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Neutrality and Many-Valued Logics

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Preamble

This book written by A. SCHUMANN & F. SMARANDACHE is devoted to advances of non-Archimedean multiple-validity idea and its applications to logical reasoning. LEIBNITZ was the first who proposed ARCHIMEDES' axiom to be rejected. He postulated infinitesimals (infinitely small numbers) of the unit interval $[0, 1]$ which are larger than zero, but smaller than each positive real number. ROBINSON applied this idea into modern mathematics in [117] and developed so-called non-standard analysis. In the framework of non-standard analysis there were obtained many interesting results examined in [37], [38], [74], [117].

There exists also a different version of mathematical analysis in that ARCHIMEDES' axiom is rejected, namely, p-adic analysis (e.g., see: [20], [86], [91], [116]). In this analysis, one investigates the properties of the completion of the field $\mathbf{Q}$ of rational numbers with respect to the metric $\rho_p(x, y) = |x - y|_p$, where the norm $|\cdot|_p$ called p-adic is defined as follows:

- $|y|_p = 0 \leftrightarrow y = 0$,
- $|x \cdot y|_p = |x|_p \cdot |y|_p$,
- $|x + y|_p \leqslant \max(|x|_p, |y|_p)$ (non-Archimedean triangular inequality).

That metric over the field $\mathbf{Q}$ is non-Archimedean, because $|n \cdot 1|_p \leqslant 1$ for all $n \in \mathbf{Z}$. This completion of the field $\mathbf{Q}$ is called the field $\mathbf{Q}_p$ of p-adic numbers. In $\mathbf{Q}_p$ there are infinitely large integers.

Nowadays there exist various many-valued logical systems (e.g., see MALINOWSKI's book [92]). However, non-Archimedean and p-adic logical multiple-validities were not yet systematically regarded. In this book, SCHUMANN & SMARANDACHE define such multiple-validities and describe the basic properties of non-Archimedean and p-adic valued logical systems proposed by them in [122], [123], [124], [125], [128], [132], [133]. At the same time, non-Archimedean valued logics are constructed on the base of t-norm approach as fuzzy ones and p-adic valued logics as discrete multi-valued systems.

Let us remember that the first logical multiple-valued system is proposed by the Polish logician Jan ŁUKASIEWICZ in [90]. For the first time he spoke about the idea of logical many-validity at Warsaw University on 7 March 1918 (*Wykład pożegnalny wygłoszony w auli Uniwersytetu Warszawskiego w dniu 7 marca 1918 r.*, page 2). However ŁUKASIEWICZ thought already about such a logic and rejection of the Aristotelian principle of contradiction in 1910 (*O zasadzie sprzeczności u Arystotelesa*, Kraków 1910). Creating many-valued logic, ŁUKASIEWICZ was inspired philosophically. In the meantime, POST designed his many-valued logic in 1921 in [105] independently and for combinatorial reasons as a generalization of Boolean algebra.

The logical multi-validity that runs the unit interval $[0, 1]$ with infinitely small numbers for the first time was proposed by SMARANDACHE in [132], [133], [134], [135], [136]. The neutrosophic logic, as he named it, is conceived for a philosophical explication of the neutrality concept. In this book, it is shown that neutrosophic logic is a generalization of non-Archimedean and p-adic valued logical systems.

In this book non-Archimedean and p-adic multiple-validities idea is regarded as one of possible approaches to explicate the neutrality concept.

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Preface

In this book, we consider various many-valued logics: standard, linear, hyperbolic, parabolic, non-Archimedean, p-adic, interval, neutrosophic, etc. We survey also results which show the tree different proof-theoretic frameworks for many-valued logics, e.g. frameworks of the following deductive calculi: HILBERT's style, sequent, and hypersequent. Recall that hypersequents are a natural generalization of GENTZEN's style sequents that was introduced independently by AVRON and POTTINGER. In particular, we examine HILBERT's style, sequent, and hypersequent calculi for infinite-valued logics based on the three fundamental continuous t-norms: ŁUKASIEWICZ's, GÖDEL's, and Product logics.

We present a general way that allows to construct systematically analytic calculi for a large family of non-Archimedean many-valued logics: hyperrational-valued, hyperreal-valued, and p-adic valued logics characterized by a special format of semantics with an appropriate rejection of ARCHIMEDES' axiom. These logics are built as different extensions of standard many-valued logics (namely, ŁUKASIEWICZ's, GÖDEL's, Product, and POST's logics).

The informal sense of ARCHIMEDES' axiom is that anything can be measured by a ruler. Also logical multiple-validity without ARCHIMEDES' axiom consists in that the set of truth values is infinite and it is not well-founded and well-ordered.

We consider two cases of non-Archimedean multi-valued logics: the first with many-validity in the interval $[0, 1]$ of hypernumbers and the second with many-validity in the ring $\mathbf{Z}_p$ of p-adic integers. Notice that in the second case we set discrete infinite-valued logics. The following logics are investigated:

- hyperrational valued ŁUKASIEWICZ's, GÖDEL's, and Product logics,
- hyperreal valued ŁUKASIEWICZ's, GÖDEL's, and Product logics,
- p-adic valued ŁUKASIEWICZ's, GÖDEL's, and POST's logics.

In [67] HÁJEK classifies truth-functional fuzzy logics as logics whose conjunction and implication are interpreted via continuous t-norms and their residua. Fundamental logics for this classification are ŁUKASIEWICZ's, GÖDEL's and

Product ones. Further, HÁJEK proposes basic fuzzy logic BL which has validity in all logics based on continuous t-norms. In this book, for the first time we consider hypervalued and p-adic valued extensions of basic fuzzy logic BL.

On the base of non-Archimedean valued logics, we construct non-Archimedean valued interval neutrosophic logic **INL** by which we can describe neutrality phenomena. This logic is obtained by adding to the truth valuation a truth triple t, i, f instead of one truth value t, where t is a truth-degree, i is an indeterminacy-degree, and f is a falsity-degree. Each parameter of this triple runs either the unit interval $[0, 1]$ of hypernumbers or the ring $\mathbf{Z}_p$ of p-adic integers.

A. Schumann & F. Smarandache

Contents

Chapter 1

Introduction

1.1 Neutrality concept in logic

Every point of view A tends to be neutralized, diminished, balanced by *Non*-A. At the same time, in between A and *Non*-A there are infinitely many points of view *Neut*-A. Let's note by A an idea, or proposition, theory, event, concept, entity, by *Non*-A what is not A, and by *Anti*-A the opposite of A. *Neut*-A means what is neither A nor *Anti*-A, i.e. neutrality is also in between the two extremes.

The classical logic, also called Boolean logic by the name of British mathematician G. Boole, is two-valued. Thus, neutralities are ignored in this logic. Peirce, before 1910, developed a semantics for three-valued logic in an unpublished note, but Post's dissertation [105] and Łukasiewicz's work [90] are cited for originating the three-valued logic. Here 1 is used for truth, $\frac{1}{2}$ for indeterminacy, and 0 for falsehood. These truth values can be understood as A, *Neut*-A, *Non*-A respectively. For example, in three valued Łukasiewicz's logic the negation of $\frac{1}{2}$ gives $\frac{1}{2}$ again, i.e. the neutrality negation is the neutrality again.

However, we can consider neutralities as degrees between truth and falsehood. In this case we must set multiple-valued logics.

The n-valued logics were developed by Łukasiewicz in [89], [90]. The practical applications of infinite valued logic, where the truth-value may be any number in the closed unit interval $[0, 1]$, are regarded by Zadeh in [153]. This logic is called fuzzy one.

In the meantime, the ancient Indian logic (nyāya) considered four possible values of a statement: 'true (only)', 'false (only)', 'both true and false', and 'neither true nor false'. The Buddhist logic added a fifth value to the previous ones, 'none of these'.

As we see, we can get the neutralities in the framework of many-valued logics. There is also an other way, when we set neutralities as the main property of two-valued logical calculi. Namely, it is possible to develop systems where the principle of classical logic, which entails that from contradictory premises any formula can be derived, in symbols: $\alpha \wedge \neg\alpha \vdash \beta$, is violated. It is called the DUNS SCOTUS law, which is valid not only in classical logic, but in almost all the known logical systems, like intuicionistic logic.

For the first time the Russian logician N. VASIL'ÈV proposed in [149] and [150] to violate the DUNS SCOTUS law, who perceived that the rejection of the law of non-contradiction could lead to a non-Aristotelian logic in the same way as the violation of the parallel postulate of Euclidean geometry had conduced to non-Euclidean geometry.

The other logician who discussed the possibility of violating the ancient Aristotelian principle of contradiction was the Polish logician J. ŁUKASIEWICZ, but he did not elaborate any logical system to cope with his intuitions. His idea was developed by S. JAŠKOWSKI in [62], who constructed a system of propositional paraconsistent logic, where he distinguished between contradictory (inconsistent) systems and trivial ones.

VASIL'ÈV and ŁUKASIEWICZ were the forerunners of paraconsistent logic in which it is devoted to the study of logical systems which can base on inconsistent theories (i.e., theories which have contradictory theses, like α and $\neg\alpha$) but which are not trivial (in the sense that not every well formed formula of their languages are also axioms).

In this book we will investigate so-called non-Archimedean multiple-valued logics, especially non-Archimedean valued fuzzy logics, in which we have the violation the DUNS SCOTUS law too.

1.2 Neutrality and non-Archimedean logical multiple-validity

The development of fuzzy logic and fuzziness was motivated in large measure by the need for a conceptual framework which can address the issue of uncertainty and lexical imprecision. Recall that fuzzy logic was introduced by Lotfi ZADEH in 1965 (see [153]) to represent data and information possessing nonstatistical uncertainties. Florentin SMARANDACHE had generalized fuzzy logic and introduced two new concepts (see [132], [133], [134]):

1. neutrosophy as study of neutralities;
2. neutrosophic logic and neutrosophic probability as a mathematical model

of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, etc.

Neutrosophy proposed by SMARANDACHE in [134] is a new branch of philosophy, which studies the nature of neutralities, as well as their logical applications. This branch represents a version of paradoxism studies. The essence of paradoxism studies is that there is a neutrality for any two extremes. For example, denote by A an idea (or proposition, event, concept), by *Anti*-A the opposite to A. Then there exists a neutrality *Neut*-A and this means that something is neither A nor *Anti*-A. It is readily seen that the paradoxical reasoning can be modelled if some elements θ_i of a frame Θ are not exclusive, but non-exclusive, i.e., here θ_i have a non-empty intersection. A mathematical model that has such a property is called the DEZERT-SMARANDACHE model (DSm model). A theory of plausible and paradoxical reasoning that studies DSm models is called the DEZERT-SMARANDACHE theory (DSmT). It is totally different from those of all existing approaches managing uncertainties and fuzziness. In this book, we consider plausible reasoning on the base of particular case of infinite DSm models, namely, on the base of non-Archimedean structures.

Let us remember that ARCHIMEDES' axiom is the formula of infinite length that has one of two following notations:

- for any ε that belongs to the interval $[0, 1]$, we have

$$(\varepsilon > 0) \rightarrow [(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots], \tag{1.1}$$

- for any positive integer ε, we have

$$[(1 \geq \varepsilon) \vee (1 + 1 \geq \varepsilon) \vee (1 + 1 + 1 \geq \varepsilon) \vee \ldots]. \tag{1.2}$$

Formulas (1.1) and (1.2) are valid in the field $\mathbf{Q}$ of rational numbers and as well as in the field $\mathbf{R}$ of real numbers. In the ring $\mathbf{Z}$ of integers, only formula (1.2) has a nontrivial sense, because $\mathbf{Z}$ doesn't contain numbers of the open interval $(0, 1)$.

Also, ARCHIMEDES' axiom affirms the existence of an integer multiple of the smaller of two numbers which exceeds the greater: for any positive real or rational number ε, there exists a positive integer n such that $\varepsilon \geq \frac{1}{n}$ or $n \cdot \varepsilon \geq 1$.

The negation of ARCHIMEDES' axiom has one of two following forms:

- there exists ε that belongs to the interval $[0, 1]$ such that

$$(\varepsilon > 0) \wedge [(\varepsilon < 1) \wedge (\varepsilon + \varepsilon < 1) \wedge (\varepsilon + \varepsilon + \varepsilon < 1) \wedge \ldots], \tag{1.3}$$

- there exists a positive integer ε such that

$$[(1 < \varepsilon) \wedge (1 + 1 < \varepsilon) \wedge (1 + 1 + 1 < \varepsilon) \wedge \ldots]. \tag{1.4}$$

Let us show that (1.3) is the negation of (1.1). Indeed,

$$\begin{gathered}
\neg\forall\varepsilon\,[(\varepsilon > 0) \to [(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots]] \leftrightarrow \\
\exists\varepsilon\,\neg\neg[(\varepsilon > 0) \wedge \neg[(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots]] \leftrightarrow \\
\exists\varepsilon\,[(\varepsilon > 0) \wedge [\neg(\varepsilon \geq 1) \wedge \neg(\varepsilon + \varepsilon \geq 1) \wedge \neg(\varepsilon + \varepsilon + \varepsilon \geq 1) \wedge \ldots]] \leftrightarrow \\
\exists\varepsilon\,[(\varepsilon > 0) \wedge [(\varepsilon < 1) \wedge (\varepsilon + \varepsilon < 1) \wedge (\varepsilon + \varepsilon + \varepsilon < 1) \wedge \ldots]]
\end{gathered}$$

It is obvious that formula (1.3) says that there exist *infinitely small numbers* (or *infinitesimals*), i.e., numbers that are smaller than all real or rational numbers of the open interval $(0, 1)$. In other words, ε is said to be an infinitesimal if and only if, for all positive integers n, we have $|\varepsilon| < \frac{1}{n}$. Further, formula (1.4) says that there exist *infinitely large integers* that are greater than all positive integers. Infinitesimals and infinitely large integers are called *nonstandard numbers* or *actual infinities*.

The field that satisfies all properties of $\mathbf{R}$ without Archimedes' axiom is called the field of *hyperreal numbers* and it is denoted by $^*\mathbf{R}$. The field that satisfies all properties of $\mathbf{Q}$ without Archimedes' axiom is called the field of hyperrational numbers and it is denoted by $^*\mathbf{Q}$. By definition of field, if $\varepsilon \in \mathbf{R}$ (respectively $\varepsilon \in \mathbf{Q}$), then $1/\varepsilon \in \mathbf{R}$ (respectively $1/\varepsilon \in \mathbf{Q}$). Therefore $^*\mathbf{R}$ and $^*\mathbf{Q}$ contain simultaneously infinitesimals and infinitely large integers: for an infinitesimal ε, we have $N = \frac{1}{\varepsilon}$, where N is an infinitely large integer.

The ring that satisfies all properties of $\mathbf{Z}$ without Archimedes' axiom is called the ring of hyperintegers and it is denoted by $^*\mathbf{Z}$. This ring includes infinitely large integers. Notice that there exists a version of $^*\mathbf{Z}$ that is called the ring of p-adic integers and is denoted by $\mathbf{Z}_p$.

We will show that nonstandard numbers (actual infinities) are non-exclusive elements. This means that their intersection isn't empty with some other elements. Therefore non-Archimedean structures of the form $^*\mathbf{S}$ (where we obtain $^*\mathbf{S}$ on the base of the set $\mathbf{S}$ of exclusive elements) are particular case of the DSm model. These structures satisfy the properties:

1. all members of $\mathbf{S}$ are exclusive and $\mathbf{S} \subset {}^*\mathbf{S}$,
2. all members of $^*\mathbf{S} \backslash \mathbf{S}$ are non-exclusive,
3. if a member a is non-exclusive, then there exists an exclusive member b such that $a \cap b \neq \emptyset$,
4. there exist non-exclusive members a, b such that $a \cap b \neq \emptyset$,
5. each positive non-exclusive member is greater (or less) than each positive exclusive member.

We will consider the following principal versions of the logic on non-Archimedean structures:

- hyperrational valued Łukasiewicz's, Gödel's, and Product logics,
- hyperreal valued Łukasiewicz's, Gödel's, and Product logics,
- p-adic valued Łukasiewicz's, Gödel's, and Post's logics.

For the first time non-Archimedean logical multiple-validities were regarded by Schumann in [122], [123], [124], [125], [128], [129].

The non-Archimedean structures are not well-founded and well-ordered. Also, **the logical neutrality may be examined as non-well-ordered multiple validity, i.e. as non-Archimedean one**. Two elements are neutral in a true sense if both are incompatible by the ordering relation.

1.3 Neutrality and neutrosophic logic

The non-Archimedean valued logic can be generalized to a *neutrosophic logic*, where the truth values of $^*[0,1]$ are extended to truth triples of the form $\langle t, i, f\rangle \subseteq (^*[0,1])^3$, where t is the truth-degree, i the indeterminacy-degree, f the falsity-degree and they are approximated by non-standard subsets of $^*[0,1]$, and these subsets may overlap and exceed the unit interval in the sense of the non-Archimedean analysis.

Neutrosophic logic was introduced by Smarandache in [132], [133]. It is an alternative to the existing logics, because it represents a mathematical model of uncertainty on non-Archimedean structures. It is a non-classical logic in which each proposition is estimated to have the percentage of truth in a subset $t \subseteq {}^*[0,1]$, the percentage of indeterminacy in a subset $i \subseteq {}^*[0,1]$, and the percentage of falsity in a subset $f \subseteq {}^*[0,1]$. Thus, neutrosophic logic is a formal frame trying to measure the truth, indeterminacy, and falsehood simultaneously, therefore it generalizes:

- Boolean logic ($i = \emptyset$, t and f consist of either 0 or 1);
- n-valued logic ($i = \emptyset$, t and f consist of members 0, 1, $\ldots$, $n-1$);
- fuzzy logic ($i = \emptyset$, t and f consist of members of $[0,1]$).

In simple neutrosophic logic, where t, i, f are singletons, the tautologies have the truth value $\langle {}^*1, {}^*0, {}^*0\rangle$, the contradictions the value $\langle {}^*0, {}^*1, {}^*1\rangle$. While for a paradox, we have the truth value $\langle {}^*1, {}^*1, {}^*1\rangle$. Indeed, the paradox is the only proposition true and false in the same time in the same world, and indeterminate as well! We can assume that some statements are indeterminate in all possible worlds, i.e. that there exists "absolute indeterminacy" $\langle {}^*1, {}^*1, {}^*1\rangle$.

The idea of tripartition (truth, indeterminacy, falsehood) appeared in 1764 when J. H. LAMBERT investigated the credibility of one witness affected by the contrary testimony of another. He generalized HOOPER's rule of combination of evidence (1680s), which was a non-Bayesian approach to find a probabilistic model. KOOPMAN in 1940s introduced the notions of lower and upper probability, followed by GOOD, and DEMPSTER (1967) gave a rule of combining two arguments. SHAFER (1976) extended it to the DEMPSTER-SHAFER theory of belief functions by defining the belief and plausibility functions and using the rule of inference of DEMPSTER for combining two evidences proceeding from two different sources. Belief function is a connection between fuzzy reasoning and probability.

The DEMPSTER-SHAFER theory of belief functions is a generalization of the Bayesian probability (BAYES 1760s, LAPLACE 1780s); this uses the mathematical probability in a more general way, and is based on probabilistic combination of evidence in artificial intelligence. In LAMBERT's opinion, "there is a chance p that the witness will be faithful and accurate, a chance q that he will be mendacious, and a chance $1-p-q$ that he will simply be careless". Therefore we have three components: accurate, mendacious, careless, which add up to 1.

ATANASSOV (see [3], [4]) used the tripartition to give five generalizations of the fuzzy set, studied their properties and applications to the neural networks in medicine:

- *Intuitionistic Fuzzy Set* (IFS): given an universe U, an IFS A over U is a set of ordered triples $\langle$universe element, degree of membership M, degree of non-membership $N\rangle$ such that $M+N \leqslant 1$ and $M, N \in [0,1]$. When $M+N=1$ one obtains the fuzzy set, and if $M+N<1$ there is an indeterminacy $I=1-M-N$.
- *Intuitionistic $\mathfrak{L}$-Fuzzy Set* (ILFS): Is similar to IFS, but M and N belong to a fixed lattice $\mathfrak{L}$.
- *Interval-valued Intuitionistic Fuzzy Set* (IVIFS): Is similar to IFS, but M and N are subsets of $[0,1]$ and $\max M + \max N \leqslant 1$.
- *Intuitionistic Fuzzy Set of Second Type* (IFS2): Is similar to IFS, but $M^2+N^2 \leqslant 1$. M and N are inside of the upper right quarter of unit circle.
- *Temporal IFS*: Is similar to IFS, but M and N are functions of the time-moment too.

However, sometimes a too large generalization may have no practical impact. Such unification theories, or attempts, have been done in the history of sciences. EINSTEIN tried in physics to build a Unifying Field Theory that seeks to unite the properties of gravitational, electromagnetic, weak, and strong interactions so that a single set of equations can be used to predict all their characteristics;

whether such a theory may be developed it is not known at the present.

Dezert suggested to develop practical applications of neutrosophic logic (see [135], [136]), e.g. for solving certain practical problems posed in the domain of research in Data/Information fusion.

Chapter 2

First-order logical language

2.1 Preliminaries

Let us remember some basic logical definitions.

Definition 1 *A first-order logical language $\mathcal{L}$ consists of the following symbols:*

1. *Variables:* (i) *Free variables: $a_0, a_1, a_2, \ldots, a_j, \ldots$ ($j \in \omega$).* (ii) *Bound variables: $x_0, x_1, x_2, \ldots, x_j, \ldots$ ($j \in \omega$)*

2. *Constants:* (i) *Function symbols of arity i ($i \in \omega$): $F_0^i, F_1^i, F_2^i, \ldots, F_j^i, \ldots$ ($j \in \omega$). Nullary function symbols are called individual constants.* (ii) *Predicate symbols of arity i ($i \in \omega$): $P_0^i, P_1^i, P_2^i, \ldots, P_j^i, \ldots$ ($j \in \omega$). Nullary predicate symbols are called truth constants.*

3. *Logical symbols:* (i) *Propositional connectives of arity n_j : $\Box_0^{n_0}$, $\Box_1^{n_1}$, $\ldots$, $\Box_r^{n_r}$.* (ii) *Quantifiers:* $\mathrm{Q}_0, \mathrm{Q}_1, \ldots, \mathrm{Q}_q$.

4. *Auxiliary symbols: (,), and , (comma).*

Terms are inductively defined as follows:

1. Every individual constant is a term.

2. Every free variable is a term.

3. If F^n is a function symbol of arity n, and $t_1, \ldots, t_n$ are terms, then $F^n(t_1, \ldots, t_n)$ is a term.

Formulas are inductively defined as follows:

1. If P^n is a predicate symbol of arity n, and $t_1, \ldots, t_n$ are terms, then $P^n(t_1, \ldots, t_n)$ is a formula. It is called atomic or an atom. It has no outermost logical symbol.

2. If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are formulas and $\Box^n$ is a propositional connective of arity n, then $\Box^n(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a formula with outermost logical symbol $\Box^n$.

3. If φ is a formula not containing the bound variable x, a is a free variable and Q is a quantifier, then $\mathrm{Q}x\varphi(x)$, where $\varphi(x)$ is obtained from φ by replacing a by x at every occurrence of a in φ, is a formula. Its outermost logical symbol is Q.

A formula is called *open* if it contains free variables, and *closed* otherwise. A formula without quantifiers is called *quantifier-free*. We denote the set of formulas of a language $\mathcal{L}$ by L.

We will write $\varphi(x)$ for a formula possibly containing the bound variable x, and $\varphi(a)$ (resp. $\varphi(t)$) for the formula obtained from φ by replacing every occurrence of the variable x by the free variable a (resp. the term t).

Hence, we shall need meta-variables for the symbols of a language $\mathcal{L}$. As a notational convention we use letters $\varphi, \phi, \psi, \ldots$ to denote formulas, letters $\Gamma, \Delta, \Lambda, \ldots$ for sequences and sets of formulas.

Definition 2 *A matrix, or matrix logic,* $\mathfrak{M}$ *for a language* $\mathcal{L}$ *is given by:*

1. *a nonempty set of truth values* V *of cardinality* $|V| = m$*,*

2. *a subset* $V_+ \subseteq V$ *of designated truth values,*

3. *an algebra with domain* V *of appropriate type: for every* n*-place connective* $\Box$ *of* $\mathcal{L}$ *there is an associated truth function* $\widetilde{\Box}$*:* $V^n \rightarrow V$ *, and*

4. *for every quantifier* Q*, an associated truth function* $\widetilde{\mathrm{Q}}$*:* $\wp(V)\backslash\emptyset \rightarrow V$

Notice that a truth function for quantifiers is a mapping from nonempty sets of truth values to truth values: for a non-empty set $M \subseteq V$, a quantified formula $\mathrm{Q}x\varphi(x)$ takes the truth value $\widetilde{\mathrm{Q}}(M)$ if, for every truth value $v \in V$, it holds that $v \in M$ iff there is a domain element d such that the truth value of φ in this point d is v (all relative to some interpretation). The set M is called the distribution of φ. For example, suppose that there are only the universal quantifier $\forall$ and the existential quantifier $\exists$ in $\mathcal{L}$. Further, we have the set of truth values $V = \{\top, \bot\}$, where $\bot$ is false and $\top$ is true, i.e. the set of designated truth values $V_+ = \{\top\}$. Then we define the truth functions for the quantifiers $\forall$ and $\exists$ as follows:

$$
\begin{aligned}
\widetilde{\forall}(\{\top\}) &= \top, \\
\widetilde{\exists}(\{\bot\}) &= \bot, \\
\widetilde{\forall}(\{\top, \bot\}) = \widetilde{\forall}(\{\bot\}) &= \bot, \\
\widetilde{\exists}(\{\top, \bot\}) = \widetilde{\exists}(\{\top\}) &= \top
\end{aligned}
$$

Also, a matrix logic $\mathfrak{M}$ for a language $\mathcal{L}$ is an algebraic system $\mathfrak{M} = \langle V$, $\widetilde{\Box}_0$, $\widetilde{\Box}_1$, …, $\widetilde{\Box}_r$, $\widetilde{\mathrm{Q}}_0$, $\widetilde{\mathrm{Q}}_1$, …, $\widetilde{\mathrm{Q}}_q$, $V_+\rangle$, where

1. V is a nonempty set of truth values for well-formed formulas of $\mathcal{L}$,

2. $\widetilde{\Box}_0$, $\widetilde{\Box}_1$, ..., $\widetilde{\Box}_r$ are a set of matrix operations defined on the set V and assigned to corresponding propositional connectives $\Box_0^{n_0}, \Box_1^{n_1}, \ldots, \Box_r^{n_r}$ of $\mathcal{L}$,

3. $\widetilde{\mathrm{Q}}_0, \widetilde{\mathrm{Q}}_1, \ldots, \widetilde{\mathrm{Q}}_q$ are a set of matrix operations defined on the set V and assigned to corresponding quantifiers $\mathrm{Q}_0, \mathrm{Q}_1, \ldots, \mathrm{Q}_q$ of $\mathcal{L}$,

4. V_+ is a set of designated truth values such that $V_+ \subseteq V$.

A *structure* $\mathcal{M} = \langle D, \Phi\rangle$ for a language $\mathcal{L}$ (an $\mathcal{L}$-structure) consists of the following:

1. A non-empty set D, called the domain (elements of D are called individuals).

2. A mapping Φ that satisfies the following:

   (a) Each n-ary function symbol F of $\mathcal{L}$ is mapped to a function $\widetilde{F}$: $D^n \to D$ if $n > 0$, or to an element of D if $n = 0$.

   (b) Each n-ary predicate symbol P of $\mathcal{L}$ is mapped to a function $\widetilde{P}$: $D^n \to V$ if $n > 0$, or to and element of V if $n = 0$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. An assignment s is a mapping from the free variables of $\mathcal{L}$ to individuals.

Definition 3 *An $\mathcal{L}$-structure $\mathcal{M} = \langle D, \Phi\rangle$ together with an assignment s is said to be an interpretation $I = \langle \mathcal{M}, s\rangle$.*

Let $I = \langle\langle D, \Phi\rangle, s\rangle$ be an interpretation. Then we can extend the mapping Φ to a mapping Φ_I from terms to individuals:

- If a is a free variable, then $\Phi_I(a) = s(a)$.
- If t is of the form $F(t_1, \ldots, t_n)$, where F is a function symbol of arity n and t_1, ..., t_n are terms, then $\Phi_I(t) = \Phi(f)\Phi_I(t_1)\ldots\Phi_I(t_n))$.

Definition 4 *Given an interpretation $I = \langle \mathcal{M}, s\rangle$, we define the valuation* val_I *to be a mapping from formulas φ of $\mathcal{L}$ to truth values as follows:*

1. *If φ is atomic, i.e., of the form $P(t_1, \ldots, t_n)$, where P is a predicate symbol of arity n and $t_1, \ldots, t_n$ are terms, then* $\mathrm{val}_I(\varphi) = \Phi(P)(\Phi_I(t_1)\ldots\Phi_I(t_n))$.

