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Theory of Hypersoft Sets: Axiomatic Properties, Aggregation Operations, Relations, Functions and Matrices

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Abstract. There are several decision-making based situations in which it is necessary to categorize the evaluating parameters into their respective sub-parametric values based non-overlapping sets. The existing soft set model is not compatible with such situations therefore hypersoft set ($\hat{H}s$-set) is developed which manages such situations by utilizing a novel mapping called multi-argument approximate mapping which broadens the domain of soft approximate mapping. This research presents the characterization of several essential axiomatic properties and set-based operations of $\hat{H}s$-set which will help the researchers to implement this emerging theory to other fields of study. The brief discussion on some hybridized structures of $\hat{H}s$-set with fuzzy set-like models is also provided.

Keywords: $\hat{H}s$-set; $\hat{H}s$-Relation; $\hat{H}s$-Function; $\hat{H}s$-Matrix.

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1. Introduction

There are several models in literature to deal uncertainties but fuzzy set [1] is the most significant in this regards. It has its own intricacies which limit it to tackle uncertain decision-making scenarios effectively. The justification behind these obstacles is, potentially, the deficiency of parameterization tool. A novel model is required for managing vagueness and uncertainties which should be liberated from all such obstacles. In 1999, Molodtsov [2] established a set-structure known as soft set ($\hat{s}$-set) in literature as a novel parameterized sub-class of universal set. In the year 2003, Maji et al. [3] broadened the idea and investigated several rudimental axiomatic properties and set-operations of $\hat{s}$-sets . They also validated several results. Later on Pei et al. [4] introduced an information system (Inf-sys) by using the idea of $\hat{s}$-sets. It is proved that $\hat{s}$-set can be considered as a particular class of Inf-sys. Afterwards, Ali et al. [5] identified many assertions in the research proposed by Maji et al. and introduced novel notions by using the concept of restricted and extended $\hat{s}$-set aggregation operations. In the same way, Babitha et al. [6, 7] made investigation on $\hat{s}$-set relation, $\hat{H}s$-set function by using the Cartesian product of $\hat{H}s$-sets. Sezgin et al. [8], Ge et al. [9], Fuli [10] provided few amendments in previous work by establishing few novel results. In order to utilize the concept of $\hat{s}$-sets in the development of algebraic structures, Saeed et al. [11] characterized the classical notions of elements and points under $\hat{s}$-set environment. Many researchers [12–23] discussed various gluing structures of $\hat{s}$-sets with other fuzzy set-like models to resolve several real-life decision making issues.

It is a matter of common observation that in various decision making problems, parameters have to be partitioned into their related sub-parametric valued sets whereas the previous researches on $\hat{s}$-set are not sufficient to manage such settings therefore Smarandache [24] initiated the notion of hypersoft set ($\hat{H}s$-set) as an extension of $\hat{s}$-set by introducing a novel multi-argument approximate mapping (maa-mapping). Any novel theory can not be implemented in real-world situations without the characterization of its elementary axiomatic-properties. Although Saeed et al. [25] made a good effort to investigate various basic properties of $\hat{H}s$-set but it does not cover many of the aspects of $\hat{H}s$-set theory. Therefore this paper aims to (i). generalize the research works described in [3, 5–10] for $\hat{H}s$-set environment and (ii). to modify the results discussed by Saeed et al. [25]. In the present work, all the necessary rudiments of $\hat{H}s$-set are investigated for its further developments. The Figure 1 explains the sectional-outlines of the paper.

Section 2	Recollects few essential definitions and results to assist the main results.	Section 3	Introduces few basic axiomatic properties of hypersoft sets and Illustrates set theoretic operations of hypersoft sets.
Section 4	Discusses some basic results and laws on hypersoft sets.	Section 5	Explains hypersoft relations and hypersoft functions.
Section 6	Presents the matrix representation of hypersoft sets with some operations	Section 7	Investigates few hybrids of hypersoft sets.
Section 8	Summarizes the paper with the provision of some future directions.		

Figure 1. Outlines of the paper

2. Preliminaries

The purpose of this section is to review some basic properties of $\hat{s}$-set for clear understanding of proposed study. The symbol $\hat{\Pi}$ will represent initial universe in the remaining parts of the article.

Definition 2.1. [2]
A $\hat{s}$-set $\mathbb{S}$ on $\hat{\Pi}$ is usually stated by a pair $(\Psi_{\mathbb{S}}, \mathfrak{G})$ in which $\Psi_{\mathbb{S}} : \mathfrak{G} \to P^{\hat{\Pi}}$ is an approximate mapping & $\mathfrak{G}$ be a sub-family of parameters. The family of $\hat{s}$-sets is symbolized as $\Sigma_{(\Psi_{\mathbb{S}}, \mathfrak{G})}$.

Definition 2.2. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}}, \mathfrak{G})}$, if $\mathfrak{G}_1 \subseteq \mathfrak{G}_2$, & $\Psi_{\mathbb{S}_1}(\hat{\epsilon}) \subseteq \Psi_{\mathbb{S}_2}(\hat{\epsilon})$ for all $\hat{\epsilon} \in \mathfrak{G}_1$ then $\hat{s}$-set $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ is a *soft-subset* of $\hat{s}$-set $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2)$.

Definition 2.3. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}}, \mathfrak{G})}$, their union is a $\hat{s}$-set $(\Psi_{\mathbb{S}_3}, \mathfrak{G}_3)$ with $\mathfrak{G}_3 = \mathfrak{G}_1 \cup \mathfrak{G}_2$ & for $\hat{\epsilon} \in \mathfrak{G}_3$,

$$\Psi_{\mathbb{S}_3}(\hat{\epsilon}) = \begin{cases} \Psi_{\mathbb{S}_1}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_1 \setminus \mathfrak{G}_2) \\ \Psi_{\mathbb{S}_2}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_2 \setminus \mathfrak{G}_1) \\ \Psi_{\mathbb{S}_1}(\hat{\epsilon}) \cup \Psi_{\mathbb{S}_2}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_1 \cap \mathfrak{G}_2) \end{cases}$$

Definition 2.4. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}}, \mathfrak{G})}$, their intersection is a $\hat{s}$-set $(\Psi_{\mathbb{S}_4}, \mathfrak{G}_4)$ with $\mathfrak{G}_4 = \mathfrak{G}_1 \cap \mathfrak{G}_2$ & for $\hat{\epsilon} \in \mathfrak{G}_4$, $\Psi_{\mathbb{S}_4}(\hat{\epsilon}) = \Psi_{\mathbb{S}_1}(\hat{\epsilon}) \cap \Psi_{\mathbb{S}_2}(\hat{\epsilon})$.

One can refer [2–10] for detailed description on $\hat{s}$-sets.

3. **Hypersoft Set**

This part of the paper provides the basic axiomatic-properties of $\hat{H}s$-set along with the modification of some notions stated in [25].

Definition 3.1. [22]
Let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, ..., \mathfrak{A}_n$ are non-overlapping sets having sub-parametric values of parameters $\ddot{a}_1, \ddot{a}_2, \ddot{a}_3, ..., \ddot{a}_n$ respectively, then a $\hat{H}s$-set on $\hat{\Pi}$, is usually stated by a pair $(\Theta, \mathfrak{A})$ in which $\Theta : \mathfrak{A} \to P^{\hat{\Pi}}$ is a maa-mapping and $\mathfrak{A} = \prod_{i=1}^{n} \mathfrak{A}_i$. The family of $\hat{H}s$-sets is symbolized by $\Sigma_{(\Theta, \mathfrak{A})}$. The model of $\hat{H}s$-set is presented in Figure 2.

Example 3.2. Mrs. Smith visits a mobile mall to purchase a mobile for her personal use. She is accompanied by her two friends who are experts in mobile purchasing. They collectively observed 8 types of mobiles which are considered as elements of universal set $\hat{\Pi} = \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}$. They have fixed some parameters for this purchase with their mutual consensus that are $\hat{e}_1$ = random only memory in giga bytes, $\hat{e}_2$ = Resolution of camera in pixels, $\hat{e}_3$ = length in inches, $\hat{e}_4$ = random access memory in giga bytes, and $\hat{e}_5$ = power of battery in mAh. These parameters have their sub-collections as:
$\mathfrak{B}_1 = \{\hat{e}_{11} = 32, \hat{e}_{12} = 64\}$, $\mathfrak{B}_2 = \{\hat{e}_{21} = 8, \hat{e}_{22} = 16\}$, $\mathfrak{B}_3 = \{\hat{e}_{31} = 6.5, \hat{e}_{32} = 6.7\}$
$\mathfrak{B}_4 = \{\hat{e}_{41} = 4, \hat{e}_{42} = 8\}$, $\mathfrak{B}_5 = \{\hat{e}_{51} = 4000\}$ then $\mathfrak{A} = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \mathfrak{B}_3 \times \mathfrak{B}_4 \times \mathfrak{B}_5$

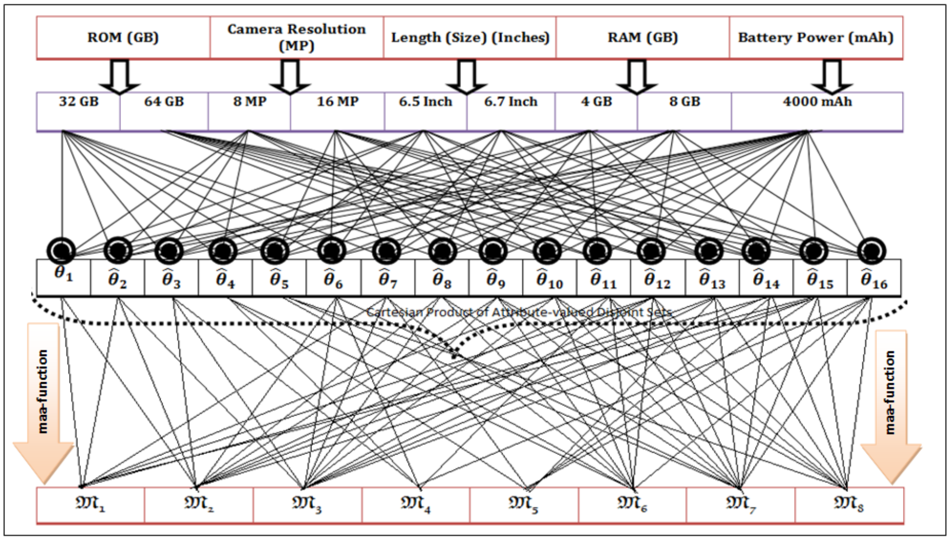

FIGURE 2. Pictorial Version of $\hat{H}s$-set

$\mathfrak{A} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, ..., \hat{\theta}_{16}\}$ and every $\hat{\theta}_i$, ${}_{(1)}i^{(16)}$, is a 5-tuple member. Then the $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is constructed as

$$(\Theta,\mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2\}\right), \left(\hat{\theta}_2, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3\}\right), \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right), \left(\hat{\theta}_4, \{\hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6\}\right), \\ \left(\hat{\theta}_5, \{\hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_6, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_7\}\right), \left(\hat{\theta}_7, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6\}\right), \\ \left(\hat{\theta}_8, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7\}\right), \left(\hat{\theta}_9, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{10}, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{11}, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{12}, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{13}, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{14}, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{15}, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{16}, \{\hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}\right) \end{array} \right\}$$

