NSS

Neutrosophic Sets and Systems, Vol. 31, 2020

University of New Mexico

Neutrosophic LI-ideals in lattice implication algebras

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Abstract. The notion of neutrosophic set theory is applied to lattice implication algebras, and the concept of neutrosophic LI-ideals and neutrosophic lattice ideals in a lattice implication algebra are introduced. Several properties are investigated. Relationships between a neutrosophic LI-ideal and a neutrosophic lattice ideal are established, and conditions for a neutrosophic lattice ideal to be a neutrosophic LI-ideal are provided. Characterizations of a neutrosophic LI-ideal are discussed. The properties of implication homomorphism of lattice implication algebras related to neutrosophic LI-ideals are studied.

Keywords: Lattice implication algebra; neutrosophic LI-ideals; neutrosophic lattice ideal; implication homomorphism.

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1. **Introduction**

Smarandache in [1, 2] introduced the notion of neutrosophic set, which is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval-valued (intuitionistic) fuzzy set. Then the neutrosophic components T, I, F were introduced, which represent the membership, indeterminacy, and non-membership values respectively, where $[0, 1]$ is the non-standard unit interval, and the neutrosophic set was defined. Then some examples were given from mathematics, physics, philosophy, and applications of the neutrosophic set. Afterward, the neutrosophic set operations (complement, intersection, union, difference, Cartesian product, inclusion, and n-ary relationship) were introduced, some generalizations and comments on them, and finally, the distinctions between the neutrosophic set and the intuitionistic fuzzy set. Jun and his colleagues in [3] applied the notion of neutrosophic set theory to BCK/BCI-algebras, and their properties and relations are investigated. Then in [4], the notion of interval neutrosophic length of a range neutrosophic set was introduced. Moreover, in [5], interval neutrosophic ideals were defined, and some properties were investigated.

Then in [6], they represented different kinds of interval neutrosophic ideals and studied some features and found the relation among them.

Borzooei et al. [7–10], appliad the neutrosophic sets to logical algebras and defined the concept of a commutative generalized neutrosophic ideal in a BCK-algebra, and proved some related properties. Characterizations of a commutative generalized neutrosophic ideal are considered. Also, some equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras are introduced. Also, Jun in [11] introduced the notion of LI-ideals, Li-maximal ideals and prime LI-ideals of lattice implication algebras, and investigated some properties of them and studied the relation among them. Since everything in the world is full of indeterminacy, and application of this notion in decision making and multicriteria decision-making method etc. We decide applied the notion of neutrosophic set theory to lattice implication algebras. We introduce the concept of neutrosophic LI-ideals and neutrosophic lattice ideals of a lattice implication algebra, and investigate several properties. We discuss relationship between a neutrosophic LI-ideal and a neutrosophic lattice ideal. We provide conditions for a neutrosophic lattice ideal to be a neutrosophic LI-ideal. We consider characterizations of a neutrosophic LI-ideal. We study the properties of implication homomorphism of lattice implication algebras related to neutrosophic LI-ideals.

2. **Preliminaries**

By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\rightarrow$ " satisfying the following axioms:

- (I1) $u \rightarrow (v \rightarrow w) = v \rightarrow (u \rightarrow w)$,
- (I2) $u \rightarrow u = 1$,
- (I3) $u \rightarrow v = v' \rightarrow u'$,
- (I4) $u \rightarrow v = v \rightarrow u = 1 \Rightarrow u = v$,
- (I5) $(u \rightarrow v) \rightarrow v = (v \rightarrow u) \rightarrow u$,
- (L1) $(u \vee v) \rightarrow w = (u \rightarrow w) \wedge (v \rightarrow w)$,
- (L2) $(u \wedge v) \rightarrow w = (u \rightarrow w) \vee (v \rightarrow w)$,

for all $u, v, w \in L$. A lattice implication algebra L is called a *lattice H-implication algebra* if it satisfies:

$$(\forall u, v, w \in L)(u \vee v \vee ((u \wedge v) \rightarrow w) = 1). \tag{1}$$

We can define a partial ordering $\leq$ on L by condition $u \leq v$ if and only if $u \rightarrow v = 1$.

In a lattice implication algebra L, the following conditions hold (see [20]):

- (a1) $0 \rightarrow u = 1$, $1 \rightarrow u = u$ and $u \rightarrow 1 = 1$.
- (a2) $u \rightarrow v \leq (v \rightarrow w) \rightarrow (u \rightarrow w)$.

(a3) $u \leq v$ implies $v \to w \leq u \to w$ and $w \to u \leq w \to v$.

(a4) $u' = u \to 0$.

(a5) $u \vee v = (u \to v) \to v$.

