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Soft Neutrosophic Quasigroups

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Abstract. This paper introduced and studied for the first time the object of neutrosophic quasigroup $\mathcal{Q}_{(Q,\cdot)}$ over a quasigroup $(Q,\cdot)$. It was shown that the direct product of any two neutrosophic quasigroups is a neutrosophic quasigroup. Also, it was established that the holomorph of any neutrosophic quasigroup is a neutrosophic quasigroup. The soft set theory which Molodtsov innovated is a potent mathematical tool used for solving mathematical problems with uncertainties and things that are not clearly defined. We broaden soft set theory by introducing soft neutrosophic quasigroup $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ over a neutrosophic quasigroup $\mathcal{Q}_{(Q,\cdot)}$. We introduced and established the order of a finite soft neutrosophic quasigroup with varied mathematical inequality expressions which exist between the order of a finite neutrosophic quasigroup and that of its soft neutrosophic quasigroup.

Keywords: Soft set, Neutrosophic set, Quasigroup, Neutrosophic quasigroup, Neutrosophic subquasigroup, Soft neutrosophic quasigroup, soft neutrosophic subquasigroup

1. **Introduction**

Molodtsov [14] introduced a better and more potent generalisation of set theory in solving classic mathematical problems represented by problems involving data structure that are deficient. Before then there exists some mathematical tools like rough set, fuzzy set, vague set, probability theory, intuitionistic set and neutrosophic sets among others. However, they have some limitations due to absence of adequate parametrisation tools that exists when solving mathematical uncertainties. Soft set has characteristics that makes it different from other tools. For instance, it's effectiveness is by using parametrisation in solving problems that involves incomplete data where grade of membership that is imperative in fuzzy set and grade of estimation in rough set are not needed.

Smarandache [18] launched the object of neutrosophic set in an attempt to generalise set theory involving uncertainties. Neutrosophic set is a mathematical tool that probe the origin, nature and the range of neutralities that is used for the generalisation of classical set, fuzzy set, interval fuzzy set and rough set etc. Neutrosophic set is used to solve mathematical problems involving imprecision, indeterminate, and inconsistencies.

In the concept of neutrosophy, each suggestion or idea is approximated to have certain degree of subset of $\mathcal{T}^\star$, indeterminacy $\mathcal{I}^\star$ and falsehood $\mathcal{F}^\star$. Neutrosophic set theory is used in solving problems involving informations that are imprecise, indeterminate, false and not properly defined which exist mainly in belief methodology. Since its introduction, many authors have published works in it, such as Vasantha and Smarandache [25] introduced certain algebraic neutrosophic structures and N-Algebraic neutrosophic structures, neutrosophic vector space, neutrosophic loops. Ali et al. [15] introduced soft neutrosophic loops and their generalizations, and Ali et al. [16] worked on soft neutrosophic groupiod and their generalisations. Maji [13] worked on neutrosophic soft sets.

Quasigroups are structures that generalises groups but they are not associative like groups. Quasigroups theory was introduced more than two centuries ago. Effectiveness of the application of the theory of quasigroups is based on its "generalized permutations" of some sort and the number of quasigroups of order n is larger than $n!$ - (Denes and Keedwell [17]). Namely, both left and right translations properties in quasigroups are permutations.

We refer readers to Aktas and Cagman [3], Molodtsov [14], Maji et. al. [13] and studies in [4, 5, 8, 12, 19, 20, 23, 24] for works on soft sets.

After the introduction of neutrosophic sets as generalization of intuitionistic fuzzy sets by Smarandache [18] in 2002, Vasantha and Smarandache [25] did a comprehensive introduction of algebraic neutrosophic structures and N-algebraic neutrosophic structures in 2006. Thereafter, Ali et al. [15, 16] studied neutrosophic groupoid, neutrosophic quasigroup and their soft sets deeply. Some recent developments in the study of 'soft neutrosophic' versions of various algebraic structures have been reported in [28, 29, 36] while some latest developments in the study of 'neutrosophic soft' versions of some algebraic structures have been reported in [30–32, 37].

The exploits done by different authors on algebraic characteristics of soft sets in general, and recent exploits on soft quasigroups in Oyem et al. [33–35] inspired us to institute the research on soft neutrosophic quasigroup structures. In this work, we introduced neutrosophic quasigroups and their soft sets.

2. Preliminaries

We start by reviewing some results concerning neutrosophic sets, quasigroups and soft sets. Various algebraic structures of neutrosophic sets have been introduced and studied in [6, 13, 15, 16, 18, 25].

Definition 2.1. *(Neutrosophic Set) If $\mathcal{X}$ is a universal set of discourse, then the set $\mathcal{A}$ on $\mathcal{X}$ is regarded as a neutrosophic set and denoted as;*

$$\mathcal{A} = \{< x, \mathcal{T}_{\mathcal{A}}(x), \mathcal{I}_{\mathcal{A}}(x)\}, \mathcal{F}_{\mathcal{A}}(x) >, x \in \mathcal{X}$$
$$\text{where } \mathcal{T}^{\star}, \mathcal{I}^{\star}, \mathcal{F}^{\star} : \mathcal{X} \longrightarrow]^{-}0, 1^{+}[\text{ and } ^{-}0 \leq \mathcal{T}_{\mathcal{A}}(x) + \mathcal{I}_{\mathcal{A}}(x) + \mathcal{F}_{\mathcal{A}}(x) \leq 3^{+}$$

Quasigroups and loops have been studied in [1, 2, 7, 9–11, 22, 26].