Definition 3.3. Let $\mathcal{F}^{\hat{\Pi}}$ be a collection consisting of fuzzy subsets on $\hat{\Pi}$. Let $\ddot{a}_i, n \geq 1$, ${}_1i^n$ are parameters having their relevant sub-parametric values in the sets $\mathfrak{A}_i$ respectively, with $\mathfrak{A}_i \cap \mathfrak{A}_j = \varnothing$, for $i \neq j$, & ${}_1i^n$, ${}_1j^n$. Then a *fuzzy $\hat{H}s$-set* $(\Theta_{fhs}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as,

$$(\Theta_{fhs}, \mathfrak{A}) = \{(\hat{\theta}, \Theta_{fhs}(\hat{\theta})) : \hat{\theta} \in \mathfrak{A}, \Theta_{fhs}(\hat{\theta}) \in \mathcal{F}^{\hat{\Pi}}\}$$

where $\Theta_{fhs} : \mathfrak{A} \rightarrow \mathcal{F}^{\hat{\Pi}}$ and for all $\hat{\theta} \in \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ..... \times \mathfrak{A}_n$

$$\Theta_{fhs}(\hat{\theta}) = \{\mu_{\Theta_{fhs}(\hat{\theta})}(\varpi)/\varpi : \varpi \in \hat{\Pi}, \mu_{\Theta_{fhs}(\hat{\theta})}(\varpi) \in \mathbb{C}(\mathbb{I}) = [0, 1]\}$$

is a fuzzy set on $\hat{\Pi}$.

One can consider this definition as modified form of fuzzy $\hat{H}s$-set stated in [22] and [24].

Example 3.4. Assuming the Example 3.2, Fuzzy $\hat{H}s$-set $(\Theta_{fhs}, \mathfrak{A})$ is constructed as

$$(\Theta_{fhs}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{0.1/\hat{\mathfrak{M}}_1, 0.2/\hat{\mathfrak{M}}_2\}\right), \left(\hat{\theta}_2, \{0.1/\hat{\mathfrak{M}}_1, 0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3\}\right), \left(\hat{\theta}_3, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.4/\hat{\mathfrak{M}}_4\}\right), \\ \left(\hat{\theta}_4, \{0.4/\hat{\mathfrak{M}}_4, 0.5/\hat{\mathfrak{M}}_5, 0.6/\hat{\mathfrak{M}}_6\}\right), \left(\hat{\theta}_5, \{0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_6, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.4/\hat{\mathfrak{M}}_4, 0.7/\hat{\mathfrak{M}}_7\}\right), \\ \left(\hat{\theta}_7, \{0.1/\hat{\mathfrak{M}}_1, 0.3/\hat{\mathfrak{M}}_3, 0.5/\hat{\mathfrak{M}}_5, 0.6/\hat{\mathfrak{M}}_6\}\right), \left(\hat{\theta}_8, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7\}\right), \\ \left(\hat{\theta}_9, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{10}, \{0.1/\hat{\mathfrak{M}}_1, 0.3/\hat{\mathfrak{M}}_3, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{11}, \{0.2/\hat{\mathfrak{M}}_2, 0.4/\hat{\mathfrak{M}}_4, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{12}, \{0.1/\hat{\mathfrak{M}}_1, 0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{13}, \{0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.5/\hat{\mathfrak{M}}_5, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{14}, \{0.1/\hat{\mathfrak{M}}_1, 0.3/\hat{\mathfrak{M}}_3, 0.5/\hat{\mathfrak{M}}_5, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right), \\ \left(\hat{\theta}_{15}, \{0.1/\hat{\mathfrak{M}}_1, 0.2/\hat{\mathfrak{M}}_2, 0.3/\hat{\mathfrak{M}}_3, 0.5/\hat{\mathfrak{M}}_5, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right), \left(\hat{\theta}_{16}, \{0.4/\hat{\mathfrak{M}}_4, 0.5/\hat{\mathfrak{M}}_5, 0.6/\hat{\mathfrak{M}}_6, 0.7/\hat{\mathfrak{M}}_7, 0.8/\hat{\mathfrak{M}}_8\}\right) \end{array} \right\}$$

Definition 3.5. Let $(\Theta_1, \mathfrak{A}_1), (\Theta_2, \mathfrak{A}_2) \in \Sigma_{(\Theta, \mathfrak{A})}$ then $(\Theta_1, \mathfrak{A}_1)$ is said to be $\hat{H}s$- *subset* of $(\Theta_2, \mathfrak{A}_2)$ if $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ and $\forall \hat{\theta} \in \mathfrak{A}_1, \Theta_1(\hat{\theta}) \subseteq \Theta_2(\hat{\theta})$.

Example 3.6. Assuming Example 3.2, if
$(\Theta_1, \mathfrak{A}_1) = \left\{ \left(\hat{\theta}_1, \{\hat{\mathfrak{M}}_1\}\right), \left(\hat{\theta}_2, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2\}\right), \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3\}\right) \right\}$
$(\Theta_2, \mathfrak{A}_2) = \left\{ \left(\hat{\theta}_1, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2\}\right), \left(\hat{\theta}_2, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3\}\right), \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right), \left(\hat{\theta}_4, \{\hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6\}\right) \right\}$
then $(\Theta_1, \mathfrak{A}_1) \subseteq (\Theta_2, \mathfrak{A}_2)$.

Definition 3.7. A set $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ..... \times \mathfrak{A}_n$ in $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is said to be *Not set* if it has the representation as $\ltimes\mathfrak{A} = \{\ltimes\hat{\theta}_1, \ltimes\hat{\theta}_2, \ltimes\hat{\theta}_3, \ltimes\hat{\theta}_4, ..., \ltimes\hat{\theta}_m\}$ where $m = \prod_{i=1}^{n} |\mathfrak{A}_i|$.

Example 3.8. Reconsidering $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4, , \mathfrak{A}_5$ from Example 3.2, we get

$$\ltimes\mathfrak{A} = \{\ltimes\hat{\theta}_1, \ltimes\hat{\theta}_2, \ltimes\hat{\theta}_3, \ltimes\hat{\theta}_4, ..., \ltimes\hat{\theta}_{16}\}$$

.

Definition 3.9. A $\hat{H}s$-set $(\Theta, \mathfrak{A}_1)$ is stated as a *relative null $\hat{H}s$-set* w.r.t $\mathfrak{A}_1 \subseteq \mathfrak{A}$, symbolized by $(\Theta, \mathfrak{A}_1)_\Phi$, if $\Theta(\hat{\theta}) = \varnothing, \forall \hat{\theta} \in \mathfrak{A}_1$.

Example 3.10. Assuming Example 3.2, if $(\Theta, \mathfrak{A}_1)_\Phi = \left\{ \left(\hat{\theta}_1, \varnothing\right), \left(\hat{\theta}_2, \varnothing\right), \left(\hat{\theta}_3, \varnothing\right) \right\}$ where $\mathfrak{A}_1 \subseteq \mathfrak{A}$.

Definition 3.11. A $\hat{H}s$-set $(\Theta, \mathfrak{A}_1)$ is stated as a *relative whole $\hat{H}s$-set* w.r.t $\mathfrak{A}_1 \subseteq \mathfrak{A}$, symbolized by $(\Theta, \mathfrak{A}_1)_{\hat{\Pi}}$, if $\Theta(\hat{\theta}) = \hat{\Pi}, \forall \hat{\theta} \in \mathfrak{A}_1$.

Example 3.12. Assuming Example 3.2, if $(\Theta, \mathfrak{A}_1)_{\hat{\Pi}} = \left\{ \left(\hat{\theta}_1, \hat{\Pi}\right), \left(\hat{\theta}_2, \hat{\Pi}\right), \left(\hat{\theta}_3, \hat{\Pi}\right) \right\}$ where $\mathfrak{A}_1 \subseteq \mathfrak{A}$.

Definition 3.13. A $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is stated as a *absolute whole $\hat{H}s$-set* on $\hat{\Pi}$, symbolized by $(\Theta, \mathfrak{A})_{\hat{\Pi}}$, if $\Theta(\hat{\theta}) = \hat{\Pi}, \forall \hat{\theta} \in \mathfrak{A}$.

Example 3.14. Assuming Example 3.2, if

$$(\Theta, \mathfrak{A})_{\hat{\Pi}} = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \hat{\Pi}\right), \left(\hat{\theta}_2, \hat{\Pi}\right), \left(\hat{\theta}_3, \hat{\Pi}\right), \left(\hat{\theta}_4, \hat{\Pi}\right), \left(\hat{\theta}_5, \hat{\Pi}\right), \left(\hat{\theta}_6, \hat{\Pi}\right), \left(\hat{\theta}_7, \hat{\Pi}\right), \left(\hat{\theta}_8, \hat{\Pi}\right), \\ \left(\hat{\theta}_9, \hat{\Pi}\right), \left(\hat{\theta}_{10}, \hat{\Pi}\right), \left(\hat{\theta}_{11}, \hat{\Pi}\right), \left(\hat{\theta}_{12}, \hat{\Pi}\right), \left(\hat{\theta}_{13}, \hat{\Pi}\right), \left(\hat{\theta}_{14}, \hat{\Pi}\right), \left(\hat{\theta}_{15}, \hat{\Pi}\right), \left(\hat{\theta}_{16}, \hat{\Pi}\right) \end{array} \right\}$$

Proposition 3.15. *Let* $(\Theta_1,\mathfrak{A}_1),(\Theta_2,\mathfrak{A}_2),(\Theta_3,\mathfrak{A}_3)\in\Sigma_{(\Theta,\mathfrak{A})}$ *with* $\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3\subseteq\mathfrak{A}$ *then*

(i) $(\Theta_1,\mathfrak{A}_1)\subseteq(\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}}$

(ii) $(\Theta_1,\mathfrak{A}_1)_{\Phi}\subseteq(\Theta_1,\mathfrak{A}_1)$

(iii) $(\Theta_1,\mathfrak{A}_1)\subseteq(\Theta_1,\mathfrak{A}_1)$

(iv) *If* $(\Theta_1,\mathfrak{A}_1)\subseteq(\Theta_2,\mathfrak{A}_2)$ *&* $(\Theta_2,\mathfrak{A}_2)\subseteq(\Theta_3,\mathfrak{A}_3)$ *then* $(\Theta_1,\mathfrak{A}_1)\subseteq(\Theta_3,\mathfrak{A}_3)$

(v) *If* $(\Theta_1,\mathfrak{A}_1)=(\Theta_2,\mathfrak{A}_2)$ *&* $(\Theta_2,\mathfrak{A}_2)=(\Theta_3,\mathfrak{A}_3)$ *then* $(\Theta_1,\mathfrak{A}_1)=(\Theta_3,\mathfrak{A}_3)$

Definition 3.16. The *complement* of a $\hat{H}s$-set $(\Theta,\mathfrak{A})$, symbolized by $(\Theta,\mathfrak{A})^{\ominus}$, is stated as $(\Theta,\mathfrak{A})^{\ominus}=(\Theta^{\ominus},\ltimes\mathfrak{A})$ where $\Theta^{\ominus}:\ltimes\mathfrak{A}\to P^{\hat{\Pi}}$ with $\Theta^{\ominus}(\ltimes\hat{\theta})=\hat{\Pi}\setminus\Theta(\hat{\theta}),\forall\hat{\theta}\in\mathfrak{A}$.