(a6) $((v \to u) \to v')' = u \wedge v = ((u \to v) \to u')'$.

(a7) $u \leq (u \to v) \to v$.

Let L_1 and L_2 be two lattice implication algebras. A mapping $f : L_1 \to L_2$ is called an *implication homomorphism* ([19]) if $f(u \to v) = f(u) \to f(v)$ for all $u, v \in L_1$. Moreover, if f satisfies the following conditions:

$$f(u \vee v) = f(u) \vee f(v),\ f(u \wedge v) = f(u) \wedge f(v),\ f(u') = (f(u))'$$

for all $u, v \in L_1$, then f is called a *lattice implication homomorphism.* For an implication homomorphism $f : L_1 \to L_2$, the *kernel* of f, written ker$f$, is defined as follows:

$$\ker f := \{u \in L_1 \mid f(u) = 0\}.$$

Note that if an implication homomorphism $f : L_1 \to L_2$ satisfies $f(0) = 0$, then f is a lattice implication homomorphism ([19]).

Definition 2.1 ([15])**.** A nonempty subset G of L is called an *LI-ideal* of L if it satisfies the following statements:

(i) $0 \in G$,

(ii) $(\forall u \in L)\ (\forall v \in G)\ ((u \to v)' \in G \implies u \in G)$.

Lemma 2.2 ([15])**.** *Every LI-ideal G of L satisfies the following implication:*

$$(\forall u \in G)\,(\forall v \in L)\,(v \leq u \implies v \in G).$$

Let L be a non-empty set. A *neutrosophic set* (NS) in L (see [1]) is a structure of the form:

$$A_\sim := \{\langle u; A_T(u), A_I(u), A_F(u)\rangle \mid u \in L\},$$

where $A_T : L \to [0,1]$ is a truth membership function, $A_I : L \to [0,1]$ is an indeterminate membership function, and $A_F : L \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T,\ A_I,\ A_F)$ for the neutrosophic set, it means

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in L\}.$$

Given a neutrosophic set $A_\sim = (A_T,\ A_I,\ A_F)$ in a lattice implication algebra L. Then we consider the following sets.

$$L(A_T;\alpha) := \{u \in L \mid A_T(u) \geq \alpha\},$$

$$L(A_I;\beta) := \{u \in L \mid A_I(u) \geq \beta\},$$

$$L(A_F;\gamma) := \{u \in L \mid A_F(u) \leq \gamma\},$$

which are called *neutrosophic level subsets* of L.

We refer the reader to the books [21] for additional details lattice implication algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

3. **Neutrosophic LI-ideals**

From now on, we let L as lattice implication algebra unless otherwise state.

Definition 3.1. A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in L is called a *neutrosophic LI-ideal* of L if the following assertions are valid.

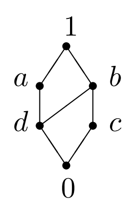
$$(\forall u \in L)\Big(\; A_T(0) \geq A_T(u),\ A_I(0) \geq A_I(u),\ A_F(0) \leq A_F(u) \; \Big) \tag{2}$$

and

$$(\forall x, y \in L) \left(\begin{array}{l} A_T(u) \geq \min\{A_T((u \to v)'), A_T(v)\} \\ A_I(u) \geq \min\{A_I((u \to v)'), A_I(v)\} \\ A_F(u) \leq \max\{A_F((u \to v)'), A_F(v)\} \end{array} \right) \tag{3}$$

The set of all neutrosophic LI-ideals of L is denoted by NLI(L).

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a poset with Hasse diagram and Cayley tables as follows:



x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Define the operations $\vee$ and $\wedge$ on L as follows:

$$u \vee v := (u \to v) \to v,\ \ u \wedge v := ((u' \to v') \to v')',$$

for all $u, v \in L$. Then L is a lattice implication algebra (see [15]). Suppose $A_\sim = (A_T, A_I, A_F)$ is a neutrosophic set in L defined by Table 1.

Table 1. Tabular representation of $A_\sim = (A_T, A_I, A_F)$

L	0	a	b	c	d	1
$A_T(u)$	0.9	0.5	0.5	0.7	0.5	0.5
$A_I(u)$	0.8	0.3	0.3	0.3	0.3	0.3
$A_F(u)$	0.2	0.4	0.6	0.6	0.4	0.6

It is routine to verify that $A_\sim = (A_T, A_I, A_F) \in$ NLI(L).