Definition 2.2. *(Groupoid, Quasigroup)*

If G is a non-empty set with a binary operation $(\cdot)$ on G, the pair $(G, \cdot)$ is called a groupoid or Magma if for all $x, y \in G$, $x \cdot y \in G$. If for any $m, n \in G$, the equations:

$$m \cdot x = n \qquad \text{and} \qquad y \cdot m = n$$

have unique solutions x and y in G respectively, then $(G, \cdot)$ is called a quasigroup.

Let $(G, \cdot)$ be a quasigroup. $(G, \cdot)$ is called a loop if $e \in G$ such that for any $x \in G$, $x \cdot e = e \cdot x = x$.

Assuming x is a member of a groupoid $(G, \cdot); x \in G$, such that the left and right translation maps of G represented as L_x and R_x are defined as

$$yL_x = x \cdot y \qquad \text{and} \qquad yR_x = y \cdot x.$$

If in the groupoid $(G, \cdot)$, the left and right translation maps are permutations, then the groupoid $(G, \cdot)$ becomes a quasigroup. Thus, their inverse mappings L_x^{-1} and R_x^{-1} exist. Therefore

$$x \backslash y = yL_x^{-1} \qquad \text{and} \qquad x/y = xR_y^{-1}$$

and note that

$$x \backslash y = z \Leftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Leftrightarrow z \cdot y = x.$$

Definition 2.3. *(Subquasigroup [22])*

Assuming $(Q, \cdot)$ is a non-empty quasigroup and $H \subset Q$. Then H will be regarded as a subquasigroup of $(Q, \cdot)$ if $(H, \cdot)$ is closed under the operation of $(Q, \cdot)$ and it is a quasigroup on its own right.

In quasigroups, the cancellation rule holds, that is, if for $x, y, z \in Q$, $x \cdot y = x \cdot z$ or $y \cdot x = z \cdot x$ then $y = z$. It means that in the Cayley table for a quasigroup, each element appears exactly once in each row and in each column, so the table forms a Latin square. The body of any finite quasigroup represented in Cayley table represents a Latin square.

The following results and definitions will be used for our main results.

Proposition 2.1. *([26])*

Take Q as a quasigroup of order n and P as a proper subquasigroup of Q or order p. Then, $2p \leq n$.

Proof. Let $x \in Q - P$, if $y \in P$, then $xy \in Q - P$. So $xP \subset Q - P$. But xP has the order p since $Q - P$ has order $n - p$; which implies that $p \leq n - p = 2p \leq n$.

Therefore the order of a subquasigroup is equal to or less than half of order of the quasigroup.

□

Proposition 2.2. *Take Q to be a quasigroup, and let P and H be subquasigroups of Q, so that $Q = P \cup H$. Then, either $P = Q$ or $H = Q$.*

Proof. Assuming $P \neq H$. If $p \in P - H$ and $h \in H$, then $ph \notin H$, and therefore $ph \in P$ and $h \in P$. So, $H \leq P$, therefore $P = Q$. □

Lemma 2.1. *([26])*

Take P to be a proper subquasigroup of $(Q, \cdot)$, so if

(1) *$a \in Q$ and $P \subset Q$, then $|P| = |a \cdot P| = |P \cdot a|$.*
(2) *$(P, \cdot)$ is a groupoid and $P \subset Q$, then $P \subset Q$.*
(3) *$a \in P$ and $P \subset Q$, $a \in P$ and $b \notin P$ means $ab \notin P$.*

Theorem 2.1. *([26])*

Take Q be a quasigroup with a proper subquasigroup P. Then,

$$2|P| \leq |Q|.$$

Definition 2.4. *(Lagrange Property [22])*

Take Q to be a finite quasigroup such that $P \subset Q$. Then the subquasigroup P of Q is said to be Lagrange-like if $|P|$ divides $|Q|$.

Definition 2.5. *(Weak Lagrange Property [22])*

Take Q to be a finite quasigroup and let $P \subseteq Q$. Then Q is said to satisfy the weak Lagrange property if every subquasigroup P of Q is Lagrange-like, that is $|P|$ divides $|Q|$ for all $P \subset Q$.

Definition 2.6. *(Strong Lagrange Property [22])*

Take Q to be a finite quasigroup and $P \subseteq Q$. Q is said to have a strong Lagrange property if P satisfies the weak Lagrange property for all the $P \subset Q$.

Remark 2.1. *The order of a subquasigroup is not necessarily a factor of the order of the quasigroup, that is the Lagrange properties does not in general hold for quasigroups. For example, let Q be a quasigroup of order 10 with only two subquasigroups H, K of orders $|H| = 2$ and $|K| = 5$ respectively such that $H \leq K \leq Q$. Since 2, 5 divide 10, then Q has the weak Lagrange property. But Q does not have the strong Lagrange property since $|H|$ is not a divisor of $|K|$.*

We now introduce soft sets and operations defined on them. Throughout this subsection, Q^* denotes an initial universe, E is the set of defined parameters and $A \subseteq E$.

Definition 2.7. *(Soft Sets, Soft Subset, Equal Soft Sets [3, 5, 12, 14, 19, 23, 33])*

Assume Q^* is a universal set of discourse and E is a set of defined parameters such that $C \subset E$. The couple (G, C) is called a soft set over Q^*, whenever $G(c) \subset Q^* \ \forall\ c \in C$; and F is a function mappings C to all the non-empty subsets of Q^*, i.e $G : C \longrightarrow 2^{Q^*} \backslash \{\emptyset\}$. A soft set (G, C) over a set Q^* is described as a set of ordered pairs: $(G, C) = \{(c, G(c)) : c \in C \ \text{ and } G(a) \in 2^{Q^*}\}$. The set of all soft sets, over Q^* under a defined set of parameters C, is denoted by $SS(Q^*_C)$.

