NSS

Neutrosophic Sets and Systems, Vol. 48, 2022

University of New Mexico

ClassicalBalanced, AntiBalanced and NeutroBalanced functions

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Abstract. In this paper, we extend the concepts of Neutrosophy to Boolean function and define ClassicalBalanced, AntiBalanced and NeutroBalanced functions. We consider functions of the form $f(x) = Tr(x^d)$, where the exponent d may be Gold exponent, Kasami exponent, Welch exponent or any arbitrary positive integer. We, for different values of d, examine nature of these functions with respect to the above stated three categories.

Keywords: Balanced function; Neutrosophy; AntiBalanced function; NeutroBalanced function; Cryptography.

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1. **Introduction**

In an algebraic structure, the axioms are valid and the operations are defined everywhere. We cannot do much mathematics just on sets. We need some sort of algebraic structures for analysis. In real life situations when we require to combine the elements of a particular domain in a certain manner, it may happen the combination is not meaningful for certain pairs. It may be undefined, indeterminate or multivalued. In such situation, we cannot have an algebraic structure and we are left with no option but to modify the combining operations.

What if we have the theoretical platform to deal with such operation the way they are. This line of thinking lead to evolution of Neutrosophy. The history of Neutrosophy is dated back to 1998 when Florentin Smarandache propounde the notion of Neutrosophy in [3]. However, the research in this area gained momentum in last couple of years. Some recent works may be found in [8–11].

Neutrosophic structures have been defined on Algebra [1–7], groups [14–16, 20, 21] and ring [12, 13] and their properties have been explored. We define neutrosophic structure on a finite field. The main reasons of choosing finite fields are:

(1) This direction remains by and large unexplored.
(2) Fields are the richest structure. The finiteness is considered to make it more computer friendly. When it is not possible to produce a rigorous logic for proving an assertion, computational tools may be utilised to establish the assertion.
(3) Finite fields are widely used in cryptography. We try to translate the concepts of Boolean function to the Neutrosophic scenario. This may lead to application of these function to cryptography.

We define three types of functions viz., ClassicalBalanced, AntiBalanced and NeutroBalanced functions. The details can be found in the subsequent sections. The paper is structured in the following manner.

In the next section we discuss preliminaries required to comprehend the paper. In section 3 we introduce three neutrosophic functions as mentioned above. In the fourth section we prove some results related to the defined function. Finally, in section 5, we conclude the paper.

2. Preliminaries

Definition 2.1. [4]

(i) A classical operation is an operation well defined for all the set's elements while a Neutro Operation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set. An AntiOperation is an operation that is outer definedfor all the set's elements.
(ii) A NeutroAlgebra is an algebra that has at least one Neutro Operation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no AntiOperation or AntiAxiom. An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

The study and analysis of cryptographic and combinatorial properties with respect to Boolean functions has been an important branch of cryptography. Boolean functions play a significant role in the construction of components used in symmetric ciphers, and cryptographic properties of such functions are of great interest. Boolean functions used in cryptographic applications provide security of a cipher against different kinds of attacks.

Over the prime field, $\mathbb{F}_2$ the n-dimensional vector space can be denoted as $\mathbb{F}_2^n$. One can identify this vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$ with the finite field $\mathbb{F}_{2^n}$ of 2^n elements, which is basically extension of the finite field $\mathbb{F}_2 = \{0, 1\}$ using some irreducible polynomial of degree n with coeffiecients either 0 or 1.

A Boolean function in n variables is an arbitrary function from $\mathbb{F}_2^n \to \mathbb{F}_2$, where $\mathbb{F} = \{0, 1\}$ is a Boolean domain and n is a non-negative integer. It is called Boolean in honor of the British mathematician and philosopher George Boole $(1815 - 1864)$.

The vectorial Boolean function is of the form $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ with range of the function being $\mathbb{F}_2^m$, where $m > 1$. It is also called an (n, m)-function. If $m = n$ then it is called as (n, n)-function. For vectorial Boolean functions we use uppercase letters, whereas Boolean functions are denoted with lowercase letters.

The finite field $\mathbb{F}_{2^n}$ of order 2^n is also denoted as $GF(2^n)$, which is due to French mathematician Evariste Galois (1811-1832). Usually, $\mathbb{F}_{2^n}^*$ is the denotion used to represent the collection of all nonzero elements in the field $\mathbb{F}_{2^n}$. With respect to multiplication, $\mathbb{F}_{2^n}^*$ acts as a cyclic group with order of the group being $2^n - 1$. For basic and recent results on finite fields, permutation polynomials, balanced functions and trace functions we refer [19, 22–24, 27].

