND) zero, for every $\epsilon > 0$ and $\lambda > 0$. That is $d(K_{\epsilon_2}) = 0$ or equivalently,

$$\lim_{n,m} \frac{1}{nm}|\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda) \geq \epsilon,\ Y(x_{kj} - L, \lambda) \geq \epsilon\}| = 0$$

Therefore, we write S_{N_2}-$\lim x_{kj} = L$ or $x_{kj} \to L(S_{N_2})$. The set of DSC-NN will be denoted by S_{N_2}. If $L = 0$, then we will write $S^0_{N_2}$.

Lemma 3.8. *Let V be a NNS. Then, the following statements are equivalent, for every $\epsilon > 0$ and $\lambda > 0$:*

(1) S_{N_2}-$\lim x_{kj} = L$.
(2) $\lim_{n,m} \frac{1}{nm}|\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon\}| = \lim_{n,m} \frac{1}{nm}|\{B(x_{kj} - L, \lambda) \geq \epsilon\}| = \lim_{n,m} \frac{1}{nm}|\{Y(x_{kj} - L, \lambda) \geq \epsilon\}| = 0$.
(3) $\lim_{n,m} \frac{1}{nm}|\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) < 1 - \epsilon$ *and* $B(x_{kj} - L, \lambda) < \epsilon,\ Y(x_{kj} - L, \lambda) < \epsilon\}| = 1$.
(4) $\lim_{n,m} \frac{1}{nm}|\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) > 1 - \epsilon\}| = \lim_{n,m} \frac{1}{nm}|\{k \leq n, j \leq m : B(x_{kj} - L, \lambda) < \epsilon\}| = \lim_{n,m} \frac{1}{nm}|\{k \leq n, j \leq m : Y(x_{kj} - L, \lambda) < \epsilon\}| = 1$.
(5) S_2-$\lim G(x_{kj} - L, \lambda) = 1$, *and* S_2-$\lim B(x_{kj} - L, \lambda) = 0$, S_2-$\lim Y(x_{kj} - L, \lambda) = 0$.

Proof: The poof of this Lemma is followed by the Definitions 3.7 and the notions showed in Section 2 .

Theorem 3.9. *Let V a NNS. If (x_{kj}) is DSC-NN, then S_{N_2}-$\lim x_{kj} = L$ is unique.*

Proof: Consider that S_{N_2}-$\lim x_{kj} = L_1$ and S_{N_2}-$\lim x_{kj} = L_2$ for $L_1 \neq L_2$. Now, take $\epsilon > 0$. Then, for a given $\mu > 0$, $(1 - \epsilon) \circ (1 - \epsilon) > 1 - \mu$ and $\epsilon \bullet \epsilon < \mu$. For any $\lambda > 0$. Let's write the following sets:

$$K_{G_1}(\epsilon, \lambda) := \{k \leq n, j \leq m : G(x_{kj} - L_1, \frac{\lambda}{2}) \leq 1 - \epsilon\},$$
$$K_{G_2}(\epsilon, \lambda) := \{k \leq n, j \leq m : G(x_{kj} - L_2, \frac{\lambda}{2}) \leq 1 - \epsilon\}$$
$$K_{B_1}(\epsilon, \lambda) := \{k \leq n, j \leq m : B(x_{kj} - L_1, \frac{\lambda}{2}) \leq 1 - \epsilon\},$$
$$K_{B_2}(\epsilon, \lambda) := \{k \leq n, j \leq m : B(x_{kj} - L_2, \frac{\lambda}{2}) \leq 1 - \epsilon\}$$
$$K_{Y_1}(\epsilon, \lambda) := \{k \leq n, j \leq m : Y(x_{kj} - L_1, \frac{\lambda}{2}) \leq 1 - \epsilon\},$$

$$K_{Y_2}(\epsilon,\lambda) := \{k \le n, j \le m : Y(x_{kj} - L_2, \frac{\lambda}{2}) \le 1-\epsilon\}$$

Since that S_{N_2}-$\lim x_{kj} = L_1$. Then, by the Lemma 3.8 , for all $\lambda > 0$,

$$d(K_{G_1}(\mu,\lambda)) = d(K_{B_1}(\mu,\lambda)) = d(K_{Y_1}(\mu,\lambda)) = 0$$