Suppose that (G^*, C) and (K^*, D) are two soft sets defined over Q^*, then (K^*, D) will be regarded as a soft subset of (G^*, C) if,

(1) *$D \subseteq C$; and*
(2) *$K^*(c) \subseteq G^*(c) \ \forall\ c \in D$.*

Definition 2.8. *(Restricted Intersection)*

Consider (G^*, C) and (K^*, D) to be two soft sets over Q^* so that $C \cap D \neq \emptyset$. We define their restricted intersection as a soft set $(G^*, C) \cap (K^*, D) = (Z^*, E)$ where (Z^*, E) is represented as $Z^*(e) = G^*(e) \cap K^*(e) \ \forall\ e \in E$ and $E = C \cap D$.

Definition 2.9. *(Extended Intersection)*

Consider (G^*, C) and (K^*, D) be two soft sets over Q^* so that $C \cap D$ is not empty. Their extended intersection is a soft set (Z^*, E), where $E = C \cup D \ \forall\ e \in E$, $Z^*(e)$ can be defined as;

$$Z^*(e) = \begin{cases} G^*(e) & \text{whenever} \quad e \in C - D \\ K^*(e) & \text{whenever} \quad e \in D - C \\ G^*(e) \cap K^*(e) & \text{whenever} \quad e \in C \cap D. \end{cases}$$

Definition 2.10. *(Union)*

The extended union of two soft sets (G^*, C) *and* (K^*, D) *over* Q^* *is defined as* $(G^*, C)\bigcup(K^*, D)$ *and it is called a soft set* (Z, E) *over* Q^**, as* $E = C \cup D \ \forall\ e \in E$ *and*

$$Z^*(e) = \begin{cases} G^*(e) & \textit{whenever} \quad e \in C - D \\ K^*(e) & \textit{whenever} \quad e \in D - C \\ G^*(e) \cup K^*(e) & \textit{whenever} \quad e \in C \cap D. \end{cases}$$

3. MAIN RESULTS

3.1. *Neutrosophic Quasigroup*

Definition 3.1. *(Neutrosophic Quasigroup)*

Let $(Q, \cdot)$ *be a quasigroup. The neutrosophic quasigroup over a quasigroup* Q *is* $\mathcal{Q} = \langle Q \cup \mathcal{N}_\rceil \rangle$ *generated by* Q *and neutrosophic element* $\mathcal{N}_\rceil$ *coupled with a binary operation* $\odot$ *such that* $\mathcal{Q} = \left(\langle Q \cup \mathcal{N}_\rceil \rangle, \odot\right)$ *is a quasigroup.* $\mathcal{Q}$ *being a neutrosophic quasigroup over* $(Q, \cdot)$ *will sometimes represented by* $\mathcal{Q}_{(Q,\cdot)}$ *or* $\mathcal{Q}_Q$*.*

Remark 3.1. *If* $\mathcal{Q}_Q$ *is neutrosophic quasigroup over* Q*, then* $\mathcal{Q}$ *contains* Q *as a subquasigroup.*

Example 3.1. *Let* $(Q, \cdot)$ *be a quasigroup of order* 4 *where* $Q = \{1, 2, 3, 4\}$ *and let*

$$\mathcal{Q} = \langle Q \cup \mathcal{N}_\rceil \rangle = \{1, 2, 3, 4, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}$$

be represented by the multiplication Table 1. Then $\mathcal{Q}_{(Q,\cdot)} = \mathcal{Q}_Q = \mathcal{Q} = \left(\langle Q \cup \mathcal{N}_\rceil \rangle, \odot\right)$ *is a the neutrosophic quasigroup over* Q*.*

TABLE 1. Neutrosophic quasigroup of order 8

$\odot$	1	2	3	4	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$
1	1	2	4	3	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$
2	2	1	3	4	$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$
3	3	4	2	1	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$
4	4	3	1	2	$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$
$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	2	1	4	3
$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	1	2	3	4
$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	3	4	1	2
$4\mathcal{N}_\rceil$	$3\mathcal{N}_\rceil$	$4\mathcal{N}_\rceil$	$2\mathcal{N}_\rceil$	$1\mathcal{N}_\rceil$	4	3	2	1

Example 3.2. *Consider* $(G, +)$ *to be a quasigroup of order* 3 *where* $G = \{i, j, k\}$ *and let*

$$\mathcal{G} = \langle G \cup \mathcal{N}_\rceil \rangle = \{i, j, k, i\mathcal{N}_\rceil, j\mathcal{N}_\rceil, k\mathcal{N}_\rceil\}$$

be represented by the multiplication Table 2 . Then $\mathcal{G}_{(G,+)} = \mathcal{G}_G = \mathcal{G} = \left(\langle G \cup \mathcal{N}_\rceil \rangle, \oplus\right)$ *is a the neutrosophic quasigroup over* G*.*

TABLE 2. Neutrosophic quasigroup of order 6

$\oplus$	i	j	k	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$
i	i	j	k	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$
j	j	k	i	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$
k	k	i	j	$k\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$
$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$	i	j	k
$j\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$	k	i	j
$k\mathcal{N}_\rceil$	$k\mathcal{N}_\rceil$	$i\mathcal{N}_\rceil$	$j\mathcal{N}_\rceil$	j	k	i