2. *If the outermost logical symbol of φ is a propositional connective $\Box$ of arity n, i.e., φ is of the form $\Box(\psi_1, \ldots, \psi_n)$, where $\psi_1, \ldots, \psi_n$ are formulas, then* $\mathrm{val}_I(\varphi) = \widetilde{\Box}(\mathrm{val}_I(\psi_1), \ldots, \mathrm{val}_I(\psi_n))$.

3. *If the outermost logical symbol of φ is a quantifier* Q, *i.e., φ is of the form* $\mathrm{Q}x\ \psi(x)$, *then*
$$\mathrm{val}_I(\varphi) = \widetilde{\mathrm{Q}}(\bigcup_{d \in D} \mathrm{val}_I(\psi(d))).$$

Suppose $|V| \geqslant 2$. A formula φ is said to be *logically valid* (e.g. it is a *many-valued tautology*) iff, for every interpretation I, it holds that $\mathrm{val}_I(\varphi) \in V_+$. Sometimes, a logically valid formula ψ is denoted by $\models \psi$.

Suppose $\mathcal{M}$ is a structure for $\mathcal{L}$, and φ a formula of $\mathcal{L}$. A formula φ is called *satisfiable* iff there is an interpretation I such that $\mathrm{val}_I(\varphi) \in V_+$. A satisfiable formula is denoted by $\mathcal{M} \models \varphi$. This means that φ is satisfiable on $\mathcal{M}$ for every assignment s. In this case $\mathcal{M}$ is called a *model* of φ.

If Γ is a set of formulas, we will write $\mathcal{M} \models \Gamma$ if $\mathcal{M} \models \gamma$ for every formula $\gamma \in \Gamma$, and say that $\mathcal{M}$ is a *model* of Γ or that $\mathcal{M}$ *satisfies* Γ.

Suppose Γ is a set of formulas of $\mathcal{L}$ and ψ is a formula of $\mathcal{L}$. Then Γ *implies* ψ, written as $\Gamma \models \psi$, if $\mathcal{M} \models \psi$ whenever $\mathcal{M} \models \Gamma$ for every structure $\mathcal{M}$ for $\mathcal{L}$. If Γ and Δ are sets of formulas of $\mathcal{L}$, then Γ *implies* Δ, written as $\Gamma \models \Delta$, if $\mathcal{M} \models \Delta$ whenever $\mathcal{M} \models \Gamma$ for every structure $\mathcal{M}$ for $\mathcal{L}$.

Suppose x is a variable, t is a term, and φ is a formula. Then the fact that *t is substitutable for x in φ* is defined as follows:

1. If φ is atomic, then t is substitutable for x in φ.
2. If φ is $\Box(\psi_1, \ldots, \psi_n)$, then t is substitutable for x in φ if and only if t is substitutable for x in ψ_1, t is substitutable for x in ψ_n, etc.
3. If φ is $\forall y\, \psi$, then t is substitutable for x in φ if and only if either
   (a) x does not occur free in φ, or
   (b) if y does not occur in t and t is substitutable for x in ψ.

2.2 Hilbert's type calculus for classical logic

Definition 5 *Classical logic is built in the framework of the language $\mathcal{L}$ and the valuation val_I of its formulas is a mapping to truth values 0, 1 that is defined as follows:*

- $\mathrm{val}_I(\neg\alpha) = 1 - \mathrm{val}_I(\alpha)$,
- $\mathrm{val}_I(\alpha \vee \beta) = \max(\mathrm{val}_I(\alpha), \mathrm{val}_I(\beta))$,
- $\mathrm{val}_I(\alpha \wedge \beta) = \min(\mathrm{val}_I(\alpha), \mathrm{val}_I(\beta))$,
- $\mathrm{val}_I(\alpha \to \beta) = \begin{cases} 0, & \textit{if } \mathrm{val}_I(\alpha) = 1 \textit{ and } \mathrm{val}_I(\beta) = 0 \\ 1, & \textit{otherwise.} \end{cases}$

Consider HILBERT's calculus for classical logic (see [71], [72]). Its axiom schemata are as follows:

$$\psi \to (\varphi \to \psi), \tag{2.1}$$
$$(\psi \to \varphi) \to ((\psi \to (\varphi \to \chi)) \to (\psi \to \chi)), \tag{2.2}$$
$$(\varphi \wedge \psi) \to \varphi, \tag{2.3}$$
$$(\varphi \wedge \psi) \to \psi, \tag{2.4}$$
$$(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to (\varphi \wedge \psi))), \tag{2.5}$$
$$\psi \to (\psi \vee \varphi), \tag{2.6}$$
$$\varphi \to (\psi \vee \varphi), \tag{2.7}$$
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)), \tag{2.8}$$
$$(\varphi \to \chi) \to (\varphi \to \neg\chi) \to \neg\varphi), \tag{2.9}$$
$$\neg\neg\varphi \to \varphi, \tag{2.10}$$
$$\forall x \varphi(x) \to \varphi[x/t], \tag{2.11}$$
$$\varphi[x/t] \to \exists x \varphi(x), \tag{2.12}$$

where the formula $\varphi[x/t]$ is the result of substituting the term t for all free occurrences of x in φ.

In HILBERT's calculus there are the following inference rules:

1. *Modus ponens*: if two formulas φ and $\varphi \to \psi$ hold, then we deduce a formula ψ:

$$\frac{\varphi, \quad \varphi \to \psi}{\psi}.$$

2. *Universal generalization*:

$$\frac{\varphi \to \psi(a)}{\varphi \to \forall x \psi(x)},$$

where a is not free in the expression φ.

3. *Existential generalization*:

$$\frac{\varphi(a) \to \psi}{\exists x \varphi(x) \to \psi}.$$

where a is not free in the expression ψ.

Definition 6 *Let Σ be a set of formulas. A deduction or proof from Σ in $\mathcal{L}$ is a finite sequence $\varphi_1 \varphi_2 \ldots \varphi_n$ of formulas such that for each $k \leq n$,*

1. *φ_k is an axiom, or*

2. *$\varphi_k \in \Sigma$, or*

3. *there are $i_1, \ldots, i_n < k$ such that φ_k follows from $\varphi_{i_1}, \ldots, \varphi_{i_n}$ by inference rules.*

A formula of Σ appearing in the deduction is called a premiss. Σ proves a formula α, written as $\Sigma \vdash \alpha$, if α is the last formula of a deduction from Σ. We'll usually write $\vdash \alpha$ for $\emptyset \vdash \alpha$.

A formula α such that $\vdash \alpha$ is called *provable*. It is evident that all formulas of the form (2.1) – (2.10) are provable.

As an example, show that $\vdash \psi \to \psi$.

At the first step, take axiom schema (2.1):

$$\psi \to ((\varphi \to \psi) \to \psi).$$

At the second step, take axiom schema (2.2):

$$(\psi \to (\varphi \to \psi)) \to ((\psi \to ((\varphi \to \psi) \to \psi)) \to (\psi \to \psi)).$$

At the third step, using axiom schema (2.1) and the formula of the second step we obtain by modus ponens the following expression:

$$(\psi \to ((\varphi \to \psi) \to \psi)) \to (\psi \to \psi).$$

At the last step, using the formula of the first step and the formula of the third step we obtain by modus ponens that

$$\psi \to \psi.$$

Further, we will consider the proof and the provability for various HILBERT's type nonclassical calculi just in the sense of definition 6.

Theorem 1 (Soundness and Completeness Theorem) *Let α be a formula and Δ be a set of formulas of classical logic.*
$\Delta \models \alpha$ if and only if $\Delta \vdash \alpha$. □

2.3 Sequent calculus for classical logic

Let us remember that the sequent calculus for usual two-valued logic was proposed by GENTZEN in [52].

By his definition, a *sequent* is an expression of the form $\Gamma_1 \hookrightarrow \Gamma_2$, where $\Gamma_1 = \{\varphi_1, \dots, \varphi_j\}$, $\Gamma_2 = \{\psi_1, \dots, \psi_i\}$ are finite sets of formulas of the language $\mathcal{L}$, that has the following interpretation: $\Gamma_1 \hookrightarrow \Gamma_2$ is logically valid iff

$$\bigwedge_j \varphi_j \to \bigvee_i \psi_i$$

is logically valid.

Also, we read such a sequent as: "φ_1 and ... and φ_j entails ψ_1 or ... or ψ_i".

A sequent of the form $\emptyset \hookrightarrow \Gamma$ is denoted by $\hookrightarrow \Gamma$. A sequent of the form $\Gamma \hookrightarrow \emptyset$ is denoted by $\Gamma \hookrightarrow$.

The inference rules are expressions containing formula variables φ, ψ, ... and multisets of formulas variables Γ, Δ, ... such that replacing these with actual formulas and multisets of formulas, gives ordered pairs consisting of a sequent S (the conclusion) and a finite set of sequents S_1, ..., S_n (the premises, written: $\frac{S_1, \dots, S_n}{S}$). Rules where $n = 0$ are called the *initial sequents* or axioms.

The only *axiom* of the sequent calculus for classical logic is as follows: $\psi \hookrightarrow \psi$.

The *inference rules*:

1. **Structural rules**:

$$\frac{\Gamma_1 \hookrightarrow \Gamma_2}{\psi, \Gamma_1 \hookrightarrow \Gamma_2}, \qquad \frac{\Gamma_1 \hookrightarrow \Gamma_2}{\Gamma_1 \hookrightarrow \Gamma_2, \psi}$$

(the left and right weakening rules respectively),

$$\frac{\psi, \psi, \Gamma_1 \hookrightarrow \Gamma_2}{\psi, \Gamma_1 \hookrightarrow \Gamma_2}, \qquad \frac{\Gamma_1 \hookrightarrow \Gamma_2, \psi, \psi}{\Gamma_1 \hookrightarrow \Gamma_2, \psi}$$

(the left and right contraction rules respectively),

$$\frac{\Gamma_1, \psi, \chi, \Delta \hookrightarrow \Gamma_2}{\Gamma_1, \chi, \psi, \Delta \hookrightarrow \Gamma_2}, \qquad \frac{\Gamma_1 \hookrightarrow \Gamma_2, \psi, \chi, \Delta}{\Gamma_1 \hookrightarrow \Gamma_2, \chi, \psi, \Delta}.$$

(the left and right exchange rules respectively).

2. **Logical rules**, where $(\# \Rightarrow)$ and $(\Rightarrow \#)$ are the left and right introduction rules for a connective $\# \in \{\neg, \rightarrow, \vee, \wedge, \forall, \exists\}$ respectively:

$$\frac{\Gamma_1 \hookrightarrow \Gamma_2, \psi}{\Gamma_1, \neg\psi \hookrightarrow \Gamma_2}\ (\neg \Rightarrow), \qquad \frac{\Gamma_1, \psi \hookrightarrow \Gamma_2}{\Gamma_1 \hookrightarrow \Gamma_2, \neg\psi}\ (\Rightarrow \neg),$$

$$\frac{\psi, \chi, \Gamma_1 \hookrightarrow \Gamma_2}{\psi \wedge \chi, \Gamma_1 \hookrightarrow \Gamma_2}\ (\wedge \Rightarrow), \qquad \frac{\Gamma_1 \hookrightarrow \Gamma_2, \psi \qquad \Gamma_1 \hookrightarrow \Gamma_2, \chi}{\Gamma_1 \hookrightarrow \Gamma_2, \psi \wedge \chi}\ (\Rightarrow \wedge),$$

$$\frac{\Gamma_1, \psi \hookrightarrow \Gamma_2 \qquad \Gamma_1, \chi \hookrightarrow \Gamma_2}{\Gamma_1, \psi \vee \chi \hookrightarrow \Gamma_2}\ (\vee \Rightarrow), \qquad \frac{\Gamma_1 \hookrightarrow \Gamma_2, \psi, \chi}{\Gamma_1 \hookrightarrow \Gamma_2, \psi \vee \chi}\ (\Rightarrow \vee),$$

$$\frac{\Gamma_1 \hookrightarrow \Gamma_2, \psi \qquad \chi, \Delta_1 \hookrightarrow \Delta_2}{\psi \rightarrow \chi, \Gamma_1, \Delta_1 \hookrightarrow \Gamma_2, \Delta_2}\ (\rightarrow\Rightarrow), \qquad \frac{\psi, \Gamma_1 \hookrightarrow \Gamma_2, \chi}{\Gamma_1 \hookrightarrow \Gamma_2, \psi \rightarrow \chi}\ (\Rightarrow\rightarrow),$$

$$\frac{\varphi[x/t], \Gamma_1 \hookrightarrow \Gamma_2}{\forall x\ \varphi(x), \Gamma_1 \hookrightarrow \Gamma_2}\ (\forall \Rightarrow), \qquad \frac{\Gamma_1 \hookrightarrow \Gamma_2, \varphi(a)}{\Gamma_1 \hookrightarrow \Gamma_2, \forall x\ \varphi(x)}\ (\Rightarrow \forall),$$

$$\frac{\varphi(a), \Gamma_1 \hookrightarrow \Gamma_2}{\exists x\ \varphi(x), \Gamma_1 \hookrightarrow \Gamma_2}\ (\exists \Rightarrow), \qquad \frac{\Gamma_1 \hookrightarrow \Gamma_2, \varphi[x/t]}{\Gamma_1 \hookrightarrow \Gamma_2, \exists x\ \varphi(x)}\ (\Rightarrow \exists),$$

where the formula $\varphi[x/t]$ is the result of substituting the term t for all free occurrences of x in φ, a has no occurrences in the below sequent.

3. **Cut rule**:

$$\frac{\Gamma_1 \hookrightarrow \Delta_1, \psi \qquad \psi, \Gamma_2 \hookrightarrow \Delta_2}{\Gamma_1, \Gamma_2 \hookrightarrow \Delta_1, \Delta_2}.$$

Definition 7 *A proof (or derivation) for a sequent calculus of a sequent S from a set of sequents U is a finite tree such that:*

- *S is the root of the tree and is called the end-sequent.*
- *The leaves of the tree are all initial sequents or members of U.*

- *Each child node of the tree is obtained from its parent nodes by an inference rule, i.e. if S is a child node of $S_1, \ldots, S_n$, then $\frac{S_1,\ldots,S_n}{S}$ is an instance of a rule.*

If we have a proof tree with the root S and $U = \emptyset$, then S is called a provable sequent. If we have a proof tree with the root S and $U \neq \emptyset$, then S is called a derivable sequent from premisses U.

It can be easily shown that for every formula ψ of $\mathcal{L}$, we have that $\vdash \psi$ if and only if $\hookrightarrow \psi$ is a provable sequent.

Further, we will consider the proof and the provability for various nonclassical sequent and hypersequent calculi just in the sense of definition 7.

As an example, using the above mentioned inference rules, show that the following proposition $(\psi \wedge (\varphi \vee \chi)) \rightarrow ((\psi \wedge \varphi) \vee (\psi \wedge \chi))$ of classical logic is provable:

$$1. \quad \frac{\psi \hookrightarrow \psi \qquad \varphi \hookrightarrow \varphi \qquad \chi \hookrightarrow \chi}{\psi, \varphi \hookrightarrow \psi \qquad \psi, \varphi \hookrightarrow \varphi \qquad \psi, \chi \hookrightarrow \psi \qquad \psi, \chi \hookrightarrow \chi},$$

$$2. \quad \frac{\psi, \varphi \hookrightarrow \psi \qquad \psi, \varphi \hookrightarrow \varphi \qquad \psi, \chi \hookrightarrow \psi \qquad \psi, \chi \hookrightarrow \chi}{\psi, \varphi \hookrightarrow \psi \wedge \varphi \qquad \psi, \chi \hookrightarrow \psi \wedge \chi},$$

$$3. \quad \frac{\psi, \varphi \hookrightarrow \psi \wedge \varphi \qquad \psi, \chi \hookrightarrow \psi \wedge \chi}{\psi, \varphi \hookrightarrow \psi \wedge \varphi, \psi \qquad \psi, \varphi \hookrightarrow \psi \wedge \varphi, \chi \qquad \psi, \chi \hookrightarrow \psi \wedge \chi, \psi \qquad \psi, \chi \hookrightarrow \psi \wedge \chi, \varphi},$$

$$4. \quad \frac{\psi, \varphi \hookrightarrow \psi \wedge \varphi, \psi \qquad \psi, \varphi \hookrightarrow \psi \wedge \varphi, \chi \qquad \psi, \chi \hookrightarrow \psi \wedge \chi, \psi \qquad \psi, \chi \hookrightarrow \psi \wedge \chi, \varphi}{\psi, \varphi \hookrightarrow \psi \wedge \varphi, \psi \wedge \chi \qquad \psi, \chi \hookrightarrow \psi \wedge \chi, \psi \wedge \varphi},$$

$$5. \quad \frac{\psi, \varphi \hookrightarrow \psi \wedge \varphi, \psi \wedge \chi \qquad \psi, \chi \hookrightarrow \psi \wedge \chi, \psi \wedge \varphi}{\psi, \varphi \vee \chi \hookrightarrow \psi \wedge \varphi, \psi \wedge \chi},$$

$$6. \quad \frac{\psi, \varphi \vee \chi \hookrightarrow \psi \wedge \varphi, \psi \wedge \chi}{\psi \wedge (\varphi \vee \chi) \hookrightarrow (\psi \wedge \varphi) \vee (\psi \wedge \chi)}.$$

$$7. \quad \frac{\psi \wedge (\varphi \vee \chi) \hookrightarrow (\psi \wedge \varphi) \vee (\psi \wedge \chi)}{\hookrightarrow (\psi \wedge (\varphi \vee \chi)) \rightarrow ((\psi \wedge \varphi) \vee (\psi \wedge \chi))}.$$

Chapter 3

n-valued Łukasiewicz's logics

3.1 Preliminaries

For the first time the Polish logician Jan Łukasiewicz began to create systems of many-valued logic, using a third value "possible" to deal with Aristotle's paradox of the sea battle (see [90], [144]). Now many-valued logic has applications in diverse fields. In the earlier years of development of multiple-validity idea, the most promising field of its application is artificial intelligence. This application concerns vague notions and commonsense reasoning, e.g. in expert systems. In this context fuzzy logic is also interesting, because multiple-validity is modelled in artificial intelligence via fuzzy sets and fuzzy logic.

Now consider n-valued Łukasiewicz's matrix logic $\mathfrak{M}_{\text{Ł}_n}$ defined as the ordered system $\langle V_n, \neg_L, \rightarrow_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{n-1\}\rangle$ for any $n \geqslant 2$, $n \in \mathbf{N}$, where

1. $V_n = \{0, 1, 2, \ldots, n-1\}$,
2. for all $x \in V_n$, $\neg_L x = (n-1) - x$,
3. for all $x, y \in V_n$, $x \rightarrow_L y = \min(n-1, (n-1) - x + y)$,
4. for all $x, y \in V_n$, $x \vee y = (x \rightarrow_L y) \rightarrow_L y = \max(x, y)$,
5. for all $x, y \in V_n$, $x \wedge y = \neg_L(\neg_L x \vee \neg_L y) = \min(x, y)$,
6. for a subset $M \subseteq V_n$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq V_n$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
8. $\{n-1\}$ is the set of designated truth values.

The truth value $0 \in V_n$ is false, the truth value $n-1 \in V_n$ is true, and other truth values $x \in V_n \backslash \{0, n-1\}$ are neutral.

By L_n denote the set of all superpositions of the functions $\neg_L, \rightarrow_L, \widetilde{\exists}, \widetilde{\forall}$.

We can construct various truth tables on the basis of n-valued Łukasiewicz's matrix logic.

1. The truth table for Łukasiewicz's negation in $\mathfrak{M}_{Ł_n}$:

p	$\neg_L p$
$n-1$	0
$n-2$	1
$n-3$	2
$\dots$	$\dots$
0	$n-1$

2. The truth table for Łukasiewicz's implication in $\mathfrak{M}_{Ł_n}$:

$\rightarrow_L$	$n-1$	$n-2$	$n-3$	$\dots$	0
$n-1$	$n-1$	$n-2$	$n-3$	$\dots$	0
$n-2$	$n-1$	$n-1$	$n-2$	$\dots$	1
$n-3$	$n-1$	$n-1$	$n-1$	$\dots$	2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
0	$n-1$	$n-1$	$n-1$	$\dots$	$n-1$

3. The truth table for the disjunction in $\mathfrak{M}_{Ł_n}$:

$\vee$	$n-1$	$n-2$	$n-3$	$\dots$	0
$n-1$	$n-1$	$n-1$	$n-1$	$\dots$	$n-1$
$n-2$	$n-1$	$n-2$	$n-2$	$\dots$	$n-2$
$n-3$	$n-1$	$n-2$	$n-3$	$\dots$	$n-3$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
0	$n-1$	$n-2$	$n-3$	$\dots$	0

4. The truth table for the conjunction in $\mathfrak{M}_{Ł_n}$:

$\wedge$	$n-1$	$n-2$	$n-3$	$\ldots$	0
$n-1$	$n-1$	$n-2$	$n-3$	$\ldots$	0
$n-2$	$n-2$	$n-2$	$n-3$	$\ldots$	0
$n-3$	$n-3$	$n-3$	$n-3$	$\ldots$	0
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
0	0	0	0	$\ldots$	0

We can define different n-valued matrix logics according to the chose of different logical operations as initial ones. As an example, we considered n-valued Łukasiewicz's matrix logic in which $\neg_L$, $\rightarrow_L$, $\vee$, $\wedge$ are basic operations.

Also, an n-valued logic $\mathfrak{M}$ is given by a set of truth values $V(\mathfrak{M}) = \{0, 1, 2, \ldots, n-1\}$, the set of designated truth values $V_+(\mathfrak{M})$, and a set of truth functions $\widetilde{\Box}_i$: $V(\mathfrak{M})^i \rightarrow V(\mathfrak{M})$ for all connectives $\Box_i$.

We denote the set of tautologies of the matrix logic $\mathfrak{M}$ by $\mathrm{Taut}(\mathfrak{M})$. We say that an n-valued logic $\mathfrak{M}_1$ is *better* than $\mathfrak{M}_2$ ($\mathfrak{M}_1 \lhd \mathfrak{M}_2$) iff $\mathrm{Taut}(\mathfrak{M}_1) \subset \mathrm{Taut}(\mathfrak{M}_2)$.

It is obvious that all n-valued propositional logics for a language $\mathcal{L}$ can be enumerated mechanically:

Proposition 1 *Assume that we have r propositional connectives $\widetilde{\Box}_j^{m_j}$ of arity m_j ($1 \leqslant j \leqslant r$). There are $\prod_{j=1}^{r} n^{n^{m_j}}$ many n-valued logics.*

Proof. The number of different truth functions $V^{m_j} \rightarrow V$ equals

$$\left|V^{V^{m_j}}\right| = n^{n^{m_j}}. \qquad \Box$$

Proposition 2 *For any truth function $\widetilde{\Box}^i$: $V(\mathfrak{M})^i \rightarrow V(\mathfrak{M})$ and for any subset of truth values $W \subseteq V$ there exists disjunctive clauses $\Phi_j(x_1, \ldots, x_i)$ of the form $(x_1 \in R_{j,1} \vee \ldots \vee x_i \in R_{j,i})$ where $R_{j,k}$ are subsets of V, with $1 \leqslant j \leqslant n^{i-1}$ and $1 \leqslant k \leqslant i$, such that for all $x_1, \ldots, x_i \in V$:*

$$\widetilde{\Box}^i(x_1, \ldots, x_i) \in W \leftrightarrow \bigwedge_{j=1}^{n^{i-1}} \Phi_j(x_1, \ldots, x_i).$$

Proof. See [16]. $\Box$

This proposition allows any truth function to be regarded as a conjunctive normal form.

3.2 Originality of $(p+1)$-valued Łukasiewicz's logics

The algebraic system $\mathfrak{M}_{P_{n+1}} = \langle V_{n+1}, \neg_P, \vee, \{n\}\rangle$, where

1. $V_{n+1} = \{0, 1, 2, \ldots, n\}$,
2. $\neg_P x = x + 1 \mod (n+1)$,
3. $x \vee y = \max(x, y)$,
4. $\{n\}$ is the set of designated truth values (verum),

is said to be the $(n+1)$-*valued Postian matrix* (it is proposed by Post in [105]).

Let P_{n+1} be the set of all functions of the $(n+1)$-ary Postian matrix logic. We say that the system F of functions is *precomplete* in P_{n+1} if F isn't a complete set, but the adding to F any function f such that $f \in P_{n+1}$ and $f \notin F$ converts F into a complete set. As an example, take the set of all functions in P_{n+1} preserving 0 and n. Denote this set by T_{n+1}. By assumption, $f(x_1, \ldots, x_m) \in T_{n+1}$ iff $f(x_1, \ldots, x_m) \in \{0, n\}$, where $x_i \in \{0, n\}$, $1 \leqslant i \leqslant m$. The class T_{n+1} of functions is *precomplete*.

It is known that we can specify each $(p+1)$-valued Łukasiewicz matrix logic for any prime number p (see [77]):

Theorem 2 *$L_{n+1} = T_{n+1}$ iff n (for any $n \geqslant 2$) is a prime number.* □

This means that the set of logical functions in the logic $\mathfrak{M}_{Ł_{p+1}}$, where p is a prime number, forms a precomplete set.

Corollary 4. 1 *Suppose there exists the infinite sequence of (p_s+1)-valued Łukasiewicz's matrix logics $\mathfrak{M}_{Ł_{p_s+1}}$ (p_s is s-th prime number). Then for each precomplete sets T_{p_s+1} of functions we have that $L_{p_s+1} = T_{p_s+1}$ for all $s = 1, 2, \ldots$* □

Take the sequence of finite-valued Łukasiewicz matrix logics

$$\mathfrak{M}_{Ł_{2+1}}, \mathfrak{M}_{Ł_{3+1}}, \mathfrak{M}_{Ł_{5+1}}, \mathfrak{M}_{Ł_{7+1}}, \ldots$$

We can show that it is sufficient if we consider just this sequence instead of the sequence of all finite-valued Łukasiewicz's logics $\mathfrak{M}_{Ł_{2+1}}$, $\mathfrak{M}_{Ł_{3+1}}$, $\mathfrak{M}_{Ł_{4+1}}$, $\mathfrak{M}_{Ł_{5+1}}$, $\mathfrak{M}_{Ł_{6+1}}$, ... (it was considered by Karpenko in [76], [77]).

Indeed, let $\varphi(n)$ be Euler's function, i.e., the function defined for all positive integers n and equal to a number k of integers such that $k \leqslant n$ and k is relatively prime to n. Now assume that $\varphi^*(n) = \varphi(n) + 1$. It is necessary to notice that if $n = p$, then $\varphi^*(n) = (p-1) + 1 = p$. Therefore we have the following algorithm, by which any natural number n is assigned to a prime number p and hence we assign any logic $\mathfrak{M}_{\text{Ł}_{n+1}}$ to a logic $\mathfrak{M}_{\text{Ł}_{p+1}}$:

0. Let $n = n_1$ and $n_1 \neq p_i$.

1. Either $\varphi_1^*(n_1) = p_i$ or $\varphi_1^*(n_1) = n_2$, where $n_2 < n_1$.

2. Either $\varphi_2^*(n_2) = p_i$ or $\varphi_2^*(n_2) = n_3$, where $n_3 < n_2$.

$$\vdots$$

k. $\varphi_k^*(n_k) = p_i$, i.e., by $\varphi_k^*(n)$ denote k-th application of the function $\varphi^*(n)$.

Since there exists the above mentioned algorithm, it follows that the function $\varphi_k^*(n_k)$ induces a partition from sets L_{n+1} into their equivalence classes:

$L_{n_1+1} \cong L_{n_2+1}$ iff there exist k and l such that $\varphi_k^*(n_1) = \varphi_l^*(n_2)$.

By $\mathcal{X}_{p_s+1}$ denote equivalence classes. Every class contains a unique precomplete set L_{p+1}. Using inverse EULER's function $\varphi^{-1}(n)$, it is possible to set the algorithm that takes each precomplete set L_{p+1} to an equivalence class $\mathcal{X}_{p+1}$.

Further, let us consider the inverse function $\varphi^{*-1}(m)$. We may assume that $m = p$, where p is a prime number.

0. We subtract 1 from p, i.e., we have $p - 1$.

1. We set the range of values for $\varphi^{-1}(p-1)$. By assumption, this family consists of two classes $\{\nu_o\}_1$ and $\{\nu_e\}_1$, where $\{\nu_o\}_1$ is the class of odd values, p isn't contained in $\{\nu_o\}_1$, and $\{\nu_e\}_1$ is the class of even values. By construction, we disregard the class $\{\nu_e\}_i$ for any i, because $\nu_e - 1$ is the odd number and consequently cannot be a value of EULER's function $\varphi(n)$. If the class $\{\nu_o\}_1$ is empty (for example, in the case $\varphi^{*-1}(3)$ or $\varphi^{*-1}(5)$), then we get the equivalence class $\mathcal{X}_{p+1}$. Conversely, if $\{\nu_o\}_1$ isn't empty, then we obtain the range of values $\varphi^{-1}(\nu_o - 1)$ for any ν_o in the class $\{\nu_o\}_1$. Here we have two subcases.