Moreover, since we have S_{N_2}-$\lim x_{kj} = L_2$, by the Lemma 3.8 , for $\lambda > 0$,

$$d(K_{G_2}(\mu,\lambda)) = d(K_{B_2}(\mu,\lambda)) = d(K_{Y_2}(\mu,\lambda)) = 0$$

Now, let

$$K_{N_2}(\mu,\lambda) := \{K_{G_1}(\mu,\lambda) \cup K_{G_2}(\mu,\lambda)\} \cap \{K_{B_1}(\mu,\lambda) \cup K_{B_2}(\mu,\lambda)\} \cap \{K_{Y_1}(\mu,\lambda) \cup K_{Y_2}(\mu,\lambda)\}.$$

Then, we can see that $d(K_{N_2}(\mu,\lambda)) = 0$ which implies $d(\mathbb{N} - K_{N_2}(\mu,\lambda)) = 1$. Then, we have the following possible situations, when we take $(k,j) \in \mathbb{N} - K_{N_2}(\mu,\lambda)$:

(1) $(k,j) \in \mathbb{N} - (K_{G_1}(\mu,\lambda) \cup K_{G_2}(\mu,\lambda))$,
(2) $(k,j) \in \mathbb{N} - (K_{B_1}(\mu,\lambda) \cup K_{B_2}(\mu,\lambda))$,
(3) $(k,j) \in \mathbb{N} - (K_{Y_1}(\mu,\lambda) \cup K_{Y_2}(\mu,\lambda))$.

First at all, consider (1). Then, we have

$$G(L_1 - L_2, \lambda) \ge G(x_{kj} - L_1, \frac{\lambda}{2}) \circ G(x_{kj} - L_2, \frac{\lambda}{2}) > 1 - \epsilon \circ (1-\epsilon)$$

And then, since $(1-\epsilon) \circ (1-\epsilon) > 1 - \mu$,

$$G(L_1 - L_2, \lambda) > 1 - \mu \tag{1}$$

By 1 , for all $\lambda > 0$, we have that $G(L_1 - L_2, \lambda) = 1$, where $\mu > 0$ is arbitrary. This is, $L_1 = L_2$.

Secondly, for (2), if we choose $(k,j) \in \mathbb{N} - (K_{B_1}(\mu,\lambda) \cup K_{B_2}(\mu,\lambda))$, then we can write

$$B(L_1 - L_2, \lambda) \le B(x_{kj} - L_1, \frac{\lambda}{2}) \bullet B(x_{kj} - L_2, \frac{\lambda}{2}) < \epsilon \bullet \epsilon$$

Using $\epsilon \bullet \epsilon < \mu$, we can see that $B(L_1 - L_2, \lambda) < \mu$. For all $\lambda > 0$, we get $B(L_1 - L_2, \lambda) = 0$, where $\mu > 0$ is arbitrary. Therefore, $L_1 = L_2$.

Finally, in the same way, for the situation (3), if we choose $(k,j) \in \mathbb{N} - (K_{Y_1}(\mu,\lambda) \cup K_{Y_2}(\mu,\lambda))$, then we can write

$$Y(L_1 - L_2, \lambda) \le Y(x_{kj} - L_1, \frac{\lambda}{2}) \bullet Y(x_{kj} - L_2, \frac{\lambda}{2}) < \epsilon \bullet \epsilon$$

Using $\epsilon \bullet \epsilon < \mu$, we can see that $Y(L_1 - L_2, \lambda) < \mu$. For all $\lambda > 0$, we get $Y(L_1 - L_2, \lambda) = 0$, where $\mu > 0$ is arbitrary. Therefore, $L_1 = L_2$. And this step ends the proof.