Definition 3.2. *(Neutrosophic Subquasigroup)*

Consider $\mathcal{Q}_Q = \left(\left\langle Q \cup \mathcal{N}_\rceil \right\rangle, \odot\right)$ *to be a neutrosophic quasigroup over* Q *and* $\emptyset \neq \mathcal{H} \subseteq \mathcal{Q}$*. Then,* $\mathcal{H}_H$ *will be regarded as a neutrosophic subquasigroup of* $\mathcal{Q}$ *if there exists* $H \leq Q$ *such that* $\mathcal{H}_H = \left(\left\langle H \cup \mathcal{N}_\rceil \right\rangle, \odot\right)$ *is a neutrosophic quasigroup over* H*. This will often be expressed as* $\mathcal{H}_H \leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$*..*

Remark 3.2. *In Definition 3.2, if* $\mathcal{H}_H = H$*, then* $\mathcal{H}_H$ *will be called a trivial neutrosophic subquasigroup of* $\mathcal{Q}$*. Also,* $\mathcal{H}_H = \mathcal{Q}_Q$ *will be regarded as a trivial neutrosophic subquasigroup of* $\mathcal{Q}$*.*

Example 3.3.

(1) *In Example 3.1,* $\mathcal{H}_H = \{1, 2, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil\}$ *is a neutrosophic subquasigroup of* $\mathcal{Q}_Q$ *i.e.* $\mathcal{H}_H \leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$ *because* $(\mathcal{H}_H, \odot)$ *is a neutrosophic quasigroup over* H *going by Table 1. However,* $\mathcal{K} = \{1, 2, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}$ *is not a neutrosophic subquasigroup of* $\mathcal{Q}_Q$ *even though* $K \leq \mathcal{Q}_Q$*. This is because by Table 1, there is no* $K \leq Q$*, such that* $\mathcal{K}_K = \left\langle K \cup \mathcal{N}_\rceil \right\rangle = \{1, 2, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}$*. Neither* $\{1, 2\}$ *nor* $\{3, 4\}$ *can be* K*.*

(2) *In Example 3.2,* $\mathcal{Q}_Q$ *has no nontrivial neutrosophic subquasigroup judging by Table 2.*

Remark 3.3. *Based on Example 3.3(1), not every subquasigroup of a neutrosophic quasigroup* $\mathcal{Q}_Q$ *is a neutrosophic subquasigroup of* $\mathcal{Q}_Q$*. Ofcourse, every neutrosophic subquasigroup of a neutrosophic quasigroup* $\mathcal{Q}_Q$ *is a subquasigroup of* $\mathcal{Q}_Q$*.*

3.2. *Direct Product of Neutrosophic Quasigroups*

The direct product $Q \times H$ of two groups (quasigroups,loops) Q, H is a group (quasigroup, loop). For group (loop), it clearly contains at least one subgroup (subloop) isomorphic to Q, namely $Q \times \{e\}$. However, this is not the case for a direct product of two quasigroups. Bruck [7] gave examples of finite nontrivial quasigroups Q and H whose direct product has no proper subquasigroup. Foguel [21] considered when the direct product $Q \times H$ of two quasigroups

Q, H contains a subquasigroup isomorphic Q. We shall now consider the direct product of two neutrosophic quasigroups.

Theorem 3.1. *(Direct Product of Neutrosophic Quasigroups)*
Take $\mathcal{Q}_{(Q,\cdot)} = \left(\left\langle Q \cup \mathcal{N}_{\rceil} \right\rangle, \odot\right)$ *and* $\mathcal{H}_{(H,*)} = \left(\left\langle H \cup \mathcal{N}_{\lfloor} \right\rangle, \circledast\right)$ *to be any two neutrosophic quasigroups. Their direct product*

$$(\mathcal{Q} \times \mathcal{H})_{(Q\times H,(\cdot,*))} = (\mathcal{Q} \times \mathcal{H})_{Q\times H} = \mathcal{Q} \times \mathcal{H} = \left(\left\langle Q \times H \cup \left(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor}\right)\right\rangle, (\cdot, *)\right)$$

is a neutrosophic quasigroup.

Proof.

$$\mathcal{Q}_Q \times \mathcal{H}_H = \left\langle Q \cup \mathcal{N}_{\rceil} \right\rangle \times \left\langle H \cup \mathcal{N}_{\lfloor} \right\rangle =$$

$$\left\{(q,h), (q\mathcal{N}_{\rceil}, h), (q, h\mathcal{N}_{\lfloor}), (q\mathcal{N}_{\rceil}, h\mathcal{N}_{\lfloor}) | q \in Q, h \in H\right\} = \left\langle Q \times H \cup \left(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor}\right)\right\rangle.$$

So, $\mathcal{Q}_Q \times \mathcal{H}_H$ is a generated by $Q \times H$ and $\left(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor}\right)$, and thus $\left(\left\langle Q \times H \cup \left(\mathcal{N}_{\rceil}, \mathcal{N}_{\lfloor}\right)\right\rangle, (\cdot, *)\right)$ is a quasigroup and has $Q \times H$ as a subquasigroup. □

Corollary 3.1. *Let* $\mathcal{Q}_{(Q,\cdot)} = \left(\left\langle Q \cup \mathcal{N}_{\rceil} \right\rangle, \odot\right)$ *and* $\mathcal{H}_{(H,*)} = \left(\left\langle H \cup \mathcal{N}_{\lfloor} \right\rangle, \circledast\right)$ *be any two neutrosophic quasigroups with neutrosophic subquasigroups* $\mathcal{Q}'_{(Q',\cdot)}$ *and* $\mathcal{H}'_{(H',*)}$ *respectively. Then,*

$$(\mathcal{Q}' \times \mathcal{H}')_{(Q'\times H',(\cdot,*))} \leq_{(\mathcal{N}_{\rceil},\mathcal{N}_{\lfloor})} (\mathcal{Q} \times \mathcal{H})_{(Q\times H,(\cdot,*))}$$