Proposition 3.3. *Every neutrosophic LI-ideal $A_\sim = (A_T,\, A_I,\, A_F)$ of L satisfies the following assertions.*

$$(\forall u, v \in L)\left(x \leq y \;\Rightarrow \begin{cases} A_T(u) \geq A_T(v) \\ A_I(u) \geq A_I(v) \\ A_F(u) \leq A_F(v) \end{cases} \right). \tag{4}$$

Proof. Let $A_\sim \in \mathrm{NLI}(L)$ and $u, v \in L$ such that $u \leq v$. Since $(u \to v)' = 0$, we have,

$$A_T(u) \geq \min\{A_T((u \to v)'), A_T(v)\} = \min\{A_T(0), A_T(v)\} = A_T(v),$$
$$A_I(u) \geq \min\{A_I((u \to v)'), A_I(v)\} = \min\{A_I(0), A_I(v)\} = A_I(v),$$
$$A_F(u) \leq \max\{A_F((u \to v)'), A_F(v)\} = \max\{A_F(0), A_F(v)\} = A_F(v).$$

□

Proposition 3.4. *Every neutrosophic LI-ideal $A_\sim = (A_T,\, A_I,\, A_F)$ of L satisfies the following assertions.*

$$(\forall u, v, w \in L)\left(u \leq v' \to w \;\Rightarrow \begin{cases} A_T(u) \geq \min\{A_T(v), A_T(w)\} \\ A_I(u) \geq \min\{A_I(v), A_I(w)\} \\ A_F(u) \leq \max\{A_F(v), A_F(w)\} \end{cases} \right). \tag{5}$$

Proof. Suppose $A_\sim \in \mathrm{NLI}(L)$ such that for all $u, v, w \in L$, $u \leq v' \to w$. Then

$$1 = u \to (v' \to w) = w' \to (u \to v) = (u \to v)' \to w,$$

and so $((u \to v)' \to w)' = 0$. By (2) and (3), we get that

$$\begin{aligned} A_T(u) &\geq \min\{A_T((u \to v)'), A_T(v)\} \\ &\geq \min\{\min\{A_T(((u \to v)' \to w)'), A_T(w)\}, A_T(v)\} \\ &= \min\{\min\{A_T(0), A_T(w)\}, A_T(v)\} \\ &= \min\{A_T(w), A_T(v)\}, \end{aligned}$$

$$\begin{aligned} A_I(u) &\geq \min\{A_I((u \to v)'), A_I(v)\} \\ &\geq \min\{\min\{A_I(((u \to v)' \to w)'), A_I(w)\}, A_I(v)\} \\ &= \min\{\min\{A_I(0), A_I(w)\}, A_I(v)\} \\ &= \min\{A_I(w), A_I(v)\}, \end{aligned}$$

and

$$\begin{aligned} A_F(u) &\geq \max\{A_F((u \to v)'), A_F(v)\} \\ &\leq \max\{\max\{A_F(((u \to v)' \to w)'), A_F(w)\}, A_F(v)\} \\ &= \max\{\max\{A_F(0), A_F(w)\}, A_F(v)\} \\ &= \max\{A_F(w), A_F(v)\}. \end{aligned}$$

Therefore, (3.4) holds. □

Definition 3.5. A neutrosophic set $A_\sim = (A_T,\, A_I,\, A_F)$ in L is called a *neutrosophic lattice ideal* of L if it satisfies (4) and

$$(\forall u, v \in L) \left(\begin{array}{l} A_T(u \vee v) \geq \min\{A_T(u), A_T(v)\} \\ A_I(u \vee v) \geq \min\{A_I(u), A_I(v)\} \\ A_F(u \vee v) \leq \max\{A_F(u), A_F(v)\} \end{array} \right) \tag{6}$$

Example 3.6. Let L be the lattice implication algebra as in Example 3.2 and $A_\sim = (A_T,\, A_I,\, A_F)$ be a neutrosophic set in L which is defined by Table 2.

Table 2. Tabular representation of $A_\sim = (A_T,\, A_I,\, A_F)$

L	0	a	b	c	d	1
$A_T(u)$	0.7	0.4	0.4	0.4	0.7	0.4
$A_I(u)$	0.8	0.5	0.5	0.5	0.8	0.5
$A_F(u)$	0.3	0.6	0.6	0.6	0.3	0.6

It is easy to see that $A_\sim = (A_T,\, A_I,\, A_F)$ is a neutrosophic lattice ideal of L.

We discussthe between a neutrosophic LI-ideal and a neutrosophic lattice ideal.