Example 3.17. From Example 3.2, we get

$$(\Theta,\mathfrak{A})^{\ominus}=\left\{\begin{array}{l}(\ltimes\hat{\theta}_1,\{\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_2,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),\\(\ltimes\hat{\theta}_4,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_5,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5\}),(\ltimes\hat{\theta}_6,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_7\}),\\(\ltimes\hat{\theta}_7,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_8,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_8\}),(\ltimes\hat{\theta}_9,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5\}),(\ltimes\hat{\theta}_{10},\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5\}),\\(\ltimes\hat{\theta}_{11},\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_5\}),(\ltimes\hat{\theta}_{12},\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5\}),(\ltimes\hat{\theta}_{13},\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}),(\ltimes\hat{\theta}_{14},\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}),\\(\ltimes\hat{\theta}_{15},\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}),(\ltimes\hat{\theta}_{16},\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\})\end{array}\right\}$$

Definition 3.18. The *relative complement* of a $\hat{H}s$-set $(\Theta,\mathfrak{A})$, symbolized by $(\Theta,\mathfrak{A})^{\circledast}$, is stated as $(\Theta,\mathfrak{A})^{\circledast}=(\Theta^{\circledast},\mathfrak{A})$ where $\Theta^{\circledast}:\mathfrak{A}\to P^{\hat{\Pi}}$ with $\Theta^{\circledast}(\hat{\theta})=\hat{\Pi}\setminus\Theta(\hat{\theta}),\forall\hat{\theta}\in\mathfrak{A}$.

Example 3.19. Reconsidering Example 3.2, we get

$$(\Theta,\mathfrak{A})^{\circledast}=\left\{\begin{array}{l}(\hat{\theta}_1,\{\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),(\hat{\theta}_2,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),(\hat{\theta}_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),\\(\hat{\theta}_4,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),(\hat{\theta}_5,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5\}),(\hat{\theta}_6,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_7\}),\\(\hat{\theta}_7,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_7,\hat{\mathfrak{M}}_8\}),(\hat{\theta}_8,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_8\}),(\hat{\theta}_9,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5\}),(\hat{\theta}_{10},\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5\}),\\(\hat{\theta}_{11},\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_5\}),(\hat{\theta}_{12},\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5\}),(\hat{\theta}_{13},\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}),(\hat{\theta}_{14},\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}),\\(\hat{\theta}_{15},\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}),(\hat{\theta}_{16},\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\})\end{array}\right\}$$

Proposition 3.20. *Let* $(\Theta,\mathfrak{A})\in\Sigma_{(\Theta,\mathfrak{A})}$ *then*

(i) $((\Theta,\mathfrak{A})^{\ominus})^{\ominus}=(\Theta,\mathfrak{A})$

(ii) $((\Theta,\mathfrak{A})^{\circledast})^{\circledast}=(\Theta,\mathfrak{A})$

(iii) $((\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}})^{\ominus}=(\Theta_1,\mathfrak{A}_1)_{\Phi}=((\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}})^{\circledast}$*;* $\mathfrak{A}_1\subseteq\mathfrak{A}$

(iv) $((\Theta_1,\mathfrak{A}_1)_{\Phi})^{\ominus}=(\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}}=((\Theta_1,\mathfrak{A}_1)_{\Phi})^{\circledast}$*;* $\mathfrak{A}_1\subseteq\mathfrak{A}$

Definition 3.21. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, the union-operation $(\Theta_1,\mathfrak{A}_1)\cup(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\cup\mathfrak{A}_2$ and for $\hat{\theta}\in\mathfrak{A}_3$,

$$\Theta_3(\hat{\theta})=\begin{cases}\Theta_1(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\setminus\mathfrak{A}_2)\\ \Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_2\setminus\mathfrak{A}_1)\\ \Theta_1(\hat{\theta})\cup\Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\cap\mathfrak{A}_2)\end{cases}.$$

Example 3.22. Let

$$(\Theta_1,\mathfrak{A}_1)=\left\{\ (\hat{\theta}_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}),(\hat{\theta}_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}),(\hat{\theta}_3,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4\})\ \right\}$$

$$(\Theta_2,\mathfrak{A}_2)=\left\{\ (\hat{\theta}_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}),(\hat{\theta}_4,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}),(\hat{\theta}_5,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\})\ \right\}$$

then

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\hat{\theta}_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\hat{\theta}_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4\}\right),\\ \left(\hat{\theta}_4,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\hat{\theta}_5,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right)\end{array}\right\}$$

Definition 3.23. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, the intersection-operation $(\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\cap\mathfrak{A}_2$ & for $\hat{\theta}\in\mathfrak{A}_3$, $\Theta_3(\hat{\theta})=\Theta_1(\hat{\theta})\cap\Theta_2(\hat{\theta})$.

Example 3.24. Reconsidering Example 3.22, we get $(\Theta_3,\mathfrak{A}_3)=\left\{\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_2\}\right)\right\}$.

Definition 3.25. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their extended-intersection $(\Theta_1,\mathfrak{A}_1)\cap_\varepsilon(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\cup\mathfrak{A}_2$ and for $\hat{\theta}\in\mathfrak{A}_3$,

$$\Theta_3(\hat{\theta})=\begin{cases}\Theta_1(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\setminus\mathfrak{A}_2)\\ \Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_2\setminus\mathfrak{A}_1)\\ \Theta_1(\hat{\theta})\cap\Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\cap\mathfrak{A}_2)\end{cases}$$

Example 3.26. Taking assumptions of Example 3.22, we get

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\hat{\theta}_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\hat{\theta}_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_2\}\right),\\ \left(\hat{\theta}_4,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\hat{\theta}_5,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right)\end{array}\right\}$$

Definition 3.27. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their AND-operation $(\Theta_1,\mathfrak{A}_1)\bigwedge(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ and for $(\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3,\hat{\theta}_i\in\mathfrak{A}_1,\hat{\theta}_j\in\mathfrak{A}_2$,

$$\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\cup\Theta_2(\hat{\theta}_j).$$

Example 3.28. Taking assumptions of Example 3.22, we get

$$\mathfrak{A}_1\times\mathfrak{A}_2=\left\{\begin{array}{l}\pi_1=\left(\hat{\theta}_1,\hat{\theta}_3\right),\pi_2=\left(\hat{\theta}_1,\hat{\theta}_4\right),\pi_3=\left(\hat{\theta}_1,\hat{\theta}_5\right),\pi_4=\left(\hat{\theta}_2,\hat{\theta}_3\right),\pi_5=\left(\hat{\theta}_2,\hat{\theta}_4\right),\\ \pi_6=\left(\hat{\theta}_2,\hat{\theta}_5\right),\pi_7=\left(\hat{\theta}_3,\hat{\theta}_3\right),\pi_8=\left(\hat{\theta}_3,\hat{\theta}_4\right),\pi_9=\left(\hat{\theta}_3,\hat{\theta}_5\right)\end{array}\right\}$$

then

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\pi_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\pi_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\left(\pi_4,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\\ \left(\pi_5,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\pi_6,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_7,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4\}\right),\left(\pi_8,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_9,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\end{array}\right\}$$

Definition 3.29. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their OR-operation $(\Theta_1,\mathfrak{A}_1)\bigvee(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ and for $(\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3,\hat{\theta}_i\in\mathfrak{A}_1,\hat{\theta}_j\in\mathfrak{A}_2$,

$$\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\cap\Theta_2(\hat{\theta}_j).$$

Example 3.30. Taking assumptions of Examples 3.22 and 3.30, we get

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\pi_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\pi_2,\{\}\right),\left(\pi_3,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_4,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\\ \left(\pi_5,\{\}\right),\left(\pi_6,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_7,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_8,\{\hat{\mathfrak{M}}_4\}\right),\left(\pi_9,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4\}\right),\end{array}\right\}$$

Definition 3.31. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-union $(\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3 = \mathfrak{A}_1 \cap \mathfrak{A}_2$ and for $\hat{\theta} \in \mathfrak{A}_3$,

$$\Theta_3(\hat{\theta}) = \Theta_1(\hat{\theta}) \cup \Theta_2(\hat{\theta}).$$

Example 3.32. Taking assumptions of Example 3.22, we get

$$(\Theta_3,\mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

Definition 3.33. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-difference $(\Theta_1,\mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3 = \mathfrak{A}_1 \cap \mathfrak{A}_2$ and for $\hat{\theta} \in \mathfrak{A}_3$,

$$\Theta_3(\hat{\theta}) = \Theta_1(\hat{\theta}) - \Theta_2(\hat{\theta}).$$

Example 3.34. Taking suppositions of Example 3.22, we get $(\Theta_3,\mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$.

Definition 3.35. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-symmetric-difference $(\Theta_1,\mathfrak{A}_1) \blacktriangle (\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ stated by

$$(\Theta_3,\mathfrak{A}_3) = \left\{ ((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) \setminus_{\mathcal{R}} ((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2)) \right\}$$

or

$$(\Theta_3,\mathfrak{A}_3) = \left\{ ((\Theta_1,\mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) \cup_{\mathcal{R}} ((\Theta_2,\mathfrak{A}_2) \setminus_{\mathcal{R}} (\Theta_1,\mathfrak{A}_1)) \right\}$$

Example 3.36. Taking suppositions of Example 3.22, we get

$$((\Theta_1,\mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

&

$$((\Theta_2,\mathfrak{A}_2) \setminus_{\mathcal{R}} (\Theta_1,\mathfrak{A}_1)) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1\}\right) \right\}$$

then

$$(\Theta_3,\mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

4. Axioms-based Results of $\hat{H}s$-sets

This part presents some classical axioms-based results of set theory that are also valid for $\hat{H}s$-settings.

