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Dombi Aggregation Operators of Linguistic Neutrosophic Numbers for Multiple Attribute Group Decision-Making Problems in Landslide Treatment Schemes

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Abstract: The landslide disaster caused huge losses to human lives and property, and the research on the selection of landslide treatment schemes has attracted global attention. Fuzzy multi-attribute decision-making methods are widely used for selecting the slope treatment schemes in engineering practice. But they do not take into account human linguistic arguments in the linguistic decision-making environment, which usually contains incomplete and uncertain information, and still lack a qualitative evaluation method. To deal with multiple attribute group decision-making (MAGDM) problems of landslide treatment schemes in the linguistic neutrosophic environment, a linguistic neutrosophic number Dombi weighted arithmetic averaging (LNNDWAA) operator and a linguistic neutrosophic number Dombi weighted geometric averaging (LNNDWGA) operator are developed to aggregate linguistic neutrosophic information. Then, a new MAGDM method using these aggregation operators is proposed in view of the Dombi operational flexibility. Finally, the proposed method is applied to select the optimal landslide treatment scheme under the linguistic neutrosophic environment. The results show that this method can effectively solve the decision-making problem of landslide treatment schemes and make the decision result more reasonable and flexible than other existing methods.

Keywords: landslide treatment scheme; multiple attribute decision; linguistic neutrosophic number; Dombi operation

1. Introduction

Landslides have caused immeasurable economic losses to human society. For example, China has an average of almost 30,000 landslides, rock falls, and debris flows every year, many of which have caused catastrophic disasters. On average, nearly 800 people are killed each year, and the direct economic loss exceeds 1 billion US dollars. In 2019, a total of 6,181 geological disasters occurred nationwide (Figure 1), where slope failure accounted for 68.27%, accounting for the vast majority [1]. Therefore, great attention has been paid to the prevention and treatment of landslides. At present, there are many prevention and control plans for slopes. To choose the most reasonable plan, some decision-making (DM) methods are needed. In recent years, research on DM methods of slope treatment schemes has received increasing attention. These DM methods include the empirical discriminant method [2], the analytic hierarchy process [3], the fuzzy multi-attribute DM method [4], the subjective and objective weighting method [5], and so on. However, most of these methods are based on the analysis or subjective judgment of expert experience, which leads to unreasonable, uneconomical, and sometimes even huge waste in the final choice of the treatment schemes. Fuzzy multi-attribute DM methods have been widely used because they can deal with fuzzy evaluation and DM problems. But they does not take into account human linguistic arguments in the DM issue of the treatment schemes, which usually contains incomplete and uncertain information.

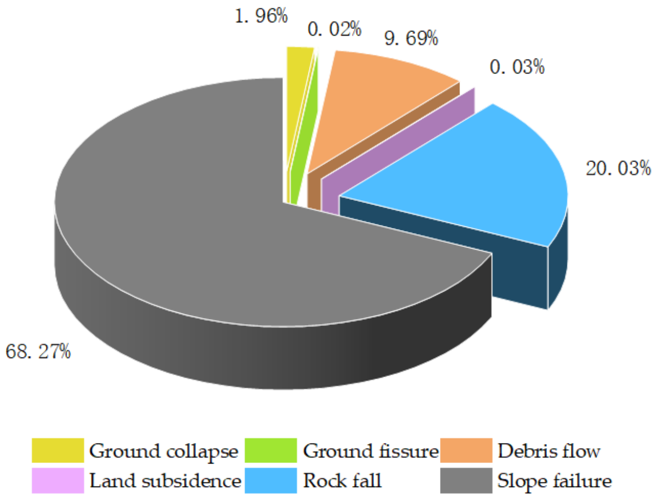

Figure 1. Types of geological disasters in 2019

Due to the indeterminacy and ambiguity of human cognition of objective things and the intricacy of multi-attribute group decision making (MAGDM) environment, linguistic variables can more effectively describe decision information than numerical values [6,7,8]. Therefore, to improve the DM effectiveness, many researchers performed extensive studies on DM challenges in linguistic environments. Zadeh [7] initially proposed the concept of linguistic variables (LVs), which can employ words or sentences to represent qualitative information. In order to solve DM problems with linguistic information, Herrera et al. [9, 10] created a technique for linguistic decision analysis. Later, Xu [11,12,13] developed linguistic aggregation operators and goal programming models to handle DM problems. To tackle incompleteness and ambiguity in DM situations more effectively, Merigó et al. [14,15,16] proposed some linguistic aggregation operators for the aggregation of LVs. Xu [17,18] proposed uncertain LVs given by interval values. Then, some scholars developed various aggregation operators of uncertain LVs for the MAGDM problems with uncertain linguistic information [19–23]. Chen et al. [24] proposed the concept of linguistic intuitionistic fuzzy numbers (LIFNs), which enables the direct description of real and false linguistic information using linguistic items. Liu et al. [25,26] put forward some LIFN aggregation operators for multi-attribute DM. However, LIFN cannot express uncertain and inconsistent linguistic information in DM problems. But the neutrosophic numbers (NNs) [27–30] and neutrosophic sets [28–30] proposed by Smarandache make up for the above shortcomings. Some scholars put forward new concepts focusing on the combination of neutrosophic set and linguistic set. Subsequently, Fang and Ye [8] introduced the linguistic neutrosophic number (LNN) which includes three independent LVs for describe true, false, and uncertain linguistic information. They also introduced the LNN weighted geometric and LNN weighted arithmetic averaging operators to handle MAGDM problems containing LNN information. However, this MAGDM method [8] can be better applicable to the expression and processing of inconsistent and uncertain linguistic information in DM problems. After that, some LNN aggregation operators were successively proposed, such as LNN normalized weighted geometric Bonferroni mean (LNNNWGBM) and LNN normalized weighted Bonferroni mean (LNNNWBM) operators [31] and linguistic neutrosophic power weighted Heronian aggregation (LNPWHA) operators [32]. These aggregation operators can effectively deal with linguistic DM problems with inconsistent and uncertain linguistic information. Furthermore, Shi and Ye developed three correlation coefficients of LNNs [33] and two cosine similarity measures of LNNs [34] for MAGDM problems with LNN information. Cui and Ye [35] defined the concept of hesitant linguistic neutrosophic number (HLNN) sets and introduced the normalized generalized distance and similarity measures of HLNNs for DM problems with HLNN information.

In 1982, Dombi [36] developed Dombi t-conorm and Dombi t-norm operations, which contain the advantage of changeability by adjusting a parameter value. Hence, Liu et al. [37] introduced the Dombi operations of intuitionistic fuzzy sets (IFSs) and proposed some Dombi aggregation operators for the MAGDM problem with intuitionistic fuzzy information. Ye and Chen [38] introduced the Dombi operations of single-valued neutrosophic numbers (SVNNs), then presented the single-valued neutrosophic Dombi weighted geometric average (SVNDWGA) operator and the single-valued neutrosophic Dombi weighted arithmetic average (SVNDWAA) operator to handle DM problems with LNNs. Ye and Lu [39] extended the Dombi operations to the environment of linguistic cubic variables (LCVs) and developed the linguistic cubic variable Dombi weighted geometric average (LCVDWGA) operator and linguistic cubic variable Dombi weighted arithmetic average (LCVDWAA) operator for MAGDM problems. However, in the available research, the Dombi operations have not yet been extended to LNNs. Therefore, the main goals of this study are (1) to propose some Dombi operations of LNNs, (2) to propose the LNN Dombi weighted geometric averaging (LNNDWGA) and LNN Dombi weighted arithmetic averaging (LNNDWAA) operators, (3) to develop a DM method based on the LNNDWAA or LNNDWGA operator for performing MAGDM problems in the LNN information environment, and (4) to validate the viability of this method through a case study.

The following sections constitute the rest of this paper. Section 2 introduces some preliminaries. In Section 3, we define the Dombi operations of LNNs and propose the LNNDWAA and LNNDWGA operators and their properties. Section 4 introduces a new MAGDM method using the LNNDWAA or LNNDWGA operator. In Section 5, the application of the proposed method is demonstrated by an application example and then a comparative analysis is given to show its superiority over existing approaches. Section 6 gives the conclusions of this article.