The trace representation is very useful in defining and analyzing various properties of Boolean functions.

Definition 1. *[25] If c is an element of $K = GF(q^n)$, its trace relative to the subfield $\mathbb{F} = GF(q)$ is defined as follows:*

$$Tr_F^K(c) = c + c^q + c^{q^2} + \ldots + c^{q^{n-1}}.$$

The values of trace functions fall into the prime field $\mathbb{F}_2$ is the most important property of the trace. Since $\mathbb{F}_2^n$ is isomorphic with $\mathbb{F}_{2^n}$, trace function can also be viewed as a Boolean function in n variables. In case of the base field $\mathbb{F}_2$, we use the notation Tr for trace.

Consider the finite field $GF(9) = (00, 10, 11, -11, 0-1, -10, -1-1, 1-1, 01)$. Trace is tabulated as below.

x	00	10	11	-11	0-1	-10	-1-1	1-1	01
Trace(x)	0	1	0	1	1	-1	0	-1	-1

A Boolean function f is said to be balanced if the output column of its truth table has same number of zero's and one's. As balanced functions would give outputs with the balanced number of zero and one, which appears more random. Hence, these functions avoid statistical dependencies between the input and output of the stream cipher, which prevents distinguishing attacks and statistical analysis [26, 27]. From cryptographic point of view, balanced functions are very important. In case of an unbalanced function, the input and output variables have considerable dependence on each other, which may cause susceptible cryptanalysis attacks.

In [17–19], one can find the construction of many such power function $f : \mathbb{F}_{2^n} \mapsto \mathbb{F}_2$ with trace representation. Some well known examples of Boolean functions $f(x) = Tr(x^d)$, where d is given by the table 1 are as follows

Table 1. Boolean functions

	Exponent "d "	Conditions
Gold function	$2^i + 1$	$\gcd(i, n) = 1$
Kasami function	$2^{2i} - 2^i + 1$	$\gcd(i, n) = 1$
Welch function	$2^t + 3$	$n = 2t + 1$

The terminology of balancedness always comes with the idea of the measurement of different conditions. The balanced characteristic of a function is defined with the classification of its co-domain. Here in the next section, we present three types of balanced Neutrosophic functions.

3. ClassicalBalanced, AntiBalanced and NeutroBalanced functions

Let ψ be a Neutrosophic functions defined on $\mathbb{F}_3^n$ to K, where K is some arbitrary set. Note that there will be three partitions of the domains say P_0, P_1, P_2 such that ψ is defined on P_0, not defined on P_1 and indeterminate on P_2. The Neutrosophic function ψ induces a generalised Boolean function f on $\mathbb{F}_3^n \to \mathbb{F}_3$ as

$$f(x) = i \text{ when } x \in P_i.$$

It can be seen easily, every function $f : \mathbb{F}_3^n \to \mathbb{F}_3$ induces a neutrosophic function $\mathbb{F}_3^n \to K$. Thus there are one to one correspondance between Neutrosophic functions $\mathbb{F}_3^n \to K$ and the generalised Boolean function from $\mathbb{F}_3^n \to \mathbb{F}_3$. We can therefore, identify a Neutrosophic function from $\mathbb{F}_3^n \to K$ by a generalised Boolean function from $\mathbb{F}_3^n \to \mathbb{F}_3$. With this identification, we proceed further and define neutrosophic functions.

Note that any $f : \mathbb{F}_3^n \to \mathbb{F}_3$ can be given as $f(x) = Tr(h(x))$, where h is a function defined on $\mathbb{F}_3^n$. We are now fully equipped to define ClassicalBalanced, AntiBalanced and NeutroBalanced functions.

Definition 3.1. A function is said to be ClassicalBalanced function if it takes equal number of 1's, 0's and -1's.

Example 3.2. Over $\mathbb{F}_{3^5}$, the function $f(x) = Tr(x^9)$ is a ClassicalBalanced function.

Definition 3.3. A function is said to be AntiBalanced function if number of 1's, 0's, and -1's are all distinct from each other.

Example 3.4. Over $\mathbb{F}_{3^2}$, the function $f(x) = Tr(x^8)$ is a AntiBalanced function.

