NSS

Neutrosophic Sets and Systems, Vol. 51, 2022

University of New Mexico

MCGDM based on TOPSIS and VIKOR using Pythagorean neutrosophic soft with aggregation operators

M.Palanikumar[1,*] and K. Arulmozhi[2]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India; palanimaths86@gmail.com

[2] Department of Mathematics, Bharath Institute of Higher and Research, Chennai-600073, India; arulmozhiems@gmail.com

*Correspondence: palanimaths86@gmail.com;

Abstract. Pythagorean neutrosophic soft set (PNSS set) is a new approach towards decision making under uncertainty. The PNSS set has much stronger abilities than the neutrosophic soft set and the Pythagorean fuzzy soft set. In this paper, we discuss aggregated operations for aggregating the PNSS decision matrix. The TOPSIS and VIKOR methods are strong approaches for multi criteria group decision making (MCGDM), which is various extensions of neutrosophic soft sets. In this approach, we propose a score function based on aggregating TOPSIS and VIKOR methods to the PNSS-positive ideal solution and the PNSS-negative ideal solution. Also, the TOPSIS and VIKOR methods provide the weights of decision-making. Afterward, a revised closeness is introduced to identify the optimal alternative.

Keywords: Pythagorean neutrosophic soft set, MCGDM, TOPSIS, VIKOR, aggregation operator.

---

1. **Introduction**

The classic article of 1965, Zadeh proposed fuzzy set theory [39]. According to this definition a fuzzy set is a function described by a membership value . It takes degrees in real unit interval. But, later it has been seen that this definition is inadequate by considering not only the degree of membership but also the degree of non-membership. Neutrosophic set is a generalization of the fuzzy set and intuitionistic fuzzy set, where the truth-membership, indeterminacy-membership, and falsity-membership are represented independently. Atanassov [3] described a set that is called an intuitionistic fuzzy set to handle mentioned ambiguity. Since this set has some problems in applications, Smarandache [31] introduced neutrosophy to deal with the problems that involves indeterminate and inconsistent information. Yager [38] as being introduced by the concept of Pythagorean fuzzy sets. It has been extended from intuitionistic fuzzy sets and is distinguished by the requirement that the square sum of their degrees of membership and non-membership not exceed unity. A neutrosophic set is used to tackle uncertainty using the truth, indeterminacy, and falsity membership grades by Smarandache [30]. The theory of soft sets was proposed by [15]. Maji et al. proposed the concepts of the fuzzy soft set [13] and the intuitionistic fuzzy soft set [14]. These two theories are applied to solve various decision making problems. In recent years, Peng et al. [29]have extended the fuzzy soft set to the Pythagorean fuzzy soft set. Smarandache et al. [5, 10] discussed the concept of Pythagorean neutrosophic set approach. A decision-making (DM) problem is the process of finding the best optional alternatives. In almost all such problems, the multiplicity of criteria for judging the alternatives is pervasive. That is, for many such problems, the decision maker wants to solve a multiple criteria decision making (MCDM) problem. A survey of the MCDM methods has been presented by Hwang and Yoon [7]. A MCDM problem can be expressed in matrix format as:

$$\mathscr{D}_{n\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_m \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \end{array}$$

where $A_1, A_2, ..., A_n$ are possible alternatives among which decision makers must choose, $\mathcal{C}_1, \mathcal{C}_2, ..., C_m$ are criteria with which alternative performance is measured, a_{ij} is the rating of alternative A_i in relation to criterion C_j.

Many researchers have studied the TOPSIS and VIKOR methods for decision making problems, including Adeel et al. [1], Akram and Arshad [2], Boran et al. [4], Eraslan and Karaaslan [6], Peng and Dai [28], Xu and Zhang [36] and Zhang and Xu [40]. In 2021, Zulqarnain discussed the TOPSIS technique as it applies to interval valued intuitionistic fuzzy soft sets (IVIFSS) information, where the mechanisms are assumed in terms of IVIFSNs. To measure the degree of dependency of IVIFSS's, [41] discussed a new correlation coefficient for IVIFSS's and examined some properties of the developed correlation coefficient. To achieve the goal accurately, the TOPSIS technique may be extended to solve MADM problems. The basic idea of TOPSIS is rather straightforward. It simultaneously considers the distances to both positive ideal solutions (PIS) and negative ideal solutions (NIS), and a preference order is ranked according to their relative closeness and a combination of these two distance measures. The VIKOR method focuses on ranking and selecting from a set of alternatives, and determining compromise solutions for a problem with conflicting criteria, which can help the decision makers reach a final decision [16,17]. Opricovic and Tzeng [18] suggested using fuzzy logic for the VIKOR method. Tzeng et al. [33] used and compared the VIKOR and TOPSIS methods in solving a public transportation problem. Newly, Pythagorean fuzzy logical with real life applications discussed many authors [8,9,32,34,35,37]. Recently, Palanikumar et al. discussed various field of applications including algebraic structures [11,12,19–27].

2. Preliminaries

Definition 2.1. [5] Let $\mathbb{U}$ be a non-empty set of the universe. A neutrosophic set A in $\mathbb{U}$ is an object having the following form : $A = \{u, \sigma_A^{\mathcal{T}}(u), \sigma_A^{\mathcal{I}}(u), \sigma_A^{\mathcal{F}}(u) | u \in \mathbb{U}\}$, where $\sigma_A^{\mathcal{T}}(u)$, $\sigma_A^{\mathcal{I}}(u)$ $\sigma_A^{\mathcal{F}}(u)$ represents the degree of truth membership, degree of indeterminacy membership and degree of falsity membership of A respectively. The mapping $\sigma_A^{\mathcal{T}}, \sigma_A^{\mathcal{I}}, \sigma_A^{\mathcal{F}} : \mathbb{U} \to [0,1]$ and $0 \leq \sigma_A^{\mathcal{T}}(u) + \sigma_A^{\mathcal{I}}(u) + \sigma_A^{\mathcal{F}}(u) \leq 3$.

Definition 2.2. [10] Let $\mathbb{U}$ be a non-empty set of the universe, Pythagorean neutrosophic set (PNSS) A in $\mathbb{U}$ is an object having the following form : $A = \{u, \sigma_A^{\mathcal{T}}(u), \sigma_A^{\mathcal{I}}(u), \sigma_A^{\mathcal{F}}(u) | u \in \mathbb{U}\}$, where $\sigma_A^{\mathcal{T}}(u)$, $\sigma_A^{\mathcal{I}}(u)$ $\sigma_A^{\mathcal{F}}(u)$ represents the degree of truth membership, degree of indeterminacy membership and degree of falsity membership of A respectively. The mapping $\sigma_A^{\mathcal{T}}, \sigma_A^{\mathcal{I}}, \sigma_A^{\mathcal{F}} :$ $\mathbb{U} \to [0,1]$ and $0 \leq (\sigma_A^{\mathcal{T}}(u))^2 + (\sigma_A^{\mathcal{I}}(u))^2 + (\sigma_A^{\mathcal{F}}(u))^2 \leq 2$. Since $A = (\sigma_A^{\mathcal{T}}, \sigma_A^{\mathcal{I}}, \sigma_A^{\mathcal{F}})$ is called a Pythagorean neutrosophic number(PNSN).

Definition 2.3. The score function for any PNSN $A = (\sigma_A^{\mathcal{T}}, \sigma_A^{\mathcal{I}}, \sigma_A^{\mathcal{F}})$ is defined as $S(A) = \sigma_A^{2T} - \sigma_A^{2I} - \sigma_A^{2F}$, where $-1 \leq S(A) \leq 1$.

3. MCGDM based on PNSS sets

Definition 3.1. Let $\mathbb{U}$ be a non-empty set of the universe and E be a set of parameter. The pair (Δ, A) or Δ_A is called a Pythagorean neutrosophic soft set (PNSS set) on $\mathbb{U}$ if $A \sqsubseteq E$ and $\Delta : A \to PNS^{\mathbb{U}}$, where $PNS^{\mathbb{U}}$ is represent the aggregate of all Pythagorean neutrosophic subsets of $\mathbb{U}$. (ie) $\Delta_A = \left\{ \left(e, \left\{ \frac{u}{\left(\sigma_{\Delta_A}^{\mathcal{T}}(u), \sigma_{\Delta_A}^{\mathcal{I}}(u), \sigma_{\Delta_A}^{\mathcal{F}}(u)\right)} \right\} \right) : e \in A, u \in \mathbb{U} \right\}$.