2. Preliminaries

2.1 Several Concepts of LNNs

Definition 1 [8]. Suppose that $Fr^{Ro}= \{Fr_0^{Ro}, Fr_1^{Ro}, \ldots, Fr_\Phi^{Ro}\}$ is a set of linguistic terms with an odd cardinality $\Phi + 1$. Then, LNN is defined as $N= <Fr_x^{Ro}, Fr_y^{Ro}, Fr_z^{Ro}>$ for $Fr_x^{Ro}, Fr_y^{Ro}, Fr_z^{Ro} \in Fr^{Ro}$ and $x, y, z \in [0, \Phi]$, where Fr_x^{Ro}, Fr_y^{Ro} and Fr_z^{Ro} independently represent truth, uncertainty, and falsity LVs, respectively.

Definition 2 [8]. Set $N = <Fr_x^{Ro}, Fr_y^{Ro}, Fr_z^{Ro}>$ as LNN in Fr^{Ro}. The score and accuracy functions of N are determined by the following eqations:

$$U(N) = (2\Phi + x - y - z)/(3\Phi) \text{ for } U(N) \in [0,1], \tag{1}$$

$$V(N) = (x - z)/\Phi \text{ for } V(N) \in [-1,1]. \tag{2}$$

Definition 3 [8]. Let $N_1 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle$ and $N_2 = \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle$ be two LNNs in Fr^{Ro}, and they imply the following ranking relations:

(1) When $U(N_1) > U(N_2) \Rightarrow N_1 \succ N_2$;

(2) When $U(N_1) < U(N_2) \Rightarrow N_1 \prec N_2$;

(3) When $V(N_1) = V(N_2)$ and $U(N_1) = U(N_2) \Rightarrow N_1 = N_2$;

(4) When $V(N_1) < V(N_2)$ and $U(N_1) = U(N_2) \Rightarrow N_1 \prec N_2$;

(5) When $V(N_1) > V(N_2)$ and $U(N_1) = U(N_2) \Rightarrow N_1 \succ N_2$.

Definition 4 [8]. Let $N_1 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle$ and $N_2 = \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle$ be two LNNs in Fr^{Ro}, and λ is a positive real number ($\lambda > 0$). Their operational laws are introduced as follows:

(1) $N_1 \oplus N_2 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle \oplus \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle = \left(Fr^{Ro}_{x_1+x_2-\frac{x_1x_2}{\Phi}}, Fr^{Ro}_{\frac{y_1y_2}{\Phi}}, Fr^{Ro}_{\frac{z_1z_2}{\Phi}} \right)$;

(2) $N_1 \otimes N_2 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle \otimes \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle = \left(Fr^{Ro}_{\frac{x_1x_2}{\Phi}}, Fr^{Ro}_{y_1+y_2-\frac{y_1y_2}{\Phi}}, Fr^{Ro}_{z_1+z_2-\frac{z_1z_2}{\Phi}} \right)$;

(3) $\lambda N_1 = \lambda \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle = \left\langle Fr^{Ro}_{\Phi-\Phi\left(1-\frac{x_1}{\Phi}\right)^{\lambda}}, Fr^{Ro}_{\Phi\left(\frac{y_1}{\Phi}\right)^{\lambda}}, Fr^{Ro}_{\Phi\left(\frac{z_1}{\Phi}\right)^{\lambda}} \right\rangle$;

(4) $N_1^{\lambda} = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle^{\lambda} = \left\langle Fr^{Ro}_{\Phi\left(\frac{x_1}{\Phi}\right)^{\lambda}}, Fr^{Ro}_{\Phi-\Phi\left(1-\frac{y_1}{\Phi}\right)^{\lambda}}, Fr^{Ro}_{\Phi-\Phi\left(1-\frac{z_1}{\Phi}\right)^{\lambda}} \right\rangle$.

2.2 Weighted Aggregation Operators of LNNs

Definition 5 [8]. Set $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) as an assemblage of LNNs in Fr^{Ro}. The LNNWAA operator is defined below:

$$LNNWAA(N_1, N_2, ..., N_h) = \sum_{g=1}^{h} \gamma_g N_g , \tag{3}$$

where γ_g is the weight of N_g (g = 1, 2, …, h) for $0 \le \gamma_g \le 1$ and $\sum_{g=1}^{h} \gamma_g = 1$.

Theorem 1 [8]. Let $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) as an assemblage of LNN in Fr^{Ro}, then the aggregation result is obtained based on the following aggregation equation:

$$LNNWAA(N_1, N_2, ..., N_h) = \sum_{g=1}^{h} \gamma_g N_g = \left\langle Fr^{Ro}_{\Phi-\Phi\prod_{g=1}^{h}(1-\frac{x_g}{\Phi})^{\gamma_g}}, Fr^{Ro}_{\Phi\prod_{g=1}^{h}(\frac{y_g}{\Phi})^{\gamma_g}}, Fr^{Ro}_{\Phi\prod_{g=1}^{h}(\frac{z_g}{\Phi})^{\gamma_g}} \right\rangle . \tag{4}$$

Definition 6 [8]. Suppose that $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) is a group of LNNs in Fr^{Ro}, the LNNWGA operator is defined by

$$LNNWGA(N_1, N_2, ..., N_h) = \prod_{g=1}^{h} N_g^{\gamma_g} , \tag{5}$$

where γ_g is the weight of N_g (g = 1, 2, …, h) for $0 \le \gamma_g \le 1$ and $\sum_{g=1}^{h}\gamma_g = 1$.

Theorem 2 [8]. Let $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) as an assemblage of linguistic neutrosophic numbers in Fr^{Ro}, then the result of aggregation is obtained based on the following aggregation equation:

$$LNNWGA(N_1, N_2, ..., N_h) = \prod_{g=1}^{h} N_g^{\gamma_g} = \left\langle Fr^{Ro}_{\Phi\prod_{g=1}^{h}(\frac{x_g}{\Phi})^{\gamma_g}}, Fr^{Ro}_{\Phi-\Phi\prod_{g=1}^{h}(1-\frac{y_g}{\Phi})^{\gamma_g}}, Fr^{Ro}_{\Phi-\Phi\prod_{g=1}^{h}(1-\frac{z_g}{\Phi})^{\gamma_g}} \right\rangle. \quad (6)$$

3. Dombi Operations of LNNs

The Dombi operations contain the advantage of flexible aggregations by modifying the value of the parameter. In 1982, Dombi [36] proposed the Dombi T-norm and T-conorm operations for the first time. Although many researchers have introduced Dombi operations in various linguistic decision-making environments and decision-making methods [37–43], the Dombi operations have not yet expanded to LNNs. Therefore, this section proposes the Dombi T-norm and T-conorm operations of LNNs, then presents the LNNDWAA and LNNDWGA operators and their properties.

3.1 Dombi Operational Laws of LNNs

Definition 7 [36]. For any two real-values *Th* and *Tj*, the Dombi T-norm and T-conorm operations between *Th* and *Tj* are defined below:

$$O_D(Th, Tj) = \frac{1}{1 + \left\{ \left(\frac{1-Th}{Th}\right)^{\rho} + \left(\frac{1-Tj}{Tj}\right)^{\rho} \right\}^{1/\rho}}, \quad (7)$$

$$O_D^c(Th, Tj) = 1 - \frac{1}{1 + \left\{ \left(\frac{Th}{1-Th}\right)^{\rho} + \left(\frac{Tj}{1-Tj}\right)^{\rho} \right\}^{1/\rho}}, \quad (8)$$

where the parameter $\rho \ge 1$ and $(Th, Tj) \in [0, 1] \times [0, 1]$.