Definition 3.5. A function is said to be NeutroBalanced function if number of 1's, 0's, and -1's are not same but exactly two of them are equal.

Example 3.6. Over $\mathbb{F}_{3^4}$, the function $f(x) = Tr(x^{14})$ is a NeutroBalanced function.

4. Some Special types of Neutrosophic functions

Composition of two functions is an intrinsic approach in the upcoming results to construct ClassicalBalanced, AntiBalanced and NeutroBalanced function. Trace of finite field is a common choice for one of the compositions of two functions. Here in the next two propositions we present the necessary and sufficient conditions for a composition of two functions to be ClassicalBalanced, AntiBalanced and NeutroBalanced function.

Proposition 1. *Let $f : \mathbb{F}_{3^n} \to \mathbb{F}_3$ be a Boolean function and h be any bijection on $f : \mathbb{F}_{3^n}$. Then f is ClasicalBalanced, AntiBalanced or NeutroBalanced if and only if the composition map fh is ClasicalBalanced, AntiBalanced or NeutroBalanced respectively.*

Proof. The proof is obvious. □

Proposition 2. *The exponential map $x \to x^a, a \in \mathbb{Z}$ on $\mathbb{F}_{3^n}$ is a bijection if and only if* $\gcd(a, 3^n - 1) = 1$.

Proof. If map $x \to x^a, a \in \mathbb{Z}$ on $\mathbb{F}_{3^n}$ is a bijection then the proof is obvious. Now let

$$\gcd(a, 3^n - 1) = 1$$

and $x_1(\neq 0)$, $x_2(\neq 0) \in \mathbb{F}_3^n$ and g be a generator of non zero elements of $\mathbb{F}_3^n$. Let if

$$f(x_1) = f(x_2),$$

then

$$x_1^a = x_2^a,$$

$$\implies (g^{u_1})^a = (g^{u_2})^a,$$

$$\implies (g^{u_1 - u_2})^a = 1,$$

$$\implies 3^n - 1 | (u_1 - u_2)(a)$$

or

$$(u_1 - u_2)(a) = 0 \mod 3^n - 1. \tag{1}$$

Now if $\gcd(a, 3^n - 1) = 1$ then,

$$(u_1 - u_2) = 0 \mod 3^n - 1. \tag{2}$$

Now since, $1 \leq u_1, u_2 \leq 3^n - 1$ therefore

$$u_1 - u_2 \leq 3^n - 1. \tag{3}$$

From (2) and (3), $u_1 = u_2$, which implies that, $x_1 = x_2$. Hence the result is proved. □

In the next theorem we prove the ClassicalBalanced property of Trace function over finite field $\mathbb{F}_{3^n}$.

Theorem 4.1. *A function of the form*

$$f(x) = Tr(x)$$

is a ClassicalBalanced function over the finite field $\mathbb{F}_{3^n}$*.*

Proof. We have

$$f(x) = Tr(x) = x + x^3 + x^9 + \ldots + x^{3^{n-1}}. \tag{4}$$

This is the absolute trace mapping the elements of $\mathbb{F}_{3^n}$ to the prime field $\mathbb{F}_3$. Therefore,

$$\begin{aligned} & f^{-1}(\mathbb{F}_3) = \mathbb{F}_{3^n}, \\ \implies \quad & f^{-1}(0) \cup f^{-1}(1) \cup f^{-1}(2) = \mathbb{F}_{3^n}, \\ \implies \quad & |f^{-1}(0)| + |f^{-1}(1)| + |f^{-1}(2)| = 3^n. \end{aligned} \tag{5}$$

Let $|f^{-1}(0)| = \alpha_1, |f^{-1}(1)| = \alpha_2$ and $|f^{-1}(2)| = \alpha_3$. Then from (5)

$$\alpha_1 + \alpha_2 + \alpha_3 = 3^n. \tag{6}$$

Now from (4) we have

$$\alpha_1 = |\{x | x + x^3 + x^9 + \ldots + x^{3^{n-1}} = 0\}|, \tag{7}$$

$$\alpha_2 = |\{x | x + x^3 + x^9 + \ldots + x^{3^{n-1}} - 1 = 0\}| \tag{8}$$

and

$$\alpha_3 = |\{x | x + x^3 + x^9 + \ldots + x^{3^{n-1}} - 2 = 0\}|. \tag{9}$$

All equations (7), (8) and (9) has a polynomial of degree 3^{n-1}. So, each can have at most 3^{n-1} roots and we conclude that

$$0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 3^{n-1}. \tag{10}$$

It is clear that (6) and (10) hold together if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 3^{n-1}$. Thus our assertion is proved. □

Before start of the proof for next theorem here we prove a lemma.

