Neutrosophic SuperHyper Bi-Topological Spaces: Original Notions and New Insights

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Abstract.

This manuscript comes as first attempt in building a new type of neutrosophic topological spaces, the aim is to shed the light on a new structure known as the n^{th}-power set $P^n(X)$ of a set, this new kind of sets enables authors to create and built new topology spaces called Neutrosophic SuperHyper Topological Spaces and Neutrosophic SuperHyper Bi-Topological Spaces , the n^{th}-power sets are the optimal representation for the applications in our real world. In this article, new concepts and theorems related to this new topologies have been discussed, which are pairwise neutrosophic open n^{th}-power set, pairwise neutrosophic closed n^{th}-power set, as well as, the closures and the interiors are defined with their properties. Many of relations for these concepts have been introduced.

Keywords: n^{th} -power set $P^n(X)$; Neutrosophic SuperHyper Topological Spaces (NSHTSs); Neutrosophic SuperHyper Bi-Topological Spaces (NSHBi-TSs).

Introduction.

The concepts of the neutrosophic n^{th}-power set of a set, SuperHyperGraph and Pliothogenic n-SuperHyperGraph, SuperHyperAlgebra, n-ary (classical-/Neutro-/Anti-) HyperAlgebra have been firstly introduced by the father of neutrosophic theory F. Smarandache in 2016 [4]. As the introduction for Neutrosophic SuperHyper Topological Spaces which is until yet is fathomless branch of science, in this section we recalling the fundamental definitions of the neutrosophic logic with preliminaries of related n^{th}-power set of a set. There is no doubt that the essential theory of neutrosophic was introduced and built by F. Smarandache in 1995 [5,6]. Any mathematician who tracking the trace of this knowledge will easily deduce that the neutrosophic theory was rapidly and broadly radiated through Neutrosophic Sets and Systems journal, and International Journal of

Neutrosophic Science, these two journals are very active and reputed journals indexed by dozens of repositories, encyclopedias, and identifications' websites especially Scopus database.
This manuscript has been organized as follow:

The authors present some basic preliminaries in section 1, while section 2 has been dedicated to submit a new structure of neutrosophic topology called Neutrosophic SuperHyper Topological Spaces, in this section and for the first time, this type of topology was discussed in details. The main core of this article is in section 3 which is contain definitions, theorems, and corollaries covered the new subject that introduced firstly in this paper which is named Neutrosophic SuperHyper Bi-Topological Spaces. The last section is the conclusion section.