Definition 8. Assume that $N_1 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle$ and $N_2 = \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle$ are two LNNs, $\lambda > 0$, and $\rho > 0$. Then, the Dombi T-norm and T-conorm operational laws of LNNs are expressed below:

$$N_1 \oplus N_2 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle \oplus \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle$$

$$= \left(Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\left(\frac{x_1/\Phi}{1-x_1/\Phi}\right)^{\rho}+\left(\frac{x_2/\Phi}{1-x_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)}, Fr^{Ro}_{\Phi\times\frac{1}{1+\left\{\left(\frac{1-y_1/\Phi}{y_1/\Phi}\right)^{\rho}+\left(\frac{1-y_2/\Phi}{y_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi\times\frac{1}{1+\left\{\left(\frac{1-z_1/\Phi}{z_1/\Phi}\right)^{\rho}+\left(\frac{1-z_2/\Phi}{z_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}} \right), \quad (9)$$

$$= \left(Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\left(\frac{x_1}{\Phi-x_1}\right)^{\rho}+\left(\frac{x_2}{\Phi-x_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi-y_1}{y_1}\right)^{\rho}+\left(\frac{\Phi-y_2}{y_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi-z_1}{z_1}\right)^{\rho}+\left(\frac{\Phi-z_2}{z_2}\right)^{\rho}\right\}^{1/\rho}}} \right)$$

$$N_1 \otimes N_2 = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle \otimes \left\langle Fr_{x_2}^{Ro}, Fr_{y_2}^{Ro}, Fr_{z_2}^{Ro} \right\rangle$$

$$= \left(Fr^{Ro}_{\Phi\times\frac{1}{1+\left\{\left(\frac{1-x_1/\Phi}{x_1/\Phi}\right)^{\rho}+\left(\frac{1-x_2/\Phi}{x_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\left(\frac{y_1/\Phi}{1-y_1/\Phi}\right)^{\rho}+\left(\frac{y_2/\Phi}{1-y_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)}, Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\left(\frac{z_1/\Phi}{1-z_1/\Phi}\right)^{\rho}+\left(\frac{z_2/\Phi}{1-z_2/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)} \right), \quad (10)$$

$$= \left(Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi-x_1}{x_1}\right)^{\rho}+\left(\frac{\Phi-x_2}{x_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\left(\frac{y_1}{\Phi-y_1}\right)^{\rho}+\left(\frac{y_2}{\Phi-y_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\left(\frac{z_1}{\Phi-z_1}\right)^{\rho}+\left(\frac{z_2}{\Phi-z_2}\right)^{\rho}\right\}^{1/\rho}}} \right)$$

$$\lambda N_1 = \lambda \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle$$

$$= \left(Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\lambda\left(\frac{x_1/\Phi}{1-x_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}\right)}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{1-y_1/\Phi}{y_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{1-z_1/\Phi}{z_1/\Phi}\right)^{\rho}\right\}^{1/\rho}}} \right), \quad (11)$$

$$= \left(Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\lambda\left(\frac{x_1}{\Phi-x_1}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{\Phi-y_1}{y_1}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{\Phi-z_1}{z_1}\right)^{\rho}\right\}^{1/\rho}}} \right)$$

$$N_1^\lambda = \left\langle Fr_{x_1}^{Ro}, Fr_{y_1}^{Ro}, Fr_{z_1}^{Ro} \right\rangle^\lambda$$
$$= \left(Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{1-x_1/\Phi}{x_1/\Phi}\right)^\rho\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\lambda\left(\frac{y_1/\Phi}{1-y_1/\Phi}\right)^\rho\right\}^{1/\rho}}\right)}, Fr^{Ro}_{\Phi\times\left(1-\frac{1}{1+\left\{\lambda\left(\frac{z_1/\Phi}{1-z_1/\Phi}\right)^\rho\right\}^{1/\rho}}\right)} \right). \quad (12)$$
$$= \left(Fr^{Ro}_{\frac{\Phi}{1+\left\{\lambda\left(\frac{\Phi-x_1}{x_1}\right)^\rho\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\lambda\left(\frac{y_1}{\Phi-y_1}\right)^\rho\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\lambda\left(\frac{z_1}{\Phi-z_1}\right)^\rho\right\}^{1/\rho}}} \right)$$

However, the operational results of the equations (9)–(12) are also LNNs.

Example 1. Let N_1 = <Fr_2^{Ro}, Fr_1^{Ro}, Fr_3^{Ro}> and N_2 = <Fr_3^{Ro}, Fr_2^{Ro}, Fr_4^{Ro}> in Fr^{Ro}= {Fr_0^{Ro}, Fr_1^{Ro}, …, Fr_6^{Ro}} be two LNNs, λ = 0.5, ρ = 1. Based on the equations (9)–(12), we have the following operational results:

$$N_1 \oplus N_2 = \left\langle Fr_2^{Ro}, Fr_1^{Ro}, Fr_3^{Ro} \right\rangle \oplus \left\langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_4^{Ro} \right\rangle$$
$$= \left(Fr^{Ro}_{6-\frac{6}{1+\left\{\left(\frac{2}{6-2}\right)^1+\left(\frac{3}{6-3}\right)^1\right\}^{1/1}}}, Fr^{Ro}_{\frac{6}{1+\left\{\left(\frac{6-1}{1}\right)^1+\left(\frac{6-2}{2}\right)^1\right\}^{1/1}}}, Fr^{Ro}_{\frac{6}{1+\left\{\left(\frac{6-3}{3}\right)^1+\left(\frac{6-4}{4}\right)^1\right\}^{1/1}}} \right) = \left\langle Fr_{3.6000}^{Ro}, Fr_{0.7500}^{Ro}, Fr_{2.4000}^{Ro} \right\rangle'$$

$$N_1 \otimes N_2 = \left\langle Fr_2^{Ro}, Fr_1^{Ro}, Fr_3^{Ro} \right\rangle \otimes \left\langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_4^{Ro} \right\rangle$$
$$= \left(Fr^{Ro}_{\frac{6}{1+\left\{\left(\frac{6-2}{2}\right)^1+\left(\frac{6-3}{3}\right)^1\right\}^{1/1}}}, Fr^{Ro}_{6-\frac{6}{1+\left\{\left(\frac{1}{6-1}\right)^1+\left(\frac{2}{6-2}\right)^1\right\}^{1/1}}}, Fr^{Ro}_{6-\frac{6}{1+\left\{\left(\frac{3}{6-3}\right)^1+\left(\frac{4}{6-4}\right)^1\right\}^{1/1}}} \right) = \left\langle Fr_{1.5000}^{Ro}, Fr_{2.4706}^{Ro}, Fr_{4.5000}^{Ro} \right\rangle'$$

$$\lambda N_1 = \lambda \left\langle Fr_2^{Ro}, Fr_1^{Ro}, Fr_3^{Ro} \right\rangle$$
$$= \left(Fr^{Ro}_{6-\frac{6}{1+\left\{0.5\times\left(\frac{2}{6-2}\right)^1\right\}^{1/1}}}, Fr^{Ro}_{\frac{6}{1+\left\{0.5\times\left(\frac{6-1}{1}\right)^1\right\}^{1/1}}}, Fr^{Ro}_{\frac{6}{1+\left\{0.5\times\left(\frac{6-3}{3}\right)^1\right\}^{1/1}}} \right) = \left\langle Fr_{1.2000}^{Ro}, Fr_{1.7143}^{Ro}, Fr_{4.0000}^{Ro} \right\rangle'$$

$$N_1^\lambda = \left\langle Fr_2^{Ro}, Fr_1^{Ro}, Fr_3^{Ro} \right\rangle^\lambda$$
$$= \left(Fr^{Ro}_{\frac{6}{1+\left\{0.5\times\left(\frac{6-2}{2}\right)^1\right\}^{1/1}}}, Fr^{Ro}_{6-\frac{6}{1+\left\{0.5\times\left(\frac{1}{6-1}\right)^1\right\}^{1/1}}}, Fr^{Ro}_{6-\frac{6}{1+\left\{0.5\times\left(\frac{3}{6-3}\right)^1\right\}^{1/1}}} \right) = \left\langle Fr_{3.0000}^{Ro}, Fr_{0.5455}^{Ro}, Fr_{2.0000}^{Ro} \right\rangle.$$

3.2 Dombi Weighted Aggregation Operators of LNNs

Definition 9. Set $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) as a group of LNNs. Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_h)$ be the weight vector of N_g such that $\gamma_g \in [0, 1]$ and $\sum_{g=1}^{h}\gamma_g = 1$. The LNNDWAA and LNNDWGA operators are proposed below:

$$LNNDWAA(N_1, N_2, ..., N_h) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_g, \tag{13}$$

$$LNNDWGA(N_1, N_2, ..., N_h) = \mathop{\otimes}_{g=1}^{h} N_g^{\gamma_g}. \tag{14}$$

Theorem 3. Let $N_g = \left\langle Fr_{x_g}^{Ro}, Fr_{y_g}^{Ro}, Fr_{z_g}^{Ro} \right\rangle$ (g = 1, 2, …, h) be an assemblage of LNNs and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_h)$ be the weight vector of N_g such that $\gamma_g \in [0, 1]$ and $\sum_{g=1}^{h}\gamma_g = 1$. The aggregated result of the LNNDWAA operator is still an LNN, which can be expressed by

$$LNNDWAA(N_1, N_2, ..., N_h) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_g = \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle. \tag{15}$$

Theorem 3 is proved through mathematical induction below.