Remark 3.2. If we write $a_{ij} = \sigma_{\Delta_A}^{\mathcal{T}}(e_j)(u_i)$, $b_{ij} = \sigma_{\Delta_A}^{\mathcal{I}}(e_j)(u_i)$ and $c_{ij} = \sigma_{\Delta_A}^{\mathcal{F}}(e_j)(u_i)$, where $i = 1,2,...,m$ and $j = 1,2,...,n$ then the PNSS set Δ_A may be represented in matrix form as

$$\Delta_A = [(a_{ij}, b_{ij}, c_{ij})]_{m \times n} = \begin{bmatrix} (a_{11}, b_{11}, c_{11}) & (a_{12}, b_{12}, c_{12}) & \ldots & (a_{1n}, b_{1n}, c_{1n}) \\ (a_{21}, b_{21}, c_{21}) & (a_{22}, b_{22}, c_{22}) & \ldots & (a_{2n}, b_{2n}, c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{m1}, b_{m1}, c_{m1}) & (a_{m2}, b_{m2}, c_{m2}) & \ldots & (a_{mn}, b_{mn}, c_{mn}) \end{bmatrix}$$

This matrix is called Pythagorean neutrosophic soft matrix (PNSSM).

Definition 3.3. The cardinal set of the PNSS set Δ_A over $\mathbb{U}$ is a PNSS set over E and is defined as $c\Delta_A = \left\{ \frac{e}{\left(\sigma^{\mathcal{T}}_{c\delta_A}(e), \sigma^{\mathcal{I}}_{c\zeta_A}(e), \sigma^{\mathcal{F}}_{c\varphi_A}(e)\right)} : e \in E \right\}$, where $\sigma^{\mathcal{T}}_{c\delta_A}$, $\sigma^{\mathcal{I}}_{c\zeta_A}$ and $\sigma^{\mathcal{F}}_{c\varphi_A}$ are mapping from E to unit interval respectively, where $\sigma^{\mathcal{T}}_{c\delta_A}(e) = \frac{|\delta_A(e)|}{|\mathbb{U}|}$, $\sigma^{\mathcal{I}}_{c\zeta_A}(e) = \frac{|\zeta_A(e)|}{|\mathbb{U}|}$ and $\sigma^{\mathcal{F}}_{c\varphi_A}(e) = \frac{|\varphi_A(e)|}{|\mathbb{U}|}$ where $|\delta_A(e)|$, $|\zeta_A(e)|$ and $|\varphi_A(e)|$ denote the scalar cardinalities of the PNSS sets $\delta_A(e)$, $\zeta_A(e)$ and $\varphi_A(e)$respectively, and $|\mathbb{U}|$ represents cardinality of the universe $\mathbb{U}$. The collection of all cardinal sets of PNSS sets of $\mathbb{U}$ is represented as $cPNS^{\mathbb{U}}$. If $A \subseteq E = \{e_i : i = 1, 2, ..., n\}$, then $c\Delta_A \in cPNS^{\mathbb{U}}$ may be represented in matrix form as $[(a_{1j}, b_{1j}, c_{1j})]_{1\times n} = [(a_{11}, b_{11}, c_{11}), (a_{12}, b_{12}, c_{12}), ..., (a_{1n}, b_{1n}, c_{1n})]$, where $(a_{1j}, b_{1j}, c_{1j}) = \mu_{c\Delta_A}(e_j)$, for $j = 1, 2, ..., n$. This matrix is termed as cardinal matrix of $c\Delta_A$ of E.

Definition 3.4. Let $\Delta_A \in PNS^{\mathbb{U}}$ and $c\Delta_A \in cPNS^{\mathbb{U}}$. The PNSS set aggregation operator $PNSS_{agg} : cPNS^{\mathbb{U}} \times PNS^{\mathbb{U}} \to PNSS(\mathbb{U}, E)$ is defined as
$PNSS_{agg}(c\Delta_A, \Delta_A) = \left\{ \frac{u}{\mu_{\Delta^*_A}(u)} : u \in \mathbb{U} \right\} = \left\{ \frac{u}{\left(\sigma^{\mathcal{T}}_{\delta^*_A}(u), \sigma^{\mathcal{I}}_{\zeta^*_A}(u), \sigma^{\mathcal{F}}_{\varphi^*_A}(u)\right)} : u \in \mathbb{U} \right\}$. This collection is called aggregate Pythagorean neutrosophic set of PNSS set Δ_A. The degree of truth membership $\sigma^{\mathcal{T}}_{\delta^*_A}(u) : \mathbb{U} \to [0, 1]$ by $\sigma^{\mathcal{T}}_{\delta^*_A}(u) = \frac{1}{|E|} \sum_{e\in E} \left(\sigma^{\mathcal{T}}_{c\delta_A}(e), \sigma^{\mathcal{T}}_{\delta_A}(e)\right)(u)$, degree of indeterminacy membership $\sigma^{\mathcal{I}}_{\zeta^*_A}(u) : \mathbb{U} \to [0, 1]$ by $\sigma^{\mathcal{I}}_{\zeta^*_A}(u) = \frac{1}{|E|} \sum_{e\in E} \left(\sigma^{\mathcal{I}}_{c\zeta_A}(e), \sigma^{\mathcal{I}}_{\zeta_A}(e)\right)(u)$ and degree of falsity membership $\sigma^{\mathcal{F}}_{\varphi^*_A}(u) : \mathbb{U} \to [0, 1]$ by $\sigma^{\mathcal{F}}_{\varphi^*_A}(u) = \frac{1}{|E|} \sum_{e\in E} \left(\sigma^{\mathcal{F}}_{c\varphi_A}(e), \sigma^{\mathcal{F}}_{\varphi_A}(e)\right)(u)$. The set $PNSS_{agg}(c\Delta_A, \Delta_A)$ is expressed in matrix form as

$$[(a_{i1}, b_{i1}, c_{i1})]_{m\times 1} = \begin{bmatrix} (a_{11}, b_{11}, c_{11}) \\ (a_{21}, b_{21}, c_{21}) \\ \vdots \\ (a_{m1}, b_{m1}, c_{m1}) \end{bmatrix}$$

where $[(a_{i1}, b_{i1}, c_{i1})] = \mu_{\Delta^*_A}(u_i)$, for $i = 1, 2, ..., m$. This matrix is called PNSS aggregate matrix of $PNSS_{agg}(c\Delta_A, \Delta_A)$ over $\mathbb{U}$.

Definition 3.5. Let $A = (\sigma^{\mathcal{T}}_{ij}, \sigma^{\mathcal{I}}_{ij}, \sigma^{\mathcal{F}}_{ij}) \in PNSSM_{m\times n}$, then the choice matrix of PNSSM A is given by $\mathscr{C}(A) = \left[\left(\frac{\sum_{j=1}^{n}(\sigma^{\mathcal{T}}_{ij})^2}{n}, \frac{\sum_{j=1}^{n}(\sigma^{\mathcal{I}}_{ij})^2}{n}, \frac{\sum_{j=1}^{n}(\sigma^{\mathcal{F}}_{ij})^2}{n}\right)\right]_{m\times 1}$ $\forall i$ when weights are equal.

Definition 3.6. Let $A = (\sigma^{\mathcal{T}}_{ij}, \sigma^{\mathcal{I}}_{ij}, \sigma^{\mathcal{F}}_{ij}) \in PNSSM_{m\times n}$, then the weighted choice matrix of PNSSM A is given by $\mathscr{C}_w(A) = \left[\left(\frac{\sum_{j=1}^{n} w_j(\sigma^{\mathcal{T}}_{ij})^2}{\sum w_j}, \frac{\sum_{j=1}^{n} w_j(\sigma^{\mathcal{I}}_{ij})^2}{\sum w_j}, \frac{\sum_{j=1}^{n} w_j(\sigma^{\mathcal{F}}_{ij})^2}{\sum w_j}\right)\right]_{m\times 1}$ $\forall i$ where $w_j > 0$ are weights (means weights are unequal).

Theorem 3.7. *Let Δ_A be a PNSS set. Suppose that $M_{\Delta_A}, M_{c\Delta_A}, M^*_{\Delta_A}$ are matrices of $\Delta_A, c\Delta_A, \Delta^*_A$ respectively, then $M_{\Delta_A} \times M^{\mathbb{T}}_{c\Delta_A} = M^*_{\Delta_A} \times |E|$, where $M^{\mathbb{T}}_{c\Delta_A}$ is the transpose of $M_{c\Delta_A}$.*

Proof. The proof follows Definition 3.3 and Definition 3.4.