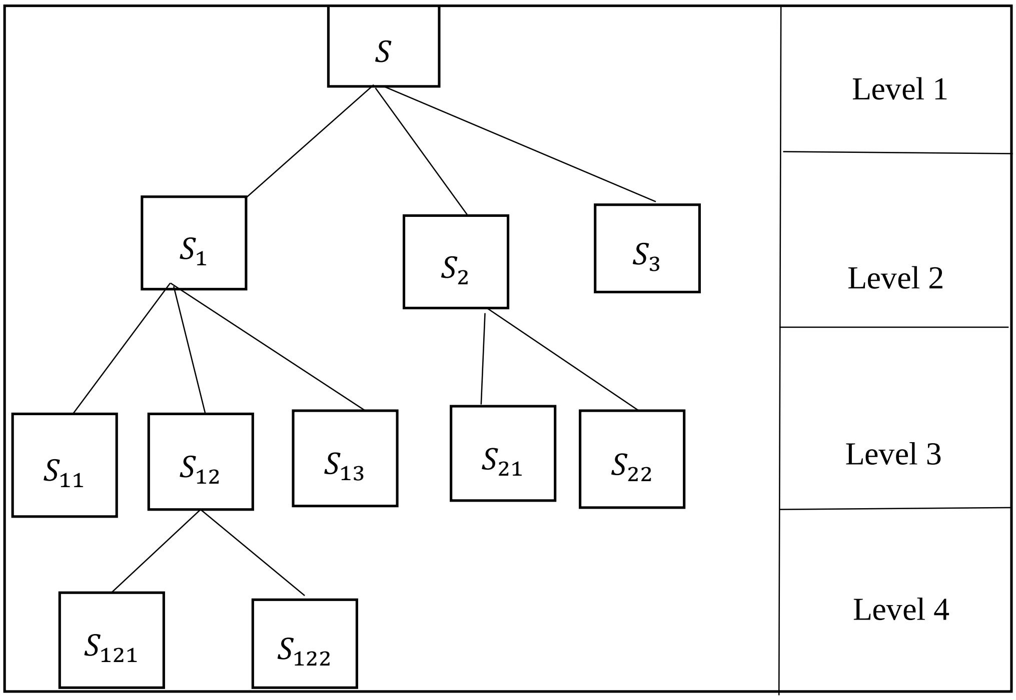
1. Preliminaries

1.1 System of Sub-System of Sub-Sub-System and so on [1]

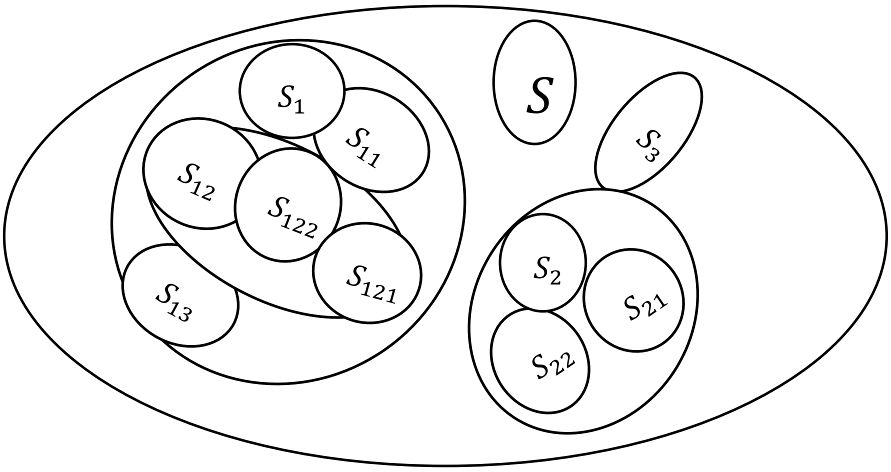
A system may be a set, space, organization, association, team, city, region, country, etc. One consider both: the static and dynamic systems.
With respect to various criteria, such as: political, religious, economic, military, educational, sportive, touristic, industrial, agricultural, etc.
A system S is made up of several sub-systems $S_1, S_2, \dots, S_p$, for integer $p \geq 1$; then each su-system S_i, for $i \in \{1,2,\dots,p\}$ is composed of many sub-sub-systems $S_{i1}, S_{i2}, \dots, S_{ip_i}$, for integer $p_i \geq 1$; then each sub-sub-systems S_{ij}, for $j \in \{1,2,\dots,p_i\}$ is composed sub-sub-sub-systems $S_{ij1}, S_{ij2}, \dots, S_{ijp_j}$, for integers p_j; and so on.

The following example of systems made of Sub-Sub-Sub-Systems (four levels)

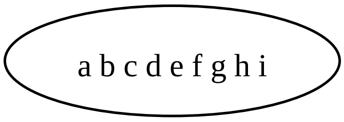
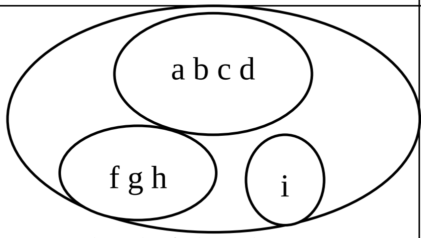
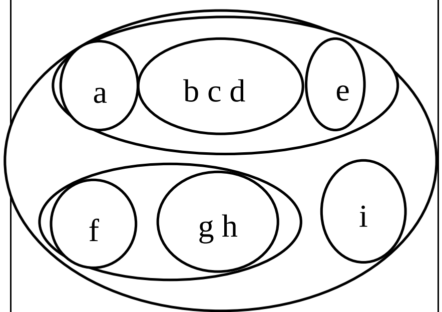
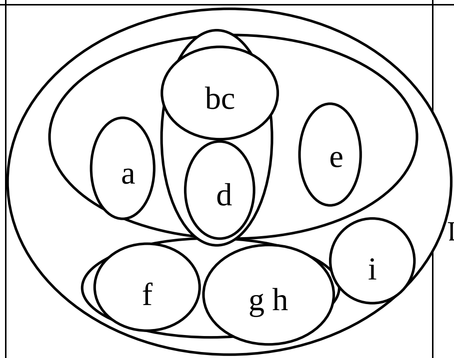
i) Using a Tree-Graph Representation, one has:



ii) Using a Geometric Representation, one has:



iii) Using an Algebraic Representation through pairs of braces {}, one has:

$P^0(S) = S = \{a, b, c, d, e, f, g, h, l\}$ 1 level of pairs of braces		Level 1
$P^1(S) = \{\{a, b, c, d, e\}, \{f, g, h\}, \{l\}\}$ 2 level of pairs of braces i.e. a pair of braces {} inside, another pair of braces {}, or {… {…} ...}	 2 level of closed curves	Level 2
$P^2(S) = P(P(S))$ $= \{\{\{a\}, \{b, c, d\}, \{e\}\}, \{\{f\}, \{g, h\}\}, \{l\}\}$ 3 levels of pairs of braces	 3 level of closed curves	Level 3
$P^3(S) = P(P^2(S))$ $= \{\{\{a\}, \{b, c\}, \{d\}, \{e\}\}, \{\{f\}, \{g, h\}\}, \{l\}\}$ 4 levels of pairs of braces	 4 level of closed curves	Level 4

1.2 Definition of nth -Power of a set [1]:

The n^{th} -Power of a set was firstly introduced by F. Smarandache at (2016) [4] by:
$P^n(S)$ as the n^{th} -PowerSet of the set S, for integer $n \geq 1$, is recursively defined as:

$P^2(S) = P(P(S))$, $P^3(S) = P\left(P(P(S))\right)$, …, $P^n(S) = P(P^{n-1}(S))$, where $P^0(S) = S$, $and\ P^1(S) = P(S)$, i.e. $P^0(S) \subset P^1(S) \subset P^2(S) \subset \cdots \subset P^{n-1}(S) \subset P^n(S)$.

The n^{th} -PowerSet of a Set is better reflect for our complex reality, since a set S (that may represent a group, a society, a country, a continent, etc.) of elements (such as: people, objects, and in general any items) is organized onto subsets $P(S)$, which on their turns are also organized onto subsets of subsets, and so on, that is our world.

