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Some Elementary Properties of Neutrosophic Integers

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Abstract: In this paper, we firstly defined a relation in the set of neutrosophic integers $Z[I]$ and proved that this relation is an equivalence relation. Thus we obtained a partition of $Z[I]$. Secondly we investigated the ordering relation in $Z[I]$ and we have seen that $Z[I]$ is not a totally ordered set. We also gave some relations of positive and negative neutrosophic integers and ordering in $Z[I]$. In the last part of the paper, we introduced the factorial of a positive neutrosophic integer.

Keywords: Neutrosophic integers; ordering in neutrosophic integers; factorial of a neutrosophic integer.

1. Introduction

Neutrosophy concept is presented by Smarandache to deal with indeterminacy in nature and science [1]. Neutrosophy has a lot of important applications in many fields and hundreds of studies have been done in these fields. One of these fields is neutrosophic number theory. Neutrosophic number theory is a mathematical way to deal with the properties of neutrosophic integers. Neutrosophic number theory was introduced in [2]. In [2], some properties of neutrosophic integers were introduced as division theorem, the form of primes in $Z[I]$.

In this study, it is obtained a partition of the set $Z[I]$ by an equivalence relation. Then it is investigated the ordering relation in $Z[I]$ and have seen that $Z[I]$ is not a totally ordered set, also given some relations of positive and negative neutrosophic integers and ordering in $Z[I]$. In the last part of the paper, we introduced the factorial of a positive neutrosophic integer.

2. Preliminaries

In the following, we give some elemantary definitions and results for emphasis.

Definition 2.1 [3] Let $(R;+,.)$ be a ring and I be an indeterminate element which satisfies $I^2 = I$. The set $R[I] = \{a+bI : a,b \in R\}$ is called a neutrosophic ring generated by I and R under the binary operations of R.

For example; $Z[I] = \{a+bI : a,b \in Z\}$ is a neutrosophic ring generated by I and Z where Z is integers ring. $Z[I]$ is called neutrosophic integers ring.

Definition 2.2 [4] Let $R[I] = \{a+bI : a,b \in R\}$ be the field of neutrosophic real numbers where R is the field of real numbers. For $a+bI, c+dI \in R[I]$,

$$a+bI \le c+dI \Leftrightarrow a \le c, a+b \le c+d .$$

Theorem 2.1 [4] The relation defined in Definition 2.2 is a partial order relation.

According to Definition 2.2, we are able to define positive neutrosophic real numbers as follows:

$$a+bI \ge 0 \Leftrightarrow a \ge 0, a+b \ge 0.$$

3. Ordering in Neutrosophic Integers

Definition 3.1 Let $a+bI, c+dI \in Z[I]$. If $a+b=c+d$, then the neutrosophic integers $a+bI$ and $c+dI$ are said to be equivalent and denoted by $a+bI \sim c+dI$. Then we write this with symbolically:

$$a+bI \sim c+dI \Leftrightarrow a+b=c+d .$$

Example 3.1 Since $-1+1=2-2$, we have $-1+I \sim 2-2I$ and since $2+3 \ne 1+2$, we have $2+3I$ is not equivalent to $1+2I$.

Theorem 3.1 The relation $"\sim"$ is an equivalence relation.

Proof. It can be proved easily.

The relation $"\sim"$ the separates the set $Z[I]$ into equivalence classes. The equivalence class of any $a+bI \in Z[I]$ denoted by $\overline{a+bI}$ and

$$\overline{a+bI} = \{x+yI : x+yI \in Z[I], x+yI \sim a+bI\} .$$

If we match $a+bI \in Z[I]$ to the point (a,b) on the cartesian plane, then the equivalence class $\overline{a+bI}$ is the set of the points (x,y) where $x,y \in Z$ on the line $x+y=a+b$.

Example 3.2

$$\begin{aligned}\overline{0+0I} &= \{x+yI : x+yI \in Z[I], x+yI \sim 0+0I\} \\ &= \{x+yI : x,y \in Z, x+y=0\} \\ &= \{\dots,-2+2I,-1+I,0+0I,1-I,2-2I,\dots\}.\end{aligned}$$

$\overline{0+0I} = \overline{0}$ is the set of the points (x,y) where $x,y \in Z$ on the line $x+y=0$.

$$\begin{aligned}\overline{1+0I} &= \{x+yI : x+yI \in Z[I], x+yI \sim 1+0I\} \\ &= \{x+yI : x,y \in Z, x+y=1\} \\ &= \{\dots,-2+3I,-1+2I,0+I,1-0I,2-I,\dots\}.\end{aligned}$$

$\overline{1+0I} = \overline{1}$ is the set of the points (x,y) where $x,y \in Z$ on the line $x+y=1$.