We can make a MCGDM based on PNSS sets by the following algorithms:

Algorithm-I

Step 1: Construct PNSS set Δ_A over the universal $\mathbb{U}$.
Step 2: Compute the cardinalities and find the cardinal set $c\Delta_A$ of Δ_A .
Step 3: Find aggregate PNSS set Δ^*_A of Δ_A.
Step 4: Compute the value of score function by $S(u) = \sigma_u^{2\mathcal{T}} - \sigma_u^{2\mathcal{I}} - \sigma_u^{2\mathcal{F}}$, $\forall\, u \in \mathbb{U}$.
Step 5: Compute $S(u)$ is maximum is the best alternative.

Example 3.8. Suppose that an automobile company produces ten different types of cars $\mathbb{U} = \{\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_{10}\}$ and lets a set of parameters $E = \{e_1, e_2, ..., e_5\}$ represent fuel economy, acceleration, top speed, ride comfort, and good power steering, respectively. Suppose that a customer has to decide which car purchase ? Following the discussion, each car is evaluated using a subset of parameters $A = \{e_1, e_2, e_4\} \subseteq E$. We apply the above algorithm as follows.

Step-1: We Construct PNSS set Δ_A of $\mathbb{U}$ is defined as below:

$$\Delta_A = \left\{ \left(e_1, \left\{\frac{\mathcal{C}_1}{(0.55,0.75,0.6)}, \frac{\mathcal{C}_4}{(0.8,0.7,0.65)}, \frac{\mathcal{C}_7}{(0.7,0.75,0.55)}, \frac{\mathcal{C}_9}{(0.9,0.5,0.8)}, \frac{\mathcal{C}_{10}}{(0.65,0.6,0.6)}\right\}\right), \right.$$
$$\left(e_2, \left\{\frac{\mathcal{C}_2}{(0.6,0.75,0.5)}, \frac{\mathcal{C}_3}{(0.65,0.55,0.8)}, \frac{\mathcal{C}_5}{(0.55,0.65,0.6)}, \frac{\mathcal{C}_8}{(0.65,0.7,0.7)}, \frac{\mathcal{C}_{10}}{(0.5,0.8,0.55)}\right\}\right),$$
$$\left.\left(e_4, \left\{\frac{\mathcal{C}_3}{(0.75,0.7,0.7)}, \frac{\mathcal{C}_4}{(0.5,0.6,0.75)}, \frac{\mathcal{C}_6}{(0.6,0.65,0.8)}, \frac{\mathcal{C}_8}{(0.7,0.75,0.7)}, \frac{\mathcal{C}_9}{(0.9,0.55,0.7)}\right\}\right) \right\}.$$

Step-2: The cardinal set of Δ_A as $c\Delta_A = \left\{\frac{e_1}{(0.36,0.33,0.32)}, \frac{e_2}{(0.295,0.345,0.315)}, \frac{e_4}{(0.345,0.325,0.365)}\right\}$.

Step-3: The aggregate PNSS set Δ^*_A of Δ_A is $M_{\Delta^*_A} = \frac{M_{\Delta_A} \times M^{\mathbb{T}}_{c\Delta_A}}{|E|}$

$$= \frac{1}{5}\left\{ \begin{bmatrix} 0.55 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0.65 & 0 & 0.75 & 0 \\ 0.8 & 0 & 0 & 0.5 & 0 \\ 0 & 0.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \\ 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.65 & 0 & 0.7 & 0 \\ 0.9 & 0 & 0 & 0.9 & 0 \\ 0.65 & 0.5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.36 \\ 0.295 \\ 0 \\ 0.345 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.75 & 0 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0.6 & 0 \\ 0 & 0.65 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.65 & 0 \\ 0.75 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.75 & 0 \\ 0.5 & 0 & 0 & 0.55 & 0 \\ 0.6 & 0.8 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.33 \\ 0.345 \\ 0 \\ 0.325 \\ 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0.7 & 0 \\ 0.65 & 0 & 0 & 0.75 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \\ 0.55 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.7 & 0 \\ 0.8 & 0 & 0 & 0.7 & 0 \\ 0.6 & 0.55 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.32 \\ 0.315 \\ 0 \\ 0.365 \\ 0 \end{bmatrix} \Bigg\} = \left\{ \begin{bmatrix} 0.0396 \\ 0.0354 \\ 0.0901 \\ 0.0921 \\ 0.03245 \\ 0.0414 \\ 0.0504 \\ 0.08665 \\ 0.1269 \\ 0.0763 \end{bmatrix}, \begin{bmatrix} 0.0495 \\ 0.05175 \\ 0.08345 \\ 0.0852 \\ 0.04485 \\ 0.04225 \\ 0.0495 \\ 0.09705 \\ 0.06875 \\ 0.0948 \end{bmatrix}, \begin{bmatrix} 0.0384 \\ 0.0315 \\ 0.1015 \\ 0.09635 \\ 0.0378 \\ 0.0584 \\ 0.0352 \\ 0.0952 \\ 0.1023 \\ 0.07305 \end{bmatrix} \right\}.$$

Hence, $\Delta^*_A = \Big\{ \frac{\mathcal{C}_1}{(0.0396, 0.0495, 0.0384)}, \frac{\mathcal{C}_2}{(0.0354, 0.05175, 0.0315)}, \frac{\mathcal{C}_3}{(0.0901, 0.08345, 0.1015)}, \frac{\mathcal{C}_4}{(0.0921, 0.0852, 0.09635)}, \frac{\mathcal{C}_5}{(0.03245, 0.04485, 0.0378)}, \frac{\mathcal{C}_6}{(0.0414, 0.04225, 0.0584)}, \frac{\mathcal{C}_7}{(0.0504, 0.0495, 0.0352)}, \frac{\mathcal{C}_8}{(0.08665, 0.09705, 0.0952)}, \frac{\mathcal{C}_9}{(0.1269, 0.06875, 0.1023)}, \frac{\mathcal{C}_{10}}{(0.0763, 0.0948, 0.07305)} \Big\}$.

Step-4: The values of the score function $S(\mathcal{C}_i)$ for each element of $\mathbb{U}$ are tabulated as follows.

Car	$S(\mathcal{C}_i)$
$\mathcal{C}_1$	-0.00236
$\mathcal{C}_2$	-0.00242
$\mathcal{C}_3$	-0.00915
$\mathcal{C}_4$	-0.00806
$\mathcal{C}_5$	-0.00239
$\mathcal{C}_6$	-0.00348
$\mathcal{C}_7$	-0.00115
$\mathcal{C}_8$	-0.01097
$\mathcal{C}_9$	0.00091
$\mathcal{C}_{10}$	-0.0085

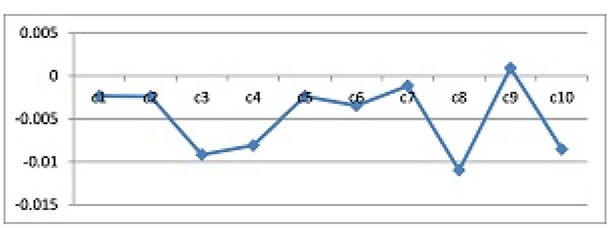

Figure 1 Graphical representation using MCGDM based on PNSS.

Step 5: Since $\max_i S(\mathcal{C}_i) = 0.00091$ which corresponds to $\mathcal{C}_9$. Therefore in this case the most suitable car $\mathcal{C}_9$ for the customer would be purchased.

Algorithm-II

Step-1: Construct Pythagorean neutrosophic soft matrix (PNSS matrix) on the basis of the parameters.

Step-2: Case-I (Equal weights) Compute the choice matrix for the positive membership, neutral membership and negative membership of PNSS matrix.

Case-II (Unequal weights) Compute the choice matrix for the positive membership, neutral membership and negative membership of PNSS matrix.

Step-3: Choose alternative with maximum score value.