2. (a) Either $\{\nu_o\}_2 = \emptyset$ or (b) $\{\nu_o\}_2 \neq \emptyset$. In the first case the process of construction $\mathcal{X}_{p+1}$ is finished. If $\{\nu_o\}_2 \neq \emptyset$, then all is repeated. Here we have also two subcases.

3. (a) Either $\{\nu_o\}_3 = \emptyset$ or (b) $\{\nu_o\}_3 \neq \emptyset$, etc.

$$\vdots$$

Thus, the sequence of finite-valued ŁUKASIEWICZ's matrix logics $\mathfrak{M}_{Ł_{2+1}}$, $\mathfrak{M}_{Ł_{3+1}}$, $\mathfrak{M}_{Ł_{5+1}}$, $\mathfrak{M}_{Ł_{7+1}}$, ... corresponds to the sequence of equivalence classes $\mathcal{X}_{2+1}$, $\mathcal{X}_{3+1}$, $\mathcal{X}_{5+1}$, $\mathcal{X}_{7+1}$, ... of all finite-valued ŁUKASIEWICZ's matrix logics $\mathfrak{M}_{Ł_{2+1}}$, $\mathfrak{M}_{Ł_{3+1}}$, $\mathfrak{M}_{Ł_{4+1}}$, $\mathfrak{M}_{Ł_{5+1}}$, ...

3.3 n-valued Łukasiewicz's calculi of Hilbert's type

Consider the n-ary ŁUKASIEWICZ propositional calculus $Ł_n$ of HILBERT's type. The axioms of this calculus are as follows:

$$(p \to_L q) \to_L ((q \to_L r) \to_L (p \to_L r)), \tag{3.1}$$

$$p \to_L (q \to_L p), \tag{3.2}$$

$$((p \to_L q) \to_L q) \to_L ((q \to_L p) \to_L p), \tag{3.3}$$

$$(p \to_L^n q) \to_L (p \to_L^{n-1} q) \tag{3.4}$$

for any $n \geqslant 1$; notice that $p \to_L^0 q = q$ and $p \to_L^{k+1} q = p \to_L (p \to_L^k q)$,

$$(p \wedge q) \to_L p, \tag{3.5}$$

$$(p \wedge q) \to_L q, \tag{3.6}$$

$$(p \to_L q) \to_L ((p \to_L r) \to_L (p \to_L (q \wedge r))), \tag{3.7}$$

$$p \to_L (p \vee q), \tag{3.8}$$

$$q \to_L (p \vee q), \tag{3.9}$$

$$((p \to_L r) \to_L ((q \to_L r) \to_L (p \vee q) \to_L r)), \tag{3.10}$$

$$(\neg_L p \to_L \neg_L q) \to_L (q \to_L p), \tag{3.11}$$

$$(p \leftrightarrow (p \to_L^{s-1} \neg_L p)) \to_L^{n-1} p \tag{3.12}$$

for any $1 \leqslant s \leqslant n-1$ such that $n-1$ doesn't divide by s and we have $p \leftrightarrow q = (p \to_L q) \wedge (q \to_L p)$.

There are two inference rules in the system $Ł_n$: modus ponens and substitution rule. This formalization of $Ł_n$ was created by TUZIAK in [146].

Let us show by means of the truth table that the formula $(p \to_L^2 q) \to_L (p \to_L q)$ isn't tautology in 3-valued ŁUKASIEWICZ's logic.

p	q	$p \to_L q$	$p \to_L (p \to_L q)$	$(p \to_L (p \to_L q)) \to_L (p \to_L q)$
2	2	2	2	2
2	1	1	1	2
2	0	0	0	2
1	2	2	2	2
1	1	2	2	2
1	0	1	2	1
0	2	2	2	2
0	1	2	2	2
0	0	2	2	2

But the formula $(p \to_L^3 q) \to_L (p \to_L^2 q)$ is a tautology:

p	q	$p \to_L q$	$p \to_L (p \to_L q)$	$p \to_L (p \to_L (p \to_L q))$	$(p \to_L^3 q) \to_L (p \to_L^2 q)$
2	2	2	2	2	2
2	1	1	1	1	2
2	0	0	0	0	2
1	2	2	2	2	2
1	1	2	2	2	2
1	0	1	2	2	2
0	2	2	2	2	2
0	1	2	2	2	2
0	0	2	2	2	2

We already know that the formula $(p \to_L^k q) \to_L (p \to_L^{k-1} q)$ is tautology for any $k \geqslant n$ in n-valued Łukasiewicz's logic. Thus, some tautologies of the classical logic are ignored in finite-valued Łukasiewicz's logic as well as the other tautologies of the classical logic are ignored in the other nonclassical logics.

3.4 Sequent calculi for n-valued Łukasiewicz's logics

Sequent calculi and natural deduction systems for n-valued logic Ł_n, where n is a finite natural number, are considered by Baaz, Fermüller, and Zach in [14], [15], [16]. An n-valued sequent is regarded as an n-tuple of finite sets Γ_i $(1 \leq i \leq n)$ of formulas, denoted by $\Gamma_1 \mid \Gamma_2 \mid \ldots \mid \Gamma_n$. It is defined to be satisfied by an interpretation iff for some $i \in \{1, \ldots, n\}$ at least one formula in Γ_i takes the truth value $i - 1 \in \{0, \ldots, n-1\}$, where $\{0, \ldots, n-1\}$ is the set of truth values for Ł_n.

By this approach, a two-valued sequent with Gentzen's standard notation $\Gamma_1 \hookrightarrow \Gamma_2$, where Γ_1 and Γ_2 are finite sequences of formulas, is interpreted truth-functionally: either one of the formulas in Γ_1 is false or one of the formulas in

Γ_2 is true. In other words, we can denote a sequent $\Gamma_1 \hookrightarrow \Gamma_2$ by $\Gamma_1 \mid \Gamma_2$ and we can define it to be satisfied by an interpretation iff for some $i \in \{1, 2\}$ at least one formula in Γ_i takes the truth value $i - 1 \in \{0, 1\}$.

Let I be an interpretation. The I $\mathfrak{p}$-*satisfies* ($\mathfrak{n}$-*satisfies*) a sequent $\Gamma_1 \mid \Gamma_2 \mid \ldots \mid \Gamma_n$ iff there is an i ($1 \leqslant i \leqslant n$) such that, for some formula $\varphi \in \Gamma_i$, $\mathrm{val}_I(\varphi) = i-1 \in V_n$ ($\mathrm{val}_I(\varphi) \neq i-1 \in V_n$). This I is called a $\mathfrak{p}$-model ($\mathfrak{n}$-model) of Γ, i.e. $I \models^p \Gamma$ ($I \models^n \Gamma$). A sequent is called $\mathfrak{p}$- ($\mathfrak{n}$)-*valid*, if it is $\mathfrak{p}$- ($\mathfrak{n}$)-satisfied by every interpretation.

Also, a sequent Γ is called $\mathfrak{p}$-satisfiable ($\mathfrak{n}$-satisfiable) iff there is an interpretation I such that $I \models^p \Gamma$ ($I \models^n \Gamma$), and $\mathfrak{p}$-valid ($\mathfrak{n}$-valid) iff for every interpretation I, $I \models^p \Gamma$ ($I \models^n \Gamma$). The concept of $\mathfrak{p}$-satisfiability was proposed by Rousseau in [119].

Notice that according to $\mathfrak{p}$-satisfiability a sequent is understood as a positive disjunction and according to $\mathfrak{n}$-satisfiability as a negative disjunction. Therefore the negation of a $\mathfrak{p}$-sequent ($\mathfrak{n}$-sequent) is equivalent to a conjunction of $\mathfrak{n}$-sequents ($\mathfrak{p}$-sequents).

By $[i : \psi]$ denote a sequent in that a formula ψ occurs at place $i + 1$.

Consider the sequent containing only one formula $i : \psi$ with the truth value i. Then we obtain the following result:

Proposition 3 *A sequent $[i : \psi]$ is $\mathfrak{p}$-unsatisfiable ($\mathfrak{n}$-unsatisfiable) iff it is $\mathfrak{n}$-valid ($\mathfrak{p}$-valid).*

Proof. The negation of the $\mathfrak{p}$-sequent ψ^i is the $\mathfrak{n}$-sequent $\neg\psi^i$.

On the other hand, the n-sequent $\neg\psi^i$ can also be written as a $\mathfrak{p}$-sequent $\bigvee_{j \neq i} \psi^j$, and hence the $\mathfrak{p}$-unsatisfiability of $[i : \psi]$ can be established by proving $[V \backslash \{i\} : \psi]$ $\mathfrak{p}$-valid. □

Proposition 4 *Let ψ be a formula. Then the following are equivalent:*

- *ψ is a tautology.*
- *The sequent $[V_+ : \psi]$ is $\mathfrak{p}$-valid*
- *The sequents $[j : \psi]$, where $j \in V \backslash V_+$, are all $\mathfrak{n}$-valid.* □

Proposition 5 *Let ψ be a formula. Then the following are equivalent:*

- *ψ is a unsatisfiable.*
- *The sequent $[V \backslash V_+ : \psi]$ is $\mathfrak{p}$-valid.*
- *The sequents $[j : \psi]$, where $j \in V_+$, are all $\mathfrak{n}$-valid.* □

Definition 8 *An introduction rule for a connective $\Box$ at place i in the n-valued* Łukasiewicz*'s logic $Ł_n$ is a schema of the form:*

$$\frac{\langle \Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_n^j, \Delta_n^j \rangle_{j \in N}}{\Gamma_1 \mid \ldots \mid \Gamma_i, \Box(\psi_1, \ldots, \psi_m) \mid \ldots \mid \Gamma_n} \Box : i$$

where the arity of $\Box$ is m, N is a finite set, $\Gamma_l = \bigcup_{j \in N} \Gamma_l^j$, $\Delta_l^j \subseteq \{\psi_1, \ldots, \psi_m\}$, and for every interpretation I the following are equivalent:

1. *$\Box(\psi_1, \ldots, \psi_n)$ takes (resp. does not take) the truth value $i - 1$.*
2. *For all $j \in N$, an interpretation I $\mathfrak{p}$- (resp. $\mathfrak{n}$)-satisfies the sequents $\Delta_1^j \mid \ldots \mid \Delta_n^j$.*

Note that these introduction rules are generated in a mechanical way from the truth table $\widetilde{\Box}^i$: $V(\mathfrak{M}_{Ł_n})^i \to V(\mathfrak{M}_{Ł_n})$ of the connective $\Box^i$ through conjunctive normal forms (see proposition 2).

Definition 9 *An introduction rule for a quantifier Q at place i in the n-valued* Łukasiewicz*'s logic $Ł_n$ is a schema of the form:*

$$\frac{\langle \Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_n^j, \Delta_n^j \rangle_{j \in N}}{\Gamma_1 \mid \ldots \mid \Gamma_i, \mathrm{Q}x\ \psi(x) \mid \ldots \mid \Gamma_n} \mathrm{Q} : i$$

where

- *N is a finite set,*
- *$\Gamma_l = \bigcup_{j \in N} \Gamma_l^j$, $\Delta_l^j \subseteq \{\psi(a_1), \ldots, \psi(a_p)\} \cup \{\psi(t_1), \ldots, \psi(t_q)\}$,*
- *the a_l are free variables satisfying the condition that they do not occur in the lower sequent,*
- *the t_k are arbitrary terms,*

and for every interpretation I the following are equivalent:

1. *$\mathrm{Q}x\ \psi(x)$ takes (does not take) the truth value $i - 1$ under I.*
2. *For all $d_1, \ldots, d_p \in D$, there are $e_1, \ldots, e_q \in D$ such that for all $j \in N$, an interpretation I $\mathfrak{p}$- ($\mathfrak{n}$)-satisfies $\Delta_1'^j \mid \ldots \mid \Delta_n'^j$ where $\Delta_l'^j$ is obtained from Δ_l^j by instantiating the eigenvariable a_k (term variable t_k) with d_k (e_k).*

A $\mathfrak{p}$-*sequent calculus* for a logic $Ł_n$ is given by:

- Axioms of the form: $\varphi \mid \varphi \mid \ldots \mid \varphi$, where φ is any formula.
- For every connective $\Box$ of the logic $Ł_n$ and every truth value $i - 1$ an appropriate introduction rule $\Box : i$.

- For every quantifier Q and every truth value $i-1$ an appropriate introduction rule Q : i.
- Weakening rules for every place i:

$$\frac{\Gamma_1 \mid \ldots \mid \Gamma_i \mid \ldots \mid \Gamma_n}{\Gamma_1 \mid \ldots \mid \Gamma_i, \psi \mid \ldots \mid \Gamma_n} w : i$$

- Cut rules for every pair of truth values $(i-1, j-1)$ such that $i \neq j$:

$$\frac{\Gamma_1 \mid \ldots \mid \Gamma_i, \psi \mid \ldots \mid \Gamma_n \quad \Delta_1 \mid \ldots \mid \Delta_j, \psi \mid \ldots \mid \Delta_n}{\Gamma_1, \Delta_1 \mid \ldots \mid \Gamma_n, \Delta_n} cut : ij$$

An $\mathfrak{n}$-*sequent calculus* for a logic $Ł_n$ is given by:

- Axioms of the form: $\Delta_1 \mid \ldots \mid \Delta_n$, where $\Delta_i = \Delta_j = \{\psi\}$ for some i, j such that $i \neq j$ and $\Delta_k = \emptyset$ otherwise (ψ is any formula).
- For every connective $\Box$ and every truth value $i-1$ an appropriate introduction rule $\Box : i$.
- For every quantifier Q and every truth value $i-1$ an appropriate introduction rule Q : i.
- Weakening rules identical to the ones of $\mathfrak{p}$-sequent calculi.
- The cut rule:

$$\frac{\langle \Gamma_1^j \mid \ldots \mid \Gamma_i^j \mid \ldots \Gamma_n^j, \Delta_n^j \rangle_{j=1}^n}{\Gamma_1 \mid \ldots \mid \Gamma_n} cut :$$

  where $\Gamma_l = \bigcup_{1 \leqslant j \leqslant n} \Gamma_l^j$.

A sequent is $\mathfrak{p}$- *($\mathfrak{n}$)-provable* if there is an upward tree of sequents such that every topmost sequent is an axiom and every other sequent is obtained from the ones standing immediately above it by an application of one of the rules of $\mathfrak{p}$-($\mathfrak{n}$)-sequent calculus.

Theorem 3 (Soundness and Completeness) *For every $\mathfrak{p}$- ($\mathfrak{n}$)-sequent calculus the following holds: A sequent is $\mathfrak{p}$- ($\mathfrak{n}$)-provable without cut rule(s) iff it is $\mathfrak{p}$- ($\mathfrak{n}$)-valid.*

Proof. See [16].

Example: 3-valued Łukasiewicz's propositional logic. Let us consider here the 3-valued ŁUKASIEWICZ logic with the set of basic connectives $\{\rightarrow_L, \neg_L\}$. Recall that the 3 values are denoted by $\{0, 1, 2\}$.

1. Introduction rules for $\neg_L$ in $\mathfrak{p}$-sequent calculus:

$$\frac{\Gamma_1 \mid \Gamma_2 \mid \Gamma_3, \psi}{\Gamma_1, \neg_L\psi \mid \Gamma_2 \mid \Gamma_3}\neg_L : 0 \qquad \frac{\Gamma_1 \mid \Gamma_2, \psi \mid \Gamma_3}{\Gamma_1 \mid \Gamma_2, \neg_L\psi \mid \Gamma_3}\neg_L : 1$$

$$\frac{\Gamma_1, \psi \mid \Gamma_2 \mid \Gamma_3}{\Gamma_1 \mid \Gamma_2 \mid \Gamma_3, \neg_L\psi}\neg_L : 2$$

2. Introduction rules for $\rightarrow_L$ in $\mathfrak{p}$-sequent calculus:

$$\frac{\Gamma_1 \mid \Gamma_2 \mid \Gamma_3, \psi \quad \Gamma_1, \varphi \mid \Gamma_2 \mid \Gamma_3}{\Gamma_1, \psi \rightarrow_L \varphi \mid \Gamma_2 \mid \Gamma_3} \rightarrow_L : 0$$

$$\frac{\Gamma_1 \mid \Gamma_2, \psi \mid \Gamma_3, \psi \quad \Gamma_1 \mid \Gamma_2, \psi, \varphi \mid \Gamma_3 \quad \Gamma_1, \varphi \mid \Gamma_2 \mid \Gamma_3, \psi}{\Gamma_1 \mid \Gamma_2, \psi \rightarrow_L \varphi \mid \Gamma_3} \rightarrow_L : 1$$

$$\frac{\Gamma_1, \psi \mid \Gamma_2, \varphi \mid \Gamma_3, \varphi \quad \Gamma_1, \psi \mid \Gamma_2, \psi \mid \Gamma_3, \varphi}{\Gamma_1 \mid \Gamma_2 \mid \Gamma_3, \psi \rightarrow_L \varphi} \rightarrow_L : 2$$

3.5 Hypersequent calculus for 3-valued Łukasiewicz's propositional logic

The hypersequent formalization of 3-valued Łukasiewicz's propositional logic Ł$_3$ was proposed by Avron in [5]. Let us remember what is a hypersequent.

Definition 10 *A hypersequent is a structure of the form:*

$$\Gamma \hookrightarrow \Delta \mid \Gamma' \hookrightarrow \Delta' \mid \ldots \mid \Gamma'' \hookrightarrow \Delta'',$$

where $\Gamma \hookrightarrow \Delta$, $\Gamma' \hookrightarrow \Delta'$,..., $\Gamma'' \hookrightarrow \Delta''$ *are finite sequences of ordinary sequents in* Gentzen*'s sense.*

We shall use G and H as variables for possibly empty hypersequents.

The standard interpretation of the | symbol is usually disjunctive, i.e. a hypersequent is true if and only if one of its components is true.

The only *axiom* of this calculus: $\psi \hookrightarrow \psi$.

The *inference rules* are as follows.

$$\frac{G|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta|\Gamma' \hookrightarrow \Delta'}, \qquad \frac{G|\Gamma \hookrightarrow \Delta|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta},$$

$$\frac{G|\Gamma' \hookrightarrow \Delta'|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta|\Gamma' \hookrightarrow \Delta'},$$

$$\frac{G|\Gamma \hookrightarrow \Delta|H}{G|\psi, \Gamma \hookrightarrow \Delta|H}, \qquad \frac{G|\Gamma \hookrightarrow \Delta|H}{G|\Gamma \hookrightarrow \Delta, \psi|H},$$

$$\frac{G|\psi, \varphi, \Gamma \hookrightarrow \Delta|H}{G|\varphi, \psi, \Gamma \hookrightarrow \Delta|H}, \qquad \frac{G|\Gamma \hookrightarrow \Delta, \psi, \varphi|H}{G|\Gamma \hookrightarrow \Delta, \varphi, \psi|H},$$

$$\frac{G|\psi, \varphi, \Gamma \hookrightarrow \Delta|H}{G|\psi \wedge \varphi, \Gamma \hookrightarrow \Delta|H}, \qquad \frac{G|\Gamma \hookrightarrow \Delta, \psi, \varphi|H}{G|\Gamma \hookrightarrow \Delta, \psi \vee \varphi|H},$$

$$\frac{G|\Gamma \hookrightarrow \Delta, \psi|H \qquad G|\Gamma \hookrightarrow \Delta, \varphi|H}{G|\Gamma \hookrightarrow \Delta, \psi \wedge \varphi|H},$$

$$\frac{G|\psi, \Gamma \hookrightarrow \Delta|H \qquad G|\varphi, \Gamma \hookrightarrow \Delta|H}{G|\psi \vee \varphi, \Gamma \hookrightarrow \Delta|H},$$

$$\frac{G|\psi, \Gamma \hookrightarrow \Delta, \varphi}{G|\Gamma \hookrightarrow \Delta, \psi \to_L \varphi}, \qquad \frac{G|\Gamma \hookrightarrow \Delta, \psi \quad G|\varphi, \Gamma \hookrightarrow \Delta}{G|\psi \to_L \varphi, \Gamma \hookrightarrow \Delta},$$

$$\frac{G|\psi, \Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta, \neg_L \psi}, \qquad \frac{G|\Gamma \hookrightarrow \Delta, \psi}{G|\neg_L \psi, \Gamma \hookrightarrow \Delta},$$

$$\frac{G|\Gamma_1, \Gamma_2, \Gamma_3 \hookrightarrow \Delta_1, \Delta_2, \Delta_3|H \qquad G|\Gamma_1', \Gamma_2', \Gamma_3' \hookrightarrow \Delta_1', \Delta_2', \Delta_3'|H}{G|\Gamma_1, \Gamma_1' \hookrightarrow \Delta_1, \Delta_1'|\Gamma_2, \Gamma_2' \hookrightarrow \Delta_2, \Delta_2'|\Gamma_3, \Gamma_3' \hookrightarrow \Delta_3, \Delta_3'|H}.$$

Chapter 4

Infinite valued Łukasiewicz's logics

4.1 Preliminaries

The ordered system $\langle V_{\mathbf{Q}}, \neg_L, \rightarrow_L, \&_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{1\}\rangle$ is called *rational valued* Łukasiewicz*'s matrix logic* $\mathfrak{M}_{\mathbf{Q}}$, where

1. $V_{\mathbf{Q}} = \{x \colon x \in \mathbf{Q}\} \cap [0, 1]$,
2. for all $x \in V_{\mathbf{Q}}$, $\neg_L x = 1 - x$,
3. for all $x, y \in V_{\mathbf{Q}}$, $x \rightarrow_L y = \min(1, 1 - x + y)$,
4. for all $x, y \in V_{\mathbf{Q}}$, $x \&_L y = \neg_L(x \rightarrow_L \neg_L y)$,
5. for all $x, y \in V_{\mathbf{Q}}$, $x \vee y = (x \rightarrow_L y) \rightarrow_L y = \max(x, y)$,
6. for all $x, y \in V_{\mathbf{Q}}$, $x \wedge y = \neg_L(\neg_L x \vee \neg_L y) = \min(x, y)$,
7. for a subset $M \subseteq V_{\mathbf{Q}}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
8. for a subset $M \subseteq V_{\mathbf{Q}}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
9. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{\mathbf{Q}}$ is false, the truth value $1 \in V_{\mathbf{Q}}$ is true, and other truth values $x \in V_{\mathbf{Q}} \backslash \{0, 1\}$ are neutral.

Real valued Łukasiewicz*'s matrix logic* $\mathfrak{M}_{\mathbf{R}}$ is the ordered system $\langle V_{\mathbf{R}}, \neg_L, \rightarrow_L, \&_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{1\}\rangle$, where

1. $V_{\mathbf{R}} = \{x \colon x \in \mathbf{R}\} \cap [0, 1]$,

2. for all $x \in V_{\mathbf{R}}$, $\neg_L x = 1 - x$,

3. for all $x, y \in V_{\mathbf{R}}$, $x \to_L y = \min(1, 1 - x + y)$,

4. for all $x, y \in V_{\mathbf{R}}$, $x \&_L y = \neg_L(x \to_L \neg_L y)$,

5. for all $x, y \in V_{\mathbf{R}}$, $x \vee y = (x \to_L y) \to_L y = \max(x, y)$,

6. for all $x, y \in V_{\mathbf{R}}$, $x \wedge y = \neg_L(\neg_L x \vee \neg_L y) = \min(x, y)$,

7. for a subset $M \subseteq V_{\mathbf{R}}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,

8. for a subset $M \subseteq V_{\mathbf{R}}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,

9. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{\mathbf{R}}$ is false, the truth value $1 \in V_{\mathbf{R}}$ is true, and other truth values $x \in V_{\mathbf{R}} \backslash \{0, 1\}$ are neutral.

The logics $\mathfrak{M}_{\mathbf{Q}}$ and $\mathfrak{M}_{\mathbf{R}}$ will be denoted by $\mathfrak{M}_{[0,1]}$. They are called *infinite valued* Łukasiewicz*'s matrix logic*.

4.2 Hilbert's type calculus for infinite valued Łukasiewicz's logic

Łukasiewicz's infinite valued logic is denoted by Ł$_\infty$. The basic operations of Ł$_\infty$ are $\perp$ (truth constant 'falsehood') and Łukasiewicz's implications $\to_L$. Other connectives are derivable:

$$\neg_L \psi =: \psi \to_L \perp,$$
$$\psi \&_L \varphi =: \neg_L(\psi \to_L \neg_L \varphi),$$
$$\psi \wedge \varphi =: \psi \&_L (\psi \to_L \varphi),$$
$$\psi \vee \varphi =: (\psi \to_L \varphi) \to_L \varphi,$$
$$\top =: \neg_L \perp.$$

The Hilbert's type calculus for Ł$_\infty$ consists of the axioms:

$$\psi \to_L (\varphi \to_L \psi), \tag{4.1}$$
$$(\psi \to_L \varphi) \to_L ((\varphi \to_L \chi) \to_L (\psi \to_L \chi)), \tag{4.2}$$
$$((\psi \to_L \varphi) \to_L \varphi) \to_L ((\varphi \to_L \psi) \to_L \psi), \tag{4.3}$$
$$(\neg_L \psi \to_L \neg_L \varphi) \to_L (\varphi \to_L \psi), \tag{4.4}$$
$$\forall x\ \varphi(x) \to_L \varphi[x/t], \tag{4.5}$$

$$\varphi[x/t] \to_L \exists x\ \varphi(x), \tag{4.6}$$

where the formula $\varphi[x/t]$ is the result of substituting the term t for all free occurrences of x in φ,

$$\forall x(\chi \to_L \varphi) \to_L (\chi \to_L \forall x\varphi), \tag{4.7}$$

$$\forall x(\varphi \to_L \chi) \to_L (\exists x\varphi \to_L \chi), \tag{4.8}$$

$$\forall x(\chi \vee \varphi) \to_L (\chi \vee \forall x\varphi), \tag{4.9}$$

where x is not free in χ.

In Ł$_\infty$ there are the following inference rules:

1. *Modus ponens*: from φ and $\varphi \to_L \psi$ infer ψ:
$$\frac{\varphi, \quad \varphi \to_L \psi}{\psi}.$$
2. *Substitution rule*: we can substitute any formulas for propositional variables.
3. *Generalization*: from φ infer $\forall x\varphi(x)$:
$$\frac{\varphi}{\forall x\ \varphi(x)}.$$

The Hilbert's type propositional calculus for Ł$_\infty$ can be obtained by extending the axiom system (10.1) – (10.8) by the expression (4.3).

4.3 Sequent calculus for infinite valued Łukasiewicz's propositional logic

An original interpretation of a sequent for infinite valued Łukasiewicz's logic Ł$_\infty$ was proposed by Metcalfe, Olivetti, and Gabbay in [96]. For setting this interpretation they used the following proposition:

Proposition 6 *Let* $\mathfrak{M}_{[-1,0]}$ *be the structure* $\langle [-1,0], \max, \min, \&_L, \to_L, 0\rangle$ *where*

- $\&_L =: \max(-1, x+y)$ *and*
- $x \to_L y =: \min(0, y-x)$,

- 0 *is the designated truth value,*

then ψ *is logically valid in* $\mathfrak{M}_{[-1,0]}$ *iff* ψ *is logically valid in* ŁUKASIEWICZ*'s matrix logic* $\mathfrak{M}_{[0,1]}$.

Proof. See [96]. □

Definition 11 *A sequent written* $\Gamma_1 \hookrightarrow \Gamma_2$*, where* $\Gamma_1 = \{\varphi_1, \ldots, \varphi_m\}$ *and* $\Gamma_2 = \{\psi_1, \ldots, \psi_n\}$*, has the following interpretation:* $\Gamma_1 \hookrightarrow \Gamma_2$ *is logically valid in* L_∞ *iff* $(\psi_1 + \ldots + \psi_n) \rightarrow_L (\varphi_1 + \ldots + \varphi_m)$ *is logically valid in* L_∞ *for all valuations* val_I *mapped from formulas of* $\mathcal{L}$ *to the structure* $\mathfrak{M}_{[-1,0]}$*, where* $\chi_1 + \ldots + \chi_k = \top$ *if* $k = 0$.

1. Axioms of the sequent calculus:

$$(ID) \qquad \psi \hookrightarrow \psi, \qquad\qquad (\Lambda) \qquad \hookrightarrow \ , \qquad (\bot) \qquad \bot \hookrightarrow \psi.$$

2. Structural rules:

$$\frac{\Gamma \hookrightarrow \Delta}{\Gamma, \psi \hookrightarrow \Delta}, \qquad\qquad \frac{\Gamma_1 \hookrightarrow \Delta_1 \quad \Gamma_2 \hookrightarrow \Delta_2}{\Gamma_1, \Gamma_2 \hookrightarrow \Delta_1, \Delta_2},$$

$$\frac{\overbrace{\Gamma, \Gamma, \ldots, \Gamma}^{n} \hookrightarrow \overbrace{\Delta, \Delta, \ldots, \Delta}^{n}}{\Gamma \hookrightarrow \Delta}, n > 0.$$

3. Logical rules:

$$\frac{\Gamma, \varphi, \varphi \rightarrow_L \psi \hookrightarrow \Delta, \psi}{\Gamma, \psi \rightarrow_L \varphi \hookrightarrow \Delta}, \qquad\qquad \frac{\Gamma \hookrightarrow \Delta \quad \Gamma, \psi \hookrightarrow \varphi, \Delta}{\Gamma \hookrightarrow \psi \rightarrow_L \varphi, \Delta},$$

$$\frac{\Gamma, \psi, \varphi \hookrightarrow \Delta \qquad \Gamma, \bot \hookrightarrow \Delta}{\Gamma, \psi \&_L \varphi \hookrightarrow \Delta}, \qquad\qquad \frac{\Gamma, (\psi \rightarrow_L \bot) \rightarrow_L \varphi \hookrightarrow \psi, \varphi, \Delta}{\Gamma \hookrightarrow \psi \&_L \varphi, \Delta}.$$

4.4 Hypersequent calculus for infinite valued Łukasiewicz's propositional logic

The concept of hypersequent calculi was introduced for non-classical logics by Avron in [8]. Hypersequents consist of multiple sequents interpreted disjunctively. Therefore hypersequent rules include, in addition to single sequent rules, external structural rules that can operate on more than one component at a time.