If we define the set $D=\{\overline{a+bI} : a+bI \in Z[I]\}$, then $D=\{,,,\overline{-2},\overline{-1},\overline{0},\overline{1},\overline{2},\dots\}=\{\overline{m} : m \in Z\}$. For $m,n \in Z$ and $m \ne n$, we see that $\overline{m} \cap \overline{n} = \varnothing$ and $\bigcup_{m \in Z} \overline{m} = Z[I]$. Then it is also obvious that the set D is a partition of $Z[I]$.

Definition 2.2 is valid for $Z[I]$. Let's rewrite it for topic integrity:

Definition 3.2 Let $a+bI, c+dI \in Z[I]$. If $a \le c$ and $a+b \le c+d$, we say that the neutrosophic integer $a+bI$ is less than or equal to $c+dI$ and denoted by $a+bI \le c+dI$. Shortly, we write:

$$a+bI \le c+dI \Leftrightarrow a \le c, a+b \le c+d .$$

Note that the relation $"\leq"$ is an partially ordering relation. Hence the set $Z[I]$ is a partially ordered set according to the relation $"\leq"$ but it is not an totally ordered set. Because, every element of $Z[I]$ can not be compared. For example; $1-2I$ and $-1+3I$ are incomparable. That is, $1-2I \nleq -1+3I$ and $-1+3I \nleq 1-2I$.

Example 3.3 The set of $x+yI \in Z[I]$ which satisfy $1+I \leq x+yI$ on the cartesian plane is drawn below:

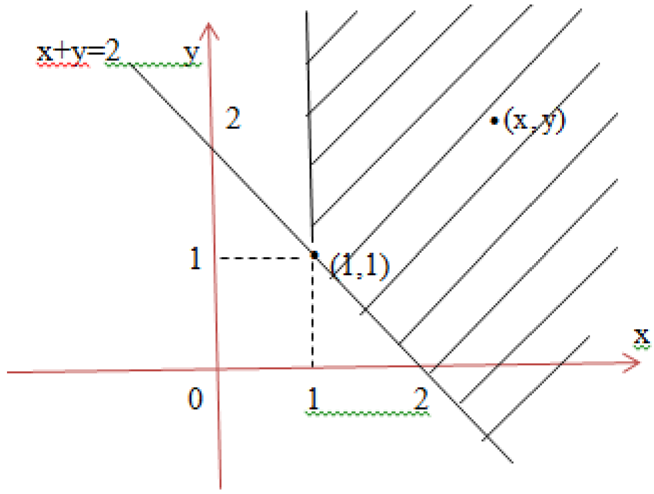

Figure 1. The set of $x+yI \in Z[I]$ which satisfy $1+I \leq x+yI$ on the cartesian plane.

Corollary 2.1 Let $a+bI \in Z[I]$.

i) $a+bI \geq 0 \Leftrightarrow a \geq 0$ and $a+b \geq 0$,

ii) $a+bI \leq 0 \Leftrightarrow a \leq 0$ and $a+b \leq 0$.

Proof. The first relation was given in [4]. (i) and (ii) can be proven using the Definition 3.2.

If we match $a+bI \in Z[I]$ to the point (a,b) on the cartesian plane, we can show the regions of positive and negative neutrosophic integers:

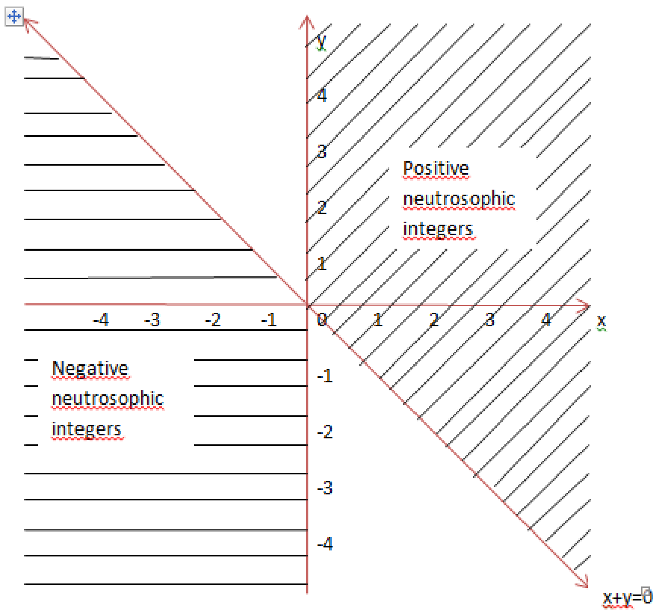

Figure 2. Positive and negative neutrosophic integers on cartesian plane.