Case-I: By Example 3.8,

$$\mathscr{C}(A) = \left\{ \begin{bmatrix} 0.0605 \\ 0.072 \\ 0.197 \\ 0.178 \\ 0.0605 \\ 0.072 \\ 0.098 \\ 0.1825 \\ 0.324 \\ 0.1345 \end{bmatrix}, \begin{bmatrix} 0.1125 \\ 0.1125 \\ 0.1585 \\ 0.17 \\ 0.0845 \\ 0.0845 \\ 0.1125 \\ 0.2105 \\ 0.1105 \\ 0.2 \end{bmatrix}, \begin{bmatrix} 0.072 \\ 0.05 \\ 0.226 \\ 0.197 \\ 0.072 \\ 0.128 \\ 0.0605 \\ 0.196 \\ 0.226 \\ 0.1325 \end{bmatrix} \right\}$$

Score value =

Car	$S(\mathcal{C}_i)$
$\mathcal{C}_1$	-0.01418
$\mathcal{C}_2$	-0.00997
$\mathcal{C}_3$	-0.03739
$\mathcal{C}_4$	-0.03603
$\mathcal{C}_5$	-0.00866
$\mathcal{C}_6$	-0.01834
$\mathcal{C}_7$	-0.00671
c_8	-0.04942
$\mathcal{C}_9$	0.04169
$\mathcal{C}_{10}$	-0.03947

Case-II: Weights $(w_j) = \{0.16, 0.19, 0.25, 0.22, 0.18\}$.

By Example 3.8,

$$\mathscr{C}_w(A) = \left\{ \begin{bmatrix} 0.0484 \\ 0.0684 \\ 0.204025 \\ 0.1574 \\ 0.057475 \\ 0.0792 \\ 0.0784 \\ 0.188075 \\ 0.3078 \\ 0.1151 \end{bmatrix}, \begin{bmatrix} 0.09 \\ 0.106875 \\ 0.165275 \\ 0.1576 \\ 0.080275 \\ 0.09295 \\ 0.09 \\ 0.21685 \\ 0.10655 \\ 0.1792 \end{bmatrix}, \begin{bmatrix} 0.0576 \\ 0.0475 \\ 0.2294 \\ 0.19135 \\ 0.0684 \\ 0.1408 \\ 0.0484 \\ 0.2009 \\ 0.2102 \\ 0.115075 \end{bmatrix} \right\}$$

Score value =

Car	$S(\mathcal{C}_i)$
$\mathcal{C}_1$	-0.00908
$\mathcal{C}_2$	-0.009
$\mathcal{C}_3$	-0.03831
$\mathcal{C}_4$	-0.03668
$\mathcal{C}_5$	-0.00782
$\mathcal{C}_6$	-0.02219
$\mathcal{C}_7$	-0.0043
$\mathcal{C}_8$	-0.05201
$\mathcal{C}_9$	0.0392
$\mathcal{C}_{10}$	-0.03211

Algorithm-III

Step-1: Obtain the aggregated Pythagorean neutrosophic weighted averaging (PNSWA) numbers $\mathscr{C}(A) = \left(\sum_{j=1}^{n} w_j \sigma_{ij}^{\mathcal{T}}, \sum_{j=1}^{n} w_j \sigma_{ij}^{\mathcal{I}}, \sum_{j=1}^{n} w_j \sigma_{ij}^{\mathcal{F}}\right)$.

Step-2: Compute the score function of $S(\mathcal{C}_i)$.

Step-3: Select the optimal alternative by $\max_i S(\mathcal{C}_i)$ value.

Weights $(w_j) = \{0.16, 0.19, 0.25, 0.22, 0.18\}$.

By Example 3.8,

$$\mathscr{C}(A) = \left\{ \begin{bmatrix} 0.088 \\ 0.114 \\ 0.2885 \\ 0.238 \\ 0.1045 \\ 0.132 \\ 0.112 \\ 0.2775 \\ 0.342 \\ 0.199 \end{bmatrix}, \begin{bmatrix} 0.12 \\ 0.1425 \\ 0.2585 \\ 0.244 \\ 0.1235 \\ 0.143 \\ 0.12 \\ 0.298 \\ 0.201 \\ 0.248 \end{bmatrix}, \begin{bmatrix} 0.096 \\ 0.095 \\ 0.306 \\ 0.269 \\ 0.114 \\ 0.176 \\ 0.088 \\ 0.287 \\ 0.282 \\ 0.2005 \end{bmatrix} \right\}$$

Score value =

Car	$S(\mathcal{C}_i)$
$\mathcal{C}_1$	-0.01587
$\mathcal{C}_2$	-0.01634
$\mathcal{C}_3$	-0.07723
$\mathcal{C}_4$	-0.07525
$\mathcal{C}_5$	-0.01733
$\mathcal{C}_6$	-0.034
$\mathcal{C}_7$	-0.0096
$\mathcal{C}_8$	-0.09417
$\mathcal{C}_9$	-0.00296
$\mathcal{C}_{10}$	-0.0621

3.1. *Analysis for PNSS-Methods:*

Analysis of final ranking as follows:

Methods	*Ranking of alternatives*	*Optimal alternatives*
$Algorithm - I$	$\mathcal{C}_8 \leq \mathcal{C}_3 \leq \mathcal{C}_4 \leq \mathcal{C}_6 \leq \mathcal{C}_2 \leq \mathcal{C}_5 \leq \mathcal{C}_1 \leq \mathcal{C}_7 \leq \mathcal{C}_{10} \leq \mathcal{C}_9$	$\mathcal{C}_9$
$Algorithm - II\ Case - (i)$	$\mathcal{C}_8 \leq \mathcal{C}_{10} \leq \mathcal{C}_3 \leq \mathcal{C}_4 \leq \mathcal{C}_6 \leq \mathcal{C}_1 \leq \mathcal{C}_2 \leq \mathcal{C}_5 \leq \mathcal{C}_7 \leq \mathcal{C}_9$	$\mathcal{C}_9$
$Algorithm - II\ Case - (ii)$	$\mathcal{C}_8 \leq \mathcal{C}_3 \leq \mathcal{C}_4 \leq \mathcal{C}_{10} \leq \mathcal{C}_6 \leq \mathcal{C}_1 \leq \mathcal{C}_2 \leq \mathcal{C}_5 \leq \mathcal{C}_7 \leq \mathcal{C}_9$	$\mathcal{C}_9$
$Algorithm - III$	$\mathcal{C}_8 \leq \mathcal{C}_3 \leq \mathcal{C}_4 \leq \mathcal{C}_{10} \leq \mathcal{C}_6 \leq \mathcal{C}_5 \leq \mathcal{C}_2 \leq \mathcal{C}_1 \leq \mathcal{C}_7 \leq \mathcal{C}_9$	$\mathcal{C}_9$

Therefore most suitable car $\mathcal{C}_9$ for the customer would be purchased.

4. **MCGDM based on PNSS-TOPSIS aggregating operator**

Algorithm-IV (PNSS-TOPSIS)

Step-1: Assume that $\mathscr{D} = \{\mathscr{D}_i : i \in \mathbb{N}\}$ is a finite set of decision makers/experts, $\mathscr{C} = \{z_i : i \in \mathbb{N}\}$ is the finite collection of alternatives and $D = \{e_i : i \in \mathbb{N}\}$ is a finite family of parameters/criterion.

Step-2: By selecting the linguistic terms and constructing weighted parameter matrix $\mathscr{P}$ can be written as

$$\mathscr{P} = [w_{ij}]_{n\times m} = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1m} \\ w_{21} & w_{22} & \dots & w_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & \dots & w_{im} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \dots & w_{nm} \end{bmatrix}$$

Where w_{ij} is the weight assigned by the expert $\mathscr{D}_i$ to the alternative $\mathscr{P}_j$ by considering linguistic variables.

Step-3: Construct weighted normalized decision matrix using the following

$$\widehat{\mathscr{N}} = [\widehat{n}_{ij}]_{n\times m} = \begin{bmatrix} \widehat{n}_{11} & \widehat{n}_{12} & \dots & \widehat{n}_{1m} \\ \widehat{n}_{21} & \widehat{n}_{22} & \dots & \widehat{n}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{n}_{i1} & \widehat{n}_{i2} & \dots & \widehat{n}_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{n}_{n1} & \widehat{n}_{n2} & \dots & \widehat{n}_{nm} \end{bmatrix}$$

where $\widehat{n}_{ij} = \frac{w_{ij}}{\sqrt{\sum_{i=1}^{n} w_{ij}^2}}$ is the normalized criteria rating and obtaining the weighted vector $\mathscr{W} = (m_1, m_2, ..., m_m)$, where $m_i = \frac{w_i}{\sqrt{\sum_{l=1}^{n} w_{li}}}$ is the relative weight of the j^{th} criterion and $w_j = \frac{\sum_{i=1}^{n} \widehat{n}_{ij}}{n}$.