Definition 12 *A hypersequent is a multiset of components written*

$$\Gamma_1 \hookrightarrow \Delta_1 | \dots | \Gamma_n \hookrightarrow \Delta_n$$

with the following interpretation: $\Gamma_1 \hookrightarrow \Delta_1 | \dots | \Gamma_n \hookrightarrow \Delta_n$ *is logically valid in* Łukasiewicz*'s logic* $Ł_\infty$ *iff for all valuations* val_I *mapped from formulas of* $\mathcal{L}$ *to the structure* $\mathfrak{M}_{[-1,0]}$ *there exists* i *such that* $\sum\limits_{\psi \in \Gamma_i} \mathrm{val}_I(\psi) \leqslant \sum\limits_{\varphi \in \Delta_i} \mathrm{val}_I(\varphi)$.

1. The axioms, i.e. the initial sequents, are as follows:

$$(ID) \qquad \psi \hookrightarrow \psi, \qquad\qquad (\Lambda) \qquad \hookrightarrow, \qquad (\bot) \qquad \bot \hookrightarrow \psi.$$

2. Let G be a variable for possibly empty hypersequents. The structural rules:

$$\frac{G|\Gamma \hookrightarrow \Delta}{G|\Gamma, \psi \hookrightarrow \Delta},$$

$$\frac{G|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta|\Gamma' \hookrightarrow \Delta'}, \qquad \frac{G|\Gamma \hookrightarrow \Delta|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta},$$

$$\frac{G|\Gamma_1, \Gamma_2 \hookrightarrow \Delta_1, \Delta_2}{G|\Gamma_1 \hookrightarrow \Delta_2|\Gamma_2 \hookrightarrow \Delta_2}, \qquad \frac{G|\Gamma_1 \hookrightarrow \Delta_1 \quad G|\Gamma_2 \hookrightarrow \Delta_2}{G|\Gamma_1, \Gamma_2 \hookrightarrow \Delta_1, \Delta_2}.$$

3. Logical rules:

$$\frac{G|\Gamma, \varphi \hookrightarrow \psi, \Delta}{G|\Gamma, \psi \rightarrow_L \varphi \hookrightarrow \Delta}, \qquad \frac{G|\Gamma \hookrightarrow \Delta \quad G|\Gamma, \psi \hookrightarrow \varphi, \Delta}{G|\Gamma \hookrightarrow \psi \rightarrow_L \varphi, \Delta},$$

$$\frac{G|\Gamma, \psi, \varphi \hookrightarrow \Delta \qquad G|\Gamma, \bot \hookrightarrow \Delta}{G|\Gamma, \psi \&_L \varphi \hookrightarrow \Delta}, \qquad \frac{G|\Gamma \hookrightarrow \psi, \varphi, \Delta | \Gamma \hookrightarrow \bot, \Delta}{G|\Gamma \hookrightarrow \psi \&_L \varphi, \Delta}.$$

As an example, consider the proof of (4.3) using just sequents:

1. $$\frac{\varphi \hookrightarrow \varphi \qquad \psi \hookrightarrow \psi \qquad \varphi \hookrightarrow \varphi \qquad \psi \hookrightarrow \psi}{\varphi, \psi \hookrightarrow \psi, \varphi \qquad \varphi, \psi \hookrightarrow \psi, \varphi},$$

2. $$\frac{\varphi, \psi \hookrightarrow \psi, \varphi \qquad \varphi, \psi \hookrightarrow \psi, \varphi}{\varphi, \varphi \to_L \psi \hookrightarrow \psi \qquad \varphi, \varphi \to_L \psi, \psi \hookrightarrow \psi, \varphi},$$

3. $$\frac{\varphi, \varphi \to_L \psi \hookrightarrow \psi \qquad \varphi, \varphi \to_L \psi, \psi \hookrightarrow \psi, \varphi}{\varphi, \varphi \to_L \psi \hookrightarrow \psi, \psi \to_L \varphi},$$

4. $$\frac{\varphi, \varphi \to_L \psi \hookrightarrow \psi, \psi \to_L \varphi}{(\psi \to_L \varphi) \to_L \varphi, \varphi \to_L \psi \hookrightarrow \psi},$$

5. $$\frac{(\psi \to_L \varphi) \to_L \varphi, \varphi \to_L \psi \hookrightarrow \psi}{(\psi \to_L \varphi) \to_L \varphi \hookrightarrow (\varphi \to_L \psi) \to_L \psi},$$

6. $$\frac{(\psi \to_L \varphi) \to_L \varphi \hookrightarrow (\varphi \to_L \psi) \to_L \psi}{\hookrightarrow ((\psi \to_L \varphi) \to_L \varphi) \to_L ((\varphi \to_L \psi) \to_L \psi)}.$$

However there exist some tautologies that can be proved only by means of hypersequents in the framework of this calculus (see [96]).

Chapter 5

Gödel's logic

5.1 Preliminaries

GÖDEL*'s matrix logic* $G_{[0,1]}$ is interesting as the logic of linear order. It is the structure $\langle [0,1], \neg_G, \rightarrow_G, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{1\} \rangle$, where

1. for all $x \in [0,1]$, $\neg_G x = x \rightarrow_G 0$,
2. for all $x, y \in [0,1]$, $x \rightarrow_G y = 1$ if $x \leqslant y$ and $x \rightarrow_G y = y$ otherwise,
3. for all $x, y \in [0,1]$, $x \vee y = \max(x, y)$,
4. for all $x, y \in [0,1]$, $x \wedge y = \min(x, y)$,
5. for a subset $M \subseteq [0,1]$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
6. for a subset $M \subseteq [0,1]$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
7. $\{1\}$ is the set of designated truth values.

The truth value $0 \in [0,1]$ is false, the truth value $1 \in [0,1]$ is true, and other truth values $x \in (0,1)$ are neutral.

5.2 Hilbert's type calculus for Gödel's logic

GÖDEL's logic denoted by $\mathbf{G}$ is one of the main intermediate between intuitionistic and classical logics. It can be obtained by adding $(\psi \rightarrow_G \varphi) \vee (\varphi \rightarrow_G \psi)$ to any axiomatization of intuitionistic logic (about intuitionism see [70], [87]). The GÖDEL logic was studied by DUMMETT in [42].

The negation of **G** is understood as follows

$$\neg_G \psi =: \psi \to_G \bot,$$

where $\bot$ is the truth constant 'falsehood'.

As an example, the HILBERT's type calculus for **G** consists of the following axioms:

$$(\psi \to_G \varphi) \to_G ((\varphi \to_G \chi) \to_G (\psi \to_G \chi)), \tag{5.1}$$

$$\psi \to_G (\psi \vee \varphi), \tag{5.2}$$

$$\varphi \to_G (\psi \vee \varphi), \tag{5.3}$$

$$(\varphi \to_G \chi) \to_G ((\psi \to_G \chi) \to_G ((\varphi \vee \psi) \to_G \chi)), \tag{5.4}$$

$$(\varphi \wedge \psi) \to_G \varphi, \tag{5.5}$$

$$(\varphi \wedge \psi) \to_G \psi, \tag{5.6}$$

$$(\chi \to_G \varphi) \to_G ((\chi \to_G \psi) \to_G (\chi \to_G (\varphi \wedge \psi))), \tag{5.7}$$

$$(\varphi \to_G (\psi \to_G \chi)) \to_G ((\varphi \wedge \psi) \to_G \chi), \tag{5.8}$$

$$((\varphi \wedge \psi) \to_G \chi) \to_G (\varphi \to_G (\psi \to_G \chi)), \tag{5.9}$$

$$(\varphi \wedge \neg_G \varphi) \to_G \psi, \tag{5.10}$$

$$(\varphi \to_G (\varphi \wedge \neg_G \varphi)) \to_G \neg_G \varphi, \tag{5.11}$$

$$(\psi \to_G \varphi) \vee (\varphi \to_G \psi), \tag{5.12}$$

$$\forall x\ \varphi(x) \to_G \varphi[x/t], \tag{5.13}$$

$$\varphi[x/t] \to_G \exists x\ \varphi(x), \tag{5.14}$$

where the formula $\varphi[x/t]$ is the result of substituting the term t for all free occurrences of x in φ,

$$\forall x(\chi \to_G \varphi) \to_G (\chi \to_G \forall x \varphi), \tag{5.15}$$

$$\forall x(\varphi \to_G \chi) \to_G (\exists x \varphi \to_G \chi), \tag{5.16}$$

$$\forall x(\chi \vee \varphi) \to_G (\chi \vee \forall x \varphi), \tag{5.17}$$

where x is not free in χ.

In **G** there are the following inference rules:

1. *Modus ponens*: from φ and $\varphi \to_G \psi$ infer ψ:

$$\frac{\varphi, \quad \varphi \to_G \psi}{\psi}.$$

2. *Substitution rule*: we can substitute any formulas for propositional variables.

3. *Generalization*: from φ infer $\forall x \varphi(x)$:

$$\frac{\varphi}{\forall x\ \varphi(x)}.$$

The HILBERT's type propositional calculus for GÖDEL's logic **G** can be obtained by extending the axiom system (10.1) – (10.8) by the following axiom:

$$\psi \to_G (\psi \wedge \psi).$$

5.3 Sequent calculus for Gödel's propositional logic

A *sequent* for GÖDEL's logic **G** is understood in the standard way, i.e. as follows: $\Gamma_1 \hookrightarrow \Gamma_2$, where $\Gamma_1 = \{\varphi_1, \ldots, \varphi_j\}$, $\Gamma_2 = \{\psi_1, \ldots, \psi_i\}$, is logically valid in **G** iff

$$\bigwedge_j \varphi_j \to_G \bigvee_i \psi_i$$

is logically valid in **G**.

1. The initial sequents in **G**:

$$(ID) \qquad \psi \hookrightarrow \psi, \qquad (\bot \hookrightarrow) \qquad \Gamma, \bot \hookrightarrow \Delta, \qquad (\hookrightarrow \bot) \qquad \Gamma \hookrightarrow \bot.$$

2. The structural rules are as follows:

$$\frac{\Gamma \hookrightarrow \Delta}{\Gamma, \psi \hookrightarrow \Delta}, \qquad \frac{\Gamma, \psi, \psi \hookrightarrow \Delta}{\Gamma, \psi \hookrightarrow \Delta},$$

$$\frac{\Gamma_1, \psi \hookrightarrow \Delta \qquad \Gamma_2 \hookrightarrow \psi}{\Gamma_1, \Gamma_2 \hookrightarrow \Delta}, \qquad \frac{\Gamma \hookrightarrow}{\Gamma \hookrightarrow \psi}.$$

3. Logical rules:

$$\frac{\Gamma, \varphi \hookrightarrow \psi}{\Gamma \hookrightarrow \varphi \to_G \psi}, \qquad \frac{\Gamma_1 \hookrightarrow \psi \quad \Gamma_2, \varphi \hookrightarrow \Delta}{\Gamma_1, \Gamma_2, \psi \to_G \varphi \hookrightarrow \Delta},$$

$$\frac{\Gamma_1, \psi \hookrightarrow \Gamma_2}{\Gamma_1, \psi \wedge \chi \hookrightarrow \Gamma_2}, \qquad \frac{\Gamma_1, \chi \hookrightarrow \Gamma_2}{\Gamma_1, \psi \wedge \chi \hookrightarrow \Gamma_2},$$

$$\frac{\Gamma_1, \psi \hookrightarrow \Gamma_2 \qquad \Gamma_1, \chi \hookrightarrow \Gamma_2}{\Gamma_1, \psi \vee \chi \hookrightarrow \Gamma_2}, \qquad \frac{\Gamma \hookrightarrow \psi \qquad \Gamma \hookrightarrow \chi}{\Gamma \hookrightarrow \psi \wedge \chi},$$

$$\frac{\Gamma_1 \hookrightarrow \psi}{\Gamma_1 \hookrightarrow \psi \vee \chi}, \qquad \frac{\Gamma_1 \hookrightarrow \chi}{\Gamma_1 \hookrightarrow \psi \vee \chi}.$$

5.4 Hypersequent calculus for Gödel's propositional logic

A *hypersequent* in **G** is interpreted in the standard way, i.e., disjunctively.

1. The initial sequents in **G** are as follows:

$$(ID) \quad \psi \hookrightarrow \psi, \qquad (\perp \hookrightarrow) \quad \Gamma, \perp \hookrightarrow \Delta, \qquad (\hookrightarrow \perp) \quad \Gamma \hookrightarrow \perp.$$

2. Let G be a variable for possibly empty hypersequents. The structural rules:

$$\frac{G|\Gamma \hookrightarrow \Delta}{G|\Gamma, \psi \hookrightarrow \Delta}, \qquad \frac{G|\Gamma, \psi, \psi \hookrightarrow \Delta}{G|\Gamma, \psi \hookrightarrow \Delta},$$

$$\frac{G|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta|\Gamma' \hookrightarrow \Delta'}, \qquad \frac{G|\Gamma \hookrightarrow \Delta|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta},$$

$$\frac{G|\Gamma_1, \Pi_2 \hookrightarrow \Delta_1 \qquad G|\Gamma_2, \Pi_2 \hookrightarrow \Delta_2}{G|\Gamma_1, \Gamma_2 \hookrightarrow \Delta_1|\Pi_1, \Pi_2 \hookrightarrow \Delta_2}, \qquad \frac{G|\Gamma \hookrightarrow}{G|\Gamma \hookrightarrow \psi}.$$

3. Logical rules:

$$\frac{G|\Gamma, \varphi \hookrightarrow \psi}{G|\Gamma \hookrightarrow \varphi \to_G \psi}, \qquad \frac{G|\Gamma_1 \hookrightarrow \psi \quad G|\Gamma_2, \varphi \hookrightarrow \Delta}{G|\Gamma_1, \Gamma_2, \psi \to_G \varphi \hookrightarrow \Delta},$$

$$\frac{G|\Gamma_1, \psi \hookrightarrow \Gamma_2}{G|\Gamma_1, \psi \wedge \chi \hookrightarrow \Gamma_2}, \qquad \frac{G|\Gamma_1, \chi \hookrightarrow \Gamma_2}{G|\Gamma_1, \psi \wedge \chi \hookrightarrow \Gamma_2},$$

$$\frac{G|\Gamma_1, \psi \hookrightarrow \Gamma_2 \qquad G|\Gamma_1, \chi \hookrightarrow \Gamma_2}{G|\Gamma_1, \psi \vee \chi \hookrightarrow \Gamma_2}, \qquad \frac{G|\Gamma \hookrightarrow \psi \qquad G|\Gamma \hookrightarrow \chi}{G|\Gamma \hookrightarrow \psi \wedge \chi},$$

$$\frac{G|\Gamma_1 \hookrightarrow \psi}{G|\Gamma_1 \hookrightarrow \psi \vee \chi}, \qquad \frac{G|\Gamma_1 \hookrightarrow \chi}{G|\Gamma_1 \hookrightarrow \psi \vee \chi}.$$

4. Cut rule:

$$\frac{G|\Gamma_1, \psi \hookrightarrow \Delta \qquad G|\Gamma_2 \hookrightarrow \psi}{G|\Gamma_1, \Gamma_2 \hookrightarrow \Delta}.$$

Chapter 6

Product logic

6.1 Preliminaries

Product matrix logic $\Pi_{[0,1]}$ behaves like $Ł_\infty$ on the interval $(0, 1]$ and like **G** at 0. It is the structure $\langle [0, 1], \neg_\Pi, \rightarrow_\Pi, \&_\Pi, \wedge, \vee, \widetilde{\exists}, \widetilde{\forall}, \{1\}\rangle$, where

1. for all $x \in [0, 1]$, $\neg_\Pi x = x \rightarrow_\Pi 0$,
2. for all $x, y \in [0, 1]$, $x \rightarrow_\Pi y = \begin{cases} \frac{y}{x}, & \text{if } x > y; \\ 1, & \text{otherwise,} \end{cases}$
3. for all $x, y \in [0, 1]$, $x \&_\Pi y = x \cdot y$,
4. for all $x, y \in [0, 1]$, $x \wedge y = x \cdot (x \rightarrow_\Pi y)$,
5. for all $x, y \in [0, 1]$, $x \vee y = ((x \rightarrow_\Pi y) \rightarrow_\Pi y) \wedge ((y \rightarrow_\Pi x) \rightarrow_\Pi x)$,
6. for a subset $M \subseteq [0, 1]$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq [0, 1]$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in [0, 1]$ is false, the truth value $1 \in [0, 1]$ is true, and other truth values $x \in (0, 1)$ are neutral.

6.2 Hilbert's type calculus for Product logic

In the Product logic denoted by $\mathbf{\Pi}$ there are the following abridged notations:

$$\neg_\Pi \psi =: \psi \rightarrow_\Pi \bot,$$

where $\perp$ is the truth constant 'falsehood',

$$\varphi \wedge \psi =: \varphi \&_{\Pi} (\varphi \rightarrow_{\Pi} \psi),$$

$$\varphi \vee \psi =: ((\varphi \rightarrow_{\Pi} \psi) \rightarrow_{\Pi} \psi) \wedge ((\psi \rightarrow_{\Pi} \varphi) \rightarrow_{\Pi} \varphi).$$

The HILBERT's type calculus for the Product logic **Π** consists of the following axioms:

$$(\varphi \rightarrow_{\Pi} \psi) \rightarrow_{\Pi} ((\psi \rightarrow_{\Pi} \chi) \rightarrow_{\Pi} (\varphi \rightarrow_{\Pi} \chi)), \tag{6.1}$$

$$(\varphi \&_{\Pi} \psi) \rightarrow_{\Pi} \varphi, \tag{6.2}$$

$$(\varphi \&_{\Pi} \psi) \rightarrow_{\Pi} (\psi \&_{\Pi} \varphi), \tag{6.3}$$

$$(\varphi \&_{\Pi} (\varphi \rightarrow_{\Pi} \psi)) \rightarrow_{\Pi} (\psi \&_{\Pi} (\psi \rightarrow_{\Pi} \varphi)), \tag{6.4}$$

$$(\varphi \rightarrow_{\Pi} (\psi \rightarrow_{\Pi} \chi)) \rightarrow_{\Pi} ((\varphi \&_{\Pi} \psi) \rightarrow_{\Pi} \chi), \tag{6.5}$$

$$((\varphi \&_{\Pi} \psi) \rightarrow_{\Pi} \chi) \rightarrow_{\Pi} (\varphi \rightarrow_{\Pi} (\psi \rightarrow_{\Pi} \chi)), \tag{6.6}$$

$$((\varphi \rightarrow_{\Pi} \psi) \rightarrow_{\Pi} \chi) \rightarrow_{\Pi} (((\psi \rightarrow_{\Pi} \varphi) \rightarrow_{\Pi} \chi) \rightarrow_{\Pi} \chi), \tag{6.7}$$

$$\perp \rightarrow_{\Pi} \psi, \tag{6.8}$$

$$\neg_{\Pi} \neg_{\Pi} \chi \rightarrow_{\Pi} (((\varphi \&_{\Pi} \chi) \rightarrow_{\Pi} (\psi \&_{\Pi} \chi)) \rightarrow_{\Pi} (\varphi \rightarrow_{\Pi} \psi)), \tag{6.9}$$

$$(\varphi \wedge \neg_{\Pi} \varphi) \rightarrow_{\Pi} \perp, \tag{6.10}$$

$$\forall x \; \varphi(x) \rightarrow_{\Pi} \varphi[x/t], \tag{6.11}$$

$$\varphi[x/t] \rightarrow_{\Pi} \exists x \; \varphi(x), \tag{6.12}$$

where the formula $\varphi[x/t]$ is the result of substituting the term t for all free occurrences of x in φ,

$$\forall x(\chi \rightarrow_{\Pi} \varphi) \rightarrow_{\Pi} (\chi \rightarrow_{\Pi} \forall x \varphi), \tag{6.13}$$

$$\forall x(\varphi \rightarrow_{\Pi} \chi) \rightarrow_{\Pi} (\exists x \varphi \rightarrow_{\Pi} \chi), \tag{6.14}$$

$$\forall x(\chi \vee \varphi) \to_\Pi (\chi \vee \forall x\varphi), \tag{6.15}$$

where x is not free in χ.

In $\mathbf{\Pi}$ there are the following inference rules:

1. *Modus ponens*: from φ and $\varphi \to_\Pi \psi$ infer ψ:

$$\frac{\varphi, \quad \varphi \to_\Pi \psi}{\psi}.$$

2. *Substitution rule*: we can substitute any formulas for propositional variables.

3. *Generalization*: from φ infer $\forall x\varphi(x)$:

$$\frac{\varphi}{\forall x\ \varphi(x)}.$$

We see that the HILBERT's type propositional calculus for Product logic is obtained by extending the axiom system (10.1) – (10.8) by the new axioms (6.9), (6.10).

6.3 Sequent calculus for Product propositional logic

A *sequent* for the Product logic $\mathbf{\Pi}$ is interpreted as follows: $\Gamma_1 \hookrightarrow \Gamma_2$, where $\Gamma_1 = \{\varphi_1, \ldots, \varphi_j\}$, $\Gamma_2 = \{\psi_1, \ldots, \psi_i\}$, is logically valid in $\mathbf{\Pi}$ iff

$$\&_{\Pi j}\varphi_j \to_\Pi \&_{\Pi i}\psi_i$$

is logically valid in $\mathbf{\Pi}$, i.e. for all valuations val_I in $\Pi_{[0,1]}$ we have $\prod_j \mathrm{val}_I(\varphi_j) \leqslant \prod_i \mathrm{val}_I(\psi_i)$.

1. The initial sequents in $\mathbf{\Pi}$:

$$(ID) \qquad \psi \hookrightarrow \psi, \qquad\qquad (\bot \hookrightarrow) \qquad \Gamma, \bot \hookrightarrow \Delta, \qquad (\Lambda) \qquad \hookrightarrow \quad .$$

2. The structural rules are as follows:

$$\frac{\Gamma \hookrightarrow \Delta}{\Gamma, \psi \hookrightarrow \Delta}, \qquad\qquad \frac{\Gamma_1 \hookrightarrow \Delta_1 \quad \Gamma_2 \hookrightarrow \Delta_2}{\Gamma_1, \Gamma_2 \hookrightarrow \Delta_1, \Delta_2},$$

$$\frac{\overbrace{\Gamma, \Gamma, \ldots, \Gamma}^{n} \hookrightarrow \overbrace{\Delta, \Delta, \ldots, \Delta}^{n}}{\Gamma \hookrightarrow \Delta}, n > 0.$$

3. Logical rules:

$$\frac{\Gamma \hookrightarrow \psi}{\Gamma \neg_{\Pi} \psi \hookrightarrow \Gamma_2}, \qquad \frac{\Gamma_1 \hookrightarrow \Gamma_2 \quad \Gamma_1, \psi \hookrightarrow \varphi, \Gamma_2}{\Gamma_1 \hookrightarrow \psi \to_{\Pi} \varphi, \Gamma_2},$$

$$\frac{\Gamma_1, \psi, \varphi \hookrightarrow \Gamma_2}{\Gamma_1, \psi \&_{\Pi} \varphi \hookrightarrow \Gamma_2}, \qquad \frac{\Gamma_1 \hookrightarrow \psi, \varphi, \Gamma_2}{\Gamma_1 \hookrightarrow \psi \&_{\Pi} \varphi, \Gamma_2},$$

$$\frac{\Gamma_1, \neg_{\Pi} \psi \hookrightarrow \Gamma_2 \qquad \Gamma_1, \varphi, \varphi \to_{\Pi} \psi \hookrightarrow \psi, \Gamma_2}{\Gamma_1, \psi \to_{\Pi} \varphi \hookrightarrow \Gamma_2}.$$

6.4 Hypersequent calculus for Product propositional logic

A *hypersequent* in $\mathbf{\Pi}$ is interpreted in the standard way, i.e., disjunctively.

1. The initial sequents in $\mathbf{\Pi}$:

$$(ID) \quad \psi \hookrightarrow \psi, \qquad (\bot \hookrightarrow) \quad \Gamma, \bot \hookrightarrow \Delta, \qquad (\Lambda) \quad \hookrightarrow \ .$$

2. Let G be a variable for possibly empty hypersequents. The structural rules:

$$\frac{G|\Gamma \hookrightarrow \Delta}{G|\Gamma, \psi \hookrightarrow \Delta},$$

$$\frac{G|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta|\Gamma' \hookrightarrow \Delta'}, \qquad \frac{G|\Gamma \hookrightarrow \Delta|\Gamma \hookrightarrow \Delta}{G|\Gamma \hookrightarrow \Delta},$$

$$\frac{G|\Gamma_1, \Gamma_2 \hookrightarrow \Delta_1, \Delta_2}{G|\Gamma_1 \hookrightarrow \Delta_2|\Gamma_2 \hookrightarrow \Delta_2}, \qquad \frac{G|\Gamma_1 \hookrightarrow \Delta_1 \quad G|\Gamma_2 \hookrightarrow \Delta_2}{G|\Gamma_1, \Gamma_2 \hookrightarrow \Delta_1, \Delta_2}.$$

3. Logical rules:

$$\frac{G|\Gamma \hookrightarrow \psi}{G|\Gamma \neg_{\Pi} \psi \hookrightarrow \Gamma_2}, \qquad \frac{G|\Gamma_1 \hookrightarrow \Gamma_2 \quad G|\Gamma_1, \psi \hookrightarrow \varphi, \Gamma_2}{G|\Gamma_1 \hookrightarrow \psi \to_{\Pi} \varphi, \Gamma_2},$$

$$\frac{G|\Gamma_1, \psi, \varphi \hookrightarrow \Gamma_2}{G|\Gamma_1, \psi \&_\Pi \varphi \hookrightarrow \Gamma_2}, \qquad \frac{G|\Gamma_1 \hookrightarrow \psi, \varphi, \Gamma_2}{G|\Gamma_1 \hookrightarrow \psi \&_\Pi \varphi, \Gamma_2},$$

$$\frac{G|\Gamma_1, \neg_\Pi \psi \hookrightarrow \Gamma_2 \qquad G|\Gamma_1, \varphi \hookrightarrow \psi, \Gamma_2}{G|\Gamma_1, \psi \rightarrow_\Pi \varphi \hookrightarrow \Gamma_2}.$$

Chapter 7

Nonlinear many valued logics

ŁUKASIEWICZ's infinite-valued logic $Ł_\infty$ is linear in the sense that its truth functions are linear. GÖDEL's infinite-valued logic **G** is linear too, but its truth function $\neg_G$ is discontinuous at the point 0.

Consider some sequences of nonlinear many-valued logics depending on natural parameter n. They can be convergent to linear many-valued logics (i.e., to $Ł_\infty$ and **G**) as $n \to \infty$. For the first time these logics were regarded by Russian logician D. BOCHVAR.

7.1 Hyperbolic logics

Definition 13 *A many-valued logic H_n is said to be hyperbolic if the following truth operations $\neg_{H_n}$ and $\to_{H_n}$ for an appropriate matrix logic $\mathfrak{M}_{H_n}$ are hyperbola functions.*

Let us consider two cases of hyperbolic matrix logic: ŁUKASIEWICZ's and GÖDEL's hyperbolic matrix logics for any positive integer n.

The ordered system $\langle V_{[0,1]}, \neg_{HL_n}, \to_{HL_n}, \vee_{HL_n}, \wedge_{HL_n}, \widetilde{\exists}, \widetilde{\forall}, \{1\}\rangle$ is called ŁUKASIEWICZ*'s hyperbolic matrix logic* $\mathfrak{M}_{HL_n}$, where for any positive integer n,

1. $V_{[0,1]} = [0,1]$,
2. for all $x \in V_{[0,1]}$, $\neg_{HL_n} x = n \cdot \frac{1-x}{1+x}$,
3. for all $x, y \in V_{[0,1]}$, $x \to_{HL_n} y = \begin{cases} 1, & \text{if } x \leqslant y; \\ n \cdot \frac{1-(x-y)}{n+(x-y)}, & \text{if } x > y. \end{cases}$,
4. for all $x, y \in V_{[0,1]}$, $x \vee_{HL_n} y = (x \to_{HL_n} y) \to_{HL_n} y$,

5. for all $x, y \in V_{[0,1]}$, $x \wedge_{HL_n} y = \neg_{HL_n}(\neg_{HL_n} x \vee \neg_{HL_n} y)$,
6. for a subset $M \subseteq V_{[0,1]}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq V_{[0,1]}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{[0,1]}$ is false, the truth value $1 \in V_{[0,1]}$ is true, and other truth values $x \in V_{[0,1]} \backslash \{0, 1\}$ are neutral.