Lemma 4.2. *Let i and n are positive integers, If $< n, i >= 1$ then $< 3^n - 1, 2^i + 1 >= 1$.*

Proof. Here n and i are positive integers therefore let $p(\neq 2, 3)$ be any prime number such that $p|(3^n - 1)$, then we can write $3^n = 1 \mod p$, which implies that $p - 1|n$ or

$$n = 0 \mod (p - 1). \tag{11}$$

Now if $p|2^i + 1$, then $2^i = -1 \mod p$ which implies that $\frac{p-1}{2}|i$, consequently $p - 1|2i$ or

$$2i = 0 \mod (p - 1). \tag{12}$$

Now combining (11) and (12), we can write $< n, 2i >= p - 1$. Therefore if $< n, i >= 1$ then at max gcd of n and $2i$ will be 2 but since $p \neq 2\, or\, 3$, hence there does not exist any prime $p \neq 2\, or\, 3$ such that $p|(3^n - 1)$ and $p|(2^i + 1)$. Hence

$$< 3^n - 1, 2^i + 1 >= 1.$$

□

Now in the next theorem we present the bijective condition for an exponent function on $\mathbb{F}_3^n$.

Theorem 4.3. *Let $f : \mathbb{F}_{3^n} \mapsto \mathbb{F}_{3^n}$ be a function defined as $f(x) = x^{2^i+1}$. If $< i, n >= 1$ for any positive integer i, then f is a bijective function.*

Proof. The proof follow from the lemma 4.2 and proposition 2. □

Corollary 4.4. *A function of the form*

$$f(x) = Tr(x^{2^i+1}),$$

for any positive integer i, is a ClassicalBalanced function with $\gcd(i, n) = 1$ *over the finite field $\mathbb{F}_{3^n}$.*

Proof. Theorem 4.1 and theorem 4.3 follows the proof of this corollary. □

If $\gcd(i, n) \neq 1$, then the functions $f(x) = Tr(x^{2^i+1})$ cannot be a ClassicalBalanced functions, which we can observe from the following examples.

Example 4.5. Over $\mathbb{F}_{3^4}$, the function $f(x) = Tr(x^{2^2+1})$ is not a ClassicalBalanced function.

Example 4.6. Over $\mathbb{F}_{3^5}$, the function $f(x) = Tr(x^{2^5+1})$ is not a ClassicalBalanced function.

Some more general results for AntiBalanced and NeutroBalanced functions are presented in following theorems.

Theorem 4.7. *A function of the form*

$$f(x) = Tr(x^{8a})$$

is a AntiBalanced function for $a \in \mathbb{N}$, $gcd(a,\, 3^2-1) = 1$ *over the finite field* $\mathbb{F}_{3^2}$.

Proof. Given function $f(x)$ is composition of two functions, $Tr(x) = x + x^3$ and $h(x) = x^a$, therefore,

$$f(x) = Tr(x^{8a}) = f_1 h(x) \tag{13}$$

where $f_1(x) = Tr(x^8)$ and $h(x) = x^a$. It is given that $\gcd(a, 3^2-1) = 1$, therefore from propositions 1 and 2, f is AntiBalanced if and only if f_1 is AntiBalanced. We now show that f_1 is AntiBalanced. From the expression of trace and f_1,

$$f_1(x) = 0 \implies x^8 + x^{24} = 0$$

Similarly,

$$f_1(x) = 1 \implies x^8 + x^{24} - 1 = 0$$

and

$$f_1(x) = 2 \implies x^8 + x^{24} - 2 = 0.$$

It can be verified computationally or otherwise that the polynomials $x^8 + x^{24}$, $x^8 + x^{24} - 1$, $x^8 + x^{24} - 2$ has 1, 0 and 8 distinct roots in $\mathbb{F}_{3^2}$. Hence, f_1 is AntiBalanced. This proves our assertion as well. □

Theorem 4.8. *A function of the form*

$$f(x) = Tr(x^{2a})$$

is a AntiBalanced function for $a \in \mathbb{N}$ *with* $2a \not\equiv 0 (\mod 13)$ *over the finite field* $\mathbb{F}_{3^3}$.