Theorem 3.10. *If N_2-$\lim x_{kj} = L$ for a NNS V. Then, S_{N_2}-$\lim x_{kj} = L$.*

Proof: Let N_2-$\lim x_{kj} = L$. Then, for every $\epsilon > 0$ and $\lambda > 0$, there exist a number $n_0 \in \mathbb{N}$ such that $G(x_{kj} - L, \lambda) > 1 - \epsilon$ and $B(x_{kj} - L, \lambda) < \epsilon$, $Y(x_{kj} - L, \lambda) < \epsilon$, for all $k, j \geq n_0$. Hence, the set

$$\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda) \geq \epsilon, Y(x_{kj} - L, \lambda) \geq \epsilon\}$$

has at most finitely many terms. Therefore, since every finite subset of $\mathbb{N}$ has double density zero,

$$\lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \epsilon \text{ or } B(x_{kj} - L, \lambda \geq \epsilon, Y(x_{kj} - L, \lambda) \geq \epsilon\}| = 0$$

And this ends the proof.

Theorem 3.11. *Let V be a NNS. S_{N_2}-$\lim x_{kj} = L$ if and only if there exists an increasing index double sequence $L_2 = \{l_1, ...l_n, ...; l_1, ..., l_m, ...\} \subset \mathbb{N} \times \mathbb{N}$, while $d(L_2) = 1$, N_2-$\lim_{n,m\to\infty} x_{l_{nm}} = L$.*

Proof: Suppose that $S_{G_{N_2}}$-$\lim x_{kj} = L$. For any $\lambda > 0$ and $\mu = 1, 2, ...$,

$$P_{N_2}(\mu, \lambda) = \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) > 1 - \frac{1}{\mu} \text{ and } B(x_{kj} - L, \lambda) < \frac{1}{\mu},$$
$$Y(x_{kj} - L, \lambda) < \frac{1}{\mu}\}$$

and

$$R_{N_2}(\mu, \lambda) = \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \frac{1}{\mu} \text{ or } B(x_{kj} - L, \lambda) \geq \frac{1}{\mu}, Y(x_{kj} - L, \lambda) \geq \frac{1}{\mu}\}.$$

Then, $d(R_{N_2}(\mu, \lambda)) = 0$, since S_{N_2}-$\lim x_{kj} = L$. Besides, for $\lambda > 0$ and $\mu = 1, 2, ...$,

$$d(P_{G_{N_2}}(\mu, \lambda)) = 1 \tag{2}$$

Now, we will prove that for $(k, j) \in PN_2(\mu, \lambda)$, N_2-$\lim x_{kj} = L$. Consider that N_2-$\lim x_{kj} \neq L$, for some $(k, j) \in P_{N_2}(\mu, \lambda)$. Then, there is $\rho > 0$ and a positive integer n_0 such that $G(x_{kj} - L, \lambda) \leq 1 - \rho$ or $B(x_{kj} - L, \lambda) \geq \rho$, $Y(x_{kj} - L, \lambda) \geq \rho$, for all $k, j \geq n_0$. Now, let $G(x_{kj} - L, \lambda) > 1 - \rho$ and $B(x_{kj} - L, \lambda) < \rho$, $Y(x_{kj} - L, \lambda) < \rho$ for all $k < n; j < m$. Hence,

$$\lim_{n,m} \frac{1}{nm} |\{k \leq n, j \leq m : G(x_{kj} - L, \lambda) > 1 - \rho \text{ and } B(x_{kj} - L, \lambda) < \rho, Y(x_{kj} - L, \lambda) < \rho\}| = 0$$

Since $\rho > \frac{1}{\mu}$, we have that $d(P_{N_2}(\mu, \lambda)) = 0$, which contradicts 2 . Therefore, N_2-$\lim x_{kj} = L$.

Now, let's assume that there exists a subset $L_2 = \{l_1, ..., l_n, ...; l_1, ..., l_m, ...\} \subset \mathbb{N} \times \mathbb{N}$ such that $d(J_2) = 1$ and N_2-*displaystyle* $\lim_{n,m\to\infty} x_{l_{nm}} = L$, this means that there exits $n_0 \in \mathbb{N}$ such that $G(x_{kj} - L, \lambda) > 1 - \mu$ and $B(x_{kj} - L, \lambda) < \mu$, $Y(x_{kj} - L, \lambda) < \mu$, for every $\mu > 0$ and $\lambda > 0$. In that case,

$$R_{N_2}(\mu, \lambda) := \{k \leq n, j \leq m : G(x_{kj} - L, \lambda) \leq 1 - \mu \text{ or } B(x_{kj} - L, \lambda) \geq \mu, Y(x_{kj} - L, \lambda) \geq \mu\}$$
$$\subseteq \mathbb{N} \times \mathbb{N} - \{l_{n+1}, l_{n+2}, ...; l_{m+1}, l_{m+2}, ...\}$$