Obviously that if $n \to \infty$, then

- $\neg_{HL_n} x = 1 - x$,
- $x \to_{HL_n} y = \min(1, 1 - x + y)$,
- $x \vee_{HL_n} y = \max(x, y)$,
- $x \wedge_{HL_n} y = \min(x, y)$,

in other words, we obtain ŁUKASIEWICZ's infinite valued logic $Ł_\infty$.

Some examples of many-valued tautologies for $\mathfrak{M}_{HL_n}$ are as follows:

$$\psi \to_{HL_n} \psi,$$

$$\neg_{HL_n} \neg_{HL_n} \psi \to_{HL_n} \psi,$$

$$\psi \to_{HL_n} \neg_{HL_n} \neg_{HL_n} \psi.$$

The ordered system $\langle V_{[0,1]}, \neg_{HG_n}, \to_{HG_n}, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{1\} \rangle$ is called GÖDEL*'s hyperbolic matrix logic* $\mathfrak{M}_{HG_n}$, where for any positive integer n,

1. $V_{[0,1]} = [0, 1]$,
2. for all $x \in V_{[0,1]}$, $\neg_{HG_n} x = (\frac{1-x}{1+x})^n$,
3. for all $x, y \in V_{[0,1]}$, $x \to_{HG_n} y = \begin{cases} 1, & \text{if } x \leqslant y; \\ \frac{(n+1) \cdot y}{n+x}, & \text{if } x > y. \end{cases}$,
4. for all $x, y \in V_{[0,1]}$, $x \vee y = \max(x, y)$,
5. for all $x, y \in V_{[0,1]}$, $x \wedge y = \min(x, y)$,
6. for a subset $M \subseteq V_{[0,1]}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq V_{[0,1]}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,

8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{[0,1]}$ is false, the truth value $1 \in V_{[0,1]}$ is true, and other truth values $x \in V_{[0,1]} \backslash \{0, 1\}$ are neutral.

It is evident that if $n \to \infty$, then

- $\neg_{HG_n} x = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \neq y, \end{cases}$
- $x \to_{HG_n} y = \begin{cases} 1, & \text{if } x \leqslant y; \\ \text{y}, & \text{if } x > y. \end{cases}$,

i.e., we obtain GÖDEL's infinite valued logic **G**.

7.2 Parabolic logics

Definition 14 *A many-valued logic P_n is said to be parabolic if the following truth operations $\neg_{P_n}$ and $\to_{P_n}$ for an appropriate matrix logic $\mathfrak{M}_{P_n}$ are parabola functions.*

The ordered system $\langle V_{[0,1]}, \neg_{P_n}, \to_{P_n}, \vee_{P_n}, \wedge_{P_n}, \widetilde{\exists}, \widetilde{\forall}, \{1\}\rangle$ is called *parabolic matrix logic* $\mathfrak{M}_{P_n}$, where for any positive integer n,

1. $V_{[0,1]} = [0, 1]$,
2. for all $x \in V_{[0,1]}$, $\neg_{P_n} x = \frac{1-x^2}{n}$,
3. for all $x, y \in V_{[0,1]}$, $x \to_{P_n} y = \begin{cases} 1, & \text{if } x^2 \leqslant y; \\ \frac{1-x^2}{n} + y, & \text{if } x^2 > y. \end{cases}$,
4. for all $x, y \in V_{[0,1]}$, $x \vee_{P_n} y = (x \to_{P_n} y) \to_{P_n} y$,
5. for all $x, y \in V_{[0,1]}$, $x \wedge_{P_n} y = \neg_{P_n}(\neg_{P_n} x \vee \neg_{P_n} y)$,
6. for a subset $M \subseteq V_{[0,1]}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq V_{[0,1]}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{[0,1]}$ is false, the truth value $1 \in V_{[0,1]}$ is true, and other truth values $x \in V_{[0,1]} \backslash \{0, 1\}$ are neutral.

The ordered system $\langle V_{[0,1]}, \neg_{P_n}, \to_{P_n}, \vee_{P_n}, \wedge_{P_n}, \widetilde{\exists}, \widetilde{\forall}, \{1\}\rangle$ is called *quasiparabolic matrix logic* $\mathfrak{M}_{P_n}$, where for any positive integer n,

1. $V_{[0,1]} = [0,1]$,
2. for all $x \in V_{[0,1]}$, $\neg_{P_n} x = \frac{1-x^2}{1+\frac{n}{n+1}\cdot x}$,
3. for all $x, y \in V_{[0,1]}$, $x \rightarrow_{P_n} y = \min(1, \frac{1-x^2}{1+\frac{n}{n+1}\cdot x} + y)$,
4. for all $x, y \in V_{[0,1]}$, $x \vee_{P_n} y = (x \rightarrow_{P_n} y) \rightarrow_{P_n} y$,
5. for all $x, y \in V_{[0,1]}$, $x \wedge_{P_n} y = \neg_{P_n}(\neg_{P_n} x \vee \neg_{P_n} y)$,
6. for a subset $M \subseteq V_{[0,1]}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M,
7. for a subset $M \subseteq V_{[0,1]}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M,
8. $\{1\}$ is the set of designated truth values.

The truth value $0 \in V_{[0,1]}$ is false, the truth value $1 \in V_{[0,1]}$ is true, and other truth values $x \in V_{[0,1]} \backslash \{0,1\}$ are neutral.

If $n \rightarrow \infty$, then the last matrix logic is transformed into Łukasiewicz's infinite valued logic $Ł_\infty$. This matrix is parabolic in a true sense just in case $n = 0$.

Chapter 8

Non-Archimedean valued logics

8.1 Standard many-valued logics

Definition 15 *Let V be a set of truth values. We will say that its members are exclusive if the powerset $\mathcal{P}(V)$ is a Boolean algebra.*

It can be easily shown that all the elements of truth value sets $\{0, 1, \ldots, n\}$, $[0, 1]$ for Łukasiewicz's, Gödel's, and Product logics are exclusive: for any their members x, y we have $\{x\} \cap \{y\} = \emptyset$ in $\mathcal{P}(V)$. Therefore Łukasiewicz's, Gödel's, and Product logics are based on the premise of existence of Shafer*'s model*. In other words, these logics are built on the families of exclusive elements (see [130], [131]).

However, for a wide class of fusion problems, "the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements θ_i cannot be properly identified and precisely separated" (see [135]). This means that if some elements θ_i of a frame Θ have non-empty intersection, then sources of evidence don't provide their beliefs with the same absolute interpretation of elements of the same frame Θ and the conflict between sources arises not only because of the possible unreliability of sources, but also because of possible different and relative interpretation of Θ (see [40], [41]).

8.2 Many-valued logics on DSm models

Definition 16 *Let V be a set of truth values. We will say that its members are non-exclusive if the powerset $\mathcal{P}(V)$ is not closed under intersection (consequently, it is not a Boolean algebra).*

A many-valued logic is said to be a *many-valued logic on non-exclusive elements* if the elements of its set V of truth values are non-exclusive. These logics are also said to be a many-valued logic on DSm model (Dezert-Smarandache model).

Recall that a DSm model (Dezert-Smarandache *model*) is formed as a hyper-power set. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set (called frame) of n non-exclusive elements. The *hyper-power set* D^Θ is defined as the set of all composite propositions built from elements of Θ with $\cap$ and $\cup$ operators such that:

1. $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$;
2. if $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$;
3. no other elements belong to D^Θ, except those obtained by using rules 1 or 2.

The cardinality of D^Θ is majored by 2^{2^n} when the cardinality of Θ equals n, i.e. $|\Theta| = n$. Since for any given finite set Θ, $|D^\Theta| \geq |2^\Theta|$, we call D^Θ the hyper-power set of Θ. Also, D^Θ constitutes what is called the *DSm model* $\mathcal{M}^f(\Theta)$. However elements θ_i can be truly exclusive. In such case, the hyper-power set D^Θ reduces naturally to the classical power set 2^Θ and this constitutes the most restricted hybrid DSm model, denoted by $\mathcal{M}^0(\Theta)$, coinciding with Shafer's model. As an example, suppose that $\Theta = \{\theta_1, \theta_2\}$ with $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, where θ_1 and θ_2 are truly exclusive (i.e., Shafer's model $\mathcal{M}^0$ holds), then because $\theta_1 \cap \theta_2 =_{\mathcal{M}^0} \emptyset$, one gets $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2 =_{\mathcal{M}^0} \emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = 2^\Theta$.

Now let us show that every non-Archimedean structure is an infinite DSm model, but no vice versa. The most easy way of setting non-Archimedean structures was proposed by Robinson in [117]. Consider a set Θ consisting only of exclusive members. Let I be any infinite index set. Then we can construct an indexed family Θ^I, i.e., we can obtain the set of all functions: $f\colon I \to \Theta$ such that $f(\alpha) \in \Theta$ for any $\alpha \in I$.

A *filter* $\mathcal{F}$ on the index set I is a family of sets $\mathcal{F} \subset \wp(I)$ for which:

1. $A \in \mathcal{F}$, $A \subset B \Rightarrow B \in \mathcal{F}$;
2. $A_1, \ldots, A_n \in \mathcal{F} \Rightarrow \bigcap\limits_{k=1}^{n} A_k \in \mathcal{F}$;
3. $\emptyset \notin \mathcal{F}$.

The set of all complements for finite subsets of I is a filter and it is called a Frechet *filter* and it is denoted by $\mathcal{U}$. A maximal filter (ultrafilter) that contains a Frechet filter is called a Frechet *ultrafilter*.

Let $\mathcal{U}$ be a Frechet ultrafilter on I. Define a new relation $\backsim$ on the set Θ^I by

$$f \backsim g \leftrightarrow \{\alpha \in I\colon f(\alpha) = g(\alpha)\} \in \mathcal{U}. \tag{8.1}$$

It is easily proved that the relation $\backsim$ is an equivalence. Notice that formula (8.1) means that f and g are equivalent iff f and g are equal on an infinite index subset. For each $f \in \Theta^I$ let $[f]$ denote the equivalence class of f under $\backsim$. The *ultrapower* $\Theta^I/\mathcal{U}$ is then defined to be the set of all equivalence classes $[f]$ as f ranges over Θ^I:

$$\Theta^I/\mathcal{U} \triangleq \{[f]\colon f \in \Theta^I\}.$$

Also, Robinson has proved that each non-empty set Θ has an ultrapower with respect to a Frechet filter/ultrafilter $\mathcal{U}$. This ultrapower $\Theta^I/\mathcal{U}$ is said to be a *proper nonstandard extension* of Θ and it is denoted by ${}^*\Theta$. Let us notice that if $\mathcal{U}$ is not ultrafilter (it is just the Frechet filter), then ${}^*\Theta$ is not well-ordered.

Proposition 7 *Let X be a non-empty set. A nonstandard extension of X consists of a mapping that assigns a set ${}^*A$ to each $A \subseteq X^m$ for all $m \geq 0$, such that ${}^*X$ is non-empty and the following conditions are satisfied for all $m, n \geq 0$:*

1. *The mapping preserves Boolean operations on subsets of X^m: if $A \subseteq X^m$, then ${}^*A \subseteq ({}^*X)^m$; if $A, B \subseteq X^m$, then ${}^*(A \cap B) = ({}^*A \cap {}^*B), {}^*(A \cup B) = ({}^*A \cup {}^*B)$, and ${}^*(A \backslash B) = ({}^*A) \backslash ({}^*B)$.*

2. *The mapping preserves Cartesian products: if $A \subseteq X^m$ and $B \subseteq X^n$, then ${}^*(A \times B) = {}^*A \times {}^*B$, where $A \times B \subseteq X^{m+n}$.* $\square$

This proposition is proved in [74].

Recall that each element of ${}^*\Theta$ is an equivalence class $[f]$ as $f\colon I \to \Theta$. There exist two groups of members of ${}^*\Theta$:

1. functions that are constant, e.g., $f(\alpha) = m \in \Theta$ for infinite index subset $\{\alpha \in I\}$. A constant function $[f = m]$ is denoted by ${}^*m$,

2. functions that aren't constant.

The set of all constant functions of ${}^*\Theta$ is called *standard set* and it is denoted by ${}^\sigma\Theta$. The members of ${}^\sigma\Theta$ are called *standard*. It is readily seen that ${}^\sigma\Theta$ and Θ are isomorphic: ${}^\sigma\Theta \simeq \Theta$.

The following proposition can be easily proved:

Proposition 8 *For any set Θ such that $|\Theta| \geq 2$, there exists a proper nonstandard extension ${}^*\Theta$ for which ${}^*\Theta \backslash {}^\sigma\Theta \neq \emptyset$.*

Proof. Let $I_1 = \{\alpha_1, \alpha_2, \dots, \alpha_n, \dots\} \subset I$ be an infinite set and let $\mathcal{U}$ be a FRECHET filter. Suppose that $\Theta_1 = \{m_1, \dots, m_n\}$ such that $|\Theta_1| \geq 1$ is the subset of Θ and there is a mapping:

$$f(\alpha) = \begin{cases} m_k & \text{if } \alpha = \alpha_k; \\ m_0 \in \Theta & \text{if } \alpha \in I \backslash I_1 \end{cases}$$

and $f(\alpha) \neq m_k$ if $\alpha = \alpha_k \mod (n+1)$, $k = 1, \dots, n$ and $\alpha \neq \alpha_k$.

Show that $[f] \in {}^*\Theta \backslash {}^\sigma\Theta$. The proof is by reductio ad absurdum. Suppose there is $m \in \Theta$ such that $m \in [f(\alpha)]$. Consider the set:

$$I_2 = \{\alpha \in I \colon f(\alpha) = m\} = \begin{cases} \{\alpha_k\} & \text{if } m = m_k,\ k = 1, \dots, n; \\ I \backslash I_1 & \text{if } m = m_0. \\ \emptyset & \text{if } m \notin \{m_0, m_1, \dots, m_n\}. \end{cases}$$

In any case $I_2 \notin \mathcal{U}$, because we have $\{\alpha_k\} \notin \mathcal{U}$, $\emptyset \notin \mathcal{U}$, $I \backslash I_1 \notin \mathcal{U}$. Thus, $[f] \in {}^*\Theta \backslash {}^\sigma\Theta$. □

The standard members of ${}^*\Theta$ are exclusive, because their intersections are empty. Indeed, the members of Θ were exclusive, therefore the members of ${}^\sigma\Theta$ are exclusive too. However the members of ${}^*\Theta \backslash {}^\sigma\Theta$ are non-exclusive. By definition, if a member $a \in {}^*\Theta$ is nonstandard, then there exists a standard member $b \in {}^*\Theta$ such that $a \cap b \neq \emptyset$ (for example, see the proof of proposition 2). We can also prove that there exist non-exclusive members $a \in {}^*\Theta$, $b \in {}^*\Theta$ such that $a \cap b \neq \emptyset$.

Proposition 9 *There exist two inconstant functions f_1, f_2 such that the intersection of f_1, f_2 isn't empty.*

Proof. Let $f_1 \colon I \to \Theta$ and $f_2 \colon I \to \Theta$. Suppose that $[f_i \neq n]$, $\forall n \in \Theta$, $i = 1, 2$, i.e., f_1, f_2 aren't constant. By proposition 2, these functions are nonstandard members of ${}^*\Theta$. Further consider an indexed family $F(\alpha)$ for all $\alpha \in I$ such that $\{\alpha \in I \colon f_i(\alpha) \in F(\alpha)\} \in \mathcal{U} \leftrightarrow [f_i] \in B$ as $i = 1, 2$. Consequently it is possible that, for some $\alpha_j \in I$, $f_1(\alpha_j) \cap f_2(\alpha_j) = n_j$ and $n_j \in F(\alpha_j)$. □

Proposition 10 *Define the structure $\langle \mathcal{P}({}^*\Theta), \cap, \cup, \neg, {}^*\Theta \rangle$ as follows*

- *for any $A, B \in \mathcal{P}({}^*\Theta)$, $A \cap B = \{f(\alpha) \colon f(\alpha) \in A \wedge f(\alpha) \in B\}$,*
- *for any $A, B \in \mathcal{P}({}^*\Theta)$, $A \cup B = \{f(\alpha) \colon f(\alpha) \in A \vee f(\alpha) \in B\}$,*
- *for any $A \in \mathcal{P}({}^*\Theta)$, $\neg A = \{f(\alpha) \colon f(\alpha) \in {}^*\Theta \backslash A\}$.*

The structure $\langle \mathcal{P}({}^*\Theta), \cap, \cup, \neg, {}^*\Theta \rangle$ is not a Boolean algebra if $|\Theta| \geq 2$.

Proof. The set $\mathcal{P}({}^*\Theta)$ is not closed under intersection. Indeed, some elements of ${}^*\Theta$ have a non-empty intersection (see the previous proposition) that doesn't belong to $\mathcal{P}({}^*\Theta)$, i.e. they aren't atoms of $\mathcal{P}({}^*\Theta)$ of the form $[f]$. □

However, the structure $\langle \mathcal{P}({}^\sigma\Theta), \cap, \cup, \neg, {}^\sigma\Theta \rangle$ is a Boolean algebra. In the meantime $\mathcal{P}({}^\sigma\Theta) \subset \mathcal{P}({}^*\Theta)$ if $|\Theta| \geq 2$.

Thus, **non-Archimedean structures are infinite DSm-models, because these contain non-exclusive members**.

In the next sections, we shall consider the following non-Archimedean structures:

1. the nonstandard extension ${}^*\mathbf{Q}$ (called the field of hyperrational numbers),
2. the nonstandard extension ${}^*\mathbf{R}$ (called the field of hyperreal numbers),
3. the nonstandard extension $\mathbf{Z}_p$ (called the ring of p-adic integers) that we obtain as follows. Let the set $\mathbf{N}$ of natural numbers be the index set and let $\Theta = \{0, \ldots, p-1\}$. Then the nonstandard extension $\Theta^{\mathbf{N}} \backslash \mathcal{U} = \mathbf{Z}_p$.

Further, we shall set the following logics on non-Archimedean structures:

- hyperrational valued Łukasiewicz's, Gödel's, and Product logics,
- hyperreal valued Łukasiewicz's, Gödel's, and Product logics,
- p-adic valued Łukasiewicz's, Gödel's, and Post's logics.

Note that these many-valued logics are the particular cases of logics on DSm models.

Recall that non-Archimedean logical multiple-validities were considered by Schumann in [122], [123], [124], [125], [128], [129].

8.3 Hyper-valued partial order structure

Assume that ${}^*\mathbf{Q}_{[0,1]} = \mathbf{Q}^{\mathbf{N}}_{[0,1]}/\mathcal{U}$ is a nonstandard extension of the subset $\mathbf{Q}_{[0,1]} = \mathbf{Q} \cap [0,1]$ of rational numbers, where $\mathcal{U}$ is the Frechet filter that may be no ultrafilter, and ${}^\sigma\mathbf{Q}_{[0,1]} \subset {}^*\mathbf{Q}_{[0,1]}$ is the subset of standard members. We can extend the usual order structure on $\mathbf{Q}_{[0,1]}$ to a partial order structure on ${}^*\mathbf{Q}_{[0,1]}$:

1. for rational numbers $x, y \in \mathbf{Q}_{[0,1]}$ we have $x \leq y$ in $\mathbf{Q}_{[0,1]}$ iff $[f] \leq [g]$ in ${}^*\mathbf{Q}_{[0,1]}$, where $\{\alpha \in \mathbf{N} \colon f(\alpha) = x\} \in \mathcal{U}$ and $\{\alpha \in \mathbf{N} \colon g(\alpha) = y\} \in \mathcal{U}$,

   i.e., f and g are constant functions such that $[f] = {}^*x$ and $[g] = {}^*y$,

2. each positive rational number ${}^*x \in {}^\sigma\mathbf{Q}_{[0,1]}$ is greater than any number $[f] \in {}^*\mathbf{Q}_{[0,1]} \backslash {}^\sigma\mathbf{Q}_{[0,1]}$,

   i.e., ${}^*x > [f]$ for any positive $x \in \mathbf{Q}_{[0,1]}$ and $[f] \in {}^*\mathbf{Q}_{[0,1]}$, where $[f]$ isn't constant function.

These conditions have the following informal sense:

1. The sets ${}^\sigma\mathbf{Q}_{[0,1]}$ and $\mathbf{Q}_{[0,1]}$ have isomorphic order structure.
2. The set ${}^*\mathbf{Q}_{[0,1]}$ contains actual infinities that are less than any positive rational number of ${}^\sigma\mathbf{Q}_{[0,1]}$.

Define this partial order structure on ${}^*\mathbf{Q}_{[0,1]}$ as follows:

$\mathcal{O}_{^*\mathbf{Q}}$ 1. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] \leq [g]$ if

$$\{\alpha \in \mathbf{N} \colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}.$$

2. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] < [g]$ if

$$\{\alpha \in \mathbf{N} \colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$$

   and $[f] \neq [g]$, i.e., $\{\alpha \in \mathbf{N} \colon f(\alpha) \neq g(\alpha)\} \in \mathcal{U}$.
3. For any hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, we set $[f] = [g]$ if $f \in [g]$.

This ordering relation is not linear, but partial, because there exist elements $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, which are incompatible.

Introduce two operations max, min in the partial order structure $\mathcal{O}_{^*\mathbf{Q}}$:

1. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\min([f],[g]) = [f]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
2. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\max([f],[g]) = [g]$ if and only if $[f] \leq [g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
3. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, $\min([f],[g]) = \max([f],[g]) = [f] = [g]$ if and only if $[f] = [g]$ under condition $\mathcal{O}_{^*\mathbf{Q}}$,
4. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{Q}}$, then $\min([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{Q}_{[0,1]}$ such that
$$\{\alpha \in \mathbf{N} \colon \min(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$
5. for all hyperrational numbers $[f], [g] \in {}^*\mathbf{Q}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{Q}}$, then $\max([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{Q}_{[0,1]}$ such that
$$\{\alpha \in \mathbf{N} \colon \max(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

It is easily seen that conditions 1 – 3 are corollaries of conditions 4, 5.

Note there exist the maximal number $^*1 \in {}^*\mathbf{Q}_{[0,1]}$ and the minimal number $^*0 \in {}^*\mathbf{Q}_{[0,1]}$ under condition $\mathcal{O}_{^*\mathbf{Q}}$. Therefore, for any $[f] \in {}^*\mathbf{Q}_{[0,1]}$, we have: $\max(^*1, [f]) = {}^*1$, $\max(^*0, [f]) = [f]$, $\min(^*1, [f]) = [f]$ and $\min(^*0, [f]) = {}^*0$.

Let us consider a nonstandard extension ${}^*\mathbf{R}_{[0,1]} = \mathbf{R}^{\mathbf{N}}_{[0,1]}/\mathcal{U}$ for the subset $\mathbf{R}_{[0,1]} = \mathbf{R} \cap [0,1]$ of real numbers. Let ${}^\sigma\mathbf{R}_{[0,1]} \subset {}^*\mathbf{R}_{[0,1]}$ be the subset of standard members. We can extend the usual order structure on $\mathbf{R}_{[0,1]}$ to a partial order structure on ${}^*\mathbf{R}_{[0,1]}$:

1. for real numbers x, $y \in \mathbf{R}_{[0,1]}$ we have $x \le y$ in $\mathbf{R}_{[0,1]}$ iff $[f] \le [g]$ in ${}^*\mathbf{R}_{[0,1]}$, where $\{\alpha \in \mathbf{N}\colon f(\alpha) = x\} \in \mathcal{U}$ and $\{\alpha \in \mathbf{N}\colon g(\alpha) = y\} \in \mathcal{U}$,

2. each positive real number ${}^*x \in {}^\sigma\mathbf{R}_{[0,1]}$ is greater than any number $[f] \in {}^*\mathbf{R}_{[0,1]} \backslash {}^\sigma\mathbf{R}_{[0,1]}$,

As before, these conditions have the following informal sense:

1. The sets ${}^\sigma\mathbf{R}_{[0,1]}$ and $\mathbf{R}_{[0,1]}$ have isomorphic order structure.

2. The set ${}^*\mathbf{R}_{[0,1]}$ contains actual infinities that are less than any positive real number of ${}^\sigma\mathbf{R}_{[0,1]}$.

Define this partial order structure on ${}^*\mathbf{R}_{[0,1]}$ as follows:

$\mathcal{O}_{^*\mathbf{R}}$ 1. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] \le [g]$ if

$$\{\alpha \in \mathbf{N}\colon f(\alpha) \le g(\alpha)\} \in \mathcal{U}.$$

2. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] < [g]$ if $\{\alpha \in \mathbf{N}\colon f(\alpha) \le g(\alpha)\} \in \mathcal{U}$ and $[f] \ne [g]$, i.e., $\{\alpha \in \mathbf{N}\colon f(\alpha) \ne g(\alpha)\} \in \mathcal{U}$.

3. For any hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, we set $[f] = [g]$ if $f \in [g]$.

Introduce two operations max, min in the partial order structure $\mathcal{O}_{^*\mathbf{R}}$:

1. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\min([f],[g]) = [f]$ if and only if $[f] \le [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,

2. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\max([f],[g]) = [g]$ if and only if $[f] \le [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,

3. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, $\min([f],[g]) = \max([f],[g]) = [f] = [g]$ if and only if $[f] = [g]$ under condition $\mathcal{O}_{^*\mathbf{R}}$,

4. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{R}}$, then $\min([f],[g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{R}_{[0,1]}$ such that
$$\{\alpha \in \mathbf{N}\colon \min(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

5. for all hyperreal numbers $[f], [g] \in {}^*\mathbf{R}_{[0,1]}$, if $[f], [g]$ are incompatible under condition $\mathcal{O}_{^*\mathbf{R}}$, then $\max([f], [g]) = [h]$ iff there exists $[h] \in {}^*\mathbf{R}_{[0,1]}$ such that
$$\{\alpha \in \mathbf{N}\colon \max(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}.$$

Note there exist the maximal number $^*1 \in {}^*\mathbf{R}_{[0,1]}$ and the minimal number $^*0 \in {}^*\mathbf{R}_{[0,1]}$ under condition $\mathcal{O}_{^*\mathbf{R}}$.

8.4 Hyper-valued matrix logics

Now define *hyperrational-valued* ŁUKASIEWICZ*'s logic* $\mathfrak{M}_{^*\mathbf{Q}}$:

Definition 17 *The ordered system* $\langle V_{^*\mathbf{Q}}, \neg_L, \rightarrow_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{^*1\}\rangle$ *is called hyperrational valued* ŁUKASIEWICZ*'s matrix logic* $\mathfrak{M}_{^*\mathbf{Q}}$*, where*

1. $V_{^*\mathbf{Q}} = {}^*\mathbf{Q}_{[0,1]}$ *is the subset of hyperrational numbers,*
2. *for all* $[x] \in V_{^*\mathbf{Q}}$*,* $\neg_L[x] = {}^*1 - [x]$*,*
3. *for all* $[x], [y] \in V_{^*\mathbf{Q}}$*,* $[x] \rightarrow_L [y] = \min({}^*1, {}^*1 - [x] + [y])$*,*
4. *for all* $[x], [y] \in V_{^*\mathbf{Q}}$*,* $[x] \vee [y] = ([x] \rightarrow_L [y]) \rightarrow_L [y] = \max([x], [y])$*,*
5. *for all* $[x], [y] \in V_{^*\mathbf{Q}}$*,* $[x] \wedge [y] = \neg_L(\neg_L[x] \vee \neg_L[y]) = \min([x], [y])$*,*
6. *for a subset* $M \subseteq V_{^*\mathbf{Q}}$*,* $\widetilde{\exists}(M) = \max(M)$*, where* $\max(M)$ *is a maximal element of* M*,*
7. *for a subset* $M \subseteq V_{^*\mathbf{Q}}$*,* $\widetilde{\forall}(M) = \min(M)$*, where* $\min(M)$ *is a minimal element of* M*,*
8. $\{^*1\}$ *is the set of designated truth values.*

The truth value $^*0 \in V_{^*\mathbf{Q}}$ is false, the truth value $^*1 \in V_{^*\mathbf{Q}}$ is true, and other truth values $x \in V_{^*\mathbf{Q}} \backslash \{^*0, {}^*1\}$ are neutral.