Proof. Going by Theorem 3.1, $(\mathcal{Q} \times \mathcal{H})_{(Q\times H,(\cdot,*))}$. Since $\mathcal{Q}'_{(Q',\cdot)}$ and $\mathcal{H}'_{(H',*)}$ are neutrosophic quasigroups, then by Theorem 3.1. Thus,

$$(\mathcal{Q}' \times \mathcal{H}')_{(Q'\times H',(\cdot,*))} \leq_{(\mathcal{N}_{\rceil},\mathcal{N}_{\lfloor})} (\mathcal{Q} \times \mathcal{H})_{(Q\times H,(\cdot,*))}$$

because $\mathcal{Q}' \times \mathcal{H}' \subseteq \mathcal{Q} \times \mathcal{H}$ and $Q' \times H' \leq Q \times H$. □

Example 3.4. *By considering the direct product of the neutrosophic quasigroups* $\mathcal{Q}_{(Q,\cdot)}$ *and* $\mathcal{G}_{(G,+)}$ *in Example 3.1 and Example 3.2 respectively, the multiplication table of the neutrosophic quasigroup* $(\mathcal{Q} \times \mathcal{G})_{(Q\times G,(\cdot,+))}$ *can be constructed by using the multiplication Table 1 and Table 2.*

3.3. *Holomorph of Neutrosophic Quasigroups*

We recall the definition of the holomorph of a quasigroup.

Definition 3.3. *If we take* $(Q, \cdot)$ *to be a quasigroup and let* $A(Q, \cdot) = A(Q) \leq AUM(Q, \cdot) = AUM(Q)$ *be the subgroup of the automorphism group of* $(Q, \cdot)$*. Let* $H(Q, \cdot) = H(Q) = A(Q) \times Q$ *and define* $\circ$ *on* $H(Q)$ *as follows* $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ *for all* x, y *in* Q *and for all* $\alpha, \beta \in A(Q)$*. Then, the pair* $(H(Q), \circ)$ *(or* $H(Q)$*) is called the* $A(Q)$*-holomorph (or holomorph)* *of Q. $(H(Q), \circ)$ is a quasigroup. The $A(Q)$-holomorph $H(Q)$ of a quasigroup Q is a semi-direct product of Q and an automorphism group $A(Q)$ of it.*

Moreover, many authors such as [38, 40–42, 44, 52, 53] have considered the holomorphs of A-loops, Bruck loops, Bol loops, conjugacy closed loops, extra loops, inverse property loops, weak inverse property loops. Jaiyéọlá [11, 47, 48] also derived some results on the holomorph of certain varieties of loops. Adeniran et. al. [39] studied the holomorph of generalized Bol Loops. Results on the holomorphy of Osborn loops and the holomorphy of middle Bol loops can be seen on Jaiyéọlá and Popoola [49]; Isere et. al. [46] and Jaiyéọlá et al. [50]. Recently, Ogunrinade et al. [51] studied the holomorphy of self distributive quasigroup, Ilojide et al. [45] studied the holomorphy of Fenyves BCI-algebras; Oyebo et al. [54] considered the Holomorphy of (r, s, t)-inverse loops while Effiong et al. [43] considered the holomorphy of Basarab loops. Specifically, we cite the work of Bruck [40] on inverse property loops (IPL), where he established that the holomorph of IPL is an IPL. Also, Huthnance Jr. [44] established that the holomorph of WIP loops is a WIP loop.

Let us now introduce the holomorph of a neutrosophic quasigroup and investigate if it is a neutrosophic quasigroup.

Theorem 3.2. *Take $(Q, \cdot)$ to be a quasigroup with $A(Q)$-holomorph $(H(Q, \cdot), \circ)$. Let $\mathcal{Q}_{(Q,\cdot)} = \left(\left\langle Q \cup \mathcal{N}_{\rceil} \right\rangle, \odot\right)$ be a neutrosophic quasigroup over $(Q, \cdot)$ then,*

1. *$\left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$ is a quasigroup and the $A\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$-holomorph of $\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$; and*
2. *$\left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$ is a neutrosophic quasigroup over $(H(Q, \cdot), \circ)$.*

Proof.

1. Note that since $A(Q, \cdot) \leq AUM(Q, \cdot)$, then $A\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right) \leq AUM\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$. Thus, since $(H(Q, \cdot), \odot)$ is a quasigroup, then with $\mathcal{H} = \left\langle Q \cup \mathcal{N}_{\rceil} \right\rangle \cup A\left(\mathcal{Q}_{(Q,\cdot)}\right) = \mathcal{Q}_{(Q,\cdot)} \cup A\left(\mathcal{Q}_{(Q,\cdot)}\right)$, $\left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$ is a quasigroup, and so, $A\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$- is holomorph of $\left(\mathcal{Q}_{(Q,\cdot)}, \odot\right)$.
2. Notice that $A(Q, \cdot) = \left\{\alpha \in A\left(\mathcal{Q}_{(Q,\cdot)}\right) | \alpha = \alpha|_{(Q,\cdot)}\right\} \leq A\left(\mathcal{Q}_{(Q,\cdot)}\right)$. Thus, $(H(Q, \cdot), \circ) \leq \left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$. Therefore, $\left(\mathcal{H}_{(H(Q,\cdot),\circ)}, \circledcirc\right)$ is a neutrosophic quasigroup over $(H(Q, \cdot), \circ)$.□

3.4. *Soft Neutrosophic Quasigroup*

Definition 3.4. *(Soft Neutrosophic Quasigroup)*

Assuming that $\mathcal{Q}_{(Q,\cdot)}$ is a neutrosophic quasigroup, and suppose that E is a set of defined parameters and $A \subset E$. The couple $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ is regarded as a soft neutrosophic quasigroup over $\mathcal{Q}_{(Q,\cdot)}$ if $N_F(a)$ is neutrosophic subquasigroup of $\mathcal{Q}_Q$ $\forall$ $a \in A$, where $N_F : A \longrightarrow 2^{\mathcal{Q}_Q}$.