Theorem 3.7. *Every neutrosophic LI-ideal is a neutrosophic lattice ideal.*

Proof. Let $A_\sim = (A_T,\, A_I,\, A_F) \in NLI(L)$. The condition (4) is valid in Proposition 3.3. Since $((u \vee v) \to v)' = (((u \to v) \to v) \to v)' = (u \to v)' \leq (u')'$ for all $u, v \in L$, by (4) and (3), we have

$$A_T(u \vee v) \geq \min\{A_T(((u \vee v) \to v)'), A_T(v)\} \geq \min\{A_T(u), A_T(v)\},$$

$$A_I(u \vee v) \geq \min\{A_I(((u \vee v) \to v)'), A_I(v)\} \geq \min\{A_I(u), A_I(v)\},$$

and

$$A_F(u \vee v) \leq \max\{A_F(((u \vee v) \to v)'), A_F(v)\} \leq \max\{A_F(u), A_F(v)\}.$$

Therefore, $A_\sim = (A_T, A_I, A_F) \in NLI(L)$. □

The converse of Theorem 3.7 is not true in general as seen in the following example.

Example 3.8. Let L be the lattice implication algebra as in Example 3.2 and $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in L defined by Table 3.

TABLE 3. Tabular representation of $A_\sim = (A_T, A_I, A_F)$

L	0	a	b	c	d	1
$A_T(x)$	0.8	0.4	0.4	0.4	0.8	0.4
$A_I(x)$	0.6	0.3	0.3	0.3	0.6	0.3
$A_F(x)$	0.3	0.5	0.5	0.5	0.3	0.5

Then $A_\sim = (A_T, A_I, A_F) \in L$, but $A_\sim \notin NLI(L)$ beacuse $A_T(a) = 0.4 < 0.8 = \min\{A_T((a \to d)'), A_T(d)\}$.

We investigate that under which condition, a neutrosophic lattice ideal can be a neutrosophic LI-ideal.

Theorem 3.9. *In a lattice H-implication algebra L, every neutrosophic lattice ideal is a neutrosophic LI-ideal.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic lattice ideal of a lattice H-implication algebra L. Moreover, since $0 \leq u$ for all $u \in L$, it follows from (4) that $A_T(0) \geq A_T(u)$, $A_I(0) \geq A_I(u)$ and $A_F(0) \leq A_F(u)$. Also, from $u \leq u \vee v$ for all $u, v \in L$, by (4) and (6) we get that,

$$A_T(u) \geq A_T(u \vee v) = A_T(v \vee (u' \vee v)') = A_T(v \vee (u \to v)') \geq \min\{A_T(v), A_T((u \to v)')\},$$

$$A_I(u) \geq A_I(u \vee v) = A_I(v \vee (u' \vee v)') = A_I(v \vee (u \to v)') \geq \min\{A_I(v), A_I((u \to v)')\},$$

and

$$A_F(u) \leq A_F(u \vee v) = A_F(v \vee (u' \vee v)') = A_F(v \vee (u \to v)') \leq \max\{A_F(v), A_F((u \to v)')\}.$$

Therefore, $A_\sim = (A_T, A_I, A_F) \in NLI(L)$. □

We consider characterizations of a neutrosophic LI-ideal.

Theorem 3.10. *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in L, the following statements are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is a neutrosophic LI-ideal of L.*
(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies (5).*

(3) $A_\sim = (A_T,\ A_I,\ A_F)$ *satisfies* (4) *and*

$$(\forall u, v \in L)\begin{pmatrix} A_T(u' \to v) \geq \min\{A_T(u), A_T(v)\} \\ A_I(u' \to v) \geq \min\{A_I(u), A_I(v)\} \\ A_F(u' \to v) \leq \max\{A_F(u), A_F(v)\} \end{pmatrix}. \tag{7}$$

(4) $A_\sim = (A_T,\ A_I,\ A_F)$ *satisfies* (2) *and*

$$(\forall u, v, w \in L)\begin{pmatrix} A_T(u' \to w) \geq \min\{A_T((u \to v)'), A_T(v' \to w)\} \\ A_I(u' \to w) \geq \min\{A_I((x \to v)'), A_I(v' \to w)\} \\ A_F(u' \to w) \leq \max\{A_F((x \to v)'), A_F(v' \to w)\} \end{pmatrix}. \tag{8}$$

(5) $A_\sim = (A_T,\ A_I,\ A_F)$ *satisfies* (2) *and*

$$(\forall u, v, w \in L)\begin{pmatrix} A_T((u \to w)') \geq \min\{A_T((u \to v)'), A_T((v \to w)')\} \\ A_I((u \to w)') \geq \min\{A_I((u \to v)'), A_I((v \to w)')\} \\ A_F((u \to w)') \leq \max\{A_F((u \to v)'), A_F((v \to w)')\} \end{pmatrix}. \tag{9}$$