Continuing in the same way, define *hyperreal valued* ŁUKASIEWICZ*'s matrix logic* $\mathfrak{M}_{^*\mathbf{R}}$:

Definition 18 *The ordered system* $\langle V_{^*\mathbf{R}}, \neg_L, \rightarrow_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{^*1\}\rangle$ *is called hyperreal valued* ŁUKASIEWICZ*'s matrix logic* $\mathfrak{M}_{^*\mathbf{R}}$*, where*

1. $V_{^*\mathbf{R}} = {}^*\mathbf{R}_{[0,1]}$ *is the subset of hyperreal numbers,*
2. *for all* $[x] \in V_{^*\mathbf{R}}$*,* $\neg_L[x] = {}^*1 - [x]$*,*
3. *for all* $[x], [y] \in V_{^*\mathbf{R}}$*,* $[x] \rightarrow_L [y] = \min({}^*1, {}^*1 - [x] + [y])$*,*
4. *for all* $[x], [y] \in V_{^*\mathbf{R}}$*,* $[x] \vee [y] = ([x] \rightarrow_L [y]) \rightarrow_L [y] = \max([x], [y])$*,*

5. *for all* $[x], [y] \in V_{^*\mathbf{R}}$, $[x] \wedge [y] = \neg_L(\neg_L[x] \vee \neg_L[y]) = \min([x], [y])$,

6. *for a subset* $M \subseteq V_{^*\mathbf{R}}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,

7. *for a subset* $M \subseteq V_{^*\mathbf{R}}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,

8. $\{^*1\}$ *is the set of designated truth values.*

The truth value $^*0 \in V_{^*\mathbf{R}}$ is false, the truth value $^*1 \in V_{^*\mathbf{R}}$ is true, and other truth values $x \in V_{^*\mathbf{R}} \backslash \{^*0, ^*1\}$ are neutral.

Definition 19 *Hyper-valued* GÖDEL*'s matrix logic* $G_{^*[0,1]}$ *is the structure* $\langle ^*[0,1]$, $\neg_G$, $\rightarrow_G$, $\vee$, $\wedge$, $\widetilde{\exists}$, $\widetilde{\forall}$, $\{^*1\}\rangle$, *where*

1. *for all* $[x] \in {^*[0,1]}$, $\neg_G[x] = [x] \rightarrow_G {^*0}$,

2. *for all* $[x], [y] \in {^*[0,1]}$, $[x] \rightarrow_G [y] = {^*1}$ *if* $[x] \leqslant [y]$ *and* $[x] \rightarrow_G [y] = [y]$ *otherwise,*

3. *for all* $[x], [y] \in {^*[0,1]}$, $[x] \vee [y] = \max([x], [y])$,

4. *for all* $[x], [y] \in {^*[0,1]}$, $[x] \wedge [y] = \min([x], [y])$,

5. *for a subset* $M \subseteq {^*[0,1]}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,

6. *for a subset* $M \subseteq {^*[0,1]}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,

7. $\{^*1\}$ *is the set of designated truth values.*

The truth value $^*0 \in {^*[0,1]}$ is false, the truth value $^*1 \in {^*[0,1]}$ is true, and other truth values $[x] \in {^*(0,1)}$ are neutral.

Definition 20 *Hyper-valued Product matrix logic* $\Pi_{^*[0,1]}$ *is the structure* $\langle ^*[0,1]$, $\neg_\Pi$, $\rightarrow_\Pi$, $\&_\Pi$, $\wedge$, $\vee$, $\widetilde{\exists}$, $\widetilde{\forall}$, $\{^*1\}\rangle$, *where*

1. *for all* $[x] \in {^*[0,1]}$, $\neg_\Pi[x] = [x] \rightarrow_\Pi {^*0}$,

2. *for all* $[x], [y] \in {^*[0,1]}$, $[x] \rightarrow_\Pi [y] = \begin{cases} ^*1, & \text{if } [x] \leqslant [y], \\ \min(^*1, \frac{[y]}{[x]}), & \text{otherwise;} \end{cases}$

3. *for all* $[x], [y] \in {^*[0,1]}$, $[x] \&_\Pi [y] = [x] \cdot [y]$,

4. *for all* $[x], [y] \in {^*[0,1]}$, $[x] \wedge [y] = [x] \cdot ([x] \rightarrow_\Pi [y])$,

5. *for all* $[x], [y] \in {^*[0,1]}$, $[x] \vee [y] = (([x] \rightarrow_\Pi [y]) \rightarrow_\Pi [y]) \wedge (([y] \rightarrow_\Pi [x]) \rightarrow_\Pi [x])$,

6. *for a subset* $M \subseteq {}^*[0,1]$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,

7. *for a subset* $M \subseteq {}^*[0,1]$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,

8. $\{{}^*1\}$ *is the set of designated truth values.*

The truth value ${}^*0 \in {}^*[0,1]$ is false, the truth value ${}^*1 \in {}^*[0,1]$ is true, and other truth values $[x] \in {}^*(0,1)$ are neutral.

8.5 Hyper-valued probability theory and hyper-valued fuzzy logic

Let X be an arbitrary set and let $\mathcal{A}$ be an *algebra of subsets* $A \subseteq X$, i.e.

1. union, intersection, and difference of two subsets of X also belong to $\mathcal{A}$;

2. $\emptyset, X$ belong to $\mathcal{A}$.

Recall that a *finitely additive probability measure* is a nonnegative set function $\mathbf{P}(\cdot)$ defined for sets $A \in \mathcal{A}$ that satisfies the following properties:

1. $\mathbf{P}(A) \geq 0$ for all $A \in \mathcal{A}$,

2. $\mathbf{P}(X) = 1$ and $\mathbf{P}(\emptyset) = 0$,

3. if $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are disjoint, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$. In particular $\mathbf{P}(\neg A) = 1 - \mathbf{P}(A)$ for all $A \in \mathcal{A}$.

The algebra $\mathcal{A}$ is called a *σ-algebra* if it is assumed to be closed under countable union (or equivalently, countable intersection), i.e. if for every n, $A_n \in \mathcal{A}$ causes $A = \bigcup\limits_n A_n \in \mathcal{A}$.

A set function $\mathbf{P}(\cdot)$ defined on a σ-algebra is called a *countable additive probability measure* (or a *σ-additive probability measure*) if in addition to satisfying equations of the definition of finitely additive probability measure, it satisfies the following countable additivity property: for any sequence of pairwise disjoint sets A_n, $\mathbf{P}(A) = \sum\limits_n \mathbf{P}(A_n)$. The ordered system $\langle X, \mathcal{A}, \mathbf{P} \rangle$ is called a *probability space*.

Now consider hyper-valued probabilities. Let I be an arbitrary set, let $\mathcal{A}$ be an algebra of subsets $A \subseteq I$, and let $\mathcal{U}$ be a FRECHET ultrafilter on I. Set for $A \in \mathcal{A}$:

$$\mu_{\mathcal{U}}(A) = \begin{cases} 1, & A \in \mathcal{U}; \\ 0, & A \notin \mathcal{U}. \end{cases}$$

Hence, there is a mapping $\mu_{\mathcal{U}} \colon \mathcal{A} \to \{0,1\}$ satisfying the following properties:

1. $\mu_{\mathcal{U}}(\emptyset) = 0$, $\mu_{\mathcal{U}}(I) = 1$;
2. if $\mu_{\mathcal{U}}(A_1) = \mu_{\mathcal{U}}(A_2) = 0$, then $\mu_{\mathcal{U}}(A_1 \cup A_2) = 0$;
3. if $A_1 \cap A_2 = \emptyset$, then $\mu_{\mathcal{U}}(A_1 \cup A_2) = \mu_{\mathcal{U}}(A_1) + \mu_{\mathcal{U}}(A_2)$.

This implies that $\mu_{\mathcal{U}}$ is a probability measure. Notice that $\mu_{\mathcal{U}}$ isn't σ-additive. As an example, if A is the set of even numbers and B is the set of odd numbers, then $A \in \mathcal{U}$ implies $B \notin \mathcal{U}$, because the filter $\mathcal{U}$ is maximal. Thus, $\mu_{\mathcal{U}}(A) = 1$ and $\mu_{\mathcal{U}}(B) = 0$, although the cardinalities of A and B are equal.

The ordered system $\langle I, \mathcal{A}, \mu_{\mathcal{U}} \rangle$ is called a probability space.

Let's consider a mapping: $f\colon I \ni \alpha \mapsto f(\alpha) \in M$. Two mappings f, g are equivalent: $f \backsim g$ if $\mu_{\mathcal{U}}(\{\alpha \in I\colon f(\alpha) = g(\alpha)\}) = 1$. An equivalence class of f is called a probabilistic events and is denoted by $[f]$. The set ${}^*M$ is the *set of all probabilistic events* of M. This ${}^*M$ is a proper nonstandard extension defined above.

Under condition 1 of proposition 7, we can obtain a nonstandard extension of an algebra $\mathcal{A}$ denoted by ${}^*\mathcal{A}$. Let ${}^*X$ be an arbitrary nonstandard extension. Then the nonstandard algebra ${}^*\mathcal{A}$ is an algebra of subsets $A \subseteq {}^*X$ if the following conditions hold:

1. union, intersection, and difference of two subsets of ${}^*X$ also belong to ${}^*\mathcal{A}$;
2. $\emptyset, {}^*X$ belong to ${}^*\mathcal{A}$.

Definition 21 *A hyperrational (respectively hyperreal) valued finitely additive probability measure is a nonnegative set function* ${}^*\mathbf{P}\colon {}^*\mathcal{A} \to V_{{}^*\mathbf{Q}}$ *(respectively* ${}^*\mathbf{P}\colon {}^*\mathcal{A} \to V_{{}^*\mathbf{R}}$*) that satisfies the following properties:*

1. ${}^*\mathbf{P}(A) \geq {}^*0$ *for all* $A \in {}^*\mathcal{A}$,
2. ${}^*\mathbf{P}({}^*X) = {}^*1$ *and* ${}^*\mathbf{P}(\emptyset) = {}^*0$,
3. *if* $A \in {}^*\mathcal{A}$ *and* $B \in {}^*\mathcal{A}$ *are disjoint, then* ${}^*\mathbf{P}(A \cup B) = {}^*\mathbf{P}(A) + {}^*\mathbf{P}(B)$. *In particular* ${}^*\mathbf{P}(\neg A) = {}^*1 - {}^*\mathbf{P}(A)$ *for all* $A \in {}^*\mathcal{A}$.

Now consider hyper-valued fuzzy logic.

Definition 22 *Suppose* ${}^*X$ *is a nonstandard extension. Then a hyperrational (respectively hyperreal) valued fuzzy set* A *in* ${}^*X$ *is a set defined by means of the membership function* ${}^*\mu_A\colon {}^*X \to V_{{}^*\mathbf{Q}}$ *(respectively by means of the membership function* ${}^*\mu_A\colon {}^*X \to V_{{}^*\mathbf{R}}$*).*

A set $A \subseteq {}^*X$ is called *crisp* if ${}^*\mu_A(u) = {}^*1$ or ${}^*\mu_A(u) = {}^*0$ for any $u \in {}^*X$.

The logical operations on hyper-valued fuzzy sets are defined as follows:

1. ${}^*\mu_{A\cap B}(x) = \min({}^*\mu_A(x), {}^*\mu_B(x))$;
2. ${}^*\mu_{A\cup B}(x) = \max({}^*\mu_A(x), {}^*\mu_B(x))$;
3. ${}^*\mu_{A+B}(x) = {}^*\mu_A(x) + {}^*\mu_B(x) - {}^*\mu_A(x) \cdot {}^*\mu_B(x)$;
4. ${}^*\mu_{\neg A}(x) = \neg{}^*\mu_A(x) = {}^*1 - {}^*\mu_A(x)$.

Chapter 9

p-Adic valued logics

9.1 Preliminaries

Let us remember that the expansion

$$n = \alpha_{-N} \cdot p^{-N} + \alpha_{-N+1} \cdot p^{-N+1} + \ldots + \alpha_{-1} \cdot p^{-1} + \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum_{k=-N}^{+\infty} \alpha_k \cdot p^k,$$

where $\alpha_k \in \{0, 1, \ldots, p-1\}$, $\forall k \in \mathbf{Z}$, and $\alpha_{-N} \neq 0$, is called the *canonical expansion of p-adic number n* (or p-adic expansion for n). The number n is called p-adic. This number can be identified with sequences of digits: $n = \ldots \alpha_2 \alpha_1 \alpha_0, \alpha_{-1} \alpha_{-2} \ldots \alpha_{-N}$. We denote the set of such numbers by $\mathbf{Q}_p$.

The expansion $n = \alpha_0 + \alpha_1 \cdot p + \ldots + \alpha_k \cdot p^k + \ldots = \sum_{k=0}^{\infty} \alpha_k \cdot p^k$, where $\alpha_k \in \{0, 1, \ldots, p-1\}$, $\forall k \in \mathbf{N} \cup \{0\}$, is called the *expansion of p-adic integer n*. The integer n is called p-adic. This number sometimes has the following notation: $n = \ldots \alpha_3 \alpha_2 \alpha_1 \alpha_0$. We denote the set of such numbers by $\mathbf{Z}_p$.

If $n \in \mathbf{Z}_p$, $n \neq 0$, and its canonical expansion contains only a finite number of nonzero digits α_j, then n is natural number (and vice versa). But if $n \in \mathbf{Z}_p$ and its expansion contains an infinite number of nonzero digits α_j, then n is an infinitely large natural number. Thus the set of p-adic integers contains actual infinities $n \in \mathbf{Z}_p \backslash \mathbf{N}$, $n \neq 0$. This is one of the most important features of non-Archimedean number systems, therefore it is natural to compare $\mathbf{Z}_p$ with the set of nonstandard numbers ${}^*\mathbf{Z}$. Also, the set $\mathbf{Z}_p$ contains *non-exclusive elements*.

It is evident that $\mathbf{Q}_p = \{0, 1, \ldots, p-1\}^{\mathbf{Z}} \backslash \mathcal{U}$ and $\mathbf{Z}_p = \{0, 1, \ldots, p-1\}^{\mathbf{N}} \backslash \mathcal{U}$.

9.2 p-Adic valued partial order structure

Extend the standard order structure on $\{0, \ldots, p-1\}$ to a partial order structure on $\mathbf{Z}_p$. Define this partial order structure on $\mathbf{Z}_p$ as follows:

$\mathcal{O}_{\mathbf{Z}_p}$ Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$.

1. We set $x \leq y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$
2. We set $x < y$ if we have $x_n \leq y_n$ for each $n = 0, 1, \ldots$ and there exists n_0 such that $x_{n_0} < y_{n_0}$.
3. We set $x = y$ if $x_n = y_n$ for each $n = 0, 1, \ldots$

Now introduce two operations max, min in the partial order structure on $\mathbf{Z}_p$:

1 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\min(x, y) = x$ if and only if $x \leq y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$,

2 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\max(x, y) = y$ if and only if $x \leq y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$,

3 for all p-adic integers $x, y \in \mathbf{Z}_p$, $\max(x, y) = \min(x, y) = x = y$ if and only if $x = y$ under condition $\mathcal{O}_{\mathbf{Z}_p}$.

The ordering relation $\mathcal{O}_{\mathbf{Z}_p}$ is not linear, but partial, because there exist elements $x, z \in \mathbf{Z}_p$, which are incompatible. As an example, let $p = 2$ and let $x = -\frac{1}{3} = \ldots 10101 \ldots 101$, $z = -\frac{2}{3} = \ldots 01010 \ldots 010$. Then the numbers x and z are incompatible.

Thus,

4 Let $x = \ldots x_n \ldots x_1 x_0$ and $y = \ldots y_n \ldots y_1 y_0$ be the canonical expansions of two p-adic integers $x, y \in \mathbf{Z}_p$ and x, y are incompatible under condition $\mathcal{O}_{\mathbf{Z}_p}$. We get $\min(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set

1. $z_n = y_n$ if $x_n \geq y_n$,
2. $z_n = x_n$ if $x_n \leq y_n$,
3. $z_n = x_n = y_n$ if $x_n = y_n$.

We get $\max(x, y) = z = \ldots z_n \ldots z_1 z_0$, where, for each $n = 0, 1, \ldots$, we set

1. $z_n = y_n$ if $x_n \leq y_n$,
2. $z_n = x_n$ if $x_n \geq y_n$,
3. $z_n = x_n = y_n$ if $x_n = y_n$.

It is important to remark that there exists the maximal number $N_{max} \in \mathbf{Z}_p$ under condition $\mathcal{O}_{\mathbf{Z}_p}$. It is easy to see:

$$N_{max} = -1 = (p-1) + (p-1) \cdot p + \ldots + (p-1) \cdot p^k + \ldots = \sum_{k=0}^{\infty} (p-1) \cdot p^k$$

Therefore

5 $\min(x, N_{max}) = x$ and $\max(x, N_{max}) = N_{max}$ for any $x \in \mathbf{Z}_p$.

9.3 p-Adic valued matrix logics

Now consider *p-adic valued* Łukasiewicz*'s matrix logic* $\mathfrak{M}_{\mathbf{Z}_p}$.

Definition 23 *The ordered system* $\langle V_{\mathbf{Z}_p}, \neg_L, \rightarrow_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{N_{max}\}\rangle$ *is called p-adic valued* Łukasiewicz*'s matrix logic* $\mathfrak{M}_{\mathbf{Z}_p}$*, where*

1. $V_{\mathbf{Z}_p} = \{0, \ldots, N_{max}\} = \mathbf{Z}_p$,
2. *for all* $x \in V_{\mathbf{Z}_p}$, $\neg_L x = N_{max} - x$,
3. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \rightarrow_L y = (N_{max} - \max(x, y) + y)$,
4. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \vee y = (x \rightarrow_L y) \rightarrow_L y = \max(x, y)$,
5. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \wedge y = \neg_L(\neg_L x \vee \neg_L y) = \min(x, y)$,
6. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,
7. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,
8. $\{N_{max}\}$ *is the set of designated truth values.*

The truth value $0 \in \mathbf{Z}_p$ is false, the truth value $N_{max} \in \mathbf{Z}_p$ is true, and other truth values $x \in \mathbf{Z}_p\{0, N_{max}\}$ are neutral.

Definition 24 *p-Adic valued* Gödel*'s matrix logic* $G_{\mathbf{Z}_p}$ *is the structure* $\langle V_{\mathbf{Z}_p}$, $\neg_G$, $\rightarrow_G$, $\vee$, $\wedge$, $\widetilde{\exists}$, $\widetilde{\forall}$, $\{N_{max}\}\rangle$, *where*

1. $V_{\mathbf{Z}_p} = \{0, \ldots, N_{max}\} = \mathbf{Z}_p$,
2. *for all* $x \in V_{\mathbf{Z}_p}$, $\neg_G x = x \rightarrow_G 0$,
3. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \rightarrow_G y = N_{max}$ *if* $x \leqslant y$ *and* $x \rightarrow_G y = y$ *otherwise,*
4. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \vee y = \max(x, y)$,

5. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \wedge y = \min(x, y)$,

6. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\exists}(M) = \max(M)$, *where* $\max(M)$ *is a maximal element of* M,

7. *for a subset* $M \subseteq V_{\mathbf{Z}_p}$, $\widetilde{\forall}(M) = \min(M)$, *where* $\min(M)$ *is a minimal element of* M,

8. $\{N_{max}\}$ *is the set of designated truth values.*

Definition 25 *p-Adic valued* POST*'s matrix logic* $P_{\mathbf{Z}_p}$ *is the structure* $\langle V_{\mathbf{Z}_p}, \neg_P, \vee, \{N_{max}\}\rangle$, *where*

1. $V_{\mathbf{Z}_p} = \mathbf{Z}_p$,

2. *for all* $\ldots x_n \ldots x_1 x_0 \in V_{\mathbf{Z}_p}$, $\neg_P \ldots x_n \ldots x_1 x_0 = \ldots y_n \ldots y_1 y_0$, *where*

$$y_j = x_j + 1 \mod p$$

for each $j = 0, 1, 2, \ldots$,

3. *for all* $x, y \in V_{\mathbf{Z}_p}$, $x \vee y = \max(x, y)$,

4. $\{N_{max}\}$ *is the set of designated truth values.*

Proposition 11 *The 2-adic valued logic* $\mathfrak{M}_{\mathbf{Z}_2} = \langle V_{\mathbf{Z}_2}, \neg_L, \rightarrow_L, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{N_{max}\}\rangle$ *is a Boolean algebra.*

Proof. Indeed, the operation $\neg_L$ in $\mathfrak{M}_{\mathbf{Z}_2}$ is the Boolean complement:

1. $\max(x, \neg_L x) = N_{max}$,

2. $\min(x, \neg_L x) = 0$. □

9.4 p-Adic probability theory and p-adic fuzzy logic

Let us remember that the frequency theory of probability was created by von MISES in [97]. This theory is based on the notion of a collective: "We will say that a collective is a mass phenomenon or a repetitive event, or simply a long sequence of observations for which there are sufficient reasons to believe that the relative frequency of the observed attribute would tend to a fixed limit if the observations were infinitely continued. This limit will be called the probability of the attribute considered within the given collective" [97].

As an example, consider a random experiment $\mathcal{S}$ and by $L = \{s_1, \ldots, s_m\}$ denote the set of all possible results of this experiment. The set $\mathcal{S}$ is called the label set, or the set of attributes. Suppose there are N realizations of

$\mathcal{S}$ and write a result x_j after each realization. Then we obtain the finite sample: $x = (x_1, \ldots, x_N), x_j \in L$. A collective is an infinite idealization of this finite sample: $x = (x_1, \ldots, x_N, \ldots), x_j \in L$. Let us compute frequencies $\nu_N(\alpha; x) = n_N(\alpha; x)/N$, where $n_N(\alpha; x)$ is the number of realizations of the attribute α in the first N tests. There exists the statistical stabilization of relative frequencies: the frequency $\nu_N(\alpha; x)$ approaches a limit as N approaches infinity for every label $\alpha \in L$. This limit $\mathbf{P}(\alpha) = \lim \nu_N(\alpha; x)$ is said to be the probability of the label α in the frequency theory of probability. Sometimes this probability is denoted by $\mathbf{P}_x(\alpha)$ to show a dependence on the collective x. Notice that the limits of relative frequencies have to be stable with respect to a place selection (a choice of a subsequence) in the collective. KHRENNIKOV developed von MISES' idea and proposed the frequency theory of p-adic probability in [79], [80]. We consider here some basic definitions of KHRENNIKOV's theory.

We shall study some ensembles $S = S_N$, which have a p-adic volume N, where N is the p-adic integer. If N is finite, then S is the ordinary finite ensemble. If N is infinite, then S has essentially p-adic structure. Consider a sequence of ensembles M_j having volumes $l_j \cdot p^j$, $j = 0, 1, \ldots$ Get $S = \bigcup\limits_{j=0}^{\infty} M_j$. Then the cardinality $|S| = N$. We may imagine an ensemble S as Get $S = \cup_{j=0}^{\infty} M_j$. Then the cardinality $|S| = N$. We may imagine an ensemble S as being the population of a tower $T = T_S$, which has an infinite number of floors with the following distribution of population through floors: population of j-th floor is M_j. Set $T_k = \cup_{j=0}^{k} M_j$. This is population of the first $k+1$ floors. Let $A \subset S$ and let there exists: $n(A) = \lim\limits_{k \to \infty} n_k(A)$, where $n_k(A) = |A \cap T_k|$. The quantity $n(A)$ is said to be a p-adic volume of the set A.

We define the probability of A by the standard proportional relation:

$$\mathbf{P}(A) \triangleq \mathbf{P}_S(A) = \frac{n(A)}{N}, \tag{9.1}$$

where $|S| = N$, $n(A) = |A \cap S|$.

We denote the family of all $A \subset S$, for which $\mathbf{P}(A)$ exists, by $\mathcal{G}_S$. The sets $A \in \mathcal{G}_S$ are said to be events. The ordered system $\langle S, \mathcal{G}_S, \mathbf{P}_S \rangle$ is called a *p-adic ensemble probability space for the ensemble S.*

Proposition 12 *Let F be the set algebra which consists of all finite subsets and their complements. Then $F \subset \mathcal{G}_S$.*

Proof. Let A be a finite set. Then $n(A) = |A|$ and the probability of A has the form:

$$\mathbf{P}(A) = \frac{|A|}{|S|}$$

Now let $B = \neg A$. Then $|B \cap T_k| = |T_k| - |A \cap T_k|$. Hence there exists $\lim\limits_{k\to\infty} |B \cap T_k| = N - |A|$. This equality implies the standard formula:

$$\mathbf{P}(\neg A) = 1 - \mathbf{P}(A)$$

In particular, we have: $\mathbf{P}(S) = 1$. $\square$

The next propositions are proved in [79]:

Proposition 13 *Let $A_1, A_2 \in \mathcal{G}_S$ and $A_1 \cap A_2 = \emptyset$. Then $A_1 \cup A_2 \in \mathcal{G}_S$ and*

$$\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2).$$

$\square$

Proposition 14 *Let $A_1, A_2 \in \mathcal{G}_S$. The following conditions are equivalent:*

1. $A_1 \cup A_2 \in \mathcal{G}_S$,
2. $A_1 \cap A_2 \in \mathcal{G}_S$,
3. $A_1 \backslash A_2 \in \mathcal{G}_S$,
4. $A_2 \backslash A_1 \in \mathcal{G}_S$. $\square$

But it is possible to find sets $A_1, A_2 \in \mathcal{G}_S$ such that, for example, $A_1 \cup A_2 \notin \mathcal{G}_S$. Thus, the family $\mathcal{G}_S$ is not an algebra, but a semi-algebra (it is closed only with respect to a finite unions of sets, which have empty intersections). $\mathcal{G}_S$ is not closed with respect to countable unions of such sets.

Proposition 15 *Let $A \in \mathcal{G}_S$, $\mathbf{P}(A) \neq 0$ and $B \in \mathcal{G}_A$. Then $B \in \mathcal{G}_S$ and the following* BAYES *formula holds:*

$$\mathbf{P}_A(B) = \frac{\mathbf{P}_S(B)}{\mathbf{P}_S(A)} \tag{9.2}$$

Proof. The tower T_A of the A has the following population structure: there are M_{A_j} elements on the j-th floor. In particular, $T_{A_k} = T_k \cap A$. Thus

$$n_{A_k}(B) = |B \cap T_{A_k}| = |B \cap T_k| = n_k(B)$$

for each $B \subset A$. Hence the existence of $n_A(B) = \lim\limits_{k\to\infty} n_{A_k}(B)$ implies the existence of $n_S(B)$ with $n_S(B) = \lim\limits_{k\to\infty} n_k(B)$. Moreover, $n_S(B) = n_A(B)$. Therefore,

$$\mathbf{P}_A(B) = \frac{n_A(B)}{n_S(A)} = \frac{n_A(B)/|S|}{n_S(A)/|S|}.$$

$\square$

Proposition 16 *Let $N \in \mathbf{Z}_p$, $N \neq 0$ and let the ensemble S_{-1} have the p-adic volume $-1 = N_{max}$ (it is the largest ensemble).*

1. *Then $S_N \in \mathcal{G}_{S_{-1}}$ and*

$$\mathbf{P}_{S_{-1}}(S_N) = \frac{|S_N|}{|S_{-1}|} = -N$$

2. *Then $\mathcal{G}_{S_N} \subset \mathcal{G}_{S_{-1}}$ and probabilities $\mathbf{P}_{S_N}(A)$ are calculated as conditional probabilities with respect to the subensemble S_N of ensemble S_{-1}:*

$$\mathbf{P}_{S_N}(A) = \mathbf{P}_{S_{-1}}(\frac{A}{S_N}) = \frac{\mathbf{P}_{S_{-1}}(A)}{\mathbf{P}_{S_{-1}}(S_N)}, A \in \mathcal{G}_{S_N}$$

□

Transform the Łukasiewicz matrix logic $\mathfrak{M}_{\mathbf{Z}_p}$ into a p-adic probability theory. Let us remember that a formula φ has the truth value $0 \in \mathbf{Z}_p$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if φ is false, a formula φ has the truth value $N_{max} \in \mathbf{Z}_p$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if φ is true, and a formula φ has other truth values $\alpha \in \mathbf{Z}_p \backslash \{0, N_{max}\}$ in $\mathfrak{M}_{\mathbf{Z}_p}$ if φ is neutral.

Definition 26 *A function $\mathbf{P}(\varphi)$ is said to be a probability measure of a formula φ in $\mathfrak{M}_{\mathbf{Z}_p}$ if $\mathbf{P}(\varphi)$ ranges over numbers of $\mathbf{Q}_p$ and satisfies the following axioms:*

1. $\mathbf{P}(\varphi) = \frac{\mathrm{val}_I(\varphi)}{N_{max}}$, *where* $\mathrm{val}_I(\varphi)$ *is a truth value of* φ;
2. *if a conjunction $\varphi \wedge \psi$ has the truth value 0, then $\mathbf{P}(\varphi \vee \psi) = \mathbf{P}(\varphi) + \mathbf{P}(\psi)$,*
3. $\mathbf{P}(\varphi \wedge \psi) = \min(\mathbf{P}(\varphi), \mathbf{P}(\psi))$.

Notice that:

1. taking into account condition 1 of our definition, if φ has the truth value N_{max} for any its interpretations, i.e. φ is a tautology, then $\mathbf{P}(\varphi) = 1$ in all possible worlds, and if φ has the truth value 0 for any its interpretations, i.e. φ is a contradiction, then $\mathbf{P}(\varphi) = 0$ in all possible worlds;
2. under condition 1, we obtain also $\mathbf{P}(\neg\varphi) = 1 - \mathbf{P}(\varphi)$.