1.3 Example

Suppose that the set of the grandparents represents the power set $P^2(S) = P(P(S))$, then the first offspring is the parents themselves which can be regarded as the power set $P(S)$, and the second offspring is the non-empty set $P^0(S) = S$, i.e. $S = P^0(S) \subset P^1(S) \subset P^2(S)$.

The following medical case study would be appropriate to demonstrate the importance of the power set concept:

There are many diseases and conditions that can be passed on through genes. Some of these diseases include Down syndrome, hemophilia, hypertension, sickle cell anemia, and cystic fibrosis. Most genetic diseases are a combination of mutations in multiple genes, often in combination with environmental factors. There are three groups of genetic diseases, each with their own causes: monogenetic diseases, multifactorial inherited diseases, and chromosomal abnormalities.

The couple of husband can be represented as PowerSet $P(S)$, it is important to know what $P(S)$ have inherited a genetic disease from their parents (i.e. represented the non-empty set $P^0(S) = S$ as grandparents) and to remember that the above mentioned genetic diseases can be passed on to their descendants (i.e. the offspring which is mathematically denoted by the power set $P^2(S) = P(P(S))$. If S & $P(S)$ are aware of possible diseases that can be inherited to $P(S)$& $P^2(S)$ respectively , contact a specialist and see what S & $P(S)$ can do to help $P(S)$& $P^2(S)$ and avoid serious problems later. By working together with the help of family and doctor, the health risks can be avoided instead of taking their toll later.

1.4 Neutrosophic HyperOperation and Neutrosophic HyperStructures [2]:

In the classical HyperOperation and classical HyperStructures, the empty-set $\emptyset$ does not belong to the power set, (i.e. $P_*(H) = P(H)/\{\emptyset\}$) . Nonetheless, in the real world we encounter many situations when HyperOperation $\#$ is indeterminate, for example $a \# b = \emptyset$ (unknown, or undefined), or partially indeterminate, for example: $a \# b = \{ [0.2, 0.3], \emptyset\}$. In our everyday life, there are many more operations and lows that have some degrees of indeterminacy (vagueness, unclearness, unknowingness, contradiction, etc.), than those that are totally determined. That's why in 2016 the scientists F. Smarandache have extended the classical HyperOperation to the Neutrosophic HyperOperation, by taking the whole power $P(H)$ (that includes the empty-set $\emptyset$ as well), instead of . $P_*(H)$(that does not include the empty-set $\emptyset$), as follow.

1.4.1 Definition of Neutrosophic HyperOperation:

Let U be a universe of discourse and H be a non-empty set, $H \subset U$.

A Neutrosophic Binary HyperOperation $\#_2$ is defined as follows:

$\#_2: H^2 \to P(H)$, where H is a discrete or continuous set, and $P(H)$ is the powerset of H that includes the empty-set $\emptyset$.

1.4.2 A Neutrosophic m-ary HyperOperation $\#_m$ is defined as:

$\#_m: H^m \to P(H)$, for integer $m \geq 1$. Similarly, for $m = 1$ one gets a Neutrosophic Unary HyperOperation.

2. Neutrosophic SuperHyper Topological Spaces

This section gives an original creativity neutrosophic mathematical structure for new notion named as Neutrosophic SuperHyper Topological Spaces (NSHTS) defined under a new kind of sets called neutrosophic n^{th}-power set $P^n(X)$.

2.1 Definition

Let X be a non-empty set, $P^n(X)$ is the neutrosophic n^{th}-power set of a set X, for integer $n \geq 1$. A Neutrosophic SuperHyper Topological space on $P^n(X)$ is a subfamily $\tau^{neutrotopo}$ of $N(P^n(X))$, and satisfying the following axioms:

1- The neutrosophic universal n^{th}-power set $1_{P^n(X)}$,and the neutrosophic empty n^{th}-power set $0_{P^n(X)}$ both are belonging to $\tau^{neutrotopo}$.

2- Any arbitrary (finite on infinite) union of members of $\tau^{neutrotopo}$ belong to $\tau^{neutrotopo}$.

3- $\tau^{neutrotopo}$ is closed under finite intersection of members of $\tau^{neutrotopo}$ (i.e. the intersection of any finite number of members of $\tau^{neutrotopo}$ belongs to $\tau^{neutrotopo}$).

Then $(\tau^{neutrotopo}, P^n(X))$ is called Neutrosophic SuperHyper Topological Spaces (NSHTS). Because of the definition of (NSHTS) via neutrosophic n^{th}-power open sets that commonly used in this manuscript, the family of neutrosophic sets $\tau^{neutrotopo}$ of the n^{th}-power sets are commonly called a (NSHTS) on the neutrosophic n^{th}-power sets $P^n(X)$.

A subpowerset $P^{m1}(C) \subseteq P^{m2}(X)$ for integers $m1 \leq m2$ is to be closed in $(\tau^{neutrotopo}, P^{n}(X))$ if its complement $P^{m2}(X)/P^{m1}(C)$ is an open set.