Proof. Suppose $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$. Then $A_\sim = (A_T,\ A_I,\ A_F)$ satisfies (5) by Proposition (3.4). Let $A_\sim = (A_T,\ A_I,\ A_F)$ be a neutrosophic set in L which satisfies the condition (3.4). Since $0 \leq u' \to u$ for all $u \in L$, we have $A_T(0) \geq \min\{A_T(u), A_T(u)\} = A_T(u)$, $A_I(0) \geq \min\{A_I(u), A_I(u)\} = A_I(u)$, and $A_F(0) \leq \max\{A_F(u), A_F(u)\} = A_F(u)$. Since $u \leq ((u \to v)')' \to v$ for all $u, v \in L$, it follows from (3.4) that $A_T(u) \geq \min\{A_T((u \to v)'), A_T(v)\}$, $A_I(u) \geq \min\{A_I((u \to v)'), A_I(v)\}$, and $A_F(u) \leq \max\{A_F((u \to v)'), A_F(v)\}$. Thus $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$. Let $u, v \in L$ such that $u \leq v$. Then $u \leq v = v \vee v \leq v' \to v$, and so $A_T(u) \geq \min\{A_T(v), A_T(v)\} = A_T(v)$, $A_I(u) \geq \min\{A_I(v), A_I(v)\} = A_I(v)$, and $A_F(u) \leq \max\{A_F(v), A_F(v)\} = A_F(v)$ by (3.4). Hence $A_\sim = (A_T,\ A_I,\ A_F)$ satisfies (4). Since $u' \to v \leq u' \to v$ for all $u, v \in L$, it follows from (3.4) that $A_T(u' \to v) \geq \min\{A_T(u), A_T(v)\}$, $A_I(u' \to v) \geq \min\{A_I(u), A_I(v)\}$, and $A_F(x' \to v) \leq \max\{A_F(u), A_F(v)\}$. Hence (7) holds.

Suppose $A_\sim = (A_T,\ A_I,\ A_F)$ satisfies (4) and (7). Since $0 \leq u$ for all $u \in L$, (2) is induced by (4). Moreover, from $u \leq ((u \to v)')' \to v$ for all $u, v \in L$, we get that,

$$u' \to w \leq (((u \to v)')' \to v)' \to w = ((u \to v)')' \to (v' \to w).$$

Thus

$$A_T(u' \to w) \geq A_T(((u \to v)')' \to (v' \to w)) \geq \min\{A_T((u \to v)'), A_T(v' \to w)\},$$

$$A_I(u' \to w) \geq A_I(((u \to v)')' \to (v' \to w)) \geq \min\{A_I((u \to v)'), A_I(v' \to w)\},$$

and

$$A_F(u' \to w) \leq A_F(((u \to v)')' \to (v' \to w)) \leq \max\{A_F((u \to v)'), A_F(v' \to w)\}.$$

Hence $A_\sim = (A_T,\ A_I,\ A_F)$ satisfies (8).

Assume $A_{\sim} = (A_T,\, A_I,\, A_F)$ satisfies (2) and (8). Let $u, v \in L$ such that $u \leq v$. Let $w = 0$ in (8) Then

$$A_T(u) = A_T(u' \to 0) \geq \min\{A_T((u \to v)'), A_T(v' \to 0)\} = \min\{A_T(0), A_T(v)\} = A_T(v),$$

$$A_I(u) = A_I(u' \to 0) \geq \min\{A_I((u \to v)'), A_I(v' \to 0)\} = \min\{A_I(0), A_I(v)\} = A_I(v),$$

and

$$A_F(u) = A_F(u' \to 0) \leq \max\{A_F((u \to v)'), A_F(v' \to 0)\} = \max\{A_F(0), A_F(v)\} = A_F(v).$$

Therefore, $A_{\sim} = (A_T,\, A_I,\, A_F)$ satisfies (5).

Suppose $A_{\sim} = (A_T,\, A_I,\, A_F) \in NLI(L)$. Since

$$((u \to w)' \to (v \to w)')' \to (u \to v)' = (u \to v) \to ((v \to w) \to (u \to w)) = 1,$$

we have, $((u \to w)' \to (v \to w)')' \leq (u \to v)'$ for all $u, v, w \in L$. By (3) and (4), we get that

$$A_T((u \to w)') \geq \min\{A_T(((u \to w)' \to (v \to w)')'), A_T((v \to w)')\} \geq \min\{A_T((u \to v)'), A_T((v \to w)')\},$$

$$A_I((u \to w)') \geq \min\{A_I(((u \to w)' \to (v \to w)')'), A_I((v \to w)')\} \geq \min\{A_I((u \to v)'), A_I((v \to w)')\},$$

and

$$A_F((u \to w)') \leq \max\{A_F(((u \to w)' \to (v \to w)')'), A_F((v \to w)')\} \leq \max\{A_F((u \to v)'), A_F((v \to w)')\}$$

for all $u, v, w \in L$. Thus $A_{\sim} = (A_T,\, A_I,\, A_F)$ satisfies (9).