Therefore, $d(R_{N_2}(\mu,\lambda)) \leq 1-1=0$. Hence, $S_{N_2}\text{-}\lim x_{kj} = L$.

Now, we show some results that we obtained on double statistical Cauchy sequences in NNS.

Definition 3.12. The double sequence (x_{kj}) is said to be statistically Cauchy with respect to NN (DSCa-NN) in a NNS V, if there exit $N = N(\epsilon)$ and $M = M(\epsilon)$, for every $\epsilon > 0$ and $\lambda > 0$ such that

$$KC_\epsilon := \{k \leq n, j \leq m : G(x_{kj} - x_{NM}, \lambda) \leq 1-\epsilon \text{ or } B(x_{kj} - x_{NM}, \lambda) \geq \epsilon, \\ Y(x_{kj} - x_{NM}, \lambda) \geq \epsilon\}$$

has DND zero. That is $d(KC_\epsilon) = 0$.

Theorem 3.13. *If a double sequence (x_{kj}) is DSC-NN in a NNS V. Then, it is DSCa-NN.*

Proof: Let (x_{kj}) be DSC-NN. We have that $(1-\epsilon)\circ(1-\epsilon) > 1-\mu$ and $\epsilon \bullet \epsilon < \mu$, for a given $\epsilon > 0$, take $\mu > 0$. Then, we have

$$d(A(\epsilon,\mu)) = d(\{k \leq n, j \leq m : G(x_{kj} - L, \frac{\lambda}{2}) \leq 1-\epsilon \text{ or } B(x_{kj} - L, \frac{\lambda}{2}) \geq \epsilon, Y(x_{kj} - L, \frac{\lambda}{2}) \geq \epsilon\}) \tag{3}$$

thus

$$d(A^c(\epsilon,\lambda)) = d(\{k \leq n, j \leq m : G(x_{kj} - L, \frac{\lambda}{2}) > 1-\epsilon \text{ and } B(x_{kj} - L, \frac{\lambda}{2}) < \epsilon, \\ Y(x_{kj} - L, \frac{\lambda}{2}) < \epsilon\}) = 1$$

for $\lambda > 0$. Let $q, w \in A^c(\epsilon,\lambda)$. Then,

$$G(x_{qw} - L, \lambda) > 1-\epsilon \text{ and } B(x_{qw} - L, \lambda) < \epsilon, \ Y(x_{qw} - L, \lambda) < \epsilon.$$

Let

$$B(\epsilon,\lambda) = \{k \leq n, j \leq m : G(x_{kj} - x_{qw}, \lambda) \leq 1-\mu \text{ or } B(x_{kj} - x_{qw}, \lambda) \geq \mu, \\ Y(x_{kj} - x_{qw}, \lambda) \geq \mu\}.$$

We claim that $B(\epsilon,\lambda) \subset A(\epsilon,\lambda)$. Let $a, s \in B(\epsilon,\lambda) - A(\epsilon,\lambda)$. Then,

$$G(x_{as} - x_{qw}, \lambda) \leq 1-\mu \text{ and } G(x_{as} - L, \frac{\lambda}{2}) > 1-\mu,$$

in particular $G(X_{qw} - L, \lambda) > 1-\epsilon$. Then,

$$1-\mu \geq G(x_{as} - x_{qw}, \lambda) \geq G(x_{as} - L, \frac{\lambda}{2}) \circ G(x_{qw} - L, \frac{\lambda}{2}) > (1-\epsilon)\circ(1-\epsilon) > 1-\mu,$$

this is not possible. Furthermore,

$$B(x_{as} - x_{qw}, \lambda) \geq \mu \text{ and } B(x_{as} - L, \frac{\lambda}{2}) < \mu,$$