2.2 Numerical Example:

What is the difference between $P^1(x)$ & $P^2(x)$ in the structured of the Neutrosophic SuperHyper topological spaces $(\tau^{neutrotopo}, P^{n}(X))$, and how it effects on the distribution of the internal elements? take a look on the following example:

Suppose $X = \{a, b, c\}$ with the following

$$P^1(x) = \left\{ \{a, T = 0.3, I = 0.1, F = 0.6\}, \{b, c\} \begin{matrix} T = \{0.7,0.4\} \\ I = \{0,0.3\} \\ F = \{0.4,0.3\} \end{matrix} \right\},$$

$$P^2(x) = \left\{ \{a, T = 0.3, I = 0.1, F = 0.6\}, \{\{b\}, \{c\}\} \begin{matrix} T = \{\{0.7\}, \{0.4\}\} \\ I = \{\{0\}, \{0.3\}\} \\ F = \{\{0.4\}, \{0.3\}\} \end{matrix} \right\}.$$

For more details, we can see that In $P^1(x)$ the element a affected by its membership functions $\{0.3, 0.1, 0.6\}$ directly, while the element(s) $\{b, c\}$ has (have) two kinds of affected (directed affect) and (indirect affect) as follow:

- The element b has a separate direct affect by its membership functions $\{0.7,0,0.4\}$, and the element c has a separate direct affect by its membership functions $\{0.4,0.3,0.3\}$.
- The structured element $\{b, c\}$ have common indirect affected by their membership functions $\{0.7,0.4\}, \{0,0.3\}, \{0.4,0.3\}$.

This is a very harmonic with the previous example (1.3) stated in section one, by expressing the elements a, b as the parents (husband and wife), each one of them can affected separately by the inherited genes from their parents, also, they will crossing their parents' gene to their offspring mutually and their descendants will be affected directly by their parents and indirectly by their grandparents.

Then $(\tau^{neutrotopo}, P^{n}(X))$ is the Neutrosophic SuperHyper Topological spaces, where:

$$\tau^{neutrotopo} = \{0_{P^n(X)}, 1_{P^n(X)}, P^1(x), P^2(x)\}$$

2.3 Definition

Let $P^n(X)$ be a neutrosophic n^{th}-power set over a non-empty set X, the neutrosophic interior and the neutrosophic closure of $P^n(X)$ are respectively defined as:

$int^n\big(P^n(X)\big) = \cup \{P^m(X): P^m(X) \subseteq P^n(X), P^m(X) \in \tau^{neutrotopo}\}$, this means that for the same collection of the neutrosophic n^{th}-power set $P^n(X)$, all $P^m(X)$ given that $m \leq n$ regarded as interior for $P^n(X)$.

$cl^n\big(P^n(X)\big) = \cap \{P^h(X): P^n(X) \subseteq P^h(X), (P^h(X))^c \in \tau^{neutrotopo}\}$.

2.4 Definition

The following mathematical phrases are true for any two neutrosophic $n1^{th}$-power set $P^{n1}(Y1)$ and $n2^{th}$-power set $P^{n2}(Y2)$ on the neutrosophic n^{th}-power set $P^n(X)$, given that $n1, n2 \leq n$, and that there is no restrictions on the relation between n_1 *and* n_2 :

1- $T_{P^{n1}(Y1)}(\{x\}) \leq T_{P^{n2}(Y2)}(\{x\}), I_{P^{n1}(Y1)}(\{x\}) \leq I_{P^{n2}(Y2)}(\{x\}), and\ F_{P^{n1}(Y1)}(\{x\}) \geq F_{P^{n2}(Y2)}(\{x\})$, for integers $n1, n2 \geq 1$, and for all $\{x\} \subseteq P^n(X)$ iff $P^{n1}(Y1) \subseteq P^{n2}(Y2)$.

2- $P^{n1}(Y1) \subseteq P^{n2}(Y2)$ and $P^{n2}(Y2) \subseteq P^{n1}(Y1)$ iff $P^{n1}(Y1) = P^{n2}(Y2)$, given that $n_1 = n_2$.

3- $P^{n1}(Y1) \cap P^{n2}(Y2) =$
$\{\langle\{x\}, \min\{T_{P^{n1}(Y1)}(\{x\}), T_{P^{n2}(Y2)}(\{x\})\}, \min\{I_{P^{n1}(Y1)}(\{x\}), I_{P^{n2}(Y2)}(\{x\})\}, \max\{F_{P^{n1}(Y1)}(\{x\}), F_{P^{n2}(Y2)}(\{x\})\}\rangle$
$: \{x\} \subseteq P^n(X)\}$

4- $P^{n1}(Y1) \cup P^{n2}(Y2) =$
$\{\langle\{x\}, \max\{T_{P^{n1}(Y1)}(\{x\}), T_{P^{n2}(Y2)}(\{x\})\}, \max\{I_{P^{n1}(Y1)}(\{x\}), I_{P^{n2}(Y2)}(\{x\})\}, \min\{F_{P^{n1}(Y1)}(\{x\}), F_{P^{n2}(Y2)}(\{x\})\}\rangle$
$: \{x\} \subseteq P^n(X)\}$

In general, the union or the intersection of any arbitrary members of neutrosophic n^{th}-power set $P^{ni}(X)_{i\in I}$ are defined by:

$\bigcap_{i \in I} P^{ni}(X) = \{\langle\{x\}, \inf\{T_{P^{ni}(\{x\})}\}, \inf\{I_{P^{ni}(\{x\})}\}, \sup\{F_{P^{ni}(\{x\})}\}\rangle : \{x\} \subseteq P^n(X)\}$,

$\bigcup_{i \in I} P^{ni}(X) = \{\langle\{x\}, \sup\{T_{P^{ni}(\{x\})}\}, \sup\{I_{P^{ni}(\{x\})}\}, \inf\{F_{P^{ni}(\{x\})}\}\rangle : \{x\} \subseteq P^n(X)\}$.