We shall sometimes write $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}} = \{N_F(a)|a \in A\}$.

Example 3.5. *Table 1 defines a finite neutrosophic quasigroup as a Latin square table; where*

$$\mathcal{Q}_{(\mathcal{Q},\cdot)} = \langle Q \cup \mathcal{N}_\rceil \rangle = \{1, 2, 3, 4, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}$$

Assume $A = \{\beta_1, \beta_2, \beta_3\}$ to be a set of parameters and let

$$N_F : A \to 2^{\mathcal{Q}} \uparrow N_F(\beta_1) = \{1, 2\},\ N_F(\beta_2) = \{1, 2, 3, 4\},\ N_F(\beta_3) = \{1, 2, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil\}.$$

Then, $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ is regarded as soft neutrosophic quasigroup over neutrosophic quasigroup $\mathcal{Q}_{(Q,\cdot)}$ because $N_F(\beta_i) \leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$ for $i = 1, 2, 3$.

Now assume $B = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ be set of defined parameters and

$$N_F : B \to 2^{\mathcal{Q}} \uparrow N_F(\beta_1) = \{1, 2\},\ N_F(\beta_2) = \{1, 2, 3, 4\},$$
$$N_F(\beta_3) = \{1, 2, 1\mathcal{N}_\rceil, 2\mathcal{N}_\rceil\},\ N_F(\beta_4) = \{1, 2, 3\mathcal{N}_\rceil, 4\mathcal{N}_\rceil\}.$$

Then, $(N_F, B)_{\mathcal{Q}_{(Q,\cdot)}}$ is not a soft neutrosophic quasigroup over neutrosophic quasigroup $\mathcal{Q}_{(Q,\cdot)}$ because $N_F(\beta_i) \leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$ for $i = 1, 2, 3$ but $N_F(\beta_4) \not\leq_{\mathcal{N}_\rceil} \mathcal{Q}_Q$.

Definition 3.5. *(Soft sub-neutrosophic quasigroup)*

If $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ and $(N_G, B)_{\mathcal{Q}_{(Q,\cdot)}}$ are two soft neutrosophic quasigroups over a common neutrosophic quasigroup $\mathcal{Q}_{(Q,\cdot)}$. $(N_F, A)_{\mathcal{Q}_{(Q,\cdot)}}$ is called soft neutrosophic subquasigroup of $(N_G, B)_{\mathcal{Q}_{(Q,\cdot)}}$ if

(1) *$A \subseteq B$, and*

(2) *$N_F(a) \leq_{\mathcal{N}_\rceil} N_G(a)$, for all $a \in A$.*

This will be expressed as $(N_F, A) \leq_{\mathcal{N}_\rceil} (N_G, B)$.

3.5. *Order of Soft Neutrosophic Quasigroup*

Pflugfelder [22] and Wall [26] established that quasigroups might not neccessarily obey Lagrange's theorem. We extend some results of Wall [26] to soft neutrosophic quasigroup. The existence of identity element in the definition of the order of soft group in Aktas [3] was considered. Hence we introduced new definition for the order of a soft neutrosophic quasigroup that is independent of identity element and associative property. We introduce the order of a soft neutrosophic quasigroup (N_F, χ) over a finite neutrosophic quasigroup $\mathcal{Q}_Q$ and to check for divisibility properties between $|\mathcal{Q}_Q|$ and $|(N_F, \chi)|$, and prove that there are some algebraic connections existing between the orders of a neutrosophic quasigroup $\mathcal{Q}_Q$ and its soft neutrosophic quasigroup (N_F, χ).

Definition 3.6. *(The Order of Soft Neutrosophic Quasigroups)*

Consider $(N_F, \chi)_{\mathcal{Q}_{(Q,\cdot)}}$ to be a soft neutrosophic quasigroup over a finite neutrosophic quasigroup $\mathcal{Q}_{(Q,\cdot)}$. $(N_F, \chi)_{\mathcal{Q}_{(Q,\cdot)}}$ will be called a finite soft neutrosophic quasigroup. The order of *a finite soft neutrosophic quasigroup* $(N_F,\chi)_{\mathcal{Q}_{(Q,\cdot)}}$*, where* χ *is the set of parameters, will be defined as;*

$$|(N_F,\chi)|_{\mathcal{Q}_{(Q,\cdot)}} = |(N_F,\chi)|_{\mathcal{Q}_Q} = |(N_F,\chi)|_{\mathcal{Q}} = |(N_F,\chi)| = \sum_{a\in\chi} |N_F(a)|,\ N_F(a) \in (N_F,\chi),\ a \in \chi.$$

Definition 3.7. *Consider* $(N_F,\chi)_{\mathcal{Q}_Q}$ *to be soft neutrosophic quasigroup over a neutrosophic quasigroup* $\mathcal{Q}_Q$*. Then, we defined the arithmetic mean of a soft neutrosophic quasigroup and the geometric mean of soft neutrosophic quasigroup* (N_F,χ)*, where* $\chi \neq 0$ *as;*

$$\mathcal{AM}_{\mathcal{Q}}(N_F,\chi) = \frac{1}{|\chi|}\sum_{a\in\chi}|N_F(a)|; \qquad \mathcal{GM}_{\mathcal{Q}}(N_F,\chi) = \sqrt[|\chi|]{\prod_{a\in\chi}|N_F(a)|};$$