Proof:

(a) Let h = 2. Based on Definition 8 we can obtain

$$LNNDWAA(N_1, N_2) = N_1 \oplus N_2$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\gamma_1\left(\frac{x_1}{\Phi - x_1}\right)^{\rho} + \gamma_2\left(\frac{x_2}{\Phi - x_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\gamma_1\left(\frac{\Phi - y_1}{y_1}\right)^{\rho} + \gamma_2\left(\frac{\Phi - y_2}{y_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\gamma_1\left(\frac{\Phi - z_1}{z_1}\right)^{\rho} + \gamma_2\left(\frac{\Phi - z_2}{z_2}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

(b) If $h = k$, we can keep the following result from the equation (15):

$$LNNDWAA(N_1, N_2, ..., N_k) = \overset{k}{\underset{g=1}{\oplus}} \gamma_g N_g$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$

(c) Set $h = k + 1$. Based on Definition 9 and the equation (15), there exists the following result:

$$LNNDWAA(N_1, N_2, ..., N_k, N_{k+1}) = \overset{k+1}{\underset{g=1}{\oplus}} \gamma_g N_g$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \oplus \gamma_{k+1} N_{k+1}.$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

In terms of the above results, the equation (15) can hold for all h.

Then, the LNNDWAA operator has some properties:

(1) Reducibility: When $\gamma = (1/h, 1/h, ..., 1/h)$, there exists

$$LNNDWAA(N_1, N_2, ..., N_h) = \overset{h}{\underset{g=1}{\oplus}} \gamma_g N_g$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$

(2) Idempotency: Let all LNNs be $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle = N$ (g = 1, 2, ..., h). Then, LNNDWAA(N_1, N_2, ..., N_h) = N.

(3) Commutativity: Let the LNN sequence (N_1', N_2', ..., N_h') be an arbitrary arrangement of (N_1, N_2, ..., N_h). Then, there is LNNDWAA(N_1', N_2', ..., N_h') = LNNDWAA(N_1, N_2, ..., N_h).

(4) Boundedness: If the maximum and minimum LNNs are $N_{\max} = \left\langle Fr^{Ro}_{\max_g(x_g)}, Fr^{Ro}_{\min_g(y_g)}, Fr^{Ro}_{\min_g(z_g)} \right\rangle$ and $N_{\min} = \left\langle Fr^{Ro}_{\min_g(x_g)}, Fr^{Ro}_{\max_g(y_g)}, Fr^{Ro}_{\max_g(z_g)} \right\rangle$, then $N_{\min} \le$ LNNDWAA(N_1, N_2, ..., N_h) $\le N_{\max}$.

Proof:

(1) Based on the equation (15), we can see that the property (1) is valid.

(2) Since $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle = N$ (g = 1, 2, ..., h), by the equation (15) we can obtain the following result:

$$LNNDWAA(N_1, N_2, ..., N_h) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_g$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\left(\frac{x}{\Phi - x}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi - y}{y}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\left(\frac{\Phi - z}{z}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

$$= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left(\frac{x}{\Phi - x}\right)}}, Fr^{Ro}_{\frac{\Phi}{1+\left(\frac{\Phi - y}{y}\right)}}, Fr^{Ro}_{\frac{\Phi}{1+\left(\frac{\Phi - z}{z}\right)}} \right\rangle = \left\langle Fr^{Ro}_{x}, Fr^{Ro}_{y}, Fr^{Ro}_{z} \right\rangle = N.$$

Hence, LNNDWAA(N_1, N_2, ..., N_h) = N holds.

(3) The property (3) is obvious.

(4) Since $\min_g(x_g) \le x_g \le \max_g(x_g), \max_g(y_g) \le y_g \le \min_g(y_g), \max_g(z_g) \le z_g \le \min_g(z_g)$, there are the following inequalities:

$$\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\min_g(x_g)}{\Phi - \min_g(x_g)}\right)^{\rho}\right\}^{1/\rho}} = \min_g(x_g) \le \Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{x_g}{\Phi - x_g}\right)^{\rho}\right\}^{1/\rho}} \le \max_g(x_g) = \Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\max_g(x_g)}{\Phi - \max_g(x_g)}\right)^{\rho}\right\}^{1/\rho}},$$

$$\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - \max_g(y_g)}{\max_g(y_g)}\right)^{\rho}\right\}^{1/\rho}} = \max_g(y_g) \le \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - y_g}{y_g}\right)^{\rho}\right\}^{1/\rho}} \le \min_g(y_g) = \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - \min_g(y_g)}{\min_g(y_g)}\right)^{\rho}\right\}^{1/\rho}},$$

$$\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - \max_g(z_g)}{\max_g(z_g)}\right)^{\rho}\right\}^{1/\rho}} = \max_g(z_g) \le \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - z_g}{z_g}\right)^{\rho}\right\}^{1/\rho}} \le \min_g(z_g) = \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi - \min_g(z_g)}{\min_g(z_g)}\right)^{\rho}\right\}^{1/\rho}}.$$

Therefore, $N_{\min}$ ≤ LNNDWAA(N_1, N_2, ..., N_h) ≤ $N_{\max}$ is true.

Theorem 4. Let $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle$ (g = 1, 2, ..., h) be a group of LNNs and $\gamma = (\gamma_1, \gamma_2, ..., \gamma_h)$ be the weight vector of N_g (g = 1, 2, ..., h) for $\gamma_g \in [0, 1]$ and $\sum_{g=1}^{h} \gamma_g = 1$. The aggregated result of the LNNDWAA operator is still an LNN, which can be expressed by

$$LNNDWGA(N_1, N_2, ..., N_h) = \overset{h}{\underset{g=1}{\otimes}} N_g^{\gamma_g}$$
$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle. \quad (16)$$

Theorem 4 is also proved based on mathematical induction, which is given below.

Proof:

(a) Let h = 2. Based on Definition 8 we can obtain

$$LNNDWGA(N_1, N_2) = N_1 \otimes N_2$$
$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\gamma_1\left(\frac{\Phi-x_1}{x_1}\right)^{\rho}+\gamma_2\left(\frac{\Phi-x_2}{x_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\gamma_1\left(\frac{y_1}{\Phi-y_1}\right)^{\rho}+\gamma_2\left(\frac{y_2}{\Phi-y_2}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\gamma_1\left(\frac{z_1}{\Phi-z_1}\right)^{\rho}+\gamma_2\left(\frac{z_2}{\Phi-z_2}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$
$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{2}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)\right\}^{1/\rho}}} \right\rangle$$

(b) If $h = k$, we can get the following equation from the equation (16):

$$LNNDWGA(N_1, N_2, ..., N_k) = \overset{k}{\underset{g=1}{\otimes}} N_g^{\gamma_g}$$
$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle.$$

(c) If $h = k + 1$, there exists the following result:

$$LNNDWGA(N_1, N_2, ..., N_k, N_{k+1}) = \overset{k+1}{\underset{g=1}{\otimes}} N_g^{\gamma_g}$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \otimes \gamma_{k+1} N_{k+1} .$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{k+1}\gamma_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle$$

In terms of the above results, the equation (16) is true for all *h*.