5- The neutrosophic n^{th}-power universal set $P^n(X)$ is denoted by $1_{P^n(X)}$,and it is exist if and only if the following conditions are holding together:
$T_{P^n(\{x\})} = 1_{P^n(X)}, I_{P^n(\{x\})} = 1_{P^n(X)}, and\ F_{P^n(\{x\})} = 0_{P^n(X)}$.

6- The neutrosophic n^{th}-power empty set $P^n(X)$ is denoted by $0_{P^n(X)}$, and it is exist if and only if the following conditions are holding together:
$T_{P^n(\{x\})} = 0_{P^n(X)}, I_{P^n(\{x\})} = 0_{P^n(X)}, and\ F_{P^n(\{x\})} = 1_{P^n(X)}$.

7- Let $P^{n1}(Y1) \subseteq P^{n2}(Y2)$, given that $n_1 \leq n_2$, then the complementary of $P^{n1}(Y1)$ concerning to $P^{n2}(Y2)$ is defined as follow:
$P^{n1}(Y1) \setminus P^{n2}(Y2) = \{\langle |T_{P^{n1}(Y1)}(\{x\}) - T_{P^{n2}(Y2)}(\{x\})|, |I_{P^{n1}(Y1)}(\{x\}) - I_{P^{n2}(Y2)}(\{x\})|, 1_{P^n(X)} - |F_{P^{n1}(Y1)}(\{x\}) - F_{P^{n2}(Y2)}(\{x\})|\rangle\}$.

8- Clearly, the neutrosophic complement of $1_{P^n(X)}$ and $0_{P^n(X)}$ are defined as:

$(1_{P^n(X)})^c = \langle T_{P^n(\{x\})} = 0_{P^n(X)}, I_{P^n(\{x\})} = 0_{P^n(X)}, F_{P^n(\{x\})} = 1_{P^n(X)}\rangle = 0_{P^n(X)}$,
$(0_{P^n(X)})^c = \langle T_{P^n(\{x\})} = 1_{P^n(X)}, I_{P^n(\{x\})} = 1_{P^n(X)}, F_{P^n(\{x\})} = 0_{P^n(X)}\rangle = 1_{P^n(X)}$.

2.5 Proposition

Let $P^{n1}(X), P^{n2}(X), P^{n3}(X), and\ P^{n4}(X) \subseteq N(P^{n}(X))$ without any restrictions on the relations between $n_1, n_2, n_3, n_4, and\ n$, then the following mathematical statements are true:

i) Let $P^{n1}(X) \subseteq P^{n2}(X), and\ P^{n3}(X) \subseteq P^{n4}(X)$, given that $n_1 \leq n_2$, & $n_3 \leq n_4$, this implies that $P^{n1}(X) \cap P^{n3}(X) \subseteq P^{n2}(X) \cap P^{n4}(X)$,

ii) $(P^{n1}(X)^c)^c = P^{n1}(X)$, also if $P^{n2}(X)^c \subseteq P^{n1}(X)^c \Rightarrow P^{n1}(X) \subseteq P^{n2}(X)$,

iii) $(P^{n1}(X) \cap P^{n2}(X))^c = P^{n1}(X)^c \cup P^{n2}(X)^c$,

iv) $(P^{n1}(X) \cup P^{n2}(X))^c = P^{n1}(X)^c \cap P^{n2}(X)^c$.

2.6 Definition

Let X be a non-empty set, $P^n(X)$ is the n^{th}-power neutrosophic set of a set X, for integer $n \geq 1$. If α, β, γ be real standard or non-standard subsets of $]^-0, 1^+[$, then the neutrosophic n^{th}-power set $P^n(x_{\alpha,\beta,\gamma})$ is called a neutrosophic n^{th}- power point, and it is defined by:

$$P^n\left(x_{\alpha,\beta,\gamma}(y)\right) = \begin{cases} \langle \alpha_{P^n(x)}, \beta_{P^n(x)}, \gamma_{P^n(x)} \rangle, & if\ P^n(x) = P^n(y) \\ \langle 0_{P^n(x)}, 0_{P^n(x)}, 1_{P^n(x)} \rangle, & if\ P^n(x) \neq P^n(y) \end{cases}$$

For $x, y \in X, and\ P^n\left(x_{\alpha,\beta,\gamma}\right), P^n(y) \subseteq P^n(X)$, here $P^n(y)$ is called the support of $P^n(x_{\alpha,\beta,\gamma})$.

2.7 Definition

Let $P^{n1}(X) \in N(P^n(X))$, the belonging operation of the neutrosophic n^{th}- power point $P^n(x_{\alpha,\beta,\gamma})$ to $P^{n1}(X)$ (i.e. $P^n(x_{\alpha,\beta,\gamma}) \in P^{n1}(X)$) is satisfied if and only if $T_{P^{n1}(\{x\})} \geq \alpha, I_{P^{n1}(\{x\})} \geq \beta, F_{P^{n1}(\{x\})} \leq \gamma$.