Let $A_{\sim} = (A_T,\, A_I,\, A_F)$ be a neutrosophic set in L satisfying (2) and (9). Since $(u \to 0)' = u$ for all $u \in L$, we have

$$A_T(u) = A_T((u \to 0)') \geq \min\{A_T((u \to v)'), A_T((v \to 0)')\} = \min\{A_T((u \to v)'), A_T(v)\},$$

$$A_I(u) = A_I((u \to 0)') \geq \min\{A_I((u \to v)'), A_I((v \to 0)')\} = \min\{A_I((u \to v)'), A_I(v)\},$$

and

$$A_F(u) = A_F((u \to 0)') \leq \max\{A_F((u \to v)'), A_F((v \to 0)')\} = \max\{A_F((u \to v)'), A_F(v)\}$$

for all $u, v \in L$. Therefore $A_{\sim} = (A_T,\, A_I,\, A_F) \in NLI(L)$. □

Theorem 3.11. *A neutrosophic set $A_{\sim} = (A_T,\, A_I,\, A_F)$ is a neutrosophic LI-ideal of L if and only if the nonempty neutrosophic level sets $L(A_T;\alpha)$, $L(A_I;\beta)$ and $L(A_F;\gamma)$ are LI-ideals of L for all $\alpha, \beta, \gamma \in [0, 1]$.*

Proof. Suppose $A_{\sim} = (A_T, A_I, A_F) \in NLI(L)$ and $\alpha, \beta, \gamma \in [0, 1]$ such that $L(A_T;\alpha)$, $L(A_I;\beta)$ and $L(A_F;\gamma)$ are nonempty. It is clear that $0 \in L(A_T;\alpha)$, $0 \in L(A_I;\beta)$ and $0 \in L(A_F;\gamma)$. Let $u, v, a, b, m, n \in L$ such that $(u \to v)' \in L(A_T;\alpha)$, $v \in L(A_T;\alpha)$, $(a \to b)' \in L(A_I;\beta)$,

$b \in L(A_I;\beta)$, $(m \to n)' \in L(A_F;\gamma)$, and $n \in L(A_F;\gamma)$. Then $A_T((u \to v)') \geq \alpha$, $A_T(v) \geq \alpha$, $A_I((a \to b)') \geq \beta$, $A_I(b) \geq \beta$, $A_F((m \to n)') \leq \gamma$, and $A_F(n) \leq \gamma$. By (2), we have

$$A_T(u) \geq \min\{A_T(u \to v)', A_T(v)\} \geq \alpha,$$

$$A_I(a) \geq \min\{A_I(a \to b)', A_I(b)\} \geq \beta,$$

and

$$A_F(m) \leq \max\{A_F(m \to n)', A_F(n)\} \leq \gamma.$$

Hence, $u \in L(A_T;\alpha)$, $a \in L(A_I;\beta)$ and $u \in L(A_F;\gamma)$. Therefore, $L(A_T;\alpha)$, $L(A_I;\beta)$ and $L(A_F;\gamma)$ are LI-ideals of L.

Conversely, let $A_\sim = (A_T,\ A_I,\ A_F)$ be a neutrosophic set in L in which the nonempty neutrosophic level sets $L(A_T;\alpha)$, $L(A_I;\beta)$ and $L(A_F;\gamma)$ are LI-ideals of L for all $\alpha, \beta, \gamma \in [0,1]$. For any $u, a, m \in L$, let $A_T(u) = \alpha$, $A_I(a) = \beta$ and $A_F(m) = \gamma$. Then $u \in L(A_T;\alpha)$, $a \in L(A_I;\beta)$ and $m \in L(A_F;\gamma)$, that is, $L(A_T;\alpha)$, $L(A_I;\beta)$ and $L(A_F;\gamma)$ are nonempty sets. Hence $0 \in L(A_T;\alpha)$, $0 \in L(A_I;\beta)$ and $0 \in L(A_F;\gamma)$ by assumption, and so $A_T(0) \geq \alpha = A_T(u)$, $A_I(0) \geq \beta = A_I(a)$ and $A_F(0) \leq \gamma = A_F(m)$. Suppose there exist $a, b \in L$ such that $A_T(a) < \min\{A_T((a \to b)'), A_T(b)\}$. Then