Since $\mathbf{P}(N_{max}) = 1$, we have

$$\mathbf{P}(\max\{x \in V_{\mathbf{Z}_p}\}) = \sum_{x \in V_{\mathbf{Z}_p}} \mathbf{P}(x) = 1$$

All events have a conditional plausibility in the logical theory of p-adic probability:

$$\mathbf{P}(\varphi) \leftrightarrow \mathbf{P}(\varphi / N_{max}), \tag{9.3}$$

i.e., for any φ, we consider the conditional plausibility that there is an event of φ, given an event N_{max},

$$\mathbf{P}(\varphi/\psi) = \frac{\mathbf{P}(\varphi \wedge \psi)}{\mathbf{P}(\psi)}. \tag{9.4}$$

The probability interpretation of the Łukasiewicz logic $\mathfrak{M}_{\mathbf{Z}_p}$ shows that this logic is a special system of fuzzy logic. Indeed, we can consider the membership function μ_A as a p-adic valued predicate.

Definition 27 *Suppose X is a non-empty set. Then a p-adic valued fuzzy set A in X is a set defined by means of the membership function $\mu_A\colon X \to \mathbf{Z}_p$, where $\mathbf{Z}_p$ is the set of all p-adic integers.*

It is obvious that the set A is completely determined by the set of tuples $\{\langle u, \mu_A(u)\rangle \colon u \in X\}$. We define a norm $|\cdot|_p \colon \mathbf{Q}_p \to \mathbf{R}$ on $\mathbf{Q}_p$ as follows:

$$|n = \sum_{k=-N}^{+\infty} \alpha_k \cdot p^k|_p \triangleq p^{-L},$$

where $L = \max\{k \colon n \equiv 0 \mod p^k\} \geq 0$, i.e. L is an index of the first number distinct from zero in p-adic expansion of n. Note that $|0|_p \triangleq 0$. The function $|\cdot|_p$ has values 0 and $\{p^\gamma\}_{\gamma \in \mathbf{Z}}$ on $\mathbf{Q}_p$. Finally, $|x|_p \geq 0$ and $|x|_p = 0 \leftrightarrow x = 0$. A set $A \subset X$ is called *crisp* if $|\mu_A(u)|_p = 1$ or $|\mu_A(u)|_p = 0$ for any $u \in X$. Notice that $|\mu_A(u) = 1|_p = 1$ and $|\mu_A(u) = 0|_p = 0$. Therefore our membership function is an extension of the classical characteristic function. Thus, $A = B$ causes $\mu_A(u) = \mu_B(u)$ for all $u \in X$ and $A \subseteq B$ causes $|\mu_A(u)|_p \leqslant |\mu_B(u)|_p$ for all $u \in X$.

In p-adic fuzzy logic, there always exists a non-empty intersection of two crisp sets. In fact, suppose the sets A, B have empty intersection and A, B are crisp. Consider two cases under condition $\mu_A(u) \neq \mu_B(u)$ for any u. First, $|\mu_A(u)|_p = 0$ or $|\mu_A(u)|_p = 1$ for all u and secondly $|\mu_B(u)|_p = 0$ or $|\mu_B(u)|_p = 1$ for all u. Assume we have $\mu_A(u_0) = N_{max}$ for some u_0, i.e., $|\mu_A(u_0)|_p = 1$. Then $\mu_B(u_0) \neq N_{max}$, but this doesn't mean that $\mu_B(u_0) = 0$. It is possible that $|\mu_A(u_0)|_p = 1$ and $|\mu_B(u_0)|_p = 1$ for u_0.

Now we set logical operations on p-adic fuzzy sets:

1. $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$;
2. $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$;
3. $\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \min(\mu_A(x), \mu_B(x))$;
4. $\mu_{\neg A}(x) = \neg\mu_A(x) = N_{max} - \mu_A(x) = -1 - \mu_A(x)$.

Chapter 10

Fuzzy logics

10.1 Preliminaries

In this section we cover the main essentials of the t-norm based approach, defining fuzzy logics as logics based on t-norms and their residua. Recall that t-*norm* is an operation $*\colon [0,1]^2 \to [0,1]$ which is commutative and associative, non-decreasing in both arguments and have 1 as unit element and 0 as zero element, i.e.

$$x * y = y * x,$$
$$(x * y) * z = x * (y * z),$$
$$x \leq x' \text{ and } y \leq y' \text{ implies } x * y \leq x' * y',$$
$$1 * x = x,\ 0 * x = 0.$$

Each t-norm determines uniquely its corresponding implication $\Rightarrow$ (*residuum*) satisfying for all $x, y, z \in [0,1]$,

$$z \leq x \Rightarrow y \text{ iff } x * z \leq y$$

The following are important examples of continuous t-norms and their residua:

1. Łukasiewicz*'s logic*:
   - $x * y = \max(x + y - 1, 0)$,
   - $x \Rightarrow y = 1$ for $x \leq y$ and $x \Rightarrow y = 1 - x + y$ otherwise.

   In this logic $*$ and $\Rightarrow$ are denoted by $\&_L$ and $\to_L$ respectively.

2. Gödel*'s logic*:
   - $x * y = \min(x, y)$,

- $x \Rightarrow y = 1$ for $x \leq y$ and $x \Rightarrow y = y$ otherwise.

In this logic $*$ and $\Rightarrow$ are denoted by $\&_G$ and $\rightarrow_G$ respectively.

3. *Product logic*:

- $x * y = x \cdot y$
- $x \Rightarrow y = 1$ for $x \leq y$ and $x \Rightarrow y = y/x$ otherwise.

In this logic $*$ and $\Rightarrow$ are denoted by $\&_\Pi$ and $\rightarrow_\Pi$ respectively.

A *regular residuated lattice* (or a *BL-algebra*) is an algebra $\mathbf{L}_L = \langle L,\ \wedge,\ \vee,\ *,\ \Rightarrow,\ 0,\ 1\rangle$ such that (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with the largest element 1 and the least element 0, (2) $\langle L, *, 1\rangle$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative, and $1 * x = x$ for all x, (3) the following conditions hold

$$z \leq (x \Rightarrow y) \text{ iff } x * z \leq y \text{ for all } x, y, z;$$

$$x \wedge y = x * (x \Rightarrow y);$$

$$x \vee y = ((x \Rightarrow y) \Rightarrow y) \wedge ((y \Rightarrow x) \Rightarrow x),$$

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1.$$

It is known the following result. Let $*$ be a continuous t-norm with residuum $\Rightarrow$, then $\mathcal{A} = \langle [0,1], \min, \max,\ *,\ \Rightarrow,\ 0,\ 1\rangle$ is a BL-algebra, called a standard BL-algebra; if $*$ is the Łukasiewicz, Gödel or Product t-norm then $\mathcal{A}$ is an Ł-algebra, **G**-algebra or **Π**-algebra respectively, called the standard Ł-algebra, **G**-algebra or **Π**-algebra.

Let us remember that every relation R of D^n can be represented by its characteristic function $c_R : D^n \to \{0,1\}$ defined by setting $c_R(x_1, \ldots, x_n) = 1$ if $\langle x_1, \ldots, x_n\rangle \in R$ and $c_R(x_1, \ldots, x_n) = 0$ if $\langle x_1, \ldots, x_n\rangle \notin R$. We can also represent the extension of a vague predicate by a generalized characteristic function assuming values in a set V. Assume that $V = \{0, 1, \ldots, n-1\}$ or $V = [0,1]$. A *fuzzy relation* or *fuzzy subset* P of D^n is any map $P : D^n \to V$. For instance, the subset $\emptyset \subset D^n$ (resp. $D^n \subseteq D^n$) can be represented by a fuzzy relation P that is considered as the constant map $P : D^n \ni x \mapsto 0$ (resp. $P : D^n \ni x \mapsto 1$), where 0 is the least element of V and 1 is the largest element of V. We denote by $\mathcal{F}(D^n)$ the class of fuzzy subsets of D^n. It is obvious that $\mathcal{P}(D^n) \subset \mathcal{F}(D^n)$, where $\mathcal{P}(D^n)$ is the powerset.

Let 1 be the top element of V. Observe that, if P and P' are fuzzy subsets of D^n, then we can define,

- the *inclusion* by setting $P \subseteq P'$ iff $P(x_1, \ldots, x_n) \leq P'(x_1, \ldots, x_n)$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- ŁUKASIEWICZ*'s relative complement* $P \to_L P'$ by setting $(P \to_L P')$ $(x_1, \ldots, x_n) = 1 - \max(P(x_1, \ldots, x_n), P'(x_1, \ldots, x_n)) + P'(x_1, \ldots, x_n)$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- ŁUKASIEWICZ*'s complement* $\neg_L P$ of P by setting $(\neg_L P)(x_1, \ldots, x_n) = 1 - P(x_1, \ldots, x_n)$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- ŁUKASIEWICZ*'s intersection* $P\&_L P'$ as $(P\&_L P')(x_1, \ldots, x_n) = \max(0, P(x_1, \ldots, x_n) + P'(x_1, \ldots, x_n) - 1)$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- the *Product relative complement* $P \to_\Pi P'$ by setting $(P \to_\Pi P')(x_1, \ldots, x_n) = 1$ if $P(x_1, \ldots, x_n) \leq P'(x_1, \ldots, x_n)$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$ and $(P \to_\Pi P')(x_1, \ldots, x_n) = \frac{P(x_1, \ldots, x_n)}{P'(x_1, \ldots, x_n)}$ otherwise,

- the *Product complement* $\neg_\Pi P$ as $\neg_\Pi P(x_1, \ldots, x_n) := P(x_1, \ldots, x_n) \to_\Pi 0$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- the *Product intersection* $P\&_\Pi P'$ as $(P\&_\Pi P')(x_1, \ldots, x_n) = P(x_1, \ldots, x_n) \cdot P'(x_1, \ldots, x_n)$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- the *union* $P \vee P'$ as $(P \vee P')(x_1, \ldots, x_n) = \max(P(x_1, \ldots, x_n), P'(x_1, \ldots, x_n))$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- the *intersection* $P \wedge P'$ by setting $(P \wedge P')(x_1, \ldots, x_n) = \min(P(x_1, \ldots, x_n), P'(x_1, \ldots, x_n))$ for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- the *union* $\bigvee\limits_{i \in I} P_i$, given a family $(P_i)_{i \in I}$ of fuzzy subsets of D^n, as

  $$\bigvee_{i \in I} P_i(x_1, \ldots, x_n) = \max\{P_i(x_1, \ldots, x_n) \colon i \in I\}$$

  for every $\langle x_1, \ldots, x_n\rangle \in D^n$,

- the *intersection* $\bigwedge\limits_{i \in I} P_i$, given a family $(P_i)_{i \in I}$ of fuzzy subsets of D^n, as

  $$\bigwedge_{i \in I} P_i(x_1, \ldots, x_n) = \min\{P_i(x_1, \ldots, x_n) \colon i \in I\}$$

  for every $\langle x_1, \ldots, x_n\rangle \in D^n$.

The structure $\mathbf{L}_V = \langle \mathcal{F}(D^n), \wedge, \vee, *, \Rightarrow, \bot(D^n), \top(D^n)\rangle$ is a BL-algebra, where $\bot(D^n)$ and $\top(D^n)$ are the constant maps $\bot(D^n) : D^n \ni x \mapsto 0 \in V$ and $\top(D^n) : D^n \ni x \mapsto 1 \in V$ respectively. It is the direct power, with index set D^n, of the structure $\langle V, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$.

10.2 Basic fuzzy logic $BL\forall$

The basic fuzzy logic denoted by $BL\forall$ has just two initial propositional operations: &, $\rightarrow$, which are understood as t-norm and its residuum respectively. The negation is derivable:

$$\neg\psi =: \psi \rightarrow \bot,$$

where $\bot$ is the truth constant 'falsehood'.

In $BL\forall$ we can define the following new operations:

- $\varphi \wedge \psi := \varphi \& (\varphi \rightarrow \psi)$,
- $\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$,
- $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$,
- $\varphi \oplus \psi := \neg\varphi \rightarrow \psi$,
- $\varphi \ominus \psi := \varphi \& \neg\psi$.

The HILBERT's type calculus for $BL\forall$ consists of the following axioms:

$$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)), \tag{10.1}$$

$$(\varphi \& \psi) \rightarrow \varphi, \tag{10.2}$$

$$(\varphi \& \psi) \rightarrow (\psi \& \varphi), \tag{10.3}$$

$$(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow \varphi)), \tag{10.4}$$

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi), \tag{10.5}$$

$$((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)), \tag{10.6}$$

$$((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi), \tag{10.7}$$

$$\bot \rightarrow \psi, \tag{10.8}$$

$$\forall x\ \varphi(x) \rightarrow \varphi[x/t], \tag{10.9}$$

$$\varphi[x/t] \rightarrow \exists x\ \varphi(x), \tag{10.10}$$

where the formula $\varphi[x/t]$ is the result of substituting the term t for all free occurrences of x in φ,

$$\forall x(\chi \to \varphi) \to (\chi \to \forall x\varphi), \tag{10.11}$$

$$\forall x(\varphi \to \chi) \to (\exists x\varphi \to \chi), \tag{10.12}$$

$$\forall x(\chi \vee \varphi) \to (\chi \vee \forall x\varphi), \tag{10.13}$$

where x is not free in χ.

In $BL\forall$ there are the following inference rules:

1. *Modus ponens*: from φ and $\varphi \to \psi$ infer ψ:

$$\frac{\varphi, \quad \varphi \to \psi}{\psi}.$$

2. *Substitution rule*: we can substitute any formulas for propositional variables.

3. *Generalization*: from φ infer $\forall x\ \varphi(x)$:

$$\frac{\varphi}{\forall x\ \varphi(x)}.$$

Theorem 4 (Soundness and Completeness) *Let φ be a formula of $BL\forall$, T a $BL\forall$-theory. Then the following conditions are equivalent:*

- *$T \vdash \varphi$;*
- $\mathrm{val}_I(\varphi) = 1$ *for each BL-algebra (with infinite intersection and infinite union) that is model for T.*

Proof. See [67]. □

10.3 Non-Archimedean valued BL-algebras

Now introduce the following new operations defined for all $[x], [y] \in {}^*\mathbf{Q}$ in the partial order structure $\mathcal{O}_{^*\mathbf{Q}}$:

- $[x] \to_L [y] = {}^*1 - \max([x], [y]) + [y]$,

- $[x] \rightarrow_{\Pi} [y] = {}^*1$ if $[x] \leq [y]$ and $[x] \rightarrow_{\Pi} [y] = \min({}^*1, \frac{[y]}{[x]})$ otherwise,

  notice that we have $\min({}^*1, \frac{[y]}{[x]}) = [h]$ iff there exists $[h] \in {}^*\mathbf{Q}_{[0,1]}$ such that $\{\alpha \in \mathbf{N}\colon \ \min(1, \frac{y(\alpha)}{x(\alpha)}) = h(\alpha)\} \in \mathcal{U}$, let us also remember that the members $[x]$, $[y]$ can be incompatible under $\mathcal{O}_{{}^*\mathbf{Q}}$,
- $\neg_L[x] = {}^*1 - [x]$, i.e. $[x] \rightarrow_L {}^*0$,
- $\neg_{\Pi}[x] = {}^*1$ if $[x] = {}^*0$ and $\neg_{\Pi}[x] = {}^*0$ otherwise, i.e. $\neg_{\Pi}[x] = [x] \rightarrow_{\Pi} {}^*0$,
- $\Delta[x] = {}^*1$ if $[x] = {}^*1$ and $\Delta[x] = {}^*0$ otherwise, i.e. $\Delta[x] = \neg_{\Pi}\neg_L[x]$,
- $[x]\&_L[y] = \max([x], {}^*1 - [y]) + [y] - {}^*1$, i.e. $[x]\&_L[y] = \neg_L([x] \rightarrow_L \neg_L[y])$,
- $[x]\&_{\Pi}[y] = [x] \cdot [y]$,
- $[x] \oplus [y] := \neg_L[x] \rightarrow_L [y]$,
- $[x] \ominus [y] := [x]\&_L\neg_L[y]$,
- $[x] \wedge [y] = \min([x], [y])$, i.e. $[x] \wedge [y] = [x]\&_L([x] \rightarrow_L [y])$,
- $[x] \vee [y] = \max([x], [y])$, i.e. $[x] \vee [y] = ([x] \rightarrow_L [y]) \rightarrow_L [y]$,
- $[x] \rightarrow_G [y] = {}^*1$ if $[x] \leq [y]$ and $[x] \rightarrow_G [y] = [y]$ otherwise, i.e. $[x] \rightarrow_G [y] = \Delta([x] \rightarrow_L [y]) \vee [y]$,
- $\neg_G[x] := [x] \rightarrow_G {}^*0$.

A *hyperrational valued BL-matrix* is a structure $\mathbf{L}_{{}^*\mathbf{Q}} = \langle {}^*\mathbf{Q}_{[0,1]}, \wedge, \vee, *, \Rightarrow, {}^*0, {}^*1\rangle$ such that (1) $\langle {}^*\mathbf{Q}_{[0,1]}, \wedge, \vee, {}^*0, {}^*1\rangle$ is a lattice with the largest element ^{*1} and the least element ^{*0}, (2) $\langle {}^*\mathbf{Q}_{[0,1]}, *, {}^*1\rangle$ is a commutative semigroup with the unit element ^{*1}, i.e. $*$ is commutative, associative, and ${}^*1 * [x] = [x]$ for all $[x] \in {}^*\mathbf{Q}_{[0,1]}$, (3) the following conditions hold

$$[z] \leq ([x] \Rightarrow [y]) \text{ iff } [x] * [z] \leq [y] \text{ for all } [x], [y], [z];$$
$$[x] \wedge [y] = [x] * ([x] \Rightarrow [y]);$$
$$[x] \vee [y] = (([x] \Rightarrow [y]) \Rightarrow [y]) \wedge (([y] \Rightarrow [x]) \Rightarrow [x]),$$
$$([x] \Rightarrow [y]) \vee ([y] \Rightarrow [x]) = {}^*1.$$

If we replace the set $\mathbf{Q}_{[0,1]}$ by $\mathbf{R}_{[0,1]}$ and the set ${}^*\mathbf{Q}_{[0,1]}$ by ${}^*\mathbf{R}_{[0,1]}$ in all above definitions, then we obtain *hyperreal valued BL-matrix* $\mathbf{L}_{{}^*\mathbf{R}}$. Matrices $\mathbf{L}_{{}^*\mathbf{Q}}$, $\mathbf{L}_{{}^*\mathbf{R}}$ are different versions of a non-Archimedean valued BL-algebra. Continuing in the same way, we can build non-Archimedean valued Ł-algebra, **G**-algebra, and **Π**-algebra.

Further consider the following new operations defined for all $x, y \in \mathbf{Z}_p$ in the partial order structure $\mathcal{O}_{\mathbf{Z}_p}$:

- $x \to_L y = N_{max} - \max(x, y) + y$,
- $x \to_\Pi y = N_{max}$ if $x \leq y$ and $x \to_\Pi y =$ integral part of $\frac{y}{x}$ otherwise,
- $\neg_L x = N_{max} - x$, i.e. $x \to_L 0$,
- $\neg_\Pi x = N_{max}$ if $x = 0$ and $\neg_\Pi x = 0$ otherwise, i.e. $\neg_\Pi x = x \to_\Pi 0$,
- $\Delta x = N_{max}$ if $x = N_{max}$ and $\Delta x = 0$ otherwise, i.e. $\Delta x = \neg_\Pi \neg_L x$,
- $x \&_L y = \max(x, N_{max} - y) + y - N_{max}$, i.e. $x \&_L y = \neg_L(x \to_L \neg_L y)$,
- $x \&_\Pi y = x \cdot y$,
- $x \oplus y := \neg_L x \to_L y$,
- $x \ominus y := x \&_L \neg_L y$,
- $x \wedge y = \min(x, y)$, i.e. $x \wedge y = x \&_L (x \to_L y)$,
- $x \vee y = \max(x, y)$, i.e. $x \vee y = (x \to_L y) \to_L y$,
- $x \to_G y = N_{max}$ if $x \leq y$ and $x \to_G y = y$ otherwise, i.e. $x \to_G y = \Delta(x \to_L y) \vee y$,
- $\neg_G x := x \to_G 0$.

A *p-adic valued BL-matrix* is a structure $\mathbf{L}_{\mathbf{Z}_p} = \langle \mathbf{Z}_p, \wedge, \vee, *, \Rightarrow, 0, N_{max} \rangle$.

10.4 Non-Archimedean valued predicate logical language

Recall that for each $i \in [0, 1]$, ${}^*i = [f = i]$, i.e. it is a constant function. Every element of ${}^*[0, 1]$ has the form of infinite tuple $[f] = \langle y_0, y_1, \ldots \rangle$, where $y_i \in [0, 1]$ for each $i = 0, 1, 2, \ldots$

Let $\mathcal{L}$ be a standard first-order language associated with p-valued (resp. infinite-valued) semantics. Then we can get an extension $\mathcal{L}'$ of first-order language $\mathcal{L}$ to set later a language of p-adic valued (resp. hyper-valued) logic.

In $\mathcal{L}'$ we build infinite sequences of well-formed formulas of $\mathcal{L}$:

$$\psi^\infty = \langle \psi_1, \ldots, \psi_N, \ldots \rangle,$$
$$\psi^i = \langle \psi_1, \ldots, \psi_i \rangle,$$

where $\psi_j \in \mathcal{L}$.

A formula ψ^∞ (resp. ψ^i) is called a *formula of infinite length* (resp. a *formula of i-th length*).

Definition 28 *Logical connectives in hyper-valued logic are defined as follows:*

1. $\psi^{\infty} \star \varphi^{\infty} = \langle \psi_1 \star \varphi_1, \ldots, \psi_N \star \varphi_N, \ldots \rangle$, *where* $\star \in \{\&, \rightarrow\}$;
2. $\neg \psi^{\infty} = \langle \neg \psi_1, \ldots, \neg \psi_N, \ldots \rangle$;
3. $\mathrm{Q}x\, \psi^{\infty} = \langle \mathrm{Q}x\, \psi_1, \ldots, \mathrm{Q}x\, \psi_N, \ldots \rangle$, $\mathrm{Q} \in \{\forall, \exists\}$;
4. $\psi^{\infty} \star \varphi^{1} = \langle \psi_1 \star \varphi, \psi_2 \star \varphi, \ldots, \psi_N \star \varphi, \ldots \rangle$, *where* $\star \in \{\&, \rightarrow\}$.

Definition 29 *Logical connectives in p-adic valued logic are defined as follows:*

1. $\psi^{\infty} \star \varphi^{\infty} = \langle \psi_1 \star \varphi_1, \ldots, \psi_N \star \varphi_N, \ldots \rangle$, *where* $\star \in \{\&, \rightarrow\}$;
2. $\neg \psi^{\infty} = \langle \neg \psi_1, \ldots, \neg \psi_N, \ldots \rangle$;
3. $\mathrm{Q}x\, \psi^{\infty} = \langle \mathrm{Q}x\, \psi_1, \ldots, \mathrm{Q}x\, \psi_N, \ldots \rangle$, $\mathrm{Q} \in \{\forall, \exists\}$.
4. $\psi^{\infty} \star \varphi^{i} = \langle \psi_1 \star \varphi_1,\ \ldots,\ \psi_i \star \varphi_i,\ \psi_{i+1} \star \bot,\ \psi_{i+2} \star \bot,\ \ldots,\ \psi_N \star \bot, \ldots \rangle$, *where* $\star \in \{\&, \rightarrow\}$.
5. *suppose* $i < j$, *then* $\psi^{i} \star \varphi^{j} = \langle \psi_1 \star \varphi_1,\ \ldots,\ \psi_i \star \varphi_i,\ \psi_{i+1} \star \bot,\ \psi_{i+2} \star \bot,\ \ldots,\ \psi_j \star \bot \rangle$, *where* $\star \in \{\&, \rightarrow\}$.

An interpretation for a language $\mathcal{L}'$ is defined in the standard way. Extend the valuation of $\mathcal{L}$ to one of $\mathcal{L}'$ as follows.

Definition 30 *Given an interpretation* $I = \langle \mathcal{M}, s \rangle$ *and a valuation* val_I *of* $\mathcal{L}$, *we define the non-Archimedean i-valuation* val_I^i *(resp.* ∞*-valuation* val_I^{∞}*) to be a mapping from formulas* φ^i *(resp.* φ^{∞}*) of* $\mathcal{L}'$ *to truth value set* V^i *(resp.* ${}^*V$*) as follows:*

1. $\mathrm{val}_I^i(\varphi^i) = \langle \mathrm{val}_I(\varphi_1), \ldots, \mathrm{val}_I(\varphi_i) \rangle$.
2. $\mathrm{val}_I^{\infty}(\varphi^{\infty}) = \langle \mathrm{val}_I(\varphi_1), \ldots, \mathrm{val}_I(\varphi_N), \ldots \rangle$.

For example, in p-adic valued case $\mathrm{val}_I^{\infty}(\psi^{\infty} \star \varphi^i) = \langle \mathrm{val}_I(\psi_1 \star \varphi_1),\ \ldots,\ \mathrm{val}_I(\psi_i \star \varphi_i),\ \mathrm{val}_I(\psi_{i+1} \star \bot),\ \mathrm{val}_I(\psi_{i+2} \star \bot),\ \ldots,\ \mathrm{val}_I(\psi_N \star \bot), \ldots \rangle$, where $\star \in \{\&, \rightarrow\}$.

Let $\mathbf{L}_{{}^*V}$ be a non-Archimedean valued BL-matrix. Then the valuations val_I^i and val_I^{∞} of $\mathcal{L}'$ to non-Archimedean valued BL-matrix gives the basic fuzzy logic with the non-Archimedean valued semantics.

We say that an $\mathbf{L}_{{}^*V}$-structure $\mathcal{M}$ is an *i-model* (resp. an ∞-*model*) of an $\mathcal{L}'$-theory T iff $\mathrm{val}_I^i(\varphi^i) = \langle 1, \ldots, 1 \rangle$ (resp. $\mathrm{val}_I^{\infty}(\varphi^{\infty}) = {}^*1$) on $\mathcal{M}$ for each $\varphi^i \in T$ (resp. $\varphi^{\infty} \in T$).

10.5 Non-Archimedean valued basic fuzzy propositional logic BL_∞

Let us construct a non-Archimedean extension of basic fuzzy propositional logic BL denoted by BL_∞. This logic is built in the language $\mathcal{L}'$ and it has a non-Archimedean valued BL-matrix as its semantics.

Remember that the logic BL has just two propositional operations: &, $\rightarrow$, which are understood as t-norm and its residuum respectively.

The logic BL_∞ is given by the following axioms:

$$(\varphi^i \rightarrow \psi^i) \rightarrow ((\psi^i \rightarrow \chi^i) \rightarrow (\varphi^i \rightarrow \chi^i)), \tag{10.14}$$

$$(\varphi^i \& \psi^i) \rightarrow \varphi^i, \tag{10.15}$$

$$(\varphi^i \& \psi^i) \rightarrow (\psi^i \& \varphi^i), \tag{10.16}$$

$$(\varphi^i \& (\varphi^i \rightarrow \psi^i)) \rightarrow (\psi^i \& (\psi^i \rightarrow \varphi^i)), \tag{10.17}$$

$$(\varphi^i \rightarrow (\psi^i \rightarrow \chi^i)) \rightarrow ((\varphi^i \& \psi^i) \rightarrow \chi^i), \tag{10.18}$$

$$((\varphi^i \& \psi^i) \rightarrow \chi^i) \rightarrow (\varphi^i \rightarrow (\psi^i \rightarrow \chi^i)), \tag{10.19}$$

$$((\varphi^i \rightarrow \psi^i) \rightarrow \chi^i) \rightarrow (((\psi^i \rightarrow \varphi^i) \rightarrow \chi^i) \rightarrow \chi^i), \tag{10.20}$$

$$\bot^i \rightarrow \psi^i, \tag{10.21}$$

$$(\varphi^\infty \rightarrow \psi^\infty) \rightarrow ((\psi^\infty \rightarrow \chi^\infty) \rightarrow (\varphi^\infty \rightarrow \chi^\infty)), \tag{10.22}$$

$$(\varphi^\infty \& \psi^\infty) \rightarrow \varphi^\infty, \tag{10.23}$$

$$(\varphi^\infty \& \psi^\infty) \rightarrow (\psi^\infty \& \varphi^\infty), \tag{10.24}$$

$$(\varphi^\infty \& (\varphi^\infty \rightarrow \psi^\infty)) \rightarrow (\psi^\infty \& (\psi^\infty \rightarrow \varphi^\infty)), \tag{10.25}$$

$$(\varphi^\infty \rightarrow (\psi^\infty \rightarrow \chi^\infty)) \rightarrow ((\varphi^\infty \& \psi^\infty) \rightarrow \chi^\infty), \tag{10.26}$$

$$((\varphi^\infty \& \psi^\infty) \to \chi^\infty) \to (\varphi^\infty \to (\psi^\infty \to \chi^\infty)), \tag{10.27}$$

$$((\varphi^\infty \to \psi^\infty) \to \chi^\infty) \to (((\psi^\infty \to \varphi^\infty) \to \chi^\infty) \to \chi^\infty), \tag{10.28}$$

$$\bot^\infty \to \psi^\infty. \tag{10.29}$$

These axioms are said to be *horizontal*. Introduce also some new axioms that show basic properties of non-Archimedean ordered structures. These express a connection between formulas of various length.