Step-4: Construct PNSS decision matrix can be calculate as follows

$$\mathscr{D}_i = [x^i_{jk}]_{l\times m} = \begin{bmatrix} x^i_{11} & x^i_{12} & \dots & x^i_{1m} \\ x^i_{21} & x^i_{22} & \dots & x^i_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{j1} & x^i_{j2} & \dots & x^i_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{l1} & x^i_{l2} & \dots & x^i_{lm} \end{bmatrix}$$

Where x^i_{jk} is a PNSS element for i^{th} decision maker so that $\mathscr{D}_i$ for each i. Then obtain the aggregating matrix $\mathscr{A} = \frac{\mathscr{D}_1 + \mathscr{D}_2 + ... + \mathscr{D}_n}{n} = [y_{jk}]_{l\times m}$.

Step-5: Find the weighted PNSS decision matrix by

$$\mathscr{Y} = [z_{jk}]_{l\times m} = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1m} \\ z_{21} & z_{22} & \dots & z_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{j1} & z_{j2} & \dots & z_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ z_{l1} & z_{l2} & \dots & z_{lm} \end{bmatrix}$$

Where $z_{jk} = m_k \times y_{jk}$.

Step-6: Calculate PNSSV-PIS and PNSSV-NIS. Now,
PNSSV-PIS $= [z_1^+, z_2^+, ..., z_l^+] = \{(\vee_k z_{jk}, \wedge_k z_{jk}, \wedge_k z_{jk}) : k = 1, 2, ..., m\}$ and PNSSV-PIS $= [z_1^-, z_2^-, ..., z_l^-] = \{(\wedge_k z_{jk}, \vee_k z_{jk}, \vee_k z_{jk}) : k = 1, 2, ..., m\}$, where $\vee$ stands for PNSS union and $\wedge$ represents PNSS intersection.

Step-7: Compute PNSS-Euclidean distances of each alternative from PNSSV-PIS and PNSSV-NIS. Now, $(d_j^+)^2 = \sum_{k=1}^{m} \left\{(\sigma_{jk}^{T+} - \sigma_j^{T+})^2 + (\sigma_{jk}^{I+} - \sigma_j^{I+})^2 + (\sigma_{jk}^{F+} - \sigma_j^{F+})^2\right\}$ and $(d_j^-)^2 = \sum_{k=1}^{m} \left\{(\sigma_{jk}^{T-} - \sigma_j^{T-})^2 + (\sigma_{jk}^{I-} - \sigma_j^{I-})^2 + (\sigma_{jk}^{F-} - \sigma_j^{F-})^2\right\}$, where $j = 1, 2, ..., n$.

Step-8: Calculate the relative closeness of each alternative to the ideal solution by $C^*(z_j) = \frac{d_j^-}{d_j^+ + d_j^-} \in [0, 1]$.

Step-9: The rank of alternatives in decreasing or increasing order of their relative closeness coefficients. The bigger $C^*(z_j)$, the more desirable alternative z_j.

Step-10: The best alternative is the one with the highest relative closeness to the ideal solution.

Example 4.1. Assume that a firm plans to invest some money in stock exchange by purchasing some shares of best five companies. In order to minimize the risk factor, they decide to invest their money 30%, 25%, 20%, 15% and 10% in accordance with the top ranked five companies.

Step-1: Assume that $\mathscr{D} = \{\mathscr{D}_i : i = 1, 2, 3, 4, 5\}$ is a finite set of decision makers/experts, $\mathscr{C} = \{z_i : i = 1, 2, ..., 10\}$ is the collection of companies/alternatives and $D = \{e_i : i = 1, 2, ..., 5\}$ is a finite family of parameters/criterion, where $e_1 =$ Momentum, $e_2 =$ Value, $e_3 =$ Growth, $e_4 =$ Volatility, $e_5 =$ Quality.

Step-2: Forms a Linguistic terms for judging alternatives as given below:

Linguistic terms	*Fuzzy weights*
$Very\ Good\ Testing(VGT)$	0.95
$Good\ Testing(GT)$	0.80
$Average\ Testing(AT)$	0.65
$Poor\ Testing(PT)$	0.50
$Very\ Poor\ Testing(VPT)$	0.35

Construct weighted parameter matrix

$$\begin{aligned} \mathscr{P} &= [w_{ij}]_{5\times 5} \\ &= \begin{bmatrix} GC & VGC & PC & VPC & AC \\ AC & GC & VPC & PC & GC \\ PC & AC & VGC & VGC & VPC \\ VGC & PC & AC & GC & PC \\ AC & VPC & VGC & GC & VPC \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0.8 & 0.95 & 0.5 & 0.35 & 0.65 \\ 0.65 & 0.8 & 0.35 & 0.5 & 0.8 \\ 0.5 & 0.65 & 0.95 & 0.95 & 0.35 \\ 0.95 & 0.5 & 0.65 & 0.8 & 0.5 \\ 0.65 & 0.35 & 0.95 & 0.8 & 0.35 \end{bmatrix}$$

Where w_{ij} is the weight provided by the specialist $\mathscr{D}_i$ to each parameter $\mathscr{P}_j$.

Step-3: The normalized weighted decision matrix is

$$\begin{aligned} \widehat{\mathscr{N}} &= [\widehat{n}_{ij}]_{5\times5} \\ &= \begin{bmatrix} 0.4926 & 0.6214 & 0.3101 & 0.219 & 0.5208 \\ 0.4002 & 0.5233 & 0.2171 & 0.3128 & 0.641 \\ 0.3079 & 0.4251 & 0.5892 & 0.5943 & 0.2804 \\ 0.585 & 0.327 & 0.4031 & 0.5005 & 0.4006 \\ 0.4002 & 0.2289 & 0.5892 & 0.5005 & 0.2804 \end{bmatrix}. \end{aligned}$$

And weighted vector is $\mathscr{W} = (0.1231, 0.1308, 0.124, 0.1251, 0.1603)$.

Step-4: The aggregated decision matrix $\mathscr{A}$ can be written as

$$\begin{aligned} \mathscr{A} &= \frac{\mathscr{D}_1 + \mathscr{D}_2 + \mathscr{D}_3 + \mathscr{D}_4 + \mathscr{D}_5}{5} \\ &= \begin{bmatrix} (0.78, 0.48, 0.7) & (0.7, 0.45, 0.6) & (0.68, 0.6, 0.65) & (0.65, 0.75, 0.9) & (0.78, 0.57, 0.6) \\ (0.8, 0.7, 0.9) & (0.65, 0.75, 0.85) & (0.64, 0.66, 0.64) & (0.69, 0.8, 0.67) & (0.68, 0.81, 0.7) \\ (0.75, 0.65, 0.75) & (0.72, 0.68, 0.42) & (0.72, 0.87, 0.45) & (0.74, 0.7, 0.59) & (0.62, 0.56, 0.85) \\ (0.8, 0.95, 0.62) & (0.9, 0.8, 0.65) & (0.85, 0.8, 0.41) & (0.81, 0.8, 0.56) & (0.9, 0.69, 0.75) \\ (0.8, 0.55, 0.95) & (0.55, 0.65, 0.9) & (0.62, 0.61, 0.68) & (0.69, 0.54, 0.67) & (0.68, 0.62, 0.7) \\ (0.84, 0.83, 0.62) & (0.9, 0.8, 0.45) & (0.9, 0.43, 0.73) & (0.83, 0.49, 0.8) & (0.9, 0.68, 0.45) \\ (0.79, 0.65, 0.75) & (0.75, 0.55, 0.65) & (0.78, 0.65, 0.55) & (0.65, 0.75, 0.9) & (0.8, 0.57, 0.6) \\ (0.75, 0.7, 0.68) & (0.76, 0.7, 0.42) & (0.8, 0.43, 0.43) & (0.47, 0.8, 0.85) & (0.83, 0.5, 0.55) \\ (0.85, 0.61, 0.74) & (0.66, 0.58, 0.65) & (0.7, 0.62, 0.78) & (0.4, 0.9, 0.64) & (0.58, 0.77, 0.6) \\ (0.9, 0.55, 0.65) & (0.63, 0.62, 0.8) & (0.69, 0.72, 0.55) & (0.83, 0.6, 0.49) & (0.62, 0.49, 0.78) \end{bmatrix} \\ &= [y_{jk}]_{10\times5} \end{aligned}$$