$$A_T(a) < \alpha_0 < \min\{A_T((a \to b)'), A_T(b)\},$$

where $\alpha_0 = \frac{1}{2}(A_T(a) + \min\{A_T((a \to b)'), A_T(b)\})$. Thus $a \notin L(A_T;\alpha_0)$, $(a \to b)' \notin L(A_T;\alpha_0)$ and $b \in L(A_T;\alpha_0)$, which is a contradiction. Hence, $A_T(u) \geq \min\{A_T((u \to v)'), A_T(v)\}$ for all $u, v \in L$. Similarly, we can verify that $A_I(u) \geq \min\{A_I((u \to v)'), A_I(v)\}$ for all $u, v \in L$. Now, suppose

$$A_F(m) > \max\{A_F((m \to n)'), A_F(n)\},$$

for some $m, n \in L$. Let $\gamma_0 := \frac{1}{2}(A_F(m) + \max\{A_F((m \to n)'), A_F(n)\})$. Then

$$A_F(m) > \gamma_0 \geq \max\{A_F((m \to n)'), A_F(n)\},$$

and so $(m \to n)' \in L(A_F;\gamma_0)$, $n \in L(A_F;\gamma_0)$, but $m \notin L(A_F;\gamma_0)$, which is a contradiction. Hence

$$A_F(m) \leq \max\{A_F((m \to n)'), A_F(n)\}$$

for all $u, v \in L$. Therefore $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$. □

Corollary 3.12. *If $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$, then $L(A_T;\alpha) \cap L(A_I;\beta) \cap L(A_F;\gamma)$ is an LI-ideal of L for all $\alpha, \beta, \gamma \in [0,1]$.*

Proof. Straightforward. □

Let $f : L_1 \to L_2$ be an implication homomorphisms of lattice implication algebras. For any neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in L_2, we define a new neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ in L_1 by $A^f_T(u) = A_T(f(u))$, $A^f_I(u) = A_I(f(u))$ and $A^f_F(u) = A_F(f(u))$ for all $u \in L_1$.

Theorem 3.13. *Let $f : L_1 \to L_2$ be an implication homomorphism of lattice implication algebras with $f(0) = 0$. If $A_\sim = (A_T, A_I, A_F) \in NLI(L_2)$, then $A^f_\sim = (A^f_T, A^f_I, A^f_F) \in NLI(L_1)$.*

Proof. Let $u, v \in L_1$. Then $A^f_T(u) = A_T(f(u)) \leq A_T(0) = A_T(f(0)) = A^f_T(0)$, $A^f_I(u) = A_I(f(u)) \leq A_I(0) = A_I(f(0)) = A^f_I(0)$, and $A^f_F(u) = A_F(f(u)) \geq A_F(0) = A_F(f(0)) = A^f_F(0)$. Thus,

$$\begin{aligned} A^f_T(u) = A_T(f(u)) &\geq \min\{A_T((f(u) \to f(v))'), A_T(f(v))\} \\ &= \min\{A_T((f(u \to v))'), A_T(f(v))\} \\ &= \min\{A_T(f((u \to v)')), A_T(f(v))\} \\ &= \min\{A^f_T((u \to v)'), A^f_T(v)\}, \end{aligned}$$

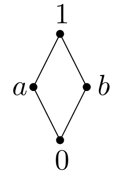
$$\begin{aligned} A^f_I(u) = A_I(f(u)) &\geq \min\{A_I((f(u) \to f(v))'), A_I(f(v))\} \\ &= \min\{A_I((f(u \to v))'), A_I(f(v))\} \\ &= \min\{A_I(f((u \to v)')), A_I(f(v))\} \\ &= \min\{A^f_I((u \to v)'), A^f_I(v)\}, \end{aligned}$$

and

$$\begin{aligned} A^f_F(u) = A_F(f(u)) &\leq \max\{A_F((f(u) \to f(v))'), A_F(f(v))\} \\ &= \max\{A_F((f(u \to v))'), A_F(f(v))\} \\ &= \max\{A_F(f((u \to v)')), A_F(f(v))\} \\ &= \max\{A^f_F((u \to v)'), A^f_F(v)\}. \end{aligned}$$

Therefore, $A^f_\sim = (A^f_T, A^f_I, A^f_F) \in NLI(L_1)$. □

Example 3.14. Let $L = \{0, a, b, 1\}$ be a poset with Hasse diagram and Cayley tables as follows:



x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Defin the operations $\vee$ and $\wedge$ on L as follows:

$$u \vee v := (u \to v) \to v \ \ and \ \ u \wedge v := ((u' \to v') \to v')',$$

for all $u, v \in L$. Then L is a lattice implication algebra (see [21]). Define a function $f : L \to L$ by $f(0) = 0$, $f(a) = b$, $f(b) = a$ and $f(1) = 1$. Then f is an implication homomorphism . Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in L defined by Table 4.