Proof. Let

$$f(x) = Tr(x^{2a}) = f_1 h(x) \tag{14}$$

where $f_1(x) = Tr(x^2)$ and $h(x) = x^a$. Now $2a \not\equiv 0 (\mod 13)$ implies that $\gcd(a, 3^3-1) = 1$. Propositions 1 and 2 confirm that f is AntiBalanced if and only if f_1 is AntiBalanced. Further we show that f_1 is AntiBalanced. From Trace $Tr(x) = x + x^3 + x^9$ over $\mathbb{F}_{3^3}$,

$$f_1(x) = 0 \implies x^2 + x^6 + x^{18} = 0.$$

Similarly,

$$f_1(x) = 1 \implies x^2 + x^6 + x^{18} - 1 = 0$$

and

$$f_1(x) = 2 \implies x^2 + x^6 + x^{18} - 2 = 0$$

After computational verification we found that the polynomials $x^2+x^6+x^{18}$, $x^2+x^6+x^{18}-1$, $x^2 + x^6 + x^{18} - 2$ has 9,6 and 12 distinct roots in $\mathbb{F}_{3^2}$. Hence, f_1 is AntiBalanced. □

Theorem 4.9. *A function of the form*

$$f(x) = Tr(x^{16a})$$

is a AntiBalanced function for $a \in \mathbb{N}$ *and* $16a \neq 0 \mod 80$ *over the finite field* $\mathbb{F}_{3^4}$*.*

Proof. Here $Tr(x) = x + x^3 + x^9 + x^{27}$ over $\mathbb{F}_{3^4}$ and

$$f(x) = Tr(x^{16a}) = f_1 h(x) \tag{15}$$

where $f_1(x) = Tr(x^{16})$ and $h(x) = x^a, \gcd(a, 3^4 - 1) = 1$. In view of propositions 1 and 2, f is AntiBalanced if and only if f_1 is AntiBalanced. We now show that f_1 is AntiBalanced.

$$f_1(x) = 0 \implies x^2 + x^6 + x^{18} = 0$$

Similarly,

$$f_1(x) = 1 \implies x^2 + x^6 + x^{18} - 1 = 0$$

and

$$f_1(x) = 2 \implies x^2 + x^6 + x^{18} - 2 = 0$$

It is verified computationally or otherwise that the polynomials $x + x^3 + x^9 + x^{27}$, $x + x^3 + x^9 + x^{27} - 1$ and $x+x^3+x^9+x^{27}-2$ has 1,16 and 64 distinct roots in $\mathbb{F}_{3^4}$ respectively. Hence, f_1 is AntiBalanced. This proves our assertion as well. □

Theorem 4.10. *A function of the form*

$$f(x) = Tr(x^{2a})$$

is a AntiBalanced function for $a \in \mathbb{N}$, $gcd(a,\ 3^5 - 1) = 1$ *over the finite field* $\mathbb{F}_{3^5}$*.*

Proof. Trace function in the finite field $\mathbb{F}_{3^5}$ is a polynomial $Tr(x) = x+x^3+x^9+x^{27}+x^{81} \in \mathbb{F}_3$ where $x \in \mathbb{F}_{3^5}$. Now given function

$$f(x) = Tr(x^{16a}) = f_1 h(x) \tag{16}$$

where $f_1(x) = Tr(x^2)$ and $h(x) = x^a$. From proposition 2, $\gcd(a, 3^5 - 1) = 1$ implies that h is a bijectiona. Now we now show that f_1 is AntiBalanced.

$$f_1(x) = 0 \implies x + x^3 + x^9 + x^{27} + x^{81} = 0$$

Similarly,

$$f_1(x) = 1 \implies x + x^3 + x^9 + x^{27} + x^{81} - 1 = 0$$

and

$$f_1(x) = 2 \implies x + x^3 + x^9 + x^{27} + x^{81} - 2 = 0$$

We found from computation search that the polynomials $x + x^3 + x^9 + x^{27} + x^{81}$, $x + x^3 + x^9 + x^{27} + x^{81} - 1$ and $x + x^3 + x^9 + x^{27} + x^{81} - 2$ has 81,90 and 72 distinct roots in $\mathbb{F}_{3^5}$ respectively. Therefore, f_1 is AntiBalanced. From proposition 1, f is AntiBalanced if and only if f_1 is AntiBalanced. Hence the theorem is proved. $\square$

Theorem 4.11. *A function of the form*

$$f(x) = Tr(x^{2a})$$

is a NeutroBalanced function for $a \in \mathbb{N}$ *with* $2a \not\equiv 0 \pmod 8$ *over the finite field* $\mathbb{F}_{3^2}$.