2.8 Definition

A sub-collection τ_n^* of neutrosophic n^{th}- power set $P^n(X)$ on a non-empty set X is said to be Neutrosophic SuperHyper Supra Topological Space on X if the n^{th}- power sets $0_{P^n(X)}, 1_{P^n(X)} \in \tau_n^*$, and $\underset{i \in I}{\overset{\infty}{\cup}} P^{ni}(X) \in \tau_n^*$ for $\{P^{ni}(X)\}_{i=1}^{\infty} \in \tau_n^*$. Then $(\tau_n^*, P^n(X))$ is called Neutrosophic SuperHyper Supra Topological Space on X.

3 Neutrosophic SuperHyper Bi-Topological Spaces

This section contains new concepts presents for the first time linking the concept of the neutrosophic n^{th}- power sets with the traditional neutrosophic bi-topological spaces.

3.3 Definition

Let $(\tau_1{}^{1stpair}, P^n(X)), and\ (\tau_2{}^{2ndpair}, P^n(X))$ be two different Neutrosophic SuperHyper topological spaces on X. Then $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ is called Neutrosophic SuperHyper Bi-Topological space (NSHBI-TS).

3.4 Definition

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be a (NSHBI-TS). A collection of a neutrosophic n^{th}- power set $N = \{\langle\{x\}: T_{P^n(\{x\})}, I_{P^n(\{x\})}, F_{P^n(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ over $P^n(X)$ is said to be a pairwise neutrosophic n^{th}- power open set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ if there exist a neutrosophic n^{th}- power open set $N_1 = \{\langle\{x\}: T_{P^{n1}(\{x\})}, I_{P^{n1}(\{x\})}, F_{P^{n1}(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ in $\tau_1{}^{1stpair}$ and a neutrosophic n^{th}- power open set $N_2 = \{\langle\{x\}: T_{P^{n2}(\{x\})}, I_{P^{n2}(\{x\})}, F_{P^{n2}(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ in $\tau_2{}^{2ndpair}$ such that $N = N_1 \cup N_2 = \{\langle\{x\},\ T_{P^n(\{x\})} = \max\{T_{P^{n1}(\{x\})}, T_{P^{n2}(\{x\})}\},\ I_{P^n(\{x\})} = \max\{I_{P^{n1}(\{x\})}, I_{P^{n2}(\{x\})}\}, F_{P^n(\{x\})} = \min\{F_{P^{n1}(\{x\})}, F_{P^{n2}(\{x\})}\}\rangle : \{x\} \subseteq P^n(X)\}$.

3.5 Definition

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be a (SHNBI-TS). A collection of a neutrosophic n^{th}- power set $C = \{\langle\{x\}: T_{P^c(\{x\})}, I_{P^c(\{x\})}, F_{P^c(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ over $P^n(X)$ is said to be a pairwise neutrosophic n^{th}- power closed set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ if its neutrosophic complement is a pairwise neutrosophic n^{th}- power open set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$. Clearly, a neutrosophic n^{th}- power set C over $P^n(X)$ is a pairwise neutrosophic n^{th}- power closed set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ if there exist a neutrosophic n^{th}- power closed set $C_1 = \{\langle\{x\}: T_{P^{c1}(\{x\})}, I_{P^{c1}(\{x\})}, F_{P^{c1}(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ in $(\tau_1{}^{1stpair})^c$,and a neutrosophic n^{th}- power closed set $C_2 = \{\langle\{x\}: T_{P^{c2}(\{x\})}, I_{P^{c2}(\{x\})}, F_{P^{c2}(\{x\})}\rangle : \{x\} \subseteq P^n(X)\}$ in $(\tau_2{}^{2ndpair})^c$ such that $C = C_1 \cap C_2 = \{\langle\{x\},\ T_{P^c(\{x\})} = \min\{T_{P^{c1}(\{x\})}, T_{P^{c2}(\{x\})}\},\ I_{P^c(\{x\})} = \min\{I_{P^{c1}(\{x\})}, I_{P^{c2}(\{x\})}\},\ F_{P^c(\{x\})} = \max\{F_{P^{c1}(\{x\})}, F_{P^{c2}(\{x\})}\}\rangle : \{x\} \subseteq P^n(X)\}$. Where $(\tau_i{}^{ipair})^c = \{(N)^c \subseteq N(P^n(X)) : N \subseteq \tau_i{}^{ipair}, i = 1st, 2nd$. The family of all pairwise neutrosophic n^{th}- power open/closed sets in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ is denoted by PN n^{th} POS in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ / PNn^{th}PCS in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$, respectively.

3.6 Theorem

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. Then,

1. $0_{P^n(x)}, and\ 1_{P^n(x)}$ are pairwise neutrosophic n^{th}- power open/closed sets.
2. An arbitrary neutrosophic union of pairwise neutrosophic n^{th}- power open sets is a pairwise neutrosophic n^{th}- power open set.
3. An arbitrary neutrosophic intersection of pairwise neutrosophic n^{th}- power closed sets is a pairwise neutrosophic n^{th}- power closed set.