1. **Non-Archimedean multiple-validity**. It is well known that there exist infinitesimals that are less than any positive number of $[0, 1]$. This property can be expressed by means of the following logical axiom:

$$(\neg(\psi^1 \leftrightarrow \psi^\infty) \& \neg(\varphi^1 \leftrightarrow \bot^\infty)) \to (\psi^\infty \to \varphi^1), \tag{10.30}$$

   where $\psi^1 = \psi_1$, i.e. it is the first member of an infinite tuple ψ^∞.

2. **p-adic multiple-validity**. There is a well known theorem according to that every equivalence class a for which $|a|_p \leq 1$ (this means that a is a p-adic integer) has exactly one representative Cauchy sequence $\{a_i\}_{i \in \omega}$ for which:

   (a) $0 \leq a_i < p^i$ for $i = 1, 2, 3, \ldots$;
   (b) $a_i \equiv a_{i+1} \mod p^i$ for $i = 1, 2, 3, \ldots$

   This property can be expressed by means of the following logical axioms:

$$\begin{gathered}((\overline{p^{i+1}-1} \ominus \overline{p^i-1}) \to_L \psi^{i+1}) \to_L \\ (\psi^{i+1} \leftrightarrow (\underbrace{\overline{p-1} \oplus \cdots \oplus \overline{p-1}}_{p^i} \oplus \psi^i),\end{gathered} \tag{10.31}$$

$$\begin{gathered}(\psi^{i+1} \leftrightarrow (\overbrace{\overline{k} \oplus \cdots \oplus \overline{k}}^{p^i}) \oplus \psi^i) \to_L \\ ((((\ldots(\overline{p^{i+1}-1} \ominus \underbrace{\overline{p^i-1}) \ominus \ldots) \ominus \overline{p^i-1})}_{0<p-k\leq p} \\ \ominus \neg_L \overline{k}) \to_L \psi^{i+1}),\end{gathered} \tag{10.32}$$

$$\begin{gathered}(\psi^{i+1} \leftrightarrow (\overbrace{\overline{k} \oplus \cdots \oplus \overline{k}}^{p^i}) \oplus \psi^i) \to_L \\ (\psi^{i+1} \to_L (((\ldots(\overline{p^{i+1}-1} \ominus \\ \underbrace{\overline{p^i-1}) \ominus \ldots) \ominus \overline{p^i-1})}_{0\leq(p-1)-k\leq p-1} \ominus \neg_L \overline{k})),\end{gathered} \tag{10.33}$$

$$(\psi^{i+1} \to_L \overline{p^i - 1}) \to_L (\psi^{i+1} \leftrightarrow \psi^i), \tag{10.34}$$

$$(\psi^{i+1} \leftrightarrow \psi^i) \vee (\psi^{i+1} \leftrightarrow (\psi^i \oplus p^i \cdot \overline{1})) \vee \dots \\ \vee(\psi^{i+1} \leftrightarrow (\psi^i \oplus p^i \cdot \overline{p-1})), \tag{10.35}$$

where $\overline{p-1}$ is a tautology at the first-order level and $\overline{p^i - 1}$ (respectively $\overline{p^{i+1} - 1}$) a tautology for formulas of i-th length (respectively of $(i+1)$-th length); $\psi^1 = \psi_1$, i.e. it is the first member of an infinite tuple ψ^∞; $\neg_L \overline{k}$ is a first-order formula that has the truth value $((p-1)-k) \in \{0, \dots, p-1\}$ for any its interpretations and $\overline{k}$ is a first-order formula that has the truth value $k \in \{0, \dots, p-1\}$ for any its interpretations; $\overline{1}$ is a first-order formula that has the truth value 1 for any its interpretations, etc. The denoting $p^i \cdot \overline{k}$ means $\underbrace{\overline{k} \oplus \dots \oplus \overline{k}}_{p^i}$.

Axioms (10.30) – (10.35) are said to be *vertical*.

The deduction rules of BL_∞ is modus ponens: from ψ, $\psi \to \varphi$ infer φ.

The notions of proof, derivability $\vdash$, theorem, and theory over BL_∞ is defined as usual.

Theorem 5 (Soundness and Completeness) *Let Φ be a formula of $\mathcal{L}'$, T an $\mathcal{L}'$-theory. Then the following conditions are equivalent:*

- $T \vdash \Phi$*;*
- $\mathrm{val}^i_I(\Phi) = \langle\underbrace{1, \dots, 1}_{i}\rangle$ *(resp. $\mathrm{val}^\infty_I(\Phi) = {}^*1$) for each $\mathbf{L}_{*V}$-model $\mathcal{M}$ of T;*

Proof. This follows from theorem 4 and semantic rules of BL_∞. $\square$

Chapter 11

Neutrosophic sets

11.1 Vague sets

Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$, with a generic element of U denoted by u_i. A vague set A in U is characterized by a truth-membership function t_A and a false-membership function f_A,

$$t_A : U \to [0, 1],$$
$$f_A : U \to [0, 1],$$

where $t_A(u_i)$ is a lower bound on the grade of membership of u_i derived from the evidence for u_i, $f_A(u_i)$ is a lower bound on the negation of u_i derived from the evidence against u_i, and $t_A(u_i) + f_A(u_i) \leqslant 1$. The grade of membership of u_i in the vague set A is bounded to a subinterval $[t_A(u_i), 1 - f_A(u_i)]$ of $[0, 1]$. The vague value $[t_A(u_i), 1 - f_A(u_i)]$ indicates that the exact grade of membership $\mu_A(u_i)$ of u_i may be unknown. But it is bounded by $t_A(u_i) \leqslant \mu_A(u_i) \leqslant 1 - f_A(u_i)$, where $t_A(u_i) + f_A(u_i) \leqslant 1$. When the universe of discourse U is continuous, a vague set A can be written as

$$A = \int_U [t_A(u_i), 1 - f_A(u_i)]/u_i, \quad u_i \in U.$$

When U is discrete, then

$$A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i, \quad u_i \in U.$$

Logical operations in vague set theory are defined as follows:

Let x and y be two vague values, $x = [t_x, 1 - f_x]$, $y = [t_y, 1 - f_y]$, where $t_x \in [0, 1]$, $f_x \in [0, 1]$, $t_y \in [0, 1]$, $f_y \in [0, 1]$, $t_x + f_x \leqslant 1$ and $t_y + f_y \leqslant 1$. Then

$$\neg_L x = [1 - t_x, f_x],$$

$$x \wedge y = [\min(t_x, t_y), \min(1 - f_x, 1 - f_y)],$$
$$x \vee y = [\max(t_x, t_y), \max(1 - f_x, 1 - f_y)].$$

11.2 Neutrosophic set operations

Definition 31 *Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$. A hyper-valued neutrosophic set A in U is characterized by a truth-membership function t_A, an indeterminacy-membership function i_A, and a false-membership function f_A*

$$t_A \ni f \colon U \to {}^*[0, 1],$$
$$i_A \ni f \colon U \to {}^*[0, 1],$$
$$f_A \ni f \colon U \to {}^*[0, 1],$$

where t_A is the degree of truth-membership function, i_A is the degree of indeterminacy-membership function, and f_A is the degree of falsity-membership function. There is no restriction on the sum of t_A, i_A, and f_A, i.e.

$${}^*0 \leqslant \max t_A(u_i) + \max i_A(u_i) + \max f_A(u_i) \leqslant {}^*3.$$

Definition 32 *Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$. A p-adic valued neutrosophic set A in U is characterized by a truth-membership function t_A, an indeterminacy-membership function i_A, and a false-membership function f_A*

$$t_A \ni f \colon U \to \mathbf{Z}_p,$$
$$i_A \ni f \colon U \to \mathbf{Z}_p,$$
$$f_A \ni f \colon U \to \mathbf{Z}_p,$$

where t_A is the degree of truth-membership function, i_A is the degree of indeterminacy-membership function, and f_A is the degree of falsity-membership function. There is no restriction on the sum of t_A, i_A, and f_A, i.e.

$$0 \leqslant \max t_A(u_i) + \max i_A(u_i) + \max f_A(u_i) \leqslant N_{max} + N_{max} + N_{max} = -3.$$

Also, a neutrosophic set A is understood as a triple $\langle t_A, i_A, f_A \rangle$ and it can be regarded as consisting of hyper-valued or p-adic valued degrees.

As we see, in neutrosophic sets, indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent. This assumption is very important in many applications such as information fusion in which we try to combine the data from different sensors. Neutrosophic sets are proposed for the first time in the framework of neutrosophy that was introduced by SMARANDACHE in 1980: “It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra" [132].

Neutrosophic set is a powerful general formal framework which generalizes the concept of the fuzzy set [153], interval valued fuzzy set [145], intuitionistic fuzzy set [3], and interval valued intuitionistic fuzzy set [4].

Suppose that t_A, i_A, f_A are subintervals of ${}^*[0,1]$. Then a neutrosophic set A is called *interval* one.

When the universe of discourse U is continuous, an interval neutrosophic set A can be written as

$$A = \int_U \langle t_A(u_i), i_A(u_i), f_A(u_i)\rangle / u_i, \quad u_i \in U.$$

When U is discrete, then

$$A = \sum_{i=1}^{n} \langle t_A(u_i), i_A(u_i), f_A(u_i)\rangle / u_i, \quad u_i \in U.$$

The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exist in real world. It can be readily seen that the interval neutrosophic set generalizes the following sets:

- the classical set, $i_A = \emptyset$, $\min t_A = \max t_A = 0$ or 1, $\min f_A = \max f_A = 0$ or 1 and $\max t_A + \max f_A = 1$.
- the fuzzy set, $i_A = \emptyset$, $\min t_A = \max t_A \in [0,1]$, $\min f_A = \max f_A \in [0,1]$ and $\max t_A + \max f_A = 1$.
- the interval valued fuzzy set, $i_A = \emptyset$, $\min t_A$, $\max t_A$, $\min f_A$, $\max f_A \in [0,1]$, $\max t_A + \min f_A = 1$ and $\min t_A + \max f_A = 1$.
- the intuitionistic fuzzy set, $i_A = \emptyset$, $\min t_A = \max t_A \in [0,1]$, $\min f_A = \max f_A \in [0,1]$ and $\max t_A + \max f_A \leqslant 1$.
- the interval valued intuitionistic fuzzy set, $i_A = \emptyset$, $\min t_A$, $\max t_A$, $\min f_A$, $\max f_A \in [0,1]$, $\max t_A + \min f_A \leqslant 1$.
- the paraconsistent set, $i_A = \emptyset$, $\min t_A = \max t_A \in [0,1]$, $\min f_A = \max f_A \in [0,1]$ and $\max t_A + \max f_A > 1$.
- the interval valued paraconsistent set, $i_A = \emptyset$, $\min t_A$, $\max t_A$, $\min f_A$, $\max f_A \in [0,1]$, $\max t_A + \min f_A > 1$.

Let S_1 and S_2 be two real standard or non-standard subsets of ${}^*[0,1]$, then $S_1 + S_2 = \{x\colon x = s_1 + s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, ${}^*a + S_2 = \{x\colon x = {}^*a + s_2, s_2 \in S_2\}$, $S_1 - S_2 = \{x\colon x = s_1 - s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $S_1 \cdot S_2 = \{x\colon x = s_1 \cdot s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $\max(S_1, S_2) = \{x\colon x = \max(s_1, s_2), s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $\min(S_1, S_2) = \{x\colon x = \min(s_1, s_2), s_1 \in S_1 \text{ and } s_2 \in S_2\}$.

1. **The complement of a neutrosophic set A is defined as follows**

   - the ŁUKASIEWICZ complement:

     $$\neg_L A = \langle {}^*1 - t_A, {}^*1 - i_A, {}^*1 - f_A \rangle, \quad t_A, i_A, f_A \subseteq ({}^*[0,1])^U,$$

     $$\neg_L A = \langle N_{max} - t_A, N_{max} - i_A, N_{max} - f_A \rangle, \quad t_A, i_A, f_A \subseteq (\mathbf{Z}_p)^U,$$

   - the GÖDEL complement:

     $$\neg_G A = \langle \neg_G t_A, \neg_G i_A, \neg_G f_A \rangle, \quad t_A, i_A, f_A \subseteq ({}^*[0,1])^U,$$

     $$\neg_G A = \langle \neg_G t_A, \neg_G i_A, \neg_G f_A \rangle, \quad t_A, i_A, f_A \subseteq (\mathbf{Z}_p)^U,$$

     where $\neg_G t_A = \{\neg_G x : x \in t_A\}$, $\neg_G i_A = \{\neg_G x : x \in i_A\}$, $\neg_G f_A = \{\neg_G x : x \in f_A\}$,

   - the Product complement:

     $$\neg_\Pi A = \langle \neg_\Pi t_A, \neg_\Pi i_A, \neg_\Pi f_A \rangle, \quad t_A, i_A, f_A \subseteq ({}^*[0,1])^U,$$

     $$\neg_\Pi A = \langle \neg_\Pi t_A, \neg_\Pi i_A, \neg_\Pi f_A \rangle, \quad t_A, i_A, f_A \subseteq (\mathbf{Z}_p)^U,$$

     where $\neg_\Pi t_A = \{\neg_\Pi x : x \in t_A\}$, $\neg_\Pi i_A = \{\neg_\Pi x : x \in i_A\}$, $\neg_\Pi f_A = \{\neg_\Pi x : x \in f_A\}$.

2. **The implication of two neutrosophic sets A and B is defined as follows**

   - the ŁUKASIEWICZ implication:

     $$A \to_L B = \langle {}^*1 - \max(t_A, t_B) + t_B, {}^*1 - \max(i_A, i_B) + i_B, {}^*1 - \max(f_A, f_B) + f_B \rangle, \quad t_A,\ i_A,\ f_A,\ t_B,\ i_B,\ f_B \subseteq ({}^*[0,1])^U,$$

     $$A \to_L B = \langle N_{max} - \max(t_A, t_B) + t_B, N_{max} - \max(i_A, i_B) + i_B, N_{max} - \max(f_A, f_B) + f_B \rangle, \quad t_A,\ i_A,\ f_A,\ t_B,\ i_B,\ f_B \subseteq (\mathbf{Z}_p)^U,$$

   - the GÖDEL implication:

     $$A \to_G B = \langle t_A \to_G t_B, i_A \to_G i_B, f_A \to_G f_B \rangle, \quad t_A,\ i_A,\ f_A,\ t_B,\ i_B,\ f_B \subseteq ({}^*[0,1])^U,$$

     $$A \to_G B = \langle t_A \to_G t_B, i_A \to_G i_B, f_A \to_G f_B \rangle, \quad t_A,\ i_A,\ f_A,\ t_B,\ i_B,\ f_B \subseteq (\mathbf{Z}_p)^U,$$

where $t_A \to_G t_B = \{x\colon x = s_1 \to_G s_2, s_1 \in t_A$ and $s_2 \in t_B\}$, $i_A \to_G i_B = \{x\colon x = s_1 \to_G s_2, s_1 \in i_A$ and $s_2 \in i_B\}$, $f_A \to_G f_B = \{x\colon x = s_1 \to_G s_2, s_1 \in f_A$ and $s_2 \in f_B\}$,

- the Product implication:

$A \to_\Pi B = \langle t_A \to_\Pi t_B, i_A \to_\Pi i_B, f_A \to_\Pi f_B\rangle,\ t_A, i_A, f_A, t_B, i_B, f_B \subseteq ({}^*[0,1])^U$,

$A \to_\Pi B = \langle t_A \to_\Pi t_B, i_A \to_\Pi i_B, f_A \to_\Pi f_B\rangle,\ t_A, i_A, f_A, t_B, i_B, f_B \subseteq (\mathbf{Z}_p)^U$,

where $t_A \to_\Pi t_B = \{x\colon x = s_1 \to_\Pi s_2, s_1 \in t_A$ and $s_2 \in t_B\}$, $i_A \to_\Pi i_B = \{x\colon x = s_1 \to_\Pi s_2, s_1 \in i_A$ and $s_2 \in i_B\}$, $f_A \to_\Pi f_B = \{x\colon x = s_1 \to_\Pi s_2, s_1 \in f_A$ and $s_2 \in f_B\}$,

3. **The intersection of two neutrosophic sets A and B is defined as follows**

- the Łukasiewicz intersection:

$A \&_L B = \langle \max(t_A, {}^*1 - t_B) + t_B - {}^*1,\ \max(i_A, {}^*1 - i_B) + i_B - {}^*1,$ $\max(f_A, {}^*1 - f_B) + f_B - {}^*1\rangle,\ \ t_A,\ i_A,\ f_A,\ t_B,\ i_B,\ f_B \subseteq ({}^*[0,1])^U$,

$A \&_L B = \langle \max(t_A, N_{max} - t_B) + t_B - N_{max}, \max(i_A, N_{max} - i_B) + i_B - N_{max}, \max(f_A, N_{max} - f_B) + f_B - N_{max}\rangle,\ \ t_A,\ i_A,\ f_A,\ t_B,\ i_B,$ $f_B \subseteq (\mathbf{Z}_p)^U$,

- the Gödel intersection:

$A \&_G B = \langle \min(t_A, t_B), \min(i_A, i_B), \min(f_A, f_B)\rangle,\ \ t_A,\ i_A,\ f_A,\ t_B,\ i_B,$ $f_B \subseteq ({}^*[0,1])^U$,

$A \&_G B = \langle \min(t_A, t_B), \min(i_A, i_B), \min(f_A, f_B)\rangle,\ \ t_A,\ i_A,\ f_A,\ t_B,\ i_B,$ $f_B \subseteq (\mathbf{Z}_p)^U$,

- the Product intersection:

$A \&_\Pi B = \langle t_A \cdot t_B, i_A \cdot i_B, f_A \cdot f_B\rangle,\ \ t_A, i_A, f_A, t_B, i_B, f_B \subseteq ({}^*[0,1])^U$,

$A \&_\Pi B = \langle t_A \cdot t_B, i_A \cdot i_B, f_A \cdot f_B\rangle,\ \ t_A, i_A, f_A, t_B, i_B, f_B \subseteq (\mathbf{Z}_p)^U$, Thus, we can extend the logical operations of fuzzy logic to the case of neutrosophic sets.

Chapter 12

Interval neutrosophic logic

12.1 Interval neutrosophic matrix logic

Interval neutrosophic logic proposed in [132], [133], [151] generalizes the interval valued fuzzy logic, the non-Archimedean valued fuzzy logic, and paraconsistent logics. In the interval neutrosophic logic, we consider not only truth-degree and falsity-degree, but also indeterminacy-degree which can reliably capture more information under uncertainty.

Now consider *hyper-valued interval neutrosophic matrix logic* INL defined as the ordered system $\langle ({}^*[0,1])^3$, $\neg_{INL}$, $\rightarrow_{INL}$, $\vee_{INL}$, $\wedge_{INL}$, $\widetilde{\exists}_{INL}$, $\widetilde{\forall}_{INL}$, $\{\langle {}^*1$, ^{*0}, ${}^*0\rangle\}\rangle$ where

1. for all $\langle t,i,f\rangle \in ({}^*[0,1])^3$, $\neg_{INL}\langle t,i,f\rangle = \langle f, 1-i, t\rangle$,
2. for all $\langle t_1,i_1,f_1\rangle, \langle t_2,i_2,f_2\rangle \in ({}^*[0,1])^3$, $\langle t_1,i_1,f_1\rangle \rightarrow_{INL} \langle t_2,i_2,f_2\rangle = \langle \min({}^*1, {}^*1 - t_1 + t_2), \max({}^*0, i_2 - i_1), \max({}^*0, f_2 - f_1)\rangle$,
3. for all $\langle t_1,\ i_1,\ f_1\rangle$, $\langle t_2,\ i_2,\ f_2\rangle \in ({}^*[0,1])^3$, $\langle t_1,\ i_1,\ f_1\rangle \wedge_{INL} \langle t_2,\ i_2,\ f_2\rangle = \langle \min(t_1,t_2), \max(i_1,i_2), \max(f_1,f_2)\rangle$,
4. for all $\langle t_1,\ i_1,\ f_1\rangle$, $\langle t_2,\ i_2,\ f_2\rangle \in ({}^*[0,1])^3$, $\langle t_1,\ i_1,\ f_1\rangle \vee_{INL} \langle t_2,\ i_2,\ f_2\rangle = \langle \max(t_1,t_2), \min(i_1,i_2), \min(f_1,f_2)\rangle$,
5. for a subset $\langle M_1,\ M_2,\ M_3\rangle \subseteq ({}^*[0,1])^3$, $\widetilde{\exists}(\langle M_1,\ M_2,\ M_3\rangle) = \langle \max(M_1), \min(M_2), \min(M_3)\rangle$,
6. for a subset $\langle M_1,\ M_2,\ M_3\rangle \subseteq ({}^*[0,1])^3$, $\widetilde{\forall}(\langle M_1,\ M_2,\ M_3\rangle) = \langle \min(M_1), \max(M_2), \max(M_3)\rangle$,
7. $\{\langle {}^*1,\ {}^*0,\ {}^*0\rangle\}$ is the set of designated truth values.

Now consider *p-adic valued interval neutrosophic matrix logic* INL defined as the ordered system $\langle (\mathbf{Z}_p)^3$, $\neg_{INL}$, $\rightarrow_{INL}$, $\vee_{INL}$, $\wedge_{INL}$, $\widetilde{\exists}_{INL}$, $\widetilde{\forall}_{INL}$, $\{\langle N_{max}$, $0,\ 0\rangle\}\rangle$ where

1. for all $\langle t, i, f \rangle \in (\mathbf{Z}_p)^3$, $\neg_{INL} \langle t, i, f \rangle = \langle f, 1 - i, t \rangle$,

2. for all $\langle t_1, i_1, f_1 \rangle, \langle t_2, i_2, f_2 \rangle \in (\mathbf{Z}_p)^3$, $\langle t_1,\ i_1,\ f_1 \rangle \rightarrow_{INL} \langle t_2,\ i_2,\ f_2 \rangle = \langle N_{max} - \max(t_1, t_2) + t_2,\ \max(0, i_2 - i_1), \max(0, f_2 - f_1) \rangle$,

3. for all $\langle t_1,\ i_1,\ f_1 \rangle$, $\langle t_2,\ i_2,\ f_2 \rangle \in (\mathbf{Z}_p)^3$, $\langle t_1,\ i_1,\ f_1 \rangle \wedge_{INL} \langle t_2,\ i_2,\ f_2 \rangle = \langle \min(t_1, t_2), \max(i_1, i_2), \max(f_1, f_2) \rangle$,

4. for all $\langle t_1,\ i_1,\ f_1 \rangle$, $\langle t_2,\ i_2,\ f_2 \rangle \in (\mathbf{Z}_p)^3$, $\langle t_1,\ i_1,\ f_1 \rangle \vee_{INL} \langle t_2,\ i_2,\ f_2 \rangle = \langle \max(t_1, t_2), \min(i_1, i_2), \min(f_1, f_2) \rangle$,

5. for a subset $\langle M_1,\ M_2,\ M_3 \rangle \subseteq (\mathbf{Z}_p)^3$, $\widetilde{\exists}(\langle M_1,\ M_2,\ M_3 \rangle) = \langle \max(M_1), \min(M_2), \min(M_3) \rangle$,

6. for a subset $\langle M_1,\ M_2,\ M_3 \rangle \subseteq (\mathbf{Z}_p)^3$, $\widetilde{\forall}(\langle M_1,\ M_2,\ M_3 \rangle) = \langle \min(M_1), \max(M_2), \max(M_3) \rangle$,

7. $\{\langle N_{max},\ 0,\ 0 \rangle\}$ is the set of designated truth values.

As we see, interval neutrosophic matrix logic INL is an extension of the non-Archimedean valued Łukasiewicz matrix logic.

12.2 Hilbert's type calculus for interval neutrosophic propositional logic

Interval neutrosophic calculus denoted by **INL** is built in the framework of the language $\mathcal{L}'$ considered in section 10.4, but its semantics is different.

An interpretation is defined in the standard way. Extend the valuation of $\mathcal{L}'$ to the valuation for interval neutrosophic calculus as follows.

Definition 33 *Given an interpretation $I = \langle \mathcal{M}, s \rangle$ and a valuation val_I^∞ of $\mathcal{L}'$, we define the hyper-valued interval neutrosophic valuation $\mathrm{val}_I^{\infty, INL}(\cdot)$ to be a mapping from formulas of the form φ^∞ of $\mathcal{L}'$ to interval neutrosophic matrix logic INL as follows:*

$$\mathrm{val}_I^{\infty, INL}(\varphi^\infty) = \langle \mathrm{val}_I^\infty(\varphi^\infty) = t(\varphi^\infty), i(\varphi^\infty), f(\varphi^\infty) \rangle \in ({}^*[0, 1])^3.$$

Definition 34 *Given an interpretation $I = \langle \mathcal{M}, s \rangle$ and a valuation val_I^∞ of $\mathcal{L}'$, we define the p-adic valued interval neutrosophic valuation $\mathrm{val}_I^{\infty, INL}(\cdot)$ to be a mapping from formulas of the form φ^∞ of $\mathcal{L}'$ to interval neutrosophic matrix logic INL as follows:*

$$\mathrm{val}_I^{\infty, INL}(\varphi^\infty) = \langle \mathrm{val}_I^\infty(\varphi^\infty) = t(\varphi^\infty), i(\varphi^\infty), f(\varphi^\infty) \rangle \in (\mathbf{Z}_p)^3.$$

We say that an INL-structure $\mathcal{M}$ is a *model* of an **INL**-theory T iff

$$\mathrm{val}_I^{\infty,INL}(\varphi^\infty) = \langle {}^*1, {}^*0, {}^*0\rangle$$

on $\mathcal{M}$ for each $\varphi^\infty \in T$.

Proposition 17 *In the matrix logic INL, modus ponens is preserved, i.e. if φ^∞ and $\varphi^\infty \rightarrow_{INL} \psi^\infty$ are INL-tautologies, then ψ^∞ is also an INL-tautology.*

Proof. Consider the hyper-valued case. Since φ^∞ and $\varphi^\infty \rightarrow_{INL} \psi^\infty$ are INL-tautologies, then

$$\mathrm{val}_I^{\infty,INL}(\varphi) = \mathrm{val}_I^{\infty,INL}(\varphi^\infty \rightarrow_{INL} \psi^\infty) = \langle {}^*1, {}^*0, {}^*0\rangle,$$

that is $\mathrm{val}_I^{\infty,INL}(\varphi^\infty) = \langle {}^*1, {}^*0, {}^*0\rangle$, $\mathrm{val}_I^{\infty,INL}(\varphi^\infty \rightarrow_{INL} \psi^\infty) = \langle \min({}^*1, {}^*1 - t(\varphi^\infty) + t(\psi)) = {}^*1, \max({}^*0, i(\psi^\infty) - i(\varphi^\infty)) = {}^*0, \max({}^*0, f(\psi^\infty) - f(\varphi^\infty)) = {}^*0\rangle$. Hence, $t(\psi^\infty) = {}^*1$, $i(\psi^\infty) = f(\psi^\infty) = {}^*0$. So ψ^∞ is an INL-tautology. □

The following axiom schemata for **INL** were regarded in [151].

$$\psi^\infty \rightarrow_{INL} (\varphi^\infty \rightarrow_{INL} \psi^\infty), \tag{12.1}$$

$$(\psi^\infty \wedge_{INL} \varphi^\infty) \rightarrow_{INL} \varphi^\infty, \tag{12.2}$$

$$\psi^\infty \rightarrow_{INL} (\psi^\infty \vee_{INL} \varphi^\infty), \tag{12.3}$$

$$\psi^\infty \rightarrow_{INL} (\varphi^\infty \rightarrow_{INL} (\psi^\infty \wedge_{INL} \varphi^\infty)), \tag{12.4}$$

$$(\psi^\infty \rightarrow_{INL} \chi^\infty) \rightarrow_{INL} ((\varphi^\infty \rightarrow_{INL} \chi^\infty) \rightarrow_{INL} ((\psi^\infty \vee_{INL} \varphi^\infty) \rightarrow_{INL} \chi^\infty)), \tag{12.5}$$

$$((\psi^\infty \vee_{INL} \varphi^\infty) \rightarrow_{INL} \chi^\infty) \leftrightarrow_{INL} ((\psi^\infty \rightarrow_{INL} \chi^\infty) \wedge_{INL} (\varphi^\infty \rightarrow_{INL} \chi^\infty)), \tag{12.6}$$

$$(\psi^\infty \rightarrow_{INL} \varphi^\infty) \leftrightarrow_{INL} (\neg_{INL}\varphi^\infty \rightarrow_{INL} \neg_{INL}\psi^\infty), \tag{12.7}$$

$$(\psi^\infty \rightarrow_{INL} \varphi^\infty) \wedge_{INL} (\varphi^\infty \rightarrow_{INL} \chi^\infty) \rightarrow_{INL} (\psi^\infty \rightarrow_{INL} \chi^\infty