(1) **Idempotent Laws**
   (a) $(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})$
   (b) $(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})$

(2) **Identity Laws**
   (a) $(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})_{\Phi}$
   (b) $(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})_{\hat{\Pi}}$
   (c) $(\Theta,\mathfrak{A}) \setminus_{\mathcal{R}} (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \blacktriangle (\Theta,\mathfrak{A})_{\Phi}$
   (d) $(\Theta,\mathfrak{A}) \setminus_{\mathcal{R}} (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) \blacktriangle (\Theta,\mathfrak{A})$

(3) **Domination Laws**

(a) $(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})_{\hat{\Pi}}$

(b) $(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})_{\Phi}$

(4) **Property of Exclusion**

$$(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A})^{\circledast} = (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})^{\circledast}$$

(5) **Property of Contradiction**

$$(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A})^{\circledast} = (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})^{\circledast}$$

(6) **Absorption Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup ((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(b) $(\Theta_1,\mathfrak{A}_1) \cap ((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(c) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} ((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} ((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(7) **Commutative Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cup (\Theta_1,\mathfrak{A}_1)$

(b) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cup_{\mathcal{R}} (\Theta_1,\mathfrak{A}_1)$

(c) $(\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cap (\Theta_1,\mathfrak{A}_1)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cap_{\varepsilon} (\Theta_1,\mathfrak{A}_1)$

(e) $(\Theta_1,\mathfrak{A}_1) \blacktriangle (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \blacktriangle (\Theta_1,\mathfrak{A}_1)$

(8) **Associative Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup ((\Theta_2,\mathfrak{A}_2) \cup (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2)) \cup (\Theta_3,\mathfrak{A}_3)$

(b) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} ((\Theta_2,\mathfrak{A}_2) \cup_{\mathcal{R}} (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) \cup_{\mathcal{R}} (\Theta_3,\mathfrak{A}_3)$

(c) $(\Theta_1,\mathfrak{A}_1) \cap ((\Theta_2,\mathfrak{A}_2) \cap (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2)) \cap (\Theta_3,\mathfrak{A}_3)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} ((\Theta_2,\mathfrak{A}_2) \cap_{\varepsilon} (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)) \cap_{\varepsilon} (\Theta_3,\mathfrak{A}_3)$

(e) $(\Theta_1,\mathfrak{A}_1) \bigvee ((\Theta_2,\mathfrak{A}_2) \bigvee (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \bigvee (\Theta_2,\mathfrak{A}_2)) \bigvee (\Theta_3,\mathfrak{A}_3)$

(f) $(\Theta_1,\mathfrak{A}_1) \bigwedge ((\Theta_2,\mathfrak{A}_2) \bigwedge (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \bigwedge (\Theta_2,\mathfrak{A}_2)) \bigwedge (\Theta_3,\mathfrak{A}_3)$

(9) **De Morgans Laws**

(a) $((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(b) $((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \cup (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(c) $((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \cap (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(d) $((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(e) $((\Theta_1,\mathfrak{A}_1) \bigvee (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \bigwedge (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(f) $((\Theta_1,\mathfrak{A}_1) \bigwedge (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \bigvee (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(g) $((\Theta_1,\mathfrak{A}_1) \bigvee (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \bigwedge (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(h) $((\Theta_1,\mathfrak{A}_1) \bigwedge (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \bigvee (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(10) **Distributive Laws**

(a) $(\Theta_1,\mathfrak{A}_1)\cup((\Theta_2,\mathfrak{A}_2)\cap(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup(\Theta_2,\mathfrak{A}_2))\cap((\Theta_1,\mathfrak{A}_1)\cup(\Theta_3,\mathfrak{A}_3))$

(b) $(\Theta_1,\mathfrak{A}_1)\cap((\Theta_2,\mathfrak{A}_2)\cup(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2))\cup((\Theta_1,\mathfrak{A}_1)\cap(\Theta_3,\mathfrak{A}_3))$

(c) $(\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}((\Theta_2,\mathfrak{A}_2)\cap_{\varepsilon}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_2,\mathfrak{A}_2))\cap_{\varepsilon}((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))$

(d) $(\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}((\Theta_2,\mathfrak{A}_2)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}(\Theta_2,\mathfrak{A}_2))\cup_{\mathcal{R}}((\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}(\Theta_3,\mathfrak{A}_3))$

(e) $(\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}((\Theta_2,\mathfrak{A}_2)\cap(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_2,\mathfrak{A}_2))\cap((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))$

(f) $(\Theta_1,\mathfrak{A}_1)\cap((\Theta_2,\mathfrak{A}_2)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2))\cup_{\mathcal{R}}((\Theta_1,\mathfrak{A}_1)\cap(\Theta_3,\mathfrak{A}_3))$

5. Relations-based Operations of $\hat{H}s$-sets

Here some relations-based classical notions and results are generalized for $\hat{H}s$-sets.

Definition 5.1. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their Cartesian product $(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ where $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ & $\Theta_3:\mathfrak{A}_3\to P(\hat{\Pi}\times\hat{\Pi})$ stated by $\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\times\Theta_2(\hat{\theta}_j)\ \forall\ (\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3$ that is $\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\{(\hat{\theta}_i,\hat{\theta}_j):\hat{\theta}_i\in\Theta_1(\hat{\theta}_i),\hat{\theta}_j\in\Theta_2(\hat{\theta}_j)\}$.

Definition 5.2. If $(\Theta_1,\mathfrak{A}_1),(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$ then a relation from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_2,\mathfrak{A}_2)$ is stated as *$\hat{H}s$-relation* $(\hat{\Xi},\mathfrak{A}_4)$ (conveniently $\hat{\Xi}$) which is the $\hat{H}s$-subset of $(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$ where $\mathfrak{A}_4\subseteq\mathfrak{A}_1\times\mathfrak{A}_2$ & $\forall\ (\hat{\theta}_1,\hat{\theta}_2)\in\mathfrak{A}_4,\hat{\Xi}(\hat{\theta}_1,\hat{\theta}_2)=\Theta_3(\hat{\theta}_1,\hat{\theta}_2)$, where $(\Theta_3,\mathfrak{A}_3)=(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$.

Definition 5.3. Let $\hat{\Xi}$ be a $\hat{H}s$-relation from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_2,\mathfrak{A}_2)$ such that $(\Theta_3,\mathfrak{A}_3)=(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$. Then

(i) The DoM $\hat{\Xi}$ (the domain of $\hat{\Xi}$) is a $\hat{H}s$-set $(\Theta,\mathfrak{W})\subset(\Theta_1,\mathfrak{A}_1)$ where $\mathfrak{W}=\{\hat{\theta}_i\in\mathfrak{A}_1:\Theta_3(\hat{\theta}_i,\hat{\theta}_j)\in\hat{\Xi}\ forsome\ \hat{\theta}_j\in\mathfrak{A}_2\}$ & $\Theta(\hat{\theta}_1)=\Theta_1(\hat{\theta}_1),\forall\ \hat{\theta}_1\in\mathfrak{W}$.

(ii) The RNG $\hat{\Xi}$ (the range of $\hat{\Xi}$) is a $\hat{H}s$-set $(\xi,\mathfrak{L})\subset(\Theta_2,\mathfrak{A}_2)$ where $\mathfrak{L}\subset\mathfrak{A}_2$ & $\mathfrak{L}=\{\hat{\theta}_j\in\mathfrak{A}_2:\Theta_3(\hat{\theta}_i,\hat{\theta}_j)\in\hat{\Xi}\ forsome\ \hat{\theta}_i\in\mathfrak{A}_1\}$ & $\xi(\hat{\theta}_2)=\Theta_1(\hat{\theta}_2),\forall\ \hat{\theta}_2\in\mathfrak{L}$.

(iii) The $\hat{\Xi}^{-1}$ (inverse of $\hat{\Xi}$) is a $\hat{H}s$-relation from $(\Theta_2,\mathfrak{A}_2)$ to $(\Theta_1,\mathfrak{A}_1)$ stated by $\hat{\Xi}^{-1}=\{\Theta_2(\hat{\theta}_j)\times\Theta_1(\hat{\theta}_i):\Theta_1(\hat{\theta}_i)\hat{\Xi}\Theta_2(\hat{\theta}_j)\}$.

Example 5.4. Let

$$(\Theta_1,\mathfrak{A}_1)=\left\{\ \Theta_1(\hat{\theta}_1),\Theta_1(\hat{\theta}_2),\Theta_1(\hat{\theta}_3)\ \right\},(\Theta_2,\mathfrak{A}_2)=\left\{\ \Theta_2(\hat{\theta}_4),\Theta_2(\hat{\theta}_5),\Theta_2(\hat{\theta}_6)\ \right\}$$

$$(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)=\left\{\begin{array}{l}(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_6)),\\(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_6)),\\(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_6))\end{array}\right\}$$

then

$$\hat{\Xi}=\left\{\ (\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_6)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_6)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_6))\ \right\}$$

(i) $DoM\,\hat{\Xi}=(\Theta,\mathfrak{W})$ where $\mathfrak{W}=\{\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3\}\subseteq\mathfrak{A}_1$ & $\Theta(\hat{\theta}_i)=\Theta_1(\hat{\theta}_i)\forall\ \hat{\theta}_i\in\mathfrak{W}$.

(ii) $RNG\,\hat{\Xi}=(\xi,\mathfrak{L})$ where $\mathfrak{L}=\{\hat{\theta}_4,\hat{\theta}_6\}\subset\mathfrak{A}_2$ & $\xi(\hat{\theta}_j)=\Theta_2(\hat{\theta}_j)\forall\ \hat{\theta}_j\in\mathfrak{L}$.

(iii)

$$\hat{\Xi}^{-1} = \left\{ \ (\Theta_2(\hat{\theta}_4) \times \Theta_1(\hat{\theta}_1)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_1)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_2)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_3)) \ \right\}.$$

Definition 5.5. Let $\hat{\Xi}$ & $\mathfrak{S}$ are two $\hat{H}s$-relations on $\hat{H}s$-set $(\Theta, \mathfrak{W})$, then we get

(i) $\hat{\Xi} \subset \mathfrak{S}$, if for all $\varpi, \varsigma \in \mathfrak{W}, \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}$.

(ii) $\hat{\Xi}^{©} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \notin \hat{\Xi}, \forall\ \varpi, \varsigma \in \mathfrak{W}\}$.

(iii) $\hat{\Xi} \cup \mathfrak{S} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}\ or\ \Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}, \forall\ \varpi, \varsigma \in \mathfrak{W}\}$.

(iv) $\hat{\Xi} \cap \mathfrak{S} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}\ \&\ \Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}, \forall\ \varpi, \varsigma \in \mathfrak{W}\}$.

Example 5.6. Let $(\Theta, \mathfrak{W}) = \left\{ \ \Theta(\hat{\theta}_1), \Theta(\hat{\theta}_2), \Theta(\hat{\theta}_3) \ \right\}$ then

$$(\Theta, \mathfrak{W}) \times (\Theta, \mathfrak{W}) = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), \\ (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), \\ (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \end{array} \right\}$$

then we get

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}$$

&

$$\mathfrak{S} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)) \ \right\}$$

now

(1) $\hat{\Xi}^{©} = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), \\ (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)) \end{array} \right\}$. &

$\mathfrak{S}^{©} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}$.