We denote the set of positive neutrosophic integers by $Z[I]^{+}$. We know that the set $Z[I]^{+}$ is not totally ordered set. We can see that $1 \le 1+I \le 2$ and $1 \le 2-I \le 2$ but $1+I$ and $2-I$ are incomparable. $0+0I$ is the smallest element of the set $Z[I]^{+} \cup \{0+0I\}$. But the set $Z[I]^{+}$ has not smallest element.

The subsemilattice of the set $Z[I]^{+} \cup \{0+0I\}$ is given the following figure:

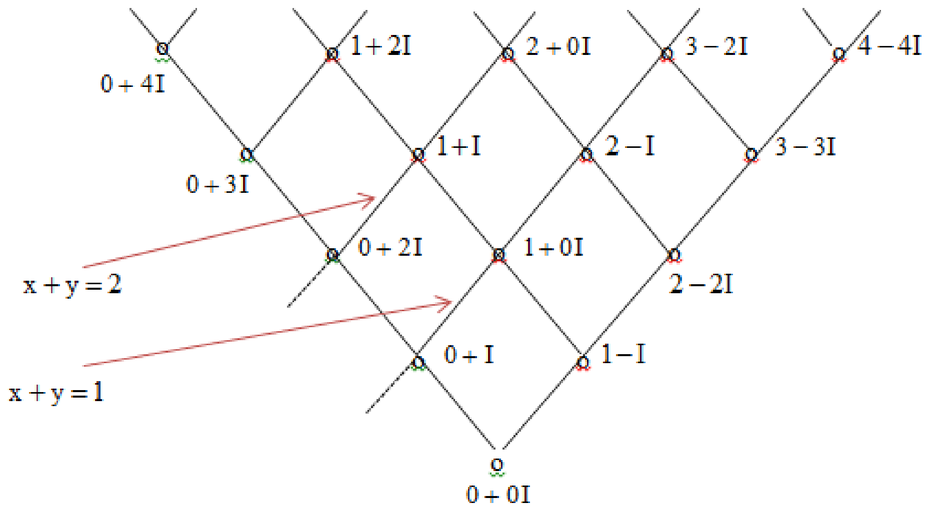

Figure 3. The subsemilattice of the set $Z[I]^{+} \cup \{0+0I\}$.

Theorem 3.2 Let $x = a + bI, y = c + dI \in Z[I]$. Then $x \le y$ if and only if there exists an $u \in Z[I]$ such that $u \ge 0$ and $x + u = y$.

Proof. Suppose that there exists an $u \in Z[I]$ such that $u \ge 0$ and $x + u = y$. Then, if $u = u_1 + u_2 I$, we get $u_1 \ge 0$ and $u_1 + u_2 \ge 0$. Also since $x + u = y$, we have $a + bI + u_1 + u_2 I = c + dI$. So $a + u_1 = c$ and $b + u_2 = d$ or $u_1 = c - a$ and $u_2 = d - b$. Since $u_1 \ge 0$, we get $c - a \ge 0$ or $a \le c$. Also since $u_1 + u_2 \ge 0$, we have $c - a + d - b \ge 0$ or $a + b \le c + d$. Hence since $a \le c$ and $a + b \le c + d$, we see that $x \le y$. Conversely, let $x \le y$. Then $a + b \le c + d$. Hence we have $a \le c$ and $a + b \le c + d$ in Z. Then if we say $c - a = u_1$ and $d - b = u_2$, we see that $u_1 \ge 0$ and $u_1 + u_2 \ge 0$. Then we have $u = u_1 + u_2 I \in Z[I]$ and $u \ge 0$.

$$
\begin{aligned}
x + u &= a + bI + u_1 + u_2 I \\
&= a + bI + c - a + (d - b)I \\
&= c + dI \\
&= y.
\end{aligned}
$$

Example 3.4 We know that $-3+2I \le 2+I$. Then $-3+2I+5-I = 2+I$ and $5-I \ge 0$.