The LNNDWGA operator also contains some properties:

(1) Reducibility: When the weight vector is γ = (1/*h*, 1/*h*, ..., 1/*h*), the equation (16) yields the following result:

$$LNNDWGA(N_1, N_2, ..., N_h) = \overset{h}{\underset{g=1}{\otimes}} N_g^{\gamma_g}$$

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\frac{1}{h}\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle .$$

(2) Idempotency: Let all LNNs be $N_g = \left\langle Fr^{Ro}_{x_g}, Fr^{Ro}_{y_g}, Fr^{Ro}_{z_g} \right\rangle$ = *N* (*g* = 1, 2, ..., *h*). Then, LNNDWGA(N_1, N_2, ..., N_h) = *N*.

(3) Commutativity: Let the LNN sequence (N_1', N_2', ..., N_h') be any permutation of (N_1, N_2, ..., N_h). Then, there is LNNDWGA(N_1', N_2', ..., N_h') = LNNDWGA(N_1, N_2, ..., N_h).

(4) Boundedness: If the maximum and minimum LNNs are $N_{\max} = \left\langle Fr^{Ro}_{\max_g(x_g)}, Fr^{Ro}_{\min_g(y_g)}, Fr^{Ro}_{\min_g(z_g)} \right\rangle$ and $N_{\min} = \left\langle Fr^{Ro}_{\min_g(x_g)}, Fr^{Ro}_{\max_g(y_g)}, Fr^{Ro}_{\max_g(z_g)} \right\rangle$, then $N_{\min}$ ≤ LNNDWGA(N_1, N_2, ..., N_h) ≤ $N_{\max}$.

Since the characteristics of the LNNDWGA operator can be easily proved by the similar proof process of the characteristics of the LNNDWAA operator, it is omitted here.

4. MAGDM Method based on the LNNDWAA and LNNDWGA Operators

This section proposed a new DM method based on the LNNDWAA and LNNDWGA operators to solve MAGDM problems in the LNN environment.

In a MAGDM problem, let *P* = {P_1, P_2, ..., P_u} be a set of alternatives and Ψ = {Ψ_1, Ψ_2, ..., Ψ_h} be a set of attributes. The weight vector of the attributes Ψ_g (*g* = 1, 2, ..., *h*) is γ = (γ_1, γ_2, ..., γ_h). Assume that there is a group of decision-makers Ω = {Ω_1, Ω_2, ..., Ω_r} with their weight vector η = (η_1, η_2, ..., η_r).

Each decision maker evaluates the value of each attribute Ψ_g (g = 1, 2, ..., h) for each alternative P_i from the set of linguistic terms Fr^{Ro} = {Fr_0^{Ro} = very low, Fr_1^{Ro} = low, Fr_2^{Ro} = slightly low, Fr_3^{Ro} = medium, Fr_4^{Ro} = slightly high, Fr_5^{Ro} = high, Fr_6^{Ro} = very high}. According to the linguistic terms, each decision-maker can assign the three linguistic values of indeterminacy, falsity, and truth to each attribute Ψ_g for the alternative P_v. Thus, LNN is composed of the obtained linguistic values. Hence, the LNN assessment information of the attributes Ψ_g (g = 1, 2, ..., h) for the alternatives P_v (v = 1, 2, ..., u) provided by each decision-maker Ω_s (s = 1, 2, ..., r) can establish the LNN decision matrix $M_s = \left(N_{vg}^s\right)_{u\times h}$, where $N_{vg}^s = \left\langle Fr_{x_{vg}^s}^{Ro}, Fr_{y_{vg}^s}^{Ro}, Fr_{z_{vg}^s}^{Ro}\right\rangle$ (s = 1, 2, ..., r; v = 1, 2, ..., u; g = 1, 2, ..., h) are LNNs.

Then, we present a MAGDM method using the score function (accuracy function) and the LNNDWAA and LNNDWGA operators to perform the MAGDM problem with LNN information. Here, the MAGDM method is described by the specific decision-making steps below.

Step 1: Aggregate all M_s (s = 1, 2, ..., r) by using the following LNNDWAA or LNNDWGA operator:

$$\begin{aligned} N_{vg} &= LNNDWAA(N_{vg}^1, N_{vg}^2, ..., N_{vg}^r) = \mathop{\oplus}_{s=1}^{r} \eta_s N_{vg}^s \\ &= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{x_{vg}^s}{\Phi - x_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{\Phi - y_{vg}^s}{y_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{\Phi - z_{vg}^s}{z_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \end{aligned} \tag{17}$$

or

$$\begin{aligned} N_{vg} &= LNNDWGA(N_{vg}^1, N_{vg}^2, ..., N_{vg}^r) = \mathop{\otimes}_{s=1}^{r} \left(N_{vg}^s\right)^{\eta_s} \\ &= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{\Phi - x_{vg}^s}{x_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{G - \frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{y_{vg}^s}{\Phi - y_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{s=1}^{r}\gamma_s\left(\frac{z_{vg}^s}{\Phi - z_{vg}^s}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \end{aligned} \tag{18}$$

to obtain the integrated matrix $R = \left(N_{vg}\right)_{u\times h}$, where $N_{vg} = \left\langle Fr_{x_{vg}}^{Ro}, Fr_{y_{vg}}^{Ro}, Fr_{z_{vg}}^{Ro}\right\rangle$ (v = 1, 2, ..., u; g = 1, 2, ..., h) are integrated LNNs.

Step 2: Use the following LNNDWAA or LNNDWGA operator to obtain the collective overall LNNs N_v for P_v (v = 1, 2, ..., u):

$$\begin{aligned} N_{v} &= LNNDWAA(N_{v1}, N_{v2}, ..., N_{vh}) = \mathop{\oplus}_{g=1}^{h} \gamma_g N_{vg} \\ &= \left\langle Fr^{Ro}_{\Phi - \frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{x_{g}}{\Phi - x_{g}}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{\Phi - y_{g}}{y_{g}}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{\Phi - z_{g}}{z_{g}}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle \end{aligned} \tag{19}$$

$$N_v = LNNDWGA(N_{v1}, N_{v2}, ..., N_{vh}) = \underset{g=1}{\otimes} N_{vg}^{\gamma_g}$$

or

$$= \left\langle Fr^{Ro}_{\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{\Phi-x_g}{x_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{y_g}{\Phi-y_g}\right)^{\rho}\right\}^{1/\rho}}}, Fr^{Ro}_{\Phi-\frac{\Phi}{1+\left\{\sum_{g=1}^{h}\eta_g\left(\frac{z_g}{\Phi-z_g}\right)^{\rho}\right\}^{1/\rho}}} \right\rangle. \quad (20)$$

Step 3: Calculate the score values of $U(N_v)$ (the accuracy values of $V(N_v)$) (v = 1, 2, ..., u) through the equation (1) (the equation (2)).

Step 4: Rank all alternatives in decreasing order, then select the more reasonable one.

Step 5: End.

5. An Illustrative Example on Slope Treatment Scheme Selection

The application of the MAGDM method proposed in this paper is illustrated by the selection of slope treatment schemes. To avoid slope instability, a set of four slope treatment options $P = \{P_1, P_2, P_3, P_4\}$ is proposed, where P_1 is gravity retaining wall + lattice protection; P_2 is anti-slide retaining wall + anti-slide pile; P_3 is anchor retaining wall + lattice protection; and P_4 is pile-plate retaining wall. The evaluation of the schemes should meet the following attribute requirements: (1) Ψ_1 is the economic status; (2) Ψ_2 is the security situation; (3) Ψ_3 is the construction feasibility; and (4) Ψ_4 is the environment situation. The weight vector of the four attributes is assigned as γ = (0.23, 0.28, 0.26, 0.23). Assume that three experts are invited as a group of decision makers $\Omega = \{\Omega_1, \Omega_2, \Omega_3\}$, then the weight vecror η = (0.29, 0.33, 0.38) is given to indicate the importance of the various decision makers.