Table 4. Tabular representation of $A_\sim = (A_T, A_I, A_F)$

L	0	a	b	1
$A_T(x)$	0.9	0.5	0.3	0.3
$A_I(x)$	0.8	0.2	0.5	0.2
$A_F(x)$	0.2	0.7	0.4	0.7

It is routine to verify that $A_\sim = (A_T, A_I, A_F) \in NLI(L)$. The neutrosophic set $A^f_\sim = (A^f_T, A^f_I, A^f_F)$ is described by Table 5.

Table 5. Tabular representation of $A^f_\sim = (A^f_T, A^f_I, A^f_F)$

L	0	a	b	1
$A^f_T(x)$	0.9	0.3	0.5	0.3
$A^f_I(x)$	0.8	0.5	0.2	0.2
$A^f_F(x)$	0.2	0.4	0.7	0.7

It is routine to verify that $A^f_\sim = (A^f_T, A^f_I, A^f_F) \in NLI(L)$.

We give additional condition for dealing with the converse of Theorem 3.13.

Theorem 3.15. *Let $f : L_1 \to L_2$ be an implication epimorphism of lattice implication algebras with $f(0) = 0$. If $A^f_\sim = (A^f_T, A^f_I, A^f_F) \in NLI(L_1)$, then $A_\sim = (A_T, A_I, A_F) \in NLI(L_2)$.*

Proof. Let $u \in L_2$. Then there exists $a \in L_1$ such that $f(a) = u$. Hence

$$A_T(u) = A_T(f(a)) = A^f_T(a) \leq A^f_T(0) = A_T(f(0)) = A_T(0),$$

$$A_I(u) = A_I(f(a)) = A^f_I(a) \leq A^f_I(0) = A_I(f(0)) = A_I(0),$$

and

$$A_F(u) = A_F(f(a)) = A^f_F(a) \geq A^f_F(0) = A_F(f(0)) = A_F(0).$$

Let $u, v \in L_2$. Then $f(a) = u$ and $f(b) = v$ for some $a, b \in L_1$. It follows that

$$\begin{aligned} A_T(u) &= A_T(f(a)) = A_T^f(a) \geq \min\{A_T^f((a \to b)'), A_T^f(b)\} \\ &= \min\{A_T(f((a \to b)')), A_T(f(b))\} \\ &= \min\{A_T((f(a) \to f(b))'), A_T(f(b))\} \\ &= \min\{A_T((u \to v)'), A_T(v)\}, \end{aligned}$$

$$\begin{aligned} A_I(u) &= A_I(f(a)) = A_I^f(a) \geq \min\{A_I^f((a \to b)'), A_I^f(b)\} \\ &= \min\{A_I(f((a \to b)')), A_I(f(b))\} \\ &= \min\{A_I((f(a) \to f(b))'), A_I(f(b))\} \\ &= \min\{A_I((u \to v)'), A_I(v)\}, \end{aligned}$$

and

$$\begin{aligned} A_F(u) &= A_F(f(a)) = A_F^f(a) \leq \max\{A_F^f((a \to b)'), A_F^f(b)\} \\ &= \max\{A_F(f((a \to b)')), A_F(f(b))\} \\ &= \max\{A_F((f(a) \to f(b))'), A_F(f(b))\} \\ &= \max\{A_F((u \to v)'), A_F(v)\}. \end{aligned}$$

Therefore, $A_\sim = (A_T, A_I, A_F)$ is a neutrosophic LI-ideal of L_2. □

4. **Conclusions**

We have applied the notion of neutrosophic set theory to lattice implication algebras. We have introduced the concepts of neutrosophic LI-ideals and neutrosophic lattice ideals of a lattice implication algebra, and investigated several properties. We have discussed the relationship between a neutrosophic LI-ideal and a neutrosophic lattice ideal, and provided conditions for a neutrosophic lattice ideal to be a neutrosophic LI-ideal. We have considered the characterizations of a neutrosophic LI-ideal. We have studied the properties of implication homomorphism of lattice implication algebras related to neutrosophic LI-ideals.

5. **Future research work**

Probing more profound, the results in this paper also provide a strong foundation for future work in logical algebric structure and in neutrosophic set. One area of future work is in combining some other kind of subalgebra like filter, implicative filter etc with neutrosophic sets. Another area is in applying the results studied here to the other algebric structures like BCI/BCK algebras. Future work will be in these two areas.

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Received: May 16, 2019 / Accepted: January 18, 2020