Proof:

1. Let $0_{P^{n1}(x)}$, $0_{P^{n2}(x)} \subseteq$ ${\tau_1}^{1stpair}, {\tau_2}^{2ndpair}$ respectively, and $n_1 + n_2 = n$, since $0_{P^{n1}(x)} \cup 0_{P^{n2}(x)} = 0_{P^n(x)}$, hence $0_{P^n(x)}$ is a PN n^{th} POS in $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$. Similarly, $1_{P^n(x)}$ is a PN n^{th} PCS in$({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$.
2. Suppose $\{N_i = \langle \{x\}: T_{P^{ni}(\{x\})}, I_{P^{ni}(\{x\})}, F_{P^{ni}(\{x\})} \rangle : i \in I\} \subseteq$ PNn^{th}POS in $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$. Then each N_i is a pairwise neutrosophic n^{th} power open set for all $i \in I$, this implies that there exist $N_i^1 \in {\tau_1}^{1stpair}$ and $N_i^2 \in {\tau_2}^{2ndtpair}$ such that $N_i = N_i^1 \cup N_i^2$ for all $i \in I$ which implies that .

$$\underset{i \in I}{\cup N_i} = \underset{i \in I}{\cup} [N_i^1 \cup N_i^2] = \begin{bmatrix} \cup N_i^1 \\ i \in I \end{bmatrix} \cup \begin{bmatrix} \cup N_i^2 \\ i \in I \end{bmatrix}$$

Now, since ${\tau_1}^{1stpair}$ and ${\tau_2}^{2ndpair}$ are both Neutrosophic SuperHyper Topological Spaces on the neutrosophic n^{th} power set $P^n(X)$, then $\begin{bmatrix} \cup N_i^1 \\ i \in I \end{bmatrix} \subseteq {\tau_1}^{1stpair}$, and $\begin{bmatrix} \cup N_i^2 \\ i \in I \end{bmatrix} \subseteq {\tau_2}^{2ndpair}$. Therefore, $\underset{i \in I}{\cup N_i}$ is a pairwise neutrosophic n^{th}- power open set.

3. It is immediate from the definition (3.3).

3.7 Corollary

Let $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. Then, the family of all pairwise neutrosophic n^{th}- power open sets is a Neutrosophic SuperHyper Supra Topological Space (NSHSTS) on X. This (NSHSTS) is denoted by τ_{12}^{supra}.

3.8 Theorem

Let $({\tau_1}^{1stpair}, {\tau_2}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. Then,

1. Every ${\tau_1}^{1stpair}, {\tau_2}^{2ndpair}$- neutrosophic n^{th}- power open set is a pairwise neutrosophic n^{th}- power open set, i.e. ${\tau_1}^{1stpair} \cup {\tau_2}^{2ndpair} \subseteq \tau_{12}^{supra}$.
2. Every ${\tau_1}^{1stpair}, {\tau_2}^{2ndpair}$- neutrosophic n^{th}- power closed set is a pairwise neutrosophic n^{th}- power closed set, i.e. $({\tau_1}^{1stpair})^c \cup ({\tau_2}^{2ndpair})^c \subseteq (\tau_{12}^{supra})^c$.
3. If ${\tau_1}^{1stpair} \subseteq {\tau_2}^{2ndpair}$, then $\tau_{12}^{supra} = {\tau_2}^{2ndpair}$ and $(\tau_{12}^{supra})^c = ({\tau_2}^{2ndpair})^c$.

Proof. Straightforward.

3.9 Definition

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space, and $P^n(X) \in N(P^n(X))$.The pairwise neutrosophic closure of $P^n(X)$, denoted by $cl_p^n(P^n(X))$, is the neutrosophic intersection of all pairwise neutrosophic closed supra n^{th}- power sets of $P^n(X)$, i.e., $cl_p^n(P^n(X)) = \cap \{P^m(X) \in (\tau_{12}^{supra})^c : P^n(X) \subseteq P^m(X)\}$. It is clear that $cl_p^n(P^n(X))$ is the smallest pairwise neutrosophic n^{th}- power closed set containing $P^n(X)$.

3.10 Theorem

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space, and $P^{n1}(X), P^{n2}(X) \in N(P^n(X))$, without restrictions on the relations between $n1, n2, n$. Then, the following mathematical statements are true:

1. $cl_p^n(0_{P^{ni}(x)}) = 0_{P^{ni}(x)}$, and $cl_p^n(1_{P^{ni}(x)}) = 1_{P^{ni}(x)}$, $i = 1,2$.
2. $P^n(X) \subseteq cl_p^n(P^n(X))$.
3. $P^n(X)$ is a pairwise neutrosophic n^{th} - power closed set if and only if $cl_p^n(P^n(X)) = P^n(X)$.
4. $P^{n1}(X) \subseteq P^{n2}(X) \Longrightarrow cl_p^n(P^{n1}(X)) \subseteq cl_p^n(P^{n2}(X))$.
5. $cl_p^n(P^{n1}(X)) \cup cl_p^n(P^{n2}(X)) \subseteq cl_p^n(P^{n1}(X) \cup P^{n2}(X))$.
6. $cl_p^n\ [\ cl_p^n(P^{ni}(X)] = cl_p^n(P^{ni}(X)$, i.e., $cl_p^n(P^{ni}(X)$ is a pairwise neutrosophic n^{th}- power closed set.

Proof. Straightforward.

3.11 Theorem

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space, and $P^{n1}(X) \in N(P^n(X))$. Then,

$$P^{n1}(x_{\alpha,\beta,\gamma}) \in cl_p^n(P^{n1}(X)) \Leftrightarrow U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \cap P^{n1}(X) \neq 0_{P^{n1}(x)},\ \forall\, U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma})),$$

Where $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right)$ is any pairwise neutrosophic n^{th}- power open set contains $P^{n1}(x_{\alpha,\beta,\gamma})$, and $\tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$ is the family of all pairwise neutrosophic supra n^{th}- power open set contains $P^{n1}(x_{\alpha,\beta,\gamma})$.