Theorem 3.3 Let $x = x_1 + x_2 I, y = y_1 + y_2 I, z = z_1 + z_2 I$ and $u = u_1 + u_2 I \in Z[I]$. Then

(i) $x \le y \Leftrightarrow x + z \le y + z$,

(ii) $x \le y$ and $z \le u \Rightarrow x + z \le y + u$,

(iii) $x \le y$ and $z \ge 0 \Rightarrow xz \le yz$,

(iv) $x \le y$ and $z \le 0 \Rightarrow xz \ge yz$,

Proof. (i) Since $x+z = x_1+z_1+(x_2+z_2)I$ and $y+z = y_1+z_1+(y_2+z_2)I$, we have

$$\begin{aligned} x \le y &\Leftrightarrow x_1+x_2I \le y_1+y_2I \\ &\Leftrightarrow x_1 \le y_1 \text{ and } x_1+x_2 \le y_1+y_2 \text{ in } Z \\ &\Leftrightarrow x_1+z_1 \le y_1+z_1 \text{ and } x_1+x_2+z_1+z_2 \le y_1+y_2+z_1+z_2 \text{ for } z_1,z_2 \in Z \\ &\Leftrightarrow x_1+z_1+(x_2+z_2)I \le y_1+z_1+(y_2+z_2)I \\ &\Leftrightarrow x+z \le y+z. \end{aligned}$$

(ii) Since $x+z = x_1+z_1+(x_2+z_2)I$ and $y+z = y_1+u_1+(y_2+u_2)I$, we have

$$\begin{aligned} x \le y \text{ and } z \le u &\Rightarrow x_1+x_2I \le y_1+y_2I \text{ and } z_1+z_2I \le u_1+u_2I \\ &\Rightarrow x_1 \le y_1, x_1+x_2 \le y_1+y_2, z_1 \le u_1 \text{ and } z_1+z_2 \le u_1+u_2 \\ &\Rightarrow x_1+z_1 \le y_1+u_1, x_1+x_2+z_1+z_2 \le y_1+y_2+u_1+u_2 \\ &\Rightarrow x_1+z_1+(x_2+z_2)I \le y_1+u_1+(y_2+u_2)I \\ &\Rightarrow x+z \le y+u. \end{aligned}$$

(iii) Let $z = z_1+z_2I \ge 0$. Then $z_1 \ge 0$ and $z_1+z_2 \ge 0$. Since $xz = x_1z_1+(x_1z_2+x_2z_1+x_2z_2)I$ and $yz = y_1z_1+(y_1z_2+y_2z_1+y_2z_2)I$, we have

$$\begin{aligned} x \le y &\Leftrightarrow x_1+x_2I \le y_1+y_2I \\ &\Leftrightarrow x_1 \le y_1, x_1+x_2 \le y_1+y_2 \\ &\Leftrightarrow x_1z_1 \le y_1z_1 \text{ and } (x_1+x_2)(z_1+z_2) \le (y_1+y_2)(z_1+z_2) \\ &\Leftrightarrow x_1z_1 \le y_1z_1 \text{ and } x_1z_1+x_1z_2+x_2z_1+x_2z_2 \le y_1z_1+y_1z_2+y_2z_1+y_2z_2 \\ &\Leftrightarrow x_1z_1+(x_1z_2+x_2z_1+x_2z_2)I \le y_1z_1+(y_1z_2+y_2z_1+y_2z_2)I \\ &\Leftrightarrow xz \le yz. \end{aligned}$$

iv) Let $z = z_1+z_2I \le 0$. Then $z_1 \le 0$ and $z_1+z_2 \le 0$. Since $xz = x_1z_1+(x_1z_2+x_2z_1+x_2z_2)I$ and $yz = y_1z_1+(y_1z_2+y_2z_1+y_2z_2)I$, we have,

$$\begin{aligned} x \le y &\Leftrightarrow x_1+x_2I \le y_1+y_2I \\ &\Leftrightarrow x_1 \le y_1, x_1+x_2 \le y_1+y_2 \\ &\Leftrightarrow x_1z_1 \ge y_1z_1 \text{ and } (x_1+x_2)(z_1+z_2) \ge (y_1+y_2)(z_1+z_2) \\ &\Leftrightarrow x_1z_1 \ge y_1z_1 \text{ and } x_1z_1+x_1z_2+x_2z_1+x_2z_2 \ge y_1z_1+y_1z_2+y_2z_1+y_2z_2 \\ &\Leftrightarrow x_1z_1+(x_1z_2+x_2z_1+x_2z_2)I \ge y_1z_1+(y_1z_2+y_2z_1+y_2z_2)I \\ &\Leftrightarrow xz \ge yz. \end{aligned}$$

4. Factorial of a Positive Neutrosophic Number

It is known that $n! = n.(n-1)...2.1$ for a $n \in Z^+$ and $0! = 1$. This is the product of all integers less than or equal to n on the positive real axis of the coordinate system.