(2) $\hat{\Xi} \cup \mathfrak{S} = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), \\ (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \end{array} \right\}$.

(3) $\hat{\Xi} \cap \mathfrak{S} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)) \ \right\}$.

Definition 5.7. Let $\hat{\Xi}$ be a $\hat{H}s$-relation on $(\Theta, \mathfrak{W})$, then

(i) if $\Theta(\varpi) \times \Theta(\varpi) \in \hat{\Xi} \forall \varpi \in \mathfrak{W}$, then $\hat{\Xi}$ is reflexive, e.g. $\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)) \ \right\}$.

(ii) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\Theta(\varsigma) \times \Theta(\varpi) \in \hat{\Xi} \forall \varpi, \varsigma \in \mathfrak{W}$, so $\hat{\Xi}$ is symmetric, e.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)) \ \right\}.$$

(iii) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ & $\Theta(\varsigma) \times \Theta(w) \in \hat{\Xi}$ then $\Theta(\varpi) \times \Theta(w) \in \hat{\Xi} \forall \varpi, \varsigma, w \in \mathfrak{W}$, so $\hat{\Xi}$ is transitive. e.g. $\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)) \ \right\}$.

(iv) if properties (i)-(iii) are satisfied then $\hat{\Xi}$ is stated as equivalence relation. E.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)) \ \right\}.$$

(v) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\varpi = \varsigma \forall \varpi, \varsigma \in \mathfrak{W}$, so $\hat{\Xi}$ is stated as identity. e.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}.$$

Definition 5.8. If $\hat{\Xi}$ is a $\hat{H}s$-relation from $(\Theta_1, \mathfrak{A}_1)$ to $(\Theta_2, \mathfrak{A}_2)$ & $\mathfrak{S}$ is a $\hat{H}s$-relation from $(\Theta_2, \mathfrak{A}_2)$ to $(\Theta_3, \mathfrak{A}_3)$ then composition of $\hat{\Xi}$ & $\mathfrak{S}$, symbolized by $\hat{\Xi} \circ \mathfrak{S}$, is also a $\hat{H}s$-relation $\mathfrak{T}$ from $(\Theta_1, \mathfrak{A}_1)$ to $(\Theta_3, \mathfrak{A}_3)$ stated as if $\Theta_1(\varpi) \in (\Theta_1, \mathfrak{A}_1)$ & $\Theta_3(w) \in (\Theta_3, \mathfrak{A}_3)$ then $\Theta_1(\varpi) \times \Theta_3(w) \in \hat{\Xi} \circ \mathfrak{S}$ i.e. $\Theta_1(\varpi) \times \Theta_3(w) \in \hat{\Xi} \circ \mathfrak{S}$ iff $\Theta_1(\varpi) \times \Theta_2(\varsigma) \in \hat{\Xi}$ & $\Theta_2(\varsigma) \times \Theta_3(w) \in \hat{\Xi}$.

Example 5.9. Let

$$\hat{\Xi} = \left\{ \begin{array}{c} (\Theta_1(\hat{\theta}_1) \times \Theta_2(\hat{\theta}_1)), (\Theta_1(\hat{\theta}_1) \times \Theta_2(\hat{\theta}_3)), \\ (\Theta_1(\hat{\theta}_2) \times \Theta_2(\hat{\theta}_3)), (\Theta_1(\hat{\theta}_3) \times \Theta_2(\hat{\theta}_3)) \end{array} \right\} \&$$

$$\mathfrak{S} = \left\{ \begin{array}{c} (\Theta_2(\hat{\theta}_1) \times \Theta_3(\hat{\theta}_1)), (\Theta_2(\hat{\theta}_1) \times \Theta_3(\hat{\theta}_2)), \\ (\Theta_2(\hat{\theta}_2) \times \Theta_3(\hat{\theta}_2)), (\Theta_2(\hat{\theta}_3) \times \Theta_3(\hat{\theta}_2)) \end{array} \right\}$$

then

$$\hat{\Xi} \circ \mathfrak{S} = \left\{ \begin{array}{c} (\Theta_1(\hat{\theta}_1) \times \Theta_3(\hat{\theta}_1)), (\Theta_1(\hat{\theta}_1) \times \Theta_3(\hat{\theta}_2)), \\ (\Theta_1(\hat{\theta}_2) \times \Theta_3(\hat{\theta}_2)), (\Theta_1(\hat{\theta}_3) \times \Theta_3(\hat{\theta}_2)) \end{array} \right\}.$$

Definition 5.10. A $\hat{H}s$-relation $\mathfrak{F}$ from $(\Theta_1, \mathfrak{A}_1)$ to $(\Theta_2, \mathfrak{A}_2)$, represented by $\mathfrak{F} : (\Theta_1, \mathfrak{A}_1) \to (\Theta_2, \mathfrak{A}_2)$, is stated as $\hat{H}s$-function when (a). $DoM\,\mathfrak{F} = \mathfrak{A}_1$, (b). $DoM\,\mathfrak{F}$ has not repeated members & (c). Element-based uniqueness exists between $RNG\,\mathfrak{F}$ & $DoM\,\mathfrak{F}$ i.e. if $\Theta_1(\varpi)\,\mathfrak{F}\,\Theta_2(\varsigma)$ (or $\Theta_1(\varpi) \times \Theta_2(\varsigma) \in \mathfrak{F}$) then $\mathfrak{F}(\Theta_1(\varpi)) = \Theta_2(\varsigma)$.

Example 5.11. Let $\mathfrak{A}_1 = \{\varpi_1, \varpi_2, \varpi_3\}$ & $\mathfrak{A}_2 = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ then
$(\Theta_1, \mathfrak{A}_1) = \left\{ \Theta_1(\varpi_1), \Theta_1(\varpi_2), \Theta_1(\varpi_3) \right\}$, $(\Theta_2, \mathfrak{A}_2) = \left\{ \Theta_2(\varsigma_1), \Theta_2(\varsigma_2), \Theta_2(\varsigma_3), \Theta_2(\varsigma_4) \right\}$
so $\hat{H}s$-functions is

$$\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_3)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_4)) \right\}$$

Definition 5.12. A $\hat{H}s$-function $\mathfrak{F} : (\Theta_1, \mathfrak{A}_1) \to (\Theta_2, \mathfrak{A}_2)$ is stated as

(i) if $RNG\,\mathfrak{F} \subset \mathfrak{A}_2$, then INTO-$\hat{H}s$-function. E.g. Let $\mathfrak{A}_1 = \{\varpi_1, \varpi_2, \varpi_3\}$ & $\mathfrak{A}_2 = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ then $\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_3)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_4)) \right\}$

(ii) if $RNG\,\mathfrak{F} = \mathfrak{A}_2$, then ONTO-$\hat{H}s$-function. E.g. Let $\mathfrak{A}_1 = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$ & $\mathfrak{A}_2 = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ then

$$\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_3)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_4)), (\Theta_1(\varpi_4) \times \Theta_2(\varsigma_2)) \right\}$$

(iii) *1-1 $\hat{H}s$-function* if $\Theta_1(\varpi_1) \neq \Theta_1(\varpi_2)$ then $\mathfrak{F}(\Theta_1(\varpi_1)) \neq \mathfrak{F}(\Theta_1(\varpi_2))$. E.g.

$$\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_4)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_2)), (\Theta_1(\varpi_4) \times \Theta_2(\varsigma_3)) \right\}$$

(iv) if it is both INTO and ONTO then bijective $\hat{H}s$-function. E.g.

$$\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_2)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_3)), (\Theta_1(\varpi_4) \times \Theta_2(\varsigma_4)) \right\}$$

Definition 5.13. The identity $\hat{H}s$-function on $\hat{H}s$-set $(\Theta, \mathfrak{L})$ is stated by $\mathfrak{I} : (\Theta, \mathfrak{L}) \to (\Theta, \mathfrak{L})$ such that $\mathfrak{I}(\Theta(l)) = \Theta(l)\ \forall\ \Theta(l) \in (\Theta, \mathfrak{L})$. E.g. Let $\mathfrak{L} = \{l_1, l_2, l_3, l_4\}$ then
$\mathfrak{I} = \left\{ (\Theta(l_1) \times \Theta(l_1)), (\Theta(l_2) \times \Theta(l_2)), (\Theta(l_3) \times \Theta(l_3)), (\Theta(l_4) \times \Theta(l_4)) \right\}$

6. Matrix-theory Based on $\hat{H}s$-sets

Here some classical matrix-based notions are generalized for $\hat{H}s$-sets.

Definition 6.1.

(i) Let $(\Theta,\mathfrak{A})$ be a $\hat{H}s$-set on $\hat{\Pi}$. A set $\mathbb{R}_{\mathfrak{A}} \subseteq \hat{\Pi}\times\mathfrak{A}$ is a relation version of $(\Theta,\mathfrak{A})$ stated as

$$\mathbb{R}_{\mathfrak{A}} = \left\{ (\varpi,\hat{\theta}) : \hat{\theta}\in\mathfrak{A}, \varpi\in\Theta(\hat{\theta}) \right\}.$$

(ii) *The characteristic function* $\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}}$ is stated by $\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}} : \hat{\Pi}\times\mathfrak{A} \to \{0,1\}$, where

$$\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}}(\varpi,\hat{\theta}) = \begin{cases} 1 & ; \ (\varpi,\hat{\theta})\in\mathbb{R}_{\mathfrak{A}} \\ 0 & ; \ (\varpi,\hat{\theta})\notin\mathbb{R}_{\mathfrak{A}} \end{cases}$$

(iii) If $|\hat{\Pi}| = m$ & $|\mathfrak{A}| = n$ then $(\ddot{s}_{ij})$ is an $m\times n$ $\hat{H}s$-matrix of $(\Theta,\mathfrak{A})$ on $\hat{\Pi}$ and stated as

$$(\ddot{s}_{ij})_{m\times n} = \begin{pmatrix} \ddot{s}_{11} & \ddot{s}_{12} & \cdots & \ddot{s}_{1n} \\ \ddot{s}_{21} & \ddot{s}_{22} & \cdots & \ddot{s}_{2n} \\ \vdots & \vdots & & \vdots \\ \ddot{s}_{m1} & \ddot{s}_{m2} & \cdots & \ddot{s}_{mn} \end{pmatrix}$$

Note: The family of all $m\times n$ $\hat{H}s$- matrices on $\hat{\Pi}$ is symbolized by $(\hat{\Pi})^{(hsm)}_{m\times n}$.