Step-5: The weighted PNSS decision matrix $\mathscr{Y}$ can be written as $\mathscr{Y} = m_k \times y_{jk} =$

$$\begin{bmatrix} (0.0961, 0.0591, 0.0862) & (0.0916, 0.0589, 0.0785) & (0.0843, 0.0744, 0.0806) & (0.0813, 0.0938, 0.1126) & (0.125, 0.0913, 0.0962) \\ (0.0985, 0.0862, 0.1108) & (0.085, 0.0981, 0.1112) & (0.0794, 0.0819, 0.0794) & (0.0863, 0.1001, 0.0838) & (0.109, 0.1298, 0.1122) \\ (0.0924, 0.08, 0.0924) & (0.0942, 0.089, 0.0549) & (0.0893, 0.1079, 0.0558) & (0.0926, 0.0876, 0.0738) & (0.0994, 0.0897, 0.1362) \\ (0.0985, 0.117, 0.0764) & (0.1177, 0.1047, 0.085) & (0.1054, 0.0992, 0.0509) & (0.1013, 0.1001, 0.0701) & (0.1442, 0.1106, 0.1202) \\ (0.0985, 0.0677, 0.117) & (0.0719, 0.085, 0.1177) & (0.0769, 0.0757, 0.0843) & (0.0863, 0.0676, 0.0838) & (0.109, 0.0994, 0.1122) \\ (0.1034, 0.1022, 0.0764) & (0.1177, 0.1047, 0.0589) & (0.1116, 0.0533, 0.0905) & (0.1039, 0.0613, 0.1001) & (0.1442, 0.109, 0.0721) \\ (0.0973, 0.08, 0.0924) & (0.0981, 0.0719, 0.085) & (0.0967, 0.0806, 0.0682) & (0.0813, 0.0938, 0.1126) & (0.1282, 0.0913, 0.0962) \\ (0.0924, 0.0862, 0.0837) & (0.0994, 0.0916, 0.0549) & (0.0992, 0.0533, 0.0533) & (0.0588, 0.1001, 0.1064) & (0.133, 0.0801, 0.0881) \\ (0.1047, 0.0751, 0.0911) & (0.0863, 0.0759, 0.085) & (0.0868, 0.0769, 0.0967) & (0.05, 0.1126, 0.0801) & (0.0929, 0.1234, 0.0962) \\ (0.1108, 0.0677, 0.08) & (0.0824, 0.0811, 0.1047) & (0.0856, 0.0893, 0.0682) & (0.1039, 0.0751, 0.0613) & (0.0994, 0.0785, 0.125) \end{bmatrix}$$

$= [z_{jk}]_{10\times5}$.

Step-6: We find PNSSV-PIS and PNSSV-NIS can be written as

z^+	$PNSSV-PIS$
z_1^+	$(0.125, 0.0589, 0.0785)$
z_2^+	$(0.109, 0.0819, 0.0794)$
z_3^+	$(0.0994, 0.08, 0.0549)$
z_4^+	$(0.1442, 0.0992, 0.0509)$
z_5^+	$(0.109, 0.0676, 0.0838)$
z_6^+	$(0.1442, 0.0533, 0.0589)$
z_7^+	$(0.1282, 0.0719, 0.0682)$
z_8^+	$(0.133, 0.0533, 0.0533)$
z_9^+	$(0.1047, 0.0751, 0.0801)$
z_{10}^+	$(0.1108, 0.0677, 0.0613)$

z^-	$PNSSV-NIS$
z_1^-	$(0.0813, 0.0938, 0.1126)$
z_2^-	$(0.0794, 0.1298, 0.1122)$
z_3^-	$(0.0893, 0.1079, 0.1362)$
z_4^-	$(0.0985, 0.117, 0.1202)$
z_5^-	$(0.0719, 0.0994, 0.1177)$
z_6^-	$(0.1034, 0.109, 0.1001)$
z_7^-	$(0.0813, 0.0938, 0.1126)$
z_8^-	$(0.0588, 0.1001, 0.1064)$
z_9^-	$(0.05, 0.1234, 0.0967)$
z_{10}^-	$(0.0824, 0.0893, 0.125)$

Step-7: We found PNSS euclidean distances of each alternative from PNSSV-PIS and PNSSV-NIS.

$Alternative\,(z_i)$	d_i^+	d_i^-
z_1	0.0979	0.0906
z_2	0.0899	0.0964
z_3	0.0979	0.1446
z_4	0.1166	0.1174
z_5	0.0863	0.0859
z_6	0.1282	0.1025
z_7	0.0983	0.0851
z_8	0.1408	0.1381
z_9	0.0906	0.1217
z_{10}	0.0937	0.1101

Step-8: We calculate closeness coefficients of each alternative from PNSSV-PIS and PNSSV-NIS.

$Alternative\,(z_i)$	C_i^*
z_1	0.4807
z_2	0.5174
z_3	0.5962
z_4	0.5017
z_5	0.4988
z_6	0.4444
z_7	0.4639
z_8	0.4951
z_9	0.5734
z_{10}	0.5404

Step-9: The order of the alternatives for C_i^* is $z_3 \geq z_9 \geq z_{10} \geq z_2 \geq z_4 \geq z_5 \geq z_8 \geq z_1 \geq z_7 \geq z_6$.

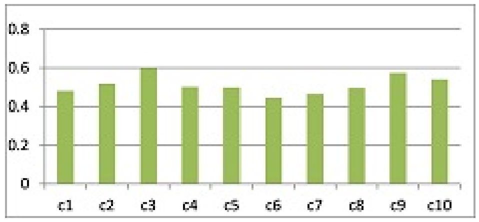

Figure 2 Graphical representation using MCGDM based on TOPSIS.

Step-10: The above ranking, it conclude that the firm should z_3 invest 30%, z_9 invest 25%, z_{10} invest 20%, z_2 invest 15% and z_4 invest 10%.

5. MCGDM based on PNSS-VIKOR aggregating operator

Algorithm-V (PNSS-VIKOR)

Step-1: Assume that $\mathscr{D} = \{\mathscr{D}_i : i \in \mathbb{N}\}$ is a finite set of decision makers/experts, $\mathscr{C} = \{z_i : i \in \mathbb{N}\}$ is the finite collection of alternatives and $D = \{e_i : i \in \mathbb{N}\}$ is a finite family of parameters/criterion.

Step-2: By selecting the linguistic terms and constructing weighted parameter matrix $\mathscr{P}$ can be written as

$$\mathscr{P} = [w_{ij}]_{n\times m} = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1m} \\ w_{21} & w_{22} & \dots & w_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & \dots & w_{im} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \dots & w_{nm} \end{bmatrix}$$

Where w_{ij} is the weight assigned by the expert $\mathscr{D}_i$ to the alternative $\mathscr{P}_j$ by considering linguistic variables.

Step-3: Construct weighted normalized decision matrix using the following

$$\widehat{\mathscr{N}} = [\widehat{n}_{ij}]_{n\times m} = \begin{bmatrix} \widehat{n}_{11} & \widehat{n}_{12} & \dots & \widehat{n}_{1m} \\ \widehat{n}_{21} & \widehat{n}_{22} & \dots & \widehat{n}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{n}_{i1} & \widehat{n}_{i2} & \dots & \widehat{n}_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{n}_{n1} & \widehat{n}_{n2} & \dots & \widehat{n}_{nm} \end{bmatrix}$$

where $\widehat{n}_{ij} = \frac{w_{ij}}{\sqrt{\sum_{i=1}^{n} w_{ij}^2}}$ is the normalized criteria rating and obtaining the weighted vector $\mathscr{W} = (m_1, m_2, ..., m_m)$, where $m_i = \frac{w_i}{\sqrt{\sum_{l=1}^{n} w_{li}}}$ is the relative weight of the j^{th} criterion and $w_j = \frac{\sum_{i=1}^{n} \widehat{n}_{ij}}{n}$.