Remark 3.4. $(N_F,\chi)_{\mathcal{Q}}$ *is a soft neutrosophic quasigroup over a finite neutrosophic quasigroup* $\mathcal{Q}$ *as in Example 3.5. We note that* $|N_F(a)|\Big||(N_F,\chi)|$ *and* $|N_F(a)|\Big||\mathcal{Q}|$ *occurred for just one case of* $a\in\chi$*,* $|N_F(a)|\Big||(N_F,\chi)|$ *occurred one case of* $a\in\chi$ *and* $|N_F(a)|\Big||\mathcal{Q}|$ *occurred in all cases for* $a\in\chi$*.*

Lemma 3.1. *Consider* $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ *to be a finite neutrosophic quasigroup, then*

- i *If* $(N_F,\chi)_{\mathcal{Q}}$ *is a finite soft neutrosophic quasigroup over* $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*. For any* $a \in \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*,* $|N_F(a)| = |\alpha\odot N_F(a)| = |N_F(a)\odot\alpha| \forall\alpha\in\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*.*
- ii *If* $(N_F,\chi)_{\mathcal{Q}}$ *is a soft set over* $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*. Then* $(N_F,\chi)_{\mathcal{Q}}$ *is soft neutrosophic quasigroup iff* $(N_F,\chi)_{\mathcal{Q}}$ *is soft neutrosophic groupoid.*
- iii *Consider* $(N_F,\chi)_{\mathcal{Q}}$ *to be soft neutrosophic quasigroup. Hence,*
  - (a) *if for any* $\alpha\in\chi,\ \alpha\in N_F(\alpha)$ *and* $\beta\notin N_F(\alpha)$ *means* $\alpha\odot\beta\notin\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$*.*
  - (b) $N_F(\alpha)\odot\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\backslash N_F(\alpha)\subset\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\backslash N_F(\alpha)$ *for all* $\alpha\in\chi$*.*

Proof.

(1) Assume (N_F,χ) to be a soft neutrosophic quasigroup over a finite neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. From (i) of Lemma 2.1; $N_F(a)\subset\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ for all $a\in\chi$, for any $\alpha\in\mathcal{Q}_{(\mathcal{Q},\cdot)}$, $|N_F(a)| = |\alpha\odot N_F(a)| = |N_F(a)\odot\alpha|\ \forall\ \alpha\in\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$.

(2) If (N_F,χ) is soft set over $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$, and $(N_F,\chi)_{\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)}$ is soft neutrosophic quasigroup, and $(N_F,\chi)_{\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)}$ a soft neutrosophic groupoid. Hence, if $(N_F,\chi)_{\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)}$ is soft neutrosophic groupoid, then $N_F(a)$ is also neutrosophic subgroupoid of $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ $\forall a\in\chi$. From (ii) of Lemma 2.1, $N_F(a)$ is neutrosophic subquasigroup of $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ $\forall\ a\in\chi$. Therefore, $(N_F,\chi)_{\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)}$ is soft neutrosophic quasigroup.

(3) If $(N_F,\chi)_{\mathcal{Q}_{(Q,\cdot)}}$ is soft neutrosophic quasigroup, therefore
   - (a) $N_F(a)\leq_{\mathcal{N}_\rceil}\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ $\forall a\in\chi$, from (iii) of Lemma 2.1, for any $a\in\chi,\ \alpha\in N_F(a)$ and $\beta\notin N_F(a)$ imply $\alpha\odot\beta\notin\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$.
   - (b) This follows from (a) above.□

Theorem 3.3. *Consider* $(N_F,\chi)_{(\mathcal{Q}_{(Q,\cdot)},\odot)}$ *to be a finite soft neutrosophic quasigroup. Then the following holds;*

1. $|(N_F,\chi)| = |\chi|\mathcal{AM}(N_F,\chi)$; 2. $2|(N_F,\chi)| \le |\chi||\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|$; 3. $|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \ge 2\mathcal{AM}(N_F,\chi)$.

Proof. We derived $|(N_F,\chi)| = |\chi|\mathcal{AM}(N_F,\chi)$ from the combination of the definition of $|(N_F,\chi)|$ and $\mathcal{AM}(N_F,\chi)$. If $(N_F,\chi)_{(\mathcal{Q}_{(Q,\cdot)},\odot)}$ is a soft neutrosophic quasigroup, then $N_F(a) \le_{\mathcal{N}_\rceil} \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ $\forall a \in \chi$. From Theorem 2.1, $2|N_F(a)| \subset |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|$ for all $a\in\chi$. So from $\chi = \{a_1,a_2,\cdots,a_n\}$,

$2|N_F(a_1)|+2|N_F(a_2)|+\cdots+2|N_F(a_n)| \le |\chi||\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \Rightarrow 2\sum_{a\in\chi}|N_F(a)| \le |\chi||\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \Rightarrow$

$$2|(N_F,\chi)| \le |\chi||\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|.$$

Also, $2\sum_{a\in\chi}|N_F(a)| \le |\chi||\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \Rightarrow |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \ge \frac{2}{|\chi|}\sum_{a\in\chi}|N_F(a)| \Rightarrow |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \ge 2\mathcal{AM}(N_F,\chi)$. □

Remark 3.5.