in particular $B(x_{qw} - L, \frac{\lambda}{2}) < \epsilon$. Then,

$$\mu \leq B(x_{as} - x_{qw}, \lambda) \leq B(x_{as} - L, \frac{\lambda}{2}) \circ B(x_{qw} - L, \frac{\lambda}{2}) < \epsilon \bullet \epsilon < \mu,$$

and, this is not possible. Similarly,

$$Y(x_{as} - x_{qw}, \lambda) \geq \mu \text{ and } Y(x_{as} - L, \frac{\lambda}{2}) < \mu,$$

in particular $Y(x_{qw} - L, \frac{\lambda}{2}) < \epsilon$. Then,

$$\mu \leq Y(x_{as} - x_{qw}, \lambda) \leq Y(x_{as} - L, \frac{\lambda}{2}) \circ Y(x_{qw} - L, \frac{\lambda}{2}) < \epsilon \bullet \epsilon < \mu,$$

and, this is not possible. In this case, $B(\epsilon, \lambda) \subset A(\epsilon, \lambda)$. Then, by 3, $d(\epsilon, \lambda) = 0$ and (x_{kj}) is DSCa-NN.

Definition 3.14. Let V be a NNS. Then, V is called double statistically complete (DSC-NN), if for every DSCa-NN is DSC-NN.

Theorem 3.15. *Every NNS V is (DSC-NN)-Complete.*

Proof: Let (x_{kj}) be DSCa-NN but bot DSC-NN. Take $\mu > 0$. We have $(1 - \epsilon) \circ (1 - \epsilon) > (1 - \mu)$ and $\epsilon \bullet \epsilon < \mu$, for a given $\epsilon > 0$ and $\lambda > 0$, since (x_{kj}) is not DSC-NN,

$$G(x_{kj} - x_{NM}, \lambda) \geq G(x_{kj} - L, \frac{\lambda}{2}) \circ G(x_{NM} - L, \frac{\lambda}{2}) > (1 - \epsilon) \circ (1 - \epsilon) > 1 - \mu,$$

$$B(x_{kj} - x_{NM}, \lambda) \leq B(x_{kj} - L, \frac{\lambda}{2}) \bullet B(x_{NM} - L, \frac{\lambda}{2}) < \epsilon \bullet \epsilon < \mu,$$

$$Y(x_{kj} - x_{NM}, \lambda) \leq Y(x_{kj} - L, \frac{\lambda}{2}) \bullet Y(x_{NM} - L, \frac{\lambda}{2}) < \epsilon \bullet \epsilon < \mu$$

For

$$T(\epsilon, \lambda) = \{k \leq N, j \leq M : B_{x_{kj} - x_{NM}}(\epsilon) \leq 1 - \mu\},$$

$d(T^c(\epsilon, \lambda)) = 0$ and hence $d(T(\epsilon, \lambda)) = 1$, and this is a contradiction, since (x_{kj}) is DSCa-NN. Therefore, (x_{kj}) must be DSC-NN. In consequence, every NNS is (DSC-NN)-complete.

Lemma 3.16. *Let V be a NNS. Then, for any double sequence $(x_{kj}) \in F$, the following conditions are equivalent:*

(1) *(x_{kj}) is DSC-NN.*
(2) *(x_{kj}) is DSCa-NN.*
(3) *NNS V is (DSC-NN)-complete.*
(4) *There exits an increasing double index sequence $L_2 = (j_{nm})$ of natural numbers such that $d(L_2) = 1$ and the double subsequence $(x_{j_{nm}})$ is a DSCa-NN.*

Proof: The proof is followed directly by the Theorems 3.11 , 3.13 and 3.15 .

4. **Conclusion**

The purpose of this paper was to define and study the notion of double statistical convergence in neutrosophic normed space. We established some of their properties and we gave some examples associated to this notion. Furthermore, statistical Cauchy double sequence and statistically double completeness for neutrosophic norm were defined. For future work, we suggest studying these notions on spaces of sequences of functions in neutrosophic normed spaces. Besides, it would be interesting to see whether these properties are satisfied on triple sequences.

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