Example 6.2. Let $\hat{\Pi} = \{\varpi_1,\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$ & $\mathfrak{A} = \{\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3,\hat{\theta}_4,\hat{\theta}_5\}$. Then $\Theta(\hat{\theta}_1) = \{\varpi_1,\varpi_2\}$, $\Theta(\hat{\theta}_2) = \varnothing$, $\Theta(\hat{\theta}_3) = \{\varpi_4,\varpi_5\}$, $\Theta(\hat{\theta}_4) = \{\varpi_2,\varpi_3,\varpi_4,\}$, $\Theta(\hat{\theta}_5) = \varnothing$, therefore we get $(\Theta,\mathfrak{A}) = \left\{ (\hat{\theta}_1,\{\varpi_1,\varpi_2\}),(\hat{\theta}_3,\{\varpi_4,\varpi_5\}),(\hat{\theta}_4,\{\varpi_2,\varpi_3,\varpi_4,\}) \right\}$ &
$\mathbb{R}_{\mathfrak{A}} = \left\{ (\varpi_1,\hat{\theta}_1),(\varpi_2,\hat{\theta}_1),(\varpi_4,\hat{\theta}_3),(\varpi_5,\hat{\theta}_3),(\varpi_2,\hat{\theta}_4),(\varpi_3,\hat{\theta}_4),(\varpi_4,\hat{\theta}_4) \right\}$. Hence $\hat{H}s$- matrix is given as

$$(\ddot{s}_{ij})_{5\times 5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} {}_1i^5,\ {}_1j^5.$$

Definition 6.3. Let $(\ddot{s}_{ij})_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$ then $(\ddot{s}_{ij})_{m\times n}$ is characterized as:

(i) The $(0)_{m\times n}$ is stated as a null $\hat{H}s$- matrix if $\ddot{s}_{ij} = 0 \ \forall\ i,j$ e.g.

$$(0)_{5\times 5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} {}_1i^5,\ {}_1j^5.$$

(ii) An $\mathfrak{A}_1$-*universal* $\hat{H}s$- *matrix*, symbolized by $(\ddot{s}_{ij})^{\mathfrak{A}_1}_{m\times n}$, if $\ddot{s}_{ij} = 1, \forall j \in J_{\mathfrak{A}_1} = \{j : \hat{\theta}_j \in \mathfrak{A}_1\}$ & i. E.g. Let $\mathfrak{A}$ be as provided in 6.2 & $\mathfrak{A}_1 = \{\hat{\theta}_2, \hat{\theta}_4, \hat{\theta}_5\} \subseteq \mathfrak{A}$ with $\Theta(\hat{\theta}_2) = \Theta(\hat{\theta}_4) = \Theta(\hat{\theta}_5) = \hat{\Pi}$ then

$$(\ddot{s}_{ij})^{\mathfrak{A}_1}_{5\times 5} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} {}_1i^5,\ {}_1j^5.$$

(iii) The $(\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}$ is stated as universal $\hat{H}s$- matrix if $\ddot{s}_{ij} = 1, \forall\ i, j$. E.g. Let $\mathfrak{A}$ as stated in 6.2 with $\Theta(\hat{\theta}_1) = \Theta(\hat{\theta}_2) = \Theta(\hat{\theta}_3) = \Theta(\hat{\theta}_4) = \Theta(\hat{\theta}_5) = \hat{\Pi}$ then

$$(\ddot{s}_{ij})^{\hat{\Pi}}_{5\times 5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} {}_1i^5,\ {}_1j^5.$$

Definition 6.4. Let $\mathfrak{L}_1 = (\ddot{s}_{ij})_{m\times n}, \mathfrak{L}_2 = \left(\ddot{t}_{ij}\right)_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$ then

(a) $\mathfrak{L}_1$ is stated as $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ if $\ddot{s}_{ij} \leq \ddot{t}_{ij}$ e.g. $\mathfrak{L}_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

(b) $\mathfrak{L}_1$ & $\mathfrak{L}_2$ are stated as comparable, symbolized by $\mathfrak{L}_1 \parallel \mathfrak{L}_2$, if $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ or $\mathfrak{L}_2 \subseteq \mathfrak{L}_1$.

(c) $\mathfrak{L}_1$ is stated as *proper* $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \subset \mathfrak{L}_2$ if for atleast one term $\ddot{s}_{ij} \leq \ddot{t}_{ij}$ e.g. $\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

(d) $\mathfrak{L}_1$ is stated as *strictly* $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \subsetneq \mathfrak{L}_2$ if for each term $\ddot{s}_{ij} \leq \ddot{t}_{ij}$ e.g. $\mathfrak{L}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

(e) *union* of $\mathfrak{L}_1$ & $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \cup \mathfrak{L}_2$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3 = (\delta_{ij})_{m\times n}$ if $\delta_{ij} = max\{\ddot{s}_{ij}, \ddot{t}_{ij}\}\ \forall\ i, j$ e.g.

Let $\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$$\mathfrak{L}_3 = \mathfrak{L}_1 \cup \mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(f) *intersection* of $\mathfrak{L}_1$ & $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \cap \mathfrak{L}_2$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3 = (\delta_{ij})_{m\times n}$ if $\delta_{ij} = min\{\ddot{s}_{ij}, \ddot{t}_{ij}\}\ \forall\ i, j$ e.g.

Let $\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$$\mathfrak{L}_3 = \mathfrak{L}_1 \cap \mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

(g) The $\mathfrak{L}^{©}$ $(\mu_{ij})_{m\times n}$ (complement of $\mathfrak{L} = (\ddot{s}_{ij})_{m\times n}$), is also a $\hat{H}s$- matrix if $\mu_{ij} = 1 - \ddot{s}_{ij}\ \forall\ i, j$ e.g.

Let $\mathfrak{L} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ then $\mathfrak{L}^{©} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$.

(h) The difference $\mathfrak{L}_2 \setminus \mathfrak{L}_1$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3$ such that $\mathfrak{L}_3 = \mathfrak{L}_2 \cap \mathfrak{L}_1^{©}$ e.g.

$\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$\mathfrak{L}_3 = \mathfrak{L}_2 \cap \mathfrak{L}_1^{\copyright}$

$$\mathfrak{L}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Proposition 6.5. *For $\mathfrak{C}_1 = (\ddot{s}_{ij})_{m\times n}, \mathfrak{C}_2 = (\ddot{t}_{ij})_{m\times n}, \mathfrak{C}_3 = (\ddot{u}_{ij})_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$, the following axiomatic results are valid:*

(1) $\mathfrak{C}_1 \cup \mathfrak{C}_1 = \mathfrak{C}_1, \;\; \mathfrak{C}_1 \cap \mathfrak{C}_1 = \mathfrak{C}_1$

(2) $\mathfrak{C}_1 \cup (0)_{m\times n} = \mathfrak{C}_1, \;\; \mathfrak{C}_1 \cap (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n} = \mathfrak{C}_1$

(3) $\mathfrak{C}_1 \cap (0)_{m\times n} = (0)_{m\times n}, \;\; \mathfrak{C}_1 \cup (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n} = (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}$

(4) $((0)_{m\times n})^{\copyright} = (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}, \;\; ((\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n})^{\copyright} = (0)_{m\times n}$

(5) $\mathfrak{C}_1 \cup \mathfrak{C}_1{}^{\copyright} = (\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}, \;\; \mathfrak{C}_1 \cap \mathfrak{C}_1{}^{\copyright} = (0)_{m\times n}$

(6) $(\mathfrak{C}_1 \cup \mathfrak{C}_2)^{\copyright} = \mathfrak{C}_1{}^{\copyright} \cap \mathfrak{C}_2{}^{\copyright}, \;\; (\mathfrak{C}_1 \cap \mathfrak{C}_2)^{\copyright} = \mathfrak{C}_1{}^{\copyright} \cup \mathfrak{C}_2{}^{\copyright}$

(7) $(\mathfrak{C}_1{}^{\copyright})^{\copyright} = \mathfrak{C}_1$

(8) $\mathfrak{C}_1 \cup \mathfrak{C}_2 = \mathfrak{C}_2 \cup \mathfrak{C}_1, \;\; \mathfrak{C}_1 \cap \mathfrak{C}_2 = \mathfrak{C}_2 \cap \mathfrak{C}_1$

(9) $\mathfrak{C}_1 \cup (\mathfrak{C}_2 \cup \mathfrak{C}_3) = (\mathfrak{C}_1 \cup \mathfrak{C}_2) \cup \mathfrak{C}_3, \;\; \mathfrak{C}_1 \cap (\mathfrak{C}_2 \cap \mathfrak{C}_3) = (\mathfrak{C}_1 \cap \mathfrak{C}_2) \cap \mathfrak{C}_3$

(10) $\mathfrak{C}_1 \cup (\mathfrak{C}_2 \cap \mathfrak{C}_3) = (\mathfrak{C}_1 \cup \mathfrak{C}_2) \cap (\mathfrak{C}_1 \cup \mathfrak{C}_3), \;\; \mathfrak{C}_1 \cap (\mathfrak{C}_2 \cup \mathfrak{C}_3) = (\mathfrak{C}_1 \cap \mathfrak{C}_2) \cup (\mathfrak{C}_1 \cap \mathfrak{C}_3)$

Definition 6.6. Let $\mathfrak{P} = (\ddot{c}_{ij})_{m\times n}, \;\; \mathfrak{Q} = (\ddot{d}_{ik})_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$, then

(i) *AND-product* is stated as
$\wedge : (\hat{\Pi})^{(hsm)}_{m\times n} \times (\hat{\Pi})^{(hsm)}_{m\times n} \to (\hat{\Pi})^{(hsm)}_{m\times n^2}$ with $(\ddot{c}_{ij}) \wedge (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = min\{\ddot{c}_{ij}, \ddot{d}_{ik}\}$ & $l = n(j-1)+k$.

(ii) *OR-product* is stated as
$\vee : (\hat{\Pi})^{(hsm)}_{m\times n} \times (\hat{\Pi})^{(hsm)}_{m\times n} \to (\hat{\Pi})^{(hsm)}_{m\times n^2}$ with $(\ddot{c}_{ij}) \vee (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = max\{\ddot{c}_{ij}, \ddot{d}_{ik}\}$.

(iii) *AND-NOT-product* is stated as
$\overline{\wedge} : (\hat{\Pi})^{(hsm)}_{m\times n} \times (\hat{\Pi})^{(hsm)}_{m\times n} \to (\hat{\Pi})^{(hsm)}_{m\times n^2}$ with $(\ddot{c}_{ij}) \overline{\wedge} (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = min\{\ddot{c}_{ij}, 1-\ddot{d}_{ik}\}$.

(iv) *OR-NOT-product* is stated as
$\underline{\vee} : (\hat{\Pi})^{(hsm)}_{m\times n} \times (\hat{\Pi})^{(hsm)}_{m\times n} \to (\hat{\Pi})^{(hsm)}_{m\times n^2}$ with $(\ddot{c}_{ij}) \underline{\vee} (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = max\{\ddot{c}_{ij}, 1-\ddot{d}_{ik}\}$.