(1) From Theorem 3.3, if equation $|(N_F,\chi)| = |\chi|\mathcal{AM}(F,\chi)$ is considered as a Lagrange's Formula for finite soft neutrosophic quasigroup. We let $|\chi|$ and $\mathcal{AM}(N_F,\chi)$ to take the character of the order of subgroup and its index in the group theory, which may not be an integer.

(2) In Theorem 3.3 both $|(N_F,\chi)| = |\chi|\mathcal{AM}(N_F,\chi)$;and $2|(N_F,\chi)| \subset |\chi||\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|$ gives both an upper and lower bound for the order of a finite soft neutrosophic quasigroup.

Also in Theorem 3.3, the second part from is proved from 1 of Lemma 3.1, such that for any $a \in \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$, $|N_F(a)| = |\alpha\odot N_F(a)| = |N_F(a)\odot\alpha| \forall\alpha \in \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. Hence, if $\alpha \in \left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ and $\alpha \notin N_F(a)$, clearly from 3 of Lemma 3.1,

$$|(N_F,\chi)| \le \sum_{a\in\chi}|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)\backslash N_F(a)| \Rightarrow |(N_F,\chi)| \le \sum_{a\in\chi}\left(|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| - |N_F(a)|\right) =$$

$$\sum_{a\in\chi}|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| - \sum_{a\in A}|N_F(a)| \Rightarrow |(N_F,\chi)| \le |\chi||\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| - |(N_F,\chi)| \Rightarrow$$

$$2|(N_F,\chi)| \le |\chi||\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|.$$

From Example 3.5, if we consider (N_F,χ) as a soft neutrosophic quasigroup over a finite neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. Then it can be observed that if $|\chi| = 3$, $|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| = 8$, $|(N_F,\chi)| = 10$, then the equations in Theorem 3.3 will be satisfied.

Theorem 3.4. *Consider $(N_F,\chi)_{(\mathcal{Q}_{(Q,\cdot)},\odot)}$ to be a finite soft neutrosophic quasigroup. We have,*

$$(i)\ |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\times \sqrt[|\chi|]{\prod_{a\in\chi}|N_F(a)|},\quad (ii)\ |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\mathcal{AM}(N_F,\chi)\quad \textit{and}$$

$$(iii)\ |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq \mathcal{AM}(N_F,\chi)+\mathcal{GM}(N_F,\chi).$$

Proof. From Theorem 2.1, we have,

$$\prod_{a\in\chi}2|N_F(a)| \leq \prod_{i=1}^{|\chi|}|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \Rightarrow 2^{|\chi|}\times\prod_{a\in\chi}|N_F(a)| \leq \prod_{i=1}^{|\chi|}|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|$$

$$\Rightarrow 2^{|\chi|}\times\prod_{a\in\chi}|N_F(a)| \leq |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|^{|\chi|} \Rightarrow \left(\frac{|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|}{2}\right)^{|\chi|} \geq \prod_{a\in\chi}|F(a)|$$

$$\Rightarrow \frac{|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)|}{2} \geq \sqrt[|\chi|]{\prod_{a\in\chi}|N_F(a)|} \Rightarrow |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\times\sqrt[|\chi|]{\prod_{a\in\chi}|N_F(a)|} \Rightarrow$$

$$|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\mathcal{GM}(N_F,\chi).$$

By Theorem 3.3, $|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\mathcal{AM}(N_F,\chi)$, therefore $2|\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq 2\mathcal{AM}(N_F,\chi) + 2\mathcal{GM}(N_F,\chi) \Rightarrow |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| \geq \mathcal{AM}(N_F,\chi)+\mathcal{GM}(N_F,\chi)$. □

Remark 3.6. *The (i) and (ii) of Theorem 3.4 defines the lower bounds of the soft neutrosophic quasigroup by taking into consideration the order of the soft neutrosophic quasigroup in relations to both the arithmetic and geometric means of the soft neutrosophic quasigroup.*

Example 3.6. *Based on Table 1 and Example 3.5, (N_F,χ) is a soft neutrosophic quasigroup over a finite neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. It will be noticed that;*

$$|\chi| = 3,\ |\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)| = 8,\ |(N_F,\chi)| = 10,\ \mathcal{AM}(N_F,\chi) = \frac{10}{3},\ \mathcal{GM}(N_F,\chi) = \sqrt[3]{32}.$$

Therefore, all the inequalities in Theorem 3.4 are satisfied.

4. Conclusion

In conclusion, we introduced and studied the abstraction of neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$ over a quasigroup $(Q,\cdot)$. It was discovered that the direct product of any two neutrosophic quasigroups is neutrosophic quasigroup and that the holomorph of any neutrosophic quasigroup is a neutrosophic quasigroup. Furthermore, soft set theory was broadened by studying soft neutrosophic quasigroup $(N_F,\chi)_{(\mathcal{Q}_{(Q,\cdot)},\odot)}$ over a neutrosophic quasigroup $\left(\mathcal{Q}_{(Q,\cdot)},\odot\right)$. From the study of order of finite soft neutrosophic quasigroup, we introduced and established the order of finite soft neutrosophic quasigroup with varied mathematical inequality expressions that exist among the order of finite neutrosophic quasigroup and the order of soft neutrosophic quasigroup over the same quasigroup. From the study of their arithmetic mean $\mathcal{AM}(N_F, \chi)$ and geometric mean $\mathcal{GM}(N_F, \chi)$ of finite soft neutrosophic quasigroup (N_F, χ), Lagrange's like Formula $|(N_F, \chi)| = |\chi|\mathcal{AM}(N_F, \chi)$ for finite soft neutrosophic quasigroup was established. In future work, Definition 2.8, Definition 2.9 and Definition 2.10 will be studied for soft neutrosophic quasigroups.

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