Decision makers need to assess the four attributes on the four alternatives from the linguistic term set $Fr^{Ro} = \{Fr_0^{Ro}$ = very low, Fr_1^{Ro} = low, Fr_2^{Ro} = slightly low, Fr_3^{Ro} = medium, Fr_4^{Ro} = slightly high, Fr_5^{Ro} = high, Fr_6^{Ro}= very high$\}$ with Φ = 6. Thus, the linguistic evaluation results of each decision-maker Ω_s (s = 1, 2, 3) can be established as the LNN decision matrices M_1, M_2, and M_3:

$$M_1 = \begin{bmatrix} \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_1^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_4^{Ro}, Fr_5^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_3^{Ro}\rangle \\ \langle Fr_5^{Ro}, Fr_2^{Ro}, Fr_1^{Ro}\rangle & \langle Fr_5^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_4^{Ro}, Fr_5^{Ro}\rangle \\ \langle Fr_4^{Ro}, Fr_4^{Ro}, Fr_5^{Ro}\rangle & \langle Fr_5^{Ro}, Fr_3^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_5^{Ro}, Fr_2^{Ro}, Fr_1^{Ro}\rangle & \langle Fr_5^{Ro}, Fr_4^{Ro}, Fr_3^{Ro}\rangle \\ \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_3^{Ro}\rangle \\ \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_1^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_4^{Ro}, Fr_4^{Ro}\rangle \\ \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_4^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle \\ \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_3^{Ro}\rangle & \langle Fr_4^{Ro}, Fr_1^{Ro}, Fr_2^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_4^{Ro}\rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_2^{Ro}\rangle \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_3^{Ro} \rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_3^{Ro} \rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_3^{Ro} \rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_1^{Ro} \rangle \\ \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_2^{Ro} \rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_2^{Ro} \rangle & \langle Fr_3^{Ro}, Fr_3^{Ro}, Fr_4^{Ro} \rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_3^{Ro} \rangle \\ \langle Fr_3^{Ro}, Fr_4^{Ro}, Fr_2^{Ro} \rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro} \rangle & \langle Fr_4^{Ro}, Fr_2^{Ro}, Fr_3^{Ro} \rangle & \langle Fr_4^{Ro}, Fr_3^{Ro}, Fr_2^{Ro} \rangle \\ \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_5^{Ro} \rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_3^{Ro} \rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_1^{Ro} \rangle & \langle Fr_3^{Ro}, Fr_2^{Ro}, Fr_3^{Ro} \rangle \end{bmatrix}$$

The decision procedures based on the LNNDWAA operator are indicated below.

Step 1: Aggregate the decision matrices M_1, M_2, and M_3 by the equation (17) for $\rho = 1$ and obtain the integrated matrix $R = \left(N_{vg}\right)_{4\times 4}$:

$$R = \begin{bmatrix} \langle Fr_{3.4249}^{Ro}, Fr_{2.0000}^{Ro}, Fr_{2.5751}^{Ro} \rangle & \langle Fr_{4.0000}^{Ro}, Fr_{2.0000}^{Ro}, Fr_{2.0033}^{Ro} \rangle & \langle Fr_{3.4790}^{Ro}, Fr_{3.2345}^{Ro}, Fr_{2.8599}^{Ro} \rangle & \langle Fr_{3.4249}^{Ro}, Fr_{2.5751}^{Ro}, Fr_{1.7045}^{Ro} \rangle \\ \langle Fr_{4.4496}^{Ro}, Fr_{2.2472}^{Ro}, Fr_{1.2346}^{Ro} \rangle & \langle Fr_{4.2808}^{Ro}, Fr_{2.5751}^{Ro}, Fr_{2.2472}^{Ro} \rangle & \langle Fr_{3.0000}^{Ro}, Fr_{2.6201}^{Ro}, Fr_{4.0000}^{Ro} \rangle & \langle Fr_{4.0000}^{Ro}, Fr_{2.5751}^{Ro}, Fr_{3.7430}^{Ro} \rangle \\ \langle Fr_{3.3799}^{Ro}, Fr_{3.6036}^{Ro}, Fr_{2.7933}^{Ro} \rangle & \langle Fr_{4.3051}^{Ro}, Fr_{3.2698}^{Ro}, Fr_{2.2140}^{Ro} \rangle & \langle Fr_{4.4496}^{Ro}, Fr_{2.2472}^{Ro}, Fr_{1.7192}^{Ro} \rangle & \langle Fr_{4.4496}^{Ro}, Fr_{3.2345}^{Ro}, Fr_{2.2140}^{Ro} \rangle \\ \langle Fr_{3.3799}^{Ro}, Fr_{2.2140}^{Ro}, Fr_{3.0211}^{Ro} \rangle & \langle Fr_{3.4249}^{Ro}, Fr_{1.5038}^{Ro}, Fr_{2.2901}^{Ro} \rangle & \langle Fr_{3.0000}^{Ro}, Fr_{2.2472}^{Ro}, Fr_{1.8692}^{Ro} \rangle & \langle Fr_{3.3799}^{Ro}, Fr_{2.5210}^{Ro}, Fr_{2.2901}^{Ro} \rangle \end{bmatrix}.$$

Step 2: Through the equation (19), obtain the collective overall LNNs of N_v for P_v (v = 1, 2, 3, 4) below:

N_1 = $<Fr_{3.6290}^{Ro}, Fr_{2.3546}^{Ro}, Fr_{2.1981}^{Ro}>$, N_2 = $<Fr_{4.0502}^{Ro}, Fr_{2.6660}^{Ro}, Fr_{2.2865}^{Ro}>$, N_3 = $<Fr_{4.2426}^{Ro}, Fr_{2.9738}^{Ro}, Fr_{2.1555}^{Ro}>$, and N_4 = $<Fr_{3.3043}^{Ro}, Fr_{2.0120}^{Ro}, Fr_{2.2835}^{Ro}>$.

Step 3: Calculate the score values of $U(N_v)$ (v = 1, 2, 3, 4) by the equation (1):

$U(N_1) = 0.6154$, $U(N_2) = 0.6165$, $U(N_3) = 0.6174$, and $U(N_4) = 0.6116$.

Step 4: Rank the four alternatives: $P_3 \succ P_2 \succ P_1 \succ P_4$. It can be seen that P_3 is the most reasonable option among the four ones.

Or the decision procedures based on the LNNDWGA operator are indicated below.

Step 1: Aggregate the decision matrices M_1, M_2, and M_3 by the equation (18) for $\rho = 1$ and obtain the integrated matrix $R = \left(N_{vg}\right)_{4\times 4}$:

$$R = \begin{bmatrix} \langle Fr_{3.2698}^{Ro}, Fr_{2.0000}^{Ro}, Fr_{2.7302}^{Ro} \rangle & \langle Fr_{4.0000}^{Ro}, Fr_{2.0000}^{Ro}, Fr_{3.1401}^{Ro} \rangle & \langle Fr_{3.3149}^{Ro}, Fr_{3.3799}^{Ro}, Fr_{3.9967}^{Ro} \rangle & \langle Fr_{3.2698}^{Ro}, Fr_{2.7302}^{Ro}, Fr_{2.4623}^{Ro} \rangle \\ \langle Fr_{4.2463}^{Ro}, Fr_{2.3964}^{Ro}, Fr_{1.4338}^{Ro} \rangle & \langle Fr_{3.7430}^{Ro}, Fr_{2.7302}^{Ro}, Fr_{2.3964}^{Ro} \rangle & \langle Fr_{3.0000}^{Ro}, Fr_{2.7655}^{Ro}, Fr_{4.0000}^{Ro} \rangle & \langle Fr_{4.0000}^{Ro}, Fr_{3.7099}^{Ro}, Fr_{4.2808}^{Ro} \rangle \\ \langle Fr_{3.2345}^{Ro}, Fr_{3.7528}^{Ro}, Fr_{3.9798}^{Ro} \rangle & \langle Fr_{3.8023}^{Ro}, Fr_{3.4249}^{Ro}, Fr_{2.3526}^{Ro} \rangle & \langle Fr_{4.2463}^{Ro}, Fr_{2.3964}^{Ro}, Fr_{2.2570}^{Ro} \rangle & \langle Fr_{4.2463}^{Ro}, Fr_{3.3799}^{Ro}, Fr_{2.3526}^{Ro} \rangle \\ \langle Fr_{3.2345}^{Ro}, Fr_{2.3526}^{Ro}, Fr_{4.2222}^{Ro} \rangle & \langle Fr_{3.2698}^{Ro}, Fr_{1.7173}^{Ro}, Fr_{2.4497}^{Ro} \rangle & \langle Fr_{3.0000}^{Ro}, Fr_{2.3964}^{Ro}, Fr_{3.4093}^{Ro} \rangle & \langle Fr_{3.2345}^{Ro}, Fr_{2.6851}^{Ro}, Fr_{2.4497}^{Ro} \rangle \end{bmatrix}.$$