Example 6.7. Let $\mathfrak{P} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ & $\mathfrak{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ then

(i) $$\mathfrak{P} \wedge \mathfrak{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\text{(ii)}\ \mathfrak{P} \vee \mathfrak{Q} = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

$$\text{(iii)}\ \mathfrak{P} \overline{\wedge} \mathfrak{Q} = \begin{pmatrix} 1&1&0&1 \\ 1&1&1&1 \\ 0&1&0&1 \\ 1&1&1&1 \end{pmatrix} \wedge \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\text{(iv)}\ \mathfrak{P} \underline{\vee} \mathfrak{Q} = \begin{pmatrix} 1&1&0&1 \\ 1&1&1&1 \\ 0&1&0&1 \\ 1&1&1&1 \end{pmatrix} \vee \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

7. Hybridized Structures of $\hat{H}s$-sets

Here the notions of some hybridized model of $\hat{H}s$-sets are presented. The set $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots\ldots \times \mathfrak{A}_m$ with $\mathfrak{A}_{\hat{\alpha}} \cap \mathfrak{A}_{\hat{\beta}} = \varnothing\ \forall\ \hat{\alpha}, \hat{\beta} = 1, 2, ..., m$ and $\mathfrak{A}_{\hat{\alpha}}$ are same as stated in Definition 3.1. The Figure 3 presents the notations and their full names that are used in this section.

Abbreviations	Used for	Abbreviations	Used for
$iv\hat{f}Hs$-set	interval-valued fuzzy hypersoft set	ivf-set	interval-valued fuzzy set
$\mathcal{F}^{ivf}(\hat{\Pi})$	collection of interval-valued fuzzy subsets over $\hat{\Pi}$	$fphs$-set	fuzzy parameterized hypersoft set
maa-function	multi-argument approximate function	iv-$fphs$-set	interval-valued fuzzy parameterized hypersoft set
$ifphs$-set	intuitionistic fuzzy parameterized hypersoft set	$nphs$-set	neutrosophic parameterized hypersoft set
if-set	intuitionistic fuzzy set	n-set	neutrosophic set

FIGURE 3. Notations

Definition 7.1. An $iv\hat{f}Hs$-set $(\Gamma, \mathfrak{A})$ on $\hat{\Pi}$ is stated by

$$(\Gamma, \mathfrak{A}) = \left\{ (\hat{\theta}, \Gamma(\hat{\theta})); \hat{\theta} \in \mathfrak{A}, \Gamma(\hat{\theta}) \in \mathcal{F}^{ivf}(\hat{\Pi}) \right\}$$

where $\Gamma : \mathfrak{A} \rightarrow \mathcal{F}^{ivf}(\hat{\Pi})$ & $\Gamma(\hat{\theta}) = \left\{ \ \ddot{\psi}_{\Gamma(\hat{\theta})}(\varpi)/\varpi : \varpi \in \hat{\Pi}, \ddot{\psi}_{\Gamma(\hat{\theta})}(\varpi) \in \mathbb{C}(\mathbb{I}) \ \right\}$ is an ivf-set on $\hat{\Pi}$.

Example 7.2. Let $\hat{\Pi} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6, \varpi_7, \varpi_8\}$ & $\mathfrak{A} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5, \hat{\theta}_6, \hat{\theta}_7, \hat{\theta}_8\}$, $iv\hat{f}Hs$-set $(\Gamma, \mathfrak{A})$ is constructed as

$$(\Gamma, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{[0.1, 0.2]/\varpi_1, [0.2, 0.3]/\varpi_2, [0.4, 0.5]/\varpi_4, [0.5, 0.6]/\varpi_5\}\right), \\ \left(\hat{\theta}_2, \{[0.1, 0.3]/\varpi_1, [0.2, 0.4]/\varpi_2, [0.3, 0.4]/\varpi_3, [0.6, 0.8]/\varpi_6\}\right), \\ \left(\hat{\theta}_3, \{[0.2, 0.3]/\varpi_2, [0.3, 0.4]/\varpi_3, [0.4, 0.5]/\varpi_4, , [0.5, 0.7]/\varpi_5\}\right), \\ \left(\hat{\theta}_4, \{[0.4, 0.5]/\varpi_4, [0.5, 0.6]/\varpi_5, [0.6, 0.7]/\varpi_6, [0.7, 0.8]/\varpi_7\}\right), \\ \left(\hat{\theta}_5, \{[0.3, 0.6]/\varpi_3, [0.6, 0.7]/\varpi_6, [0.7, 0.8]/\varpi_7, [0.8, 0.9]/\varpi_8\}\right), \\ \left(\hat{\theta}_6, \{[0.2, 0.4]/\varpi_2, [0.3, 0.5]/\varpi_3, [0.4, 0.6]/\varpi_4, [0.7, 0.8]/\varpi_7\}\right), \\ \left(\hat{\theta}_7, \{[0.1, 0.4]/\varpi_1, [0.3, 0.4]/\varpi_3, [0.5, 0.7]/\varpi_5, [0.6, 0.8]/\varpi_6\}\right), \\ \left(\hat{\theta}_8, \{[0.2, 0.5]/\varpi_2, [0.3, 0.6]/\varpi_3, [0.6, 0.8]/\varpi_6, [0.7, 0.8]/\varpi_7\}\right) \end{array} \right\}$$

Definition 7.3. A $fphs$-set $(\mathcal{D}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{D}, \mathfrak{A}) = \left\{ \ \left(\varphi_{\mathcal{F}}(\hat{\theta})/\hat{\theta}, \Theta_{\mathcal{F}}(\hat{\theta})\right), \hat{\theta} \in \mathfrak{A}, \ \Theta_{\mathcal{F}}(\hat{\theta}) \in P^{\hat{\Pi}}, \ \varphi_{\mathcal{F}}(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \ \right\}$$

where $\mathcal{F}$ is a fuzzy set with $\varphi_{\mathcal{F}} : \mathfrak{A} \rightarrow \mathbb{C}(\mathbb{I})$ as membership function of $fphs$-set & $\Theta_{\mathcal{F}} : \mathfrak{A} \rightarrow P^{\hat{\Pi}}$ is maa-function of $fphs$-set.

Example 7.4. From Example 7.2, we get

$$(\mathcal{D}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(0.1/\hat{\theta}_1, \{\varpi_1, \varpi_2\}\right), \left(0.2/\hat{\theta}_2, \{\varpi_1, \varpi_2, \varpi_3\}\right), \left(0.3/\hat{\theta}_3, \{\varpi_2, \varpi_3, \varpi_4\}\right), \\ \left(0.4/\hat{\theta}_4, \{\varpi_4, \varpi_5, \varpi_6\}\right), \left(0.5/\hat{\theta}_5, \{\varpi_6, \varpi_7, \varpi_8\}\right), \left(0.6/\hat{\theta}_6, \{\varpi_2, \varpi_3, \varpi_4, \varpi_7\}\right), \\ \left(0.7/\hat{\theta}_7, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}\right), \left(0.8/\hat{\theta}_8, \{\varpi_2, \varpi_3, \varpi_6, \varpi_7\}\right) \end{array} \right\}$$

Definition 7.5. An iv-$fphs$-set $(\mathcal{E}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{E}, \mathfrak{A}) = \left\{ \ \left(\Psi_{\mathcal{F}^{iv}}(\hat{\theta})/\hat{\theta}, \psi_{\mathcal{F}^{iv}}(\hat{\theta})\right), \hat{\theta} \in \mathfrak{A}, \ \psi_{\mathcal{F}_{iv}}(\hat{\theta}) \in P^{\hat{\Pi}}, \ \Psi_{\mathcal{F}^{iv}}(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \ \right\}$$

where $\mathcal{F}^{iv}$ is an ivf-set with $\Psi_{\mathcal{F}^{iv}} : \mathfrak{A} \rightarrow \mathbb{C}(\mathbb{I})$ as membership function of $fphs$-set and $\psi_{\mathcal{F}_{iv}} : \mathfrak{A} \rightarrow P^{\hat{\Pi}}$ is maa-function of iv-$fphs$-set.

Example 7.6. From Example 7.2, we get

$$(\mathcal{E}, \mathfrak{A}) = \left\{ \begin{array}{l} \left([0.1, 0.2]/\hat{\theta}_1, \{\varpi_1, \varpi_2\}\right), \left([0.2, 0.3]/\hat{\theta}_2, \{\varpi_1, \varpi_2, \varpi_3\}\right), \\ \left([0.3, 0.4]/\hat{\theta}_3, \{\varpi_2, \varpi_3, \varpi_4\}\right), \left([0.4, 0.5]/\hat{\theta}_4, \{\varpi_4, \varpi_5, \varpi_6\}\right), \\ \left([0.5, 0.6]/\hat{\theta}_5, \{\varpi_6, \varpi_7, \varpi_8\}\right), \left([0.6, 0.7]/\hat{\theta}_6, \{\varpi_2, \varpi_3, \varpi_4, \varpi_7\}\right), \\ \left([0.7, 0.8]/\hat{\theta}_7, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}\right), \left([0.8, 0.9]/\hat{\theta}_8, \{\varpi_2, \varpi_3, \varpi_6, \varpi_7\}\right) \end{array} \right\}$$

Definition 7.7. An $ifphs$-set $(\mathcal{H}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{H}, \mathfrak{A}) = \left\{ \ \left(< \varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) > /\hat{\theta}, \psi^{\mathcal{IF}}(\hat{\theta})\right); \hat{\theta} \in \mathfrak{A}, \ \psi^{\mathcal{IF}}(\hat{\theta}) \in P^{\hat{\Pi}}, \ \varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \ \right\}$$

where $\mathcal{IF}$ is an if-set with $\varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) : \mathfrak{A} \rightarrow \mathbb{C}(\mathbb{I})$ as membership and non-membership functions of $ifphs$-set and $\psi^{\mathcal{IF}} : \mathfrak{A} \rightarrow P^{\hat{\Pi}}$ is maa-function of $ifphs$-set.