Proof. The trace of $\mathbb{F}_{3^2}$ is $Tr(x) = x + x^3 \in \mathbb{F}_3$ where $x \in \mathbb{F}_{3^2}$. Given function $f(x)$ can be written as,

$$f(x) = Tr(x^{2a}) = f_1 h(x) \tag{17}$$

where $f_1(x) = Tr(x^2)$ and $h(x) = x^a$. Now from the given condition $\gcd(a, 3^2 - 1) = 1$ and proposition 2, h is a bijective function. Now We show that f_1 is NeutroBalanced. Using $Tr(x)$ on $\mathbb{F}_{3^2}$, we can write

$$f_1(x) = 0 \implies x + x^3 = 0$$

Similarly,

$$f_1(x) = 1 \implies x + x^3 - 1 = 0$$

and

$$f_1(x) = 2 \implies x + x^3 - 2 = 0$$

Count of the roots of above three polynomials can settle the proof of Neutrobalanced property of f. It can be verified computationally or otherwise that the polynomials $x+x^3$, $x+x^3-1$ and $x + x^3 - 2$ has 5, 2 and 2 distinct roots in $\mathbb{F}_{3^2}$ respectively. Hence, f_1 is NeutroBalanced. It is already proved in proposition 1 that f is NeutroBalanced if and only if f_1 is NeutroBalanced. Hence the theorem is proved $\square$

Theorem 4.12. *A function of the form*

$$f(x) = Tr(x^{2a})$$

is a NeutroBalanced function for $a \in \mathbb{N}$ *with* $2a \not\equiv 0 \pmod{16}$ *over the finite field* $\mathbb{F}_{3^4}$*.*

Proof. Here the trace function, $Tr(x)$, on the extension field $\mathbb{F}_{3^4}$ is

$$Tr(x) = x + x^3 + x^9 + x^{27} \in \mathbb{F}_3,$$

where $x \in \mathbb{F}_{3^4}$. Now given function

$$f(x) = Tr(x^{2a}) = f_1 h(x), \tag{18}$$

where $f_1(x) = Tr(x^2)$ and $h(x) = x^a$, $\gcd(a, 3^4 - 1) = 1$. In view of propositions 1 and 2, f is NeutroBalanced if and only if f_1 is NeutroBalanced. We now show that f_1 is NeutroBalanced. From the expression of $Tr(x)$ on $\mathbb{F}_{3^4}$,

$$f_1(x) = 0 \implies x + x^3 + x^9 + x^{27} = 0$$

Similarly,

$$f_1(x) = 1 \implies x + x^3 + x^9 + x^{27} - 1 = 0$$

and

$$f_1(x) = 2 \implies x + x^3 + x^9 + x^{27} - 2 = 0$$

Now it can be observe from proposition 1 that f is NeutroBalanced if enumeration of roots of any two polynomials from $x+x^3+x^9+x^{27}$, $x+x^3+x^9+x^{27}-1$ and $x+x^3+x^9+x^{27}-2$ are same. We found computationally that the polynomials $x+x^3+x^9+x^{27}$, $x+x^3+x^9+x^{27}-1$ and $x+x^3+x^9+x^{27}-2$ has 21, 30 and 30 distinct roots in $\mathbb{F}_{3^4}$ respectively. Hence, f_1 is NeutroBalanced. This proves our assertion as well. □

5. Conclusions

In this paper, we have defined ClassicalBalanced, AntiBalanced and NeutroBalanced functions. So far a function over finite field is classified into balanced function and unbalanced function. With this work it is a new approach to define a class of functions which lie between these two, which are called as NeutroBalanced functions. NeutroBalanced functions are defined with the logic of neutrosophy. NeutroBalanced functions may lead to a new direction with its application in point of view.

Funding: "Fourth author received funding from VGST/KSTePS and DST, GoK, India"

Acknowledgments: Fourth author thanks VGST/KSTePS and DST, GoK, India for sanctioning the project [VGST/GRD-897/2019-20/2020-21]. The authors are grateful to Prof. Dr. Florentin Smarandache from University of New Mexico for his valuable ideas and guidance.

Conflicts of Interest: "The authors declare no conflict of interest."

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Received: Nov 20, 2021. Accepted: Feb 4, 2022