Step-4: Construct PNSS decision matrix can be calculated as

$$\mathscr{D}_i = [x^i_{jk}]_{l\times m} = \begin{bmatrix} x^i_{11} & x^i_{12} & \dots & x^i_{1m} \\ x^i_{21} & x^i_{22} & \dots & x^i_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{j1} & x^i_{j2} & \dots & x^i_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ x^i_{l1} & x^i_{l2} & \dots & x^i_{lm} \end{bmatrix}$$

Where x^i_{jk} is a PNSS element for i^{th} decision maker so that $\mathscr{D}_i$ for each i. Then obtain the aggregating matrix $\mathscr{A} = \frac{\mathscr{D}_1+\mathscr{D}_2+\ldots+\mathscr{D}_n}{n} = [y_{jk}]_{l\times m}$.

Step-5: Construct the weighted PNSS decision matrix by

$$\mathscr{Y} = [z_{jk}]_{l\times m} = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1m} \\ z_{21} & z_{22} & \dots & z_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{j1} & z_{j2} & \dots & z_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ z_{l1} & z_{l2} & \dots & z_{lm} \end{bmatrix}$$

Where $z_{jk} = m_k \times y_{jk}$.

Step-6: Calculate the values of PNSSV-PIS and PNSSV-NIS. Now, PNSSV-PIS $= [z^+_1, z^+_2, ..., z^+_l] = \{(\vee_k z_{jk}, \wedge_k z_{jk}, \wedge_k z_{jk}) : j = 1, 2, ..., l\}$ and PNSSV-PIS $= [z^-_1, z^-_2, ..., z^-_l] = \{(\wedge_k z_{jk}, \vee_k z_{jk}, \vee_k z_{jk}) : j = 1, 2, ..., l\}$, where $\vee$ stands for PNSS union and $\wedge$ represents PNSS intersection.

Step-7: Find the values of utility $\mathscr{S}_i$, individual regret $\mathscr{R}_i$ and compromise $\mathscr{Q}_i$, where $\mathscr{S}_i = \sum_{j=1}^{m} m_j \left(\frac{d(z_{ij}, z^+_j)}{d(z^+_j, z^-_j)}\right)$, $\mathscr{R}_i = \max_{j=1}^{m} m_j \left(\frac{d(z_{ij}, z^+_j)}{d(z^+_j, z^-_j)}\right)$ and $\mathscr{Q}_i = \kappa\left(\frac{\mathscr{S}_i - \mathscr{S}^-}{\mathscr{S}^+ - \mathscr{S}^-}\right) + (1-\kappa)\left(\frac{\mathscr{R}_i - \mathscr{R}^-}{\mathscr{R}^+ - \mathscr{R}^-}\right)$. Where $\mathscr{S}^+ = \max_i \mathscr{S}_i$, $\mathscr{S}^- = \min_i \mathscr{S}_i$, $\mathscr{R}^+ = \max_i \mathscr{R}_i$ and $\mathscr{R}^- = \min_i \mathscr{R}_i$. The real number κ is called a coefficient of decision mechanism. The role of κ is that if compromise solution is to be selected by majority if $\kappa > 0.5$; for consensus if $\kappa = 0.5$ and $\kappa < 0.5$ represents veto. Let m_j represents the weight of the j^{th} criteria.

Step-8: The rank of choices and derive compromise solution. Arrange $\mathscr{S}_i$, $\mathscr{R}_i$ and $\mathscr{Q}_i$ in increasing order to make these three ranking lists. The alternative z_α will be declared compromise solution if it ranks the best in $\mathscr{Q}_i$ (having least value) and satisfies the following two requirements simultaneously:

$[C-1]$ acceptable: If z_α and z_β represent top alternatives in $\mathscr{Q}_i$, then $\mathscr{Q}(z_\beta) - \mathscr{Q}(z_\alpha) \geq \frac{1}{n-1}$, where n is the number of parameters.

$[C-2]$ acceptable: The alternative z_α should be best ranked by $\mathscr{S}_i$ and /or $\mathscr{R}_i$.

If above two conditions are not met simultaneously, then there exist multiple compromise solutions:

(i) If only condition $C-1$ is satisfied, then both alternatives z_α and z_β are called the compromise solutions:

(ii) If condition $C-1$ is not satisfied, then the alternatives z_α, z_β,..., z_ζ are called the compromise solutions, where z_ζ is founded by $\mathscr{Q}(z_\zeta) - \mathscr{Q}(z_\alpha) \geq \frac{1}{n-1}$.

Example 5.1. We resolve Example 4.1 using VIKOR method. The first five steps are the same as in Example 4.1. So we start with step 6.

Step-6: We compute PNSSV-PIS and PNSSV-NIS are listed as follows.

z^+	$PNSSV-PIS$
z_1^+	$(0.1108, 0.0591, 0.0764)$
z_2^+	$(0.1177, 0.0589, 0.0549)$
z_3^+	$(0.1116, 0.0533, 0.0509)$
z_4^+	$(0.1039, 0.0613, 0.0613)$
z_5^+	$(0.1442, 0.0785, 0.0721)$

z^-	$PNSSV-NIS$
z_1^-	$(0.0924, 0.117, 0.117)$
z_2^-	$(0.0719, 0.1047, 0.1177)$
z_3^-	$(0.0769, 0.1079, 0.0967)$
z_4^-	$(0.05, 0.1126, 0.1126)$
z_5^-	$(0.0929, 0.1298, 0.1362)$

Step-7: Taking $\kappa = 0.5$, we found that the values of utility $\mathscr{S}_i$, individual regret $\mathscr{R}_i$ and compromise $\mathscr{Q}_i$ for each alternative z_i.

$Alternative\,(z)$	$\mathscr{S}_i$	$\mathscr{R}_i$	$\mathscr{Q}_i$
z_1	0.2972	0.0897	0.2208
z_2	0.457	0.1225	0.9271
z_3	0.3763	0.1309	0.7881
z_4	0.4024	0.0997	0.5843
z_5	0.4104	0.1189	0.7732
z_6	0.3065	0.0737	0.1049
z_7	0.3031	0.0897	0.2364
z_8	0.2666	0.1033	0.2591
z_9	0.4212	0.1196	0.8079
z_{10}	0.3184	0.1148	0.4958

Step-8: The rank of alternatives for $\mathscr{Q}_i$: $z_6 \leq z_1 \leq z_7 \leq z_8 \leq z_{10} \leq z_4 \leq z_5 \leq z_3 \leq z_9 \leq z_2$. Now, $\mathscr{Q}(z_1) - \mathscr{Q}(z_6) = 0.1159 \ngeq \frac{1}{4}$. Thus, the condition C-1 is not satisfied. Further $\mathscr{Q}(z_{10}) - \mathscr{Q}(z_6) = 0.3909 \geq \frac{1}{4}$. Therefore, we decide $z_6, z_1, z_7, z_8, z_{10}$ are multiple compromise solutions. Hence the firm should invest 30% on z_6, 25% on z_1, 20% on z_7, 15% on z_8 and 10% on z_{10}.

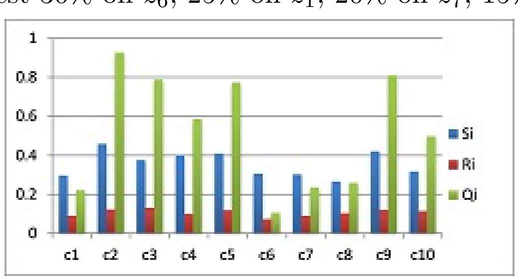

Figure 3 Graphical representation using MCGDM based on VIKOR.

6. Analysis and discussion

We used the above example to analyse the two methods in the literature. The ranking results of all ten alternatives were obtained using these two approaches. These two methods assume a scale component for each criterion. The normalisation approach is different in these two methods. The TOPSIS method utilises a vector normalisation approach and the VIKOR method utilises a linear normalisation approach. The TOPSIS method uses "n"- dimensional Euclidean distance that by itself could constitute some balance between total and individual contentment, but the VIKOR method uses a different way by which weight "κ" is introduced. The major difference between the two methods is in the aggregation function. We can find the ranking of values using an aggregating function. The best ranked alternative by VIKOR is closest to the ideal solution. However, the best ranked alternative by TOPSIS is the one using the ranking index, which does not mean the closest to the ideal solution. Hence, the advantage of the VIKOR method gives a compromise solution.

7. Conclusion:

In this communication, we studied various properties of PNSSS and PNSSM that occur in investment decision making. We proposed the first four algorithms, followed by MCGDM under PNSS. The last two algorithms are based on PNSS linguistic TOPSIS and VIKOR approaches using aggregation operators. Again, we interact with the PNSS aggregation operator and score function values based on some technique. Also, we made use of various sorts of statistical charts to imagine the rankings of different alternatives under consideration. We have analyzed an application of the new approach in a DM problem regarding the selection of particulars where we can see the different conclusions obtained by using different types of aggregation operators.