Proof:

Let $P^{n1}(x_{\alpha,\beta,\gamma}) \in cl_p^n(P^{n1}(X))$, and suppose that there exist $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$, such that $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \cap P^{n1}(X) = 0_{P^{n1}(x)}$. Then $P^{n1}(X) \subseteq (\ U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right))^c$, thus $cl_p^n(P^{n1}(X)) \subseteq$

$cl_p^n\left(U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right)\right)^c = (U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right))^c$ which implies that $cl_p^n(P^{n1}(X)) \cap U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) = 0_{P^{n1}(x)}$, this is a contradiction. hence $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \cap P^{n1}(X) \neq 0_{P^{n1}(x)}$.

Conversely, assume that $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl_p^n(P^{n1}(X))$, then $P^{n1}(x_{\alpha,\beta,\gamma}) \in (cl_p^n(P^{n1}(X)))^c$. Thus, $(cl_p^n(P^{n1}(X)))^c \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$, therefore, by hypothesis, $(cl_p^n(P^{n1}(X)))^c \cap P^{n1}(X) \neq 0_{P^{n1}(x)}$, this is a contradiction. Hence we get $P^{n1}(x_{\alpha,\beta,\gamma}) \in cl_p^n(P^{n1}(X))$.

3.12 Theorem

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. A neutrosophic n^{th}- power set $P^{n1}(X)$ over $P^n(X)$ is a pairwise neutrosophic n^{th}- power closed set if and only if $P^{n1}(X) = cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X)) \cap cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X))$.

Proof:

Suppose that $P^{n1}(X)$ is a pairwise neutrosophic n^{th}-power closed set and $P^{n1}(x_{\alpha,\beta,\gamma}) \notin P^{n1}(X)$. Then $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl_p^n(\,P^{n1}(X))$. Thus, by theorem (3.9), there exists $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$ such that $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \cap P^{n1}(X) = 0_{P^{n1}(x)}$. Again, since $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) \in \tau_{12}^{supra}(P^{n1}(x_{\alpha,\beta,\gamma}))$, then there exists $P^{m1}(X) \in \tau_1{}^{1stpair}$ and $P^{m2}(X) \in \tau_2{}^{2ndpair}$ such that $U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right) = P^{m1}(X) \cup P^{m2}(X)$. consequently, $(\,P^{m1}(X) \cup P^{m2}(X)) \cap P^{n1}(X) = 0_{P^{n1}(x)}$, this implies that $P^{m1}(X) \cap P^{n1}(X) = 0_{P^{n1}(x)}$, and $P^{m2}(X) \cap P^{n1}(X) = 0_{P^{n1}(x)}$. Since $P^{n1}(x_{\alpha,\beta,\gamma}) \in U\left(P^{n1}(x_{\alpha,\beta,\gamma})\right)$, then either $P^{n1}(x_{\alpha,\beta,\gamma}) \in P^{m1}(X)$ or $P^{n1}(x_{\alpha,\beta,\gamma}) \in P^{m2}(X)$, this implies that either $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X))$ or $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X))$. Therefore, $P^{n1}(x_{\alpha,\beta,\gamma}) \notin cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X)) \cap cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X))$. Thus, $cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X)) \cap cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X)) \subseteq P^{n1}(X)$. On the other hand, we have $P^{n1}(X) \subseteq cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X)) \cap cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X))$. Hence, $P^{n1}(X) = cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X)) \cap cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X))$.

Conversely, suppose that $P^{n1}(X) = cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X)) \cap cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X))$. Since, $cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X))$ is a neutrosophic n^{th} -power closed set in $(\tau_1{}^{1stpair}, P^n(X))$, and $cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X))$ is a neutrosophic n^{th}-power closed set in $(\tau_2{}^{2ndpair}, P^n(X))$, so, by definition (3.3), $cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X)) \cap cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X))$ is a pairwise neutrosophic n^{th}- power closed set in $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$, consequently, $P^{n1}(X)$ is a pairwise neutrosophic n^{th}- power closed set.

3.13 Corollary

Let $(\tau_1{}^{1stpair}, \tau_2{}^{2ndpair}, P^n(X))$ be Neutrosophic SuperHyper Bi-Topological space. Then, $cl_p^n(P^n(X)) = cl^n_{\tau_1{}^{1stpair}}(P^{n1}(X)) \cap cl^n_{\tau_2{}^{2ndpair}}(P^{n1}(X)), \forall\, P^{n1}(X) \in N(P^n(X))$.

4 Conclusion

The types of the topological spaces in neutrosophic theory are always changed depending upon the structure of the sets, in this article, the Neutrosophic SuperHyper Topological Spaces has been fathomed especially Neutrosophic SuperHyper Bi-Topological Spaces. The definitions of the neutrosophic interior and the neutrosophic closure of $P^n(X)$ have been presented. Also, the neutrosophic universal n^{th}-power set $P^n(X)$ and the neutrosophic empty n^{th}-power set $P^n(X)$ were discussed. The union and the intersection operations have been defined. As well as, the authors presented pairwise neutrosophic n^{th}-power open set, pairwise neutrosophic closed n^{th} - power set, many of theorems, propositions and examples to support the new notion.

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