Now we want to extend the factorial concept in Z to $Z[I]$. For $n \in Z^+$, we have $n = n+0I \in Z[I]$. The we can write $(n+0I)! = (n+0I).(n-1+0I)...(2+0I).(1+0I)$. The numbers $n+0I, n-1+0I, ...2+0I, 1+0I$ are some positive neutrosophic integers less than or equal to $n+0I$. If we match these numbers to the points $(n,0),(n-1,0),...,(2,0),(1,0)$, we see that they are on the half line $y = 0.x = 0$.

Now we take $5+5I \in Z[I]$. Then the numbers $5+5I, 4+4I, 3+3I, 2+2I, 1+I$ are some positive neutrosophic integers less than or equal to $5+5I$. If we match these numbers to the points

$(5,5),(4,4),(3,3),(2,2),(1,1)$, we see that they are on the half line $y = x$. We can write

$$\begin{aligned}(5+5I)! &= (5+5I)(4+4I)(3+3I)(2+2I)(1+I)\\ &= 5.4.3.2.1.(1+I)^5\\ &= 5!(1+I)^5\end{aligned}$$

Now we construct $(12+16I)!$ similarly. The points $(12,16),(9,12),(6,8),(3,4)$ are on the half line $y = \frac{16}{12}x = \frac{4}{3}x$. The corresponding neutrosophic integers $12+16I, 9+12I, 6+8I, 3+4I$ are less than or equal to $12+16I$. So we can write

$$\begin{aligned}(12+16I)! &= (12+16I)(9+12I)(6+8I)(3+4I)\\ &= 4.3.2.1.(3+4I)^4\\ &= 4!(3+4I)^4\end{aligned}$$

Now we are ready to define the factorial of a positive neutrosophic integer:

Definition 4.1 Let $a+bI \in Z[I]$. Then

$$(a+bI)! = d!\left(\frac{a}{d}+\frac{b}{d}I\right)^d$$

where $d = \gcd\{a,b\}$ (gcd:greatest common divisor).

Example 4.1

i) $5! = (5+0I)! = 5!\left(\frac{5}{5}+\frac{0}{5}I\right)^5 = 5!+0I$ since $\gcd\{5,0\} = 5$.

ii) $(0+5I)! = 5!\left(\frac{0}{5}+\frac{5}{5}I\right)^5 = 0+5!I$ since $\gcd\{0,5\} = 5$.

iii) $(9-3I)! = 3!\left(\frac{9}{3}-\frac{3}{3}I\right)^3 = 3!(3-I)^3$ since $\gcd\{9,-3\} = 3$.

The following Theorem and its proof were given for the neutrosophic n square matrices in [5, Theorem 3.6].

Theorem 4.1 Let $a+bI \in Z[I]$. Then,

$$(a+bI)^n = a^n + \left((a+b)^n - a^n\right)I$$

for $n \in Z^+$.

Proof. We use induction on n. For n=1, the above equality is true. Suppose that the claim is true for n-1. That is, $(a+bI)^{n-1} = a^{n-1} + \left((a+b)^{n-1} - a^{n-1}\right)I$. Then we have

$$\begin{aligned}(a+bI)^n &= (a+bI)^{n-1}(a+bI)\\ &= \left(a^{n-1} + \left((a+b)^{n-1} - a^{n-1}\right)I\right)\left(a+bI\right)\\ &= a^n + \left(a^{n-1}b + (a+b)^{n-1}a - a^n + (a+b)^{n-1}b - a^{n-1}b\right)I\end{aligned}$$

$$= a^n + \left((a+b)^{n-1}(a+b) - a^n\right)I$$
$$= a^n + \left((a+b)^n - a^n\right)I$$

Therefore Theorem is true.

Corollary 4.1 Let $a + bI \in Z[I]^+$. Then

$$(a+bI)! = d!\left\{\left(\frac{a}{d}\right)^d + \left[\left(\frac{a}{d}+\frac{b}{d}\right)^d - \left(\frac{a}{d}\right)^d\right]I\right\}$$

where $d = \gcd\{a, b\}$.

Proof. It is clear by Definition 4.1 and Theorem 4.1.

5. Conclusions

In this paper, it is obtained a partition of the set $Z[I]$ by an equivalence relation. Then, it is investigated the ordering relation in $Z[I]$ and have seen that $Z[I]$ is not a totally ordered set, also given some relations of positive and negative neutrosophic integers and ordering in $Z[I]$. In the last part of the paper, we introduced the factorial of a positive neutrosophic integer. In our future studies, we intend to continue to examine the properties of $Z[I]$.

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Conflicts of Interest: The authors declare no conflict of interest.

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