Acknowledgments: The authors would like to thank the Editor-InChief and the anonymous referees for their various suggestions and helpful comments that have led to the improved in the quality and clarity version of the paper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Adeel, A.; Akram. M; and Koam, A.N.A.; Group decision-making based on m-polar fuzzy linguistic TOPSIS method, Symmetry; 2019, Vol. 11(735), pp. 1–20.
2. Akram, M. and Arshad, M.; A novel trapezoidal bipolar fuzzy TOPSIS method for group decision making, Group Decision and Negotiation; 2018.

3. Atanassov, K.; Intuitionistic fuzzy sets, Fuzzy sets and Systems; 1986, Vol. 20(1), pp. 87–96.
4. Boran, F.E.; Genc, S.; Kurt, M. and Akay, D.; A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method, Expert Systems with Applications; 2009, Vol. 36(8), pp. 11363–11368.
5. Broumi, S.; Deli. I. and Smarandache, F.; Neutrosophic Parametrized Soft Set theory and its decision making problem, International Frontier Science Letters; 2014, Vol. 1(1), pp. 1–11.
6. Eraslan, S. and Karaaslan, F.; A group decision making method based on TOPSIS under fuzzy soft environment, Journal of New Theory; 2015, Vol. 3, pp. 30–40.
7. Hwang, C.L.and Yoon,K.; Multiple Attributes Decision Making Methods and Applications, Springer Berlin Heidelberg; 1981.
8. Jana, C.; Senapati, T. and Pal, M.; Pythagorean fuzzy Dombi aggregation operators and its applications in multiple attribute decision making, International Journal of Intelligent Systems; 2019, Vol. 34(9), pp. 2019–2038.
9. Jana, C. and Pal, M.; Application of bipolar intuitionistic fuzzy soft sets in decision making problem, International Journal of Fuzzy System Applications, 2018, Vol. 7(3), pp. 32–55.
10. Jansi, R.; Mohana, K. and Smarandache, F.; Correlation measure for Pythagorean neutrosophic sets with T and F as dependent neutrosophic components, Neutrosophic Sets and Systems; 2019, Vol. 30, pp. 202–212.
11. Mohanraj, G.; and Palanikumar, M.; On Prime k-ideals in Semirings, Nonlinear studies; 2021, Vol. 27(3), pp. 769–774.
12. Mohanraj, G.; and Palanikumar, M.; On Various Prime and Semiprime bi-ideals of Rings, Nonlinear studies; 2021, Vol. 27(3), 811–815.
13. Maji, P.K.; Biswas, R. and Roy, A.R.; Fuzzy Soft Set, Journal of Fuzzy Mathematics; 2001, Vol. 9(3), pp. 589–602.
14. Maji, P.K.; Biswas, R. and Roy, A.R.; On intuitionistic Fuzzy Soft Set, Journal of Fuzzy Mathematics; 2001, Vol. 9(3), pp. 677–692.
15. Molodtsov, D.; Soft set theory First results, Computers and Mathematics with Applications; 1999, Vol. 37, pp. 19–31.
16. Opricovic, S.; Tzeng, G.H.; Extended VIKOR method in comparison with outranking methods, Eur. J. Oper. Res.; 2007, Vol. 178, pp. 514–529.
17. Opricovic, S.; Fuzzy VIKOR with an application to water resources planning, Expert Syst. Appl.; 2011, Vol. 38, pp. 12983–12990.
18. Opricovic, S. and Tzeng, G.H.; Fuzzy multicriteria model for post-earthquake land-use planning, Natural Hazards Review; 2003, Vol. 4, pp. 59–64.
19. Palanikumar, M.; Arulmozhi, K.; On Various ideals and its Applications of Bisemirings, 2020; Gedrag and Organisatie Review, Vol. 33(2). pp. 522–533.
20. Palanikumar,M.; Arulmozhi, K.; On New Ways of various ideals in ternary semigroups, 2020; Matrix Science Mathematic, Vol. 4(1). pp. 06-09.
21. Palanikumar, M.; Arulmozhi,K.; On New Approach of Various fuzzy ideals in Ordered gamma semigroups, 2020; Gedrag and Organisatie Review, Vol. 33(2). pp. 331-342.
22. Palanikumar, M.; Arulmozhi, K.; On Various Tri-ideals in ternary Semirings, 2021; Bulletin of the International Mathematical Virtual Institute, Vol. 11(1). pp. 79-90.
23. Palanikumar, M.; Arulmozhi, K.; and Jana, K.; Multiple attribute decision-making approach for Pythagorean neutrosophic normal interval-valued aggregation operators, 2022; Comp. Appl. Math., Vol. 41(90). pp. 1–27.
24. Palanikumar, M.; Arulmozhi, K.; On intuitionistic fuzzy normal subbisemirings of bisemirings, 2021, Nonlinear studies; Vol. 28(3), 717–721.

25. Palanikumar, M.; Iampan, A.; Spherical Fermatean Interval Valued Fuzzy Soft Set Based On Multi Criteria Group Decision Making, International Journal of Innovative Computing, Information and Control; 2022, Vol. 18(2), 607–619.
26. Palanikumar, M.; Arulmozhi, K.; (α, β) neutrosophic subbisemiring of bisemiring, Neutrosophic Sets and Systems; 2022, Vol. 48, 368–385.
27. Palanikumar, M.; Iampan, A.; A Novel Approach to Decision Making Based on Type-II Generalized Fermatean Bipolar Fuzzy Soft Sets, International Journal of Innovative Computing, Information and Control; 2022, Vol. 18(3), 769–783
28. Peng, X.D. and Dai, J.; Approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function, Neural Computing and Applications; 2018, Vol. 29(10), pp. 939–954.
29. Peng, X.D.; Yang, Y and Song, J.P.; Pythagorean fuzzy soft set and its application, Computer Engineering; 2015, Vol. 41(7), pp. 224–229.
30. Smarandache, F.; Neutrosophic set- generalization of the intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics; 2005, Vol. 24(3), pp. 287–297.
31. Smarandache,F.; A unifying field in logics Neutrosophy Neutrosophic Probability, 1999; Set and Logic, Rehoboth American Research Press.
32. Ting Yu Chen; An Interval Valued Pythagorean Fuzzy Compromise Approach with Correlation Based Closeness Indices for Multiple Criteria Decision Analysis of Bridge Construction Methods, Complexity; 2018, pp. 1–29.
33. Tzeng, G.H.; Lin, C.W. and Opricovic, S.; Multi-criteria analysis of alternative-fuel buses for public transportation, Energy Policy; 2005, Vol. 33, pp. 1373–1383.
34. Ullah, K.; Mahmood, T.; Ali, Z. and Jan, N.; On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition, Complex and Intelligent Systems; 2019, pp. 1–13.
35. Xiao, Z.; Chen, W.J. and Li, L.L; A method based on interval-valued fuzzy soft set for multi attribute group decision making problems under uncertain environment, Knowledge and Information Systems, 2013, Vol. 34, pp. 653–669.
36. Xu, Z. and Zhang, X.; Hesitant fuzzy multi-attribute decision-making based on TOPSIS with incomplete weight information, Knowledge-Based Systems; 2013, Vol. 52, pp. 53–64.
37. Yager, R.R. and Abbasov, A.M.; Pythagorean membership grades, complex numbers, and decision making, International Journal of Intelligent Systems; 2014, Vol. 28, pp. 436–452.
38. Yager, R.R.; Pythagorean membership grades in multi criteria decision making, IEEE Trans. Fuzzy Systems; 2014, Vol. 22, pp. 958–965.
39. Zadeh, L. A.; Fuzzy sets, Information and control; 1965, Vol. 8(3), pp. 338–353.
40. Zhang, X. and Xu, Z.; Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, International Journal of Intelligent Systems; 2014, Vol. 29, pp. 1061–1078.
41. Zulqarnain, R.M.; Xiao Long Xin; Muhammad Saqlain and Waseem Asghar Khan, TOPSIS Method Based on the Correlation Coefficient of Interval-Valued Intuitionistic Fuzzy Soft Sets and Aggregation Operators with Their Application in Decision-Making, Journal of Mathematics; 2021, pp. 1-16.

Received: June 1, 2022. Accepted: September 25, 2022.