Step 2: Through the equation (20), obtain the collective overall LNNs N_v for P_v (v = 1, 2, 3, 4) below:

N_1 = $<Fr_{3.4588}^{Ro}, Fr_{2.6338}^{Ro}, Fr_{3.2455}^{Ro}>$, N_2 = $<Fr_{3.6611}^{Ro}, Fr_{2.9722}^{Ro}, Fr_{3.4480}^{Ro}>$, N_3 = $<Fr_{3.8440}^{Ro}, Fr_{3.3047}^{Ro}, Fr_{2.9054}^{Ro}>$, and N_4 = $<Fr_{3.1795}^{Ro}, Fr_{2.2959}^{Ro}, Fr_{3.3218}^{Ro}>$.

Step 3: Obtain the score values of $U(N_v)$ (v = 1, 2, 3, 4) by the equation (1):

$U(N_1) = 0.5322$, $U(N_2) = 0.5134$, $U(N_3) = 0.5352$, and $U(N_4) = 0.5312$.

Step 4: Rank the four alternatives: $P_3 \succ P_1 \succ P_4 \succ P_2$. It can be seen that P_3 is the most reasonable choice among the four ones.

We can repeat the above decision process by changing the parameter ρ from 2 to 4. The sorting results obtained by using the LNNDWAA operator are shown in Figure 2. Then, the ranking orders based on the LNNDWGA operator are indicated in Figure 3.

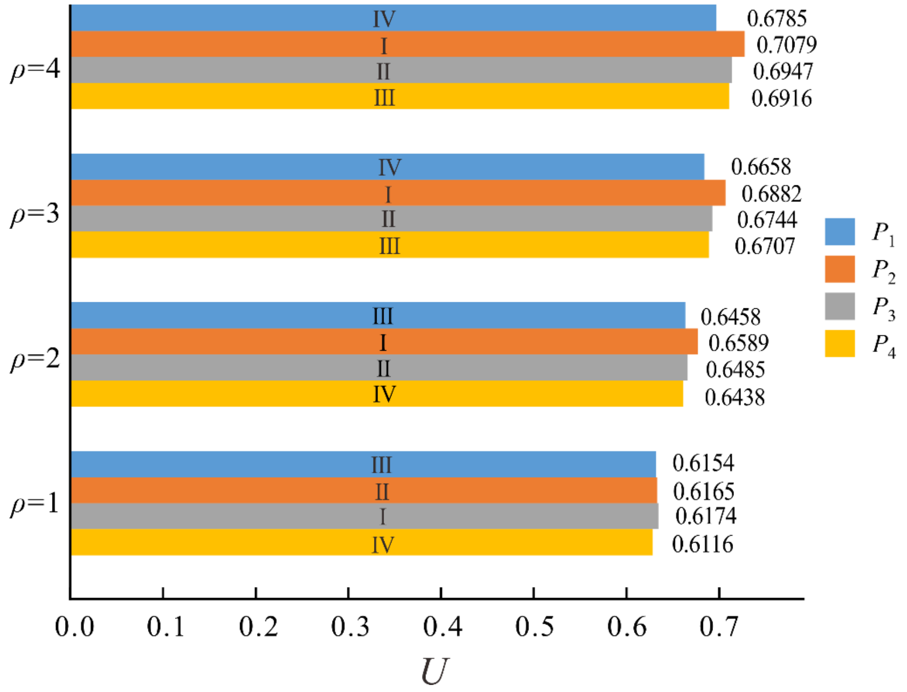

Figure 2. Ranking orders of the four alternatives based on the LNNDWAA operator (I, II, III, IV are ranking numbers)

As shown in Figure 2, the sorting results obtained based on the LNNDWAA operator change with the change of the parameter values of ρ. With an increase of ρ, the score values of the four alternatives gradually increase. However, the ranking orders tend to robustness when $\rho > 3$.

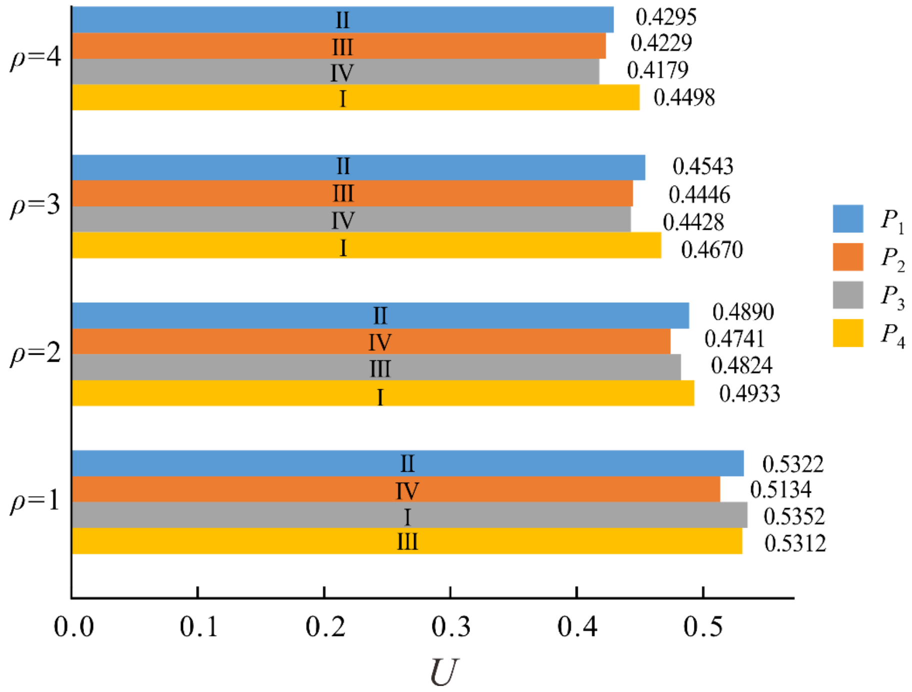

Figure 3. Ranking orders of the four alternatives based on the LNNDWGA operator

For the evaluation results using the LNNDWGA operator in Figure 3, the ranking orders also change with the change of ρ. With an increase of ρ, the score values of the four alternatives gradually decrease. The ranking orders tend to robustness when the value of the parameter ρ exceeds 3.

Furthermore, a comparison is made between the new MAGDM method and the existing relative MAGDM methods by the operators of LNNWAA and LNNWGA proposed by Fan and Ye [24]. According to the calculational steps given by Fan and Ye [24], the alternatives are evaluated as follows.