Example 7.8. From Example 7.2, we get

$$(\mathcal{H},\mathfrak{A}) = \left\{ \begin{array}{l} \left(< 0.1, 0.2 > /\hat{\theta}_1, \{\varpi_1, \varpi_2\}\right), \left(< 0.2, 0.3 > /\hat{\theta}_2, \{\varpi_1, \varpi_2, \varpi_3\}\right), \\ \left(< 0.3, 0.4 > /\hat{\theta}_3, \{\varpi_2, \varpi_3, \varpi_4\}\right), \left(< 0.4, 0.5 > /\hat{\theta}_4, \{\varpi_4, \varpi_5, \varpi_6\}\right), \\ \left(< 0.5, 0.6 > /\hat{\theta}_5, \{\varpi_6, \varpi_7, \varpi_8\}\right), \left(< 0.6, 0.7 > /\hat{\theta}_6, \{\varpi_2, \varpi_3, \varpi_4, \varpi_7\}\right), \\ \left(< 0.7, 0.8 > /\hat{\theta}_7, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}\right), \left(< 0.8, 0.9 > /\hat{\theta}_8, \{\varpi_2, \varpi_3, \varpi_6, \varpi_7\}\right) \end{array} \right\}$$

Definition 7.9. A *nphs*-set $(\mathcal{N},\mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{N},\mathfrak{A}) = \left\{ \begin{array}{l} \left(< \lambda_1(\hat{\theta}), \lambda_2(\hat{\theta}), \lambda_3(\hat{\theta}) > /\hat{\theta}, \psi^{\mathcal{N}}(\hat{\theta})\right); \hat{\theta} \in \mathfrak{A}, \psi^{\mathcal{N}}(\hat{\theta}) \in P^{\hat{\Pi}}, \\ \lambda_1(\hat{\theta}) \in \mathbb{C}(\mathbb{I}), \lambda_2(\hat{\theta}) \in \mathbb{C}(\mathbb{I}), \lambda_3(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \end{array} \right\}$$

where $\mathcal{N}$ is a neutrosophic set with $\lambda_1(\hat{\theta}), \lambda_2(\hat{\theta}), \lambda_3(\hat{\theta}) : \mathfrak{A} \to \mathbb{C}(\mathbb{I})$ as membership, indeterminate and falsity of *nphs*-set and $\psi^{\mathcal{N}} : \mathfrak{A} \to P^{\hat{\Pi}}$ is maa-function of *nphs*-set.

Example 7.10. From Example 7.2, we get

$$(\mathcal{N},\mathfrak{A}) = \left\{ \begin{array}{l} \left(< 0.1, 0.2, 0.2 > /\hat{\theta}_1, \{\varpi_1, \varpi_2\}\right), \left(< 0.2, 0.3, 0.3 > /\hat{\theta}_2, \{\varpi_1, \varpi_2, \varpi_3\}\right), \\ \left(< 0.3, 0.4, 0.4 > /\hat{\theta}_3, \{\varpi_2, \varpi_3, \varpi_4\}\right), \left(< 0.4, 0.5, 0.5 > /\hat{\theta}_4, \{\varpi_4, \varpi_5, \varpi_6\}\right), \\ \left(< 0.5, 0.6, 0.6 > /\hat{\theta}_5, \{\varpi_6, \varpi_7, \varpi_8\}\right), \left(< 0.6, 0.7, 0.7 > /\hat{\theta}_6, \{\varpi_2, \varpi_3, \varpi_4, \varpi_7\}\right), \\ \left(< 0.7, 0.5, 0.8 > /\hat{\theta}_7, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}\right), \left(< 0.8, 0.4, 0.9 > /\hat{\theta}_8, \{\varpi_2, \varpi_3, \varpi_6, \varpi_7\}\right) \end{array} \right\}$$

Definition 7.11. A $\hat{H}s$-set $(\ddot{\mathfrak{B}},\mathfrak{A})$ is known as *bijective $\hat{H}s$-set* (*bhs*-set) on $\hat{\Pi}$ if

(i) $\bigcup\limits_{j \in \mathfrak{A}} \ddot{\mathfrak{B}}(\hat{\theta}) = \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.12. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}},\mathfrak{A}) = \left\{ \left(\hat{\theta}_1, \{\varpi_1\}\right), \left(\hat{\theta}_2, \{\varpi_2\}\right), \left(\hat{\theta}_3, \{\varpi_3\}\right), \left(\hat{\theta}_4, \{\varpi_4\}\right), \left(\hat{\theta}_5, \{\varpi_5\}\right), \left(\hat{\theta}_6, \{\varpi_6\}\right), \left(\hat{\theta}_7, \{\varpi_7\}\right), \left(\hat{\theta}_8, \{\varpi_8\}\right) \right\}$$

Definition 7.13. A $f\hat{h}s$-set $(\ddot{\mathfrak{B}}^f,\mathfrak{A})$ is stated as *bijective $f\hat{h}s$-set* on $\hat{\Pi}$ if

(i) $\bigcup\limits_{\hat{\theta} \in \mathfrak{A}} \ddot{\mathfrak{B}}^f(\hat{\theta}) = \hat{\Pi}$ with $\sum\limits_{\varpi \in \hat{\Pi}} \ddot{\psi}_f(\varpi) \in \mathbb{C}(\mathbb{I})$ where $\ddot{\psi}_f(\varpi)$ is a f-membership for each $\varpi \in \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^f(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^f(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.14. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}}^f,\mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{0.1/\varpi_1\}\right), \left(\hat{\theta}_2, \{0.2/\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{0.13/\varpi_3\}\right), \left(\hat{\theta}_4, \{0.14/\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{0.05/\varpi_5\}\right), \left(\hat{\theta}_6, \{0.06/\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{0.07/\varpi_7\}\right), \left(\hat{\theta}_8, \{0.08/\varpi_8\}\right) \end{array} \right\}$$

Definition 7.15. An $iv\hat{f}hs$-set $(\ddot{\mathfrak{B}}^{ivf},\mathfrak{A})$ is stated as *bijective $iv\hat{f}hs$-set* on $\hat{\Pi}$ if

(i) $\bigcup\limits_{\hat{\theta} \in \mathfrak{A}} \ddot{\mathfrak{B}}^{ivf}(\hat{\theta}) = \hat{\Pi}$ with $\sum\limits_{\varpi \in \hat{\Pi}} Sup(\ddot{\psi}_f(\varpi)) \in \mathbb{C}(\mathbb{I})$ where $\ddot{\psi}_f(\varpi)$ is an ivf-membership for each $\varpi \in \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{ivf}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{ivf}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.16. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{ivf}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{[0.01, 0.1]/\varpi_1\}\right), \left(\hat{\theta}_2, \{[0.02, 0.2]/\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{[0.03, 0.13]/\varpi_3\}\right), \left(\hat{\theta}_4, \{[0.04, 0.14]/\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{[0.03, 0.05]/\varpi_5\}\right), \left(\hat{\theta}_6, \{[0.02, 0.06]/\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{[0.03, 0.07]/\varpi_7\}\right), \left(\hat{\theta}_8, \{[0.04, 0.08]/\varpi_8\}\right) \end{array} \right\}$$

Definition 7.17. An $if\hat{h}s$-set $(\ddot{\mathfrak{B}}^{if}, \mathfrak{A})$ is known as *bijective* $if\hat{h}s$*-set* on $\hat{\Pi}$ if

(i) $\bigcup_{\hat{\theta} \in \mathfrak{A}} \ddot{\mathfrak{B}}^{if}(\hat{\theta}) = \hat{\Pi}$ with $\sum_{\varpi \in \hat{\Pi}} T_{if}(\varpi)$ & $\sum_{\varpi \in \hat{\Pi}} F_{if}(\varpi) \in \mathbb{C}(\mathbb{I})$ where $T_{if}(\varpi)$ & $F_{if}(\varpi)$ are membership and non-membership grades for each $\varpi \in \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{if}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{if}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.18. Reassuming Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{if}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{< 0.01, 0.1 > /\varpi_1\}\right), \left(\hat{\theta}_2, \{< 0.02, 0.2 > /\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{< 0.03, 0.13 > /\varpi_3\}\right), \left(\hat{\theta}_4, \{< 0.04, 0.14 > /\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{< 0.03, 0.05 > /\varpi_5\}\right), \left(\hat{\theta}_6, \{< 0.02, 0.06 > /\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{< 0.03, 0.07 > /\varpi_7\}\right), \left(\hat{\theta}_8, \{< 0.04, 0.08 > /\varpi_8\}\right) \end{array} \right\}$$

Definition 7.19. A $n\hat{h}s$-set $(\ddot{\mathfrak{B}}^{\mathcal{N}}, \mathfrak{A})$ is known as *bijective* $n\hat{h}s$*-set* on $\hat{\Pi}$ if

(i) $\bigcup_{\hat{\theta} \in \mathfrak{A}} \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}) = \hat{\Pi}$ with $\sum_{\varpi \in \hat{\Pi}} T_{\mathcal{N}}(\varpi)$, $\sum_{\varpi \in \hat{\Pi}} I_{\mathcal{N}}(\varpi)$ & $\sum_{\varpi \in \hat{\Pi}} F_{\mathcal{N}}(\varpi) \in \mathbb{C}(\mathbb{I})$ where $T_{\mathcal{N}}(\varpi), I_{\mathcal{N}}(\varpi)$ & $F_{\mathcal{N}}(\varpi)$ are membership, indeterminacy and non-membership grades for each $\varpi \in \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.20. Reassuming Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{\mathcal{N}}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{< 0.01, 0.02, 0.1 > /\varpi_1\}\right), \left(\hat{\theta}_2, \{< 0.02, 0.03, 0.2 > /\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{< 0.03, 0.04, 0.13 > /\varpi_3\}\right), \left(\hat{\theta}_4, \{< 0.04, 0.05, 0.14 > /\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{< 0.03, 0.04, 0.05 > /\varpi_5\}\right), \left(\hat{\theta}_6, \{< 0.02, 0.05, 0.06 > /\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{< 0.03, 0.04, 0.07 > /\varpi_7\}\right), \left(\hat{\theta}_8, \{< 0.04, 0.05, 0.08 > /\varpi_8\}\right) \end{array} \right\}$$

8. **Conclusions**

In this research work, several important rudiments (i.e. axioms-based properties, set-based aggregations etc.,) of $\hat{H}s$-set are investigated and explained with the support of real-scenarios based examples. In order to attract the intellectual attention of researchers, definitions of some glued models of $\hat{H}s$-set are also presented which will motivate them to extend the theory to other branches of mathematical-cum-computational sciences. Some future directions and scope of $\hat{H}s$-sets are presented in Figure 4.

Conflicts of Interest: The authors declare no conflict of interest.

Discipline	Scope
Fuzzy sets and systems	Development of intutionistic neutrosophic hypersoft set , spherical hypersoft set, picture fuzzy hypersoft set, geometric hypersoft set etc.
Graph Theory	Development of fuzzy hypersoft graph, intutionistic fuzzy hypersoft graph, neutrosophic fuzzy hypersoft graph, intutionistic neutrosophic hypersoft graph, possibility fuzzy hypersoft graph, possibility intutionistic fuzzy hypersoft graph, possibility neutrosophic hypersoft graph etc.
Algebra	Development of hypersoft groups, hypersoft rings, hypersoft vector spaces and their related structures.
Functional Analysis	Characterization of hypersoft metric spaces, hypersoft inner product spaces, normed hypersoft spaces, hypersoft measure theory, hypersoft Hilbert spaces etc.
Topology	Characterization of topological spaces, separation axioms, connected spaces and their relevant spaces.
Mathematical Analysis	Development of hypersoft fixed point theory, hypersoft real analysis, hypersoft modular inequalities, hypersoft complex analysis etc.

FIGURE 4. Future Directions and Scope of $\hat{H}s$-sets

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