Step 1: By the LNNWAA operator of the equation (4), we can obtain the integrated matrix $R = \left(N_{vg}\right)_{u\times h}$:

$$R = \begin{bmatrix} \left\langle Fr^{Ro}_{3.3757}, Fr^{Ro}_{2.0000}, Fr^{Ro}_{2.6243} \right\rangle & \left\langle Fr^{Ro}_{4.0000}, Fr^{Ro}_{2.0000}, Fr^{Ro}_{2.3988} \right\rangle & \left\langle Fr^{Ro}_{3.4284}, Fr^{Ro}_{3.2610}, Fr^{Ro}_{3.0433} \right\rangle & \left\langle Fr^{Ro}_{3.3757}, Fr^{Ro}_{2.6243}, Fr^{Ro}_{1.9761} \right\rangle \\ \left\langle Fr^{Ro}_{4.3642}, Fr^{Ro}_{2.2863}, Fr^{Ro}_{1.3013} \right\rangle & \left\langle Fr^{Ro}_{4.0917}, Fr^{Ro}_{2.6243}, Fr^{Ro}_{2.2863} \right\rangle & \left\langle Fr^{Ro}_{3.0000}, Fr^{Ro}_{2.6672}, Fr^{Ro}_{4.0000} \right\rangle & \left\langle Fr^{Ro}_{4.0000}, Fr^{Ro}_{3.5858}, Fr^{Ro}_{3.8255} \right\rangle \\ \left\langle Fr^{Ro}_{3.3328}, Fr^{Ro}_{3.6377}, Fr^{Ro}_{2.9822} \right\rangle & \left\langle Fr^{Ro}_{4.1300}, Fr^{Ro}_{3.2988}, Fr^{Ro}_{2.2496} \right\rangle & \left\langle Fr^{Ro}_{4.3642}, Fr^{Ro}_{2.2863}, Fr^{Ro}_{1.9083} \right\rangle & \left\langle Fr^{Ro}_{4.3642}, Fr^{Ro}_{3.2610}, Fr^{Ro}_{2.2496} \right\rangle \\ \left\langle Fr^{Ro}_{3.3328}, Fr^{Ro}_{2.2496}, Fr^{Ro}_{3.3286} \right\rangle & \left\langle Fr^{Ro}_{3.3757}, Fr^{Ro}_{1.5911}, Fr^{Ro}_{2.3332} \right\rangle & \left\langle Fr^{Ro}_{3.0000}, Fr^{Ro}_{2.2863}, Fr^{Ro}_{2.3620} \right\rangle & \left\langle Fr^{Ro}_{3.3328}, Fr^{Ro}_{2.5716}, Fr^{Ro}_{2.3332} \right\rangle \end{bmatrix}.$$

Step 2: The collective overall linguistic neutrosophic numbers of N_v for P_v (v = 1, 2, 3, 4) was determined below:

$N_1 = \langle Fr^{Ro}_{3.5807}, Fr^{Ro}_{2.4175}, Fr^{Ro}_{2.4916}\rangle$, $N_2 = \langle Fr^{Ro}_{3.9258}, Fr^{Ro}_{2.7432}, Fr^{Ro}_{2.6147}\rangle$, $N_3 = \langle Fr^{Ro}_{4.0996}, Fr^{Ro}_{3.0590}, Fr^{Ro}_{2.2998}\rangle$, and $N_4 = \langle Fr^{Ro}_{3.2625}, Fr^{Ro}_{2.1144}, Fr^{Ro}_{2.5240}\rangle$.

Step 3: Calculate the score values of $U(N_v)$ (v = 1, 2, 3, 4) for the collective overall linguistic neutrosophic numbers of N_v:

$U(N_1) = 0.5929$, $U(N_2) = 0.5860$, $U(N_3) = 0.5967$, and $U(N_4) = 0.5902$.

Step 4: We can get the ranking of the four alternatives: $P_3 \succ P_1 \succ P_4 \succ P_2$. It can be seen that P_3 is the most reasonable choice among the four ones.

Or by the LNNWGA operator of the equation (5), the calculational steps are given below.

Step 1: This step is the same as Step 1 mentioned above.

Step 2: Through the equation (5), the collective overall LNNs of N_v for P_v (v = 1, 2, 3, 4) below:

$N_1 = \langle Fr^{Ro}_{3.5543}, Fr^{Ro}_{2.5138}, Fr^{Ro}_{2.5421}\rangle$, $N_2 = \langle Fr^{Ro}_{3.8110}, Fr^{Ro}_{2.8160}, Fr^{Ro}_{3.0491}\rangle$, $N_3 = \langle Fr^{Ro}_{4.0389}, Fr^{Ro}_{3.1457}, Fr^{Ro}_{2.3507}\rangle$, and $N_4 = \langle Fr^{Ro}_{3.2545}, Fr^{Ro}_{2.1658}, Fr^{Ro}_{2.5717}\rangle$.

Step 3: Calculate the score values of $U(N_v)$ (v = 1, 2, 3, 4):

$U(N_1) = 0.5832$, $U(N_2) = 0.5525$, $U(N_3) = 0.5857$, and $U(N_4) = 0.5843$.

Step 4: We can get the ranking of the four alternatives: $P_3 \succ P_4 \succ P_2 \succ P_1$. It can be seen that P_3 is the most reasonable option among the four ones.

Figure 4 shows the comparison of the decision results obtained using the LNNWGA and LNNWAA operators [24] and the proposed LNNDWAA and LNNDWGA operators in this study. The ranking orders in this MAGDM example are influenced by different aggregation operators and values of the parameter ρ. According to the results obtained using the LNNWGA and LNNWAA operators, the scheme P_3 is the most reasonable option among the four alternatives. It is the same as the result based on the proposed LNNDWAA and LNNDWGA operators when $\rho = 1$. However, the best alternative is P_2 according to the proposed LNNDWAA operator when $\rho = 2, 3, 4$. According to the result of the proposed LNNDWGA operator, when $\rho = 2, 3, 4$, the best alternative is P_4.

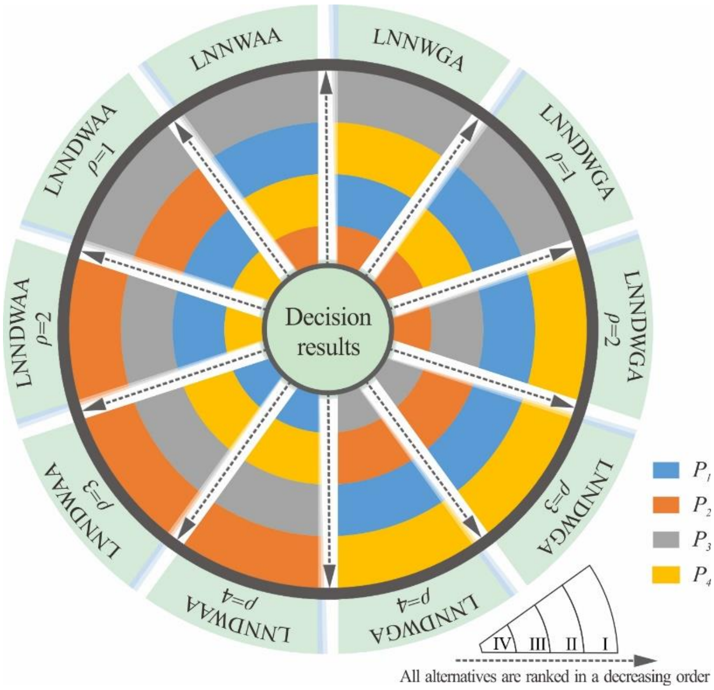

Figure 4. Comparison of the decision results based on different aggregation operators and values of ρ

6. Conclusion

In this study, the LNNDWAA and LNNDWGA operators and their properties were proposed in view of the Dombi operations in the LNN environment. A novel technique for MAGDM problems was proposed using the LNNDWAA or LNNDWGA operator. In the proposed MAGDM process, regarding the satisfactory assessment of alternatives over multiple attributes, we established a decision matrix based on the suitable evaluation results given by the decision makers. Then, we used the LNNDWAA/LNNDWGA operator to aggregate LNN information. Finally, the score values (accuracy values if necessary) was calculated and the ranking results of alternatives are given in a descending order to obtain the optimal choice. In the DM application, an illustrative example of the selection of landslide treatment schemes was presented to verify the feasibility of the proposed method. Compared with the related MAGDM methods in previous studies, this new method can influence the sorting order of alternatives by changing the parameter values of ρ. Thus, it can overcome the insufficiency of decision flexibility in the existing MAGDM method with LNNs. Therefore, we can more effectively deal with the DM problem of landslide treatment schemes by specifying various parameter values according to the preferences and demands of decision makers. It is obvious that this new method can better solve the DM problem of landslide treatment schemes and make the DM results more reasonable and flexible in the uncertainty and inconsistency of human linguistic judgments.

Data Availability: The data used to support the findings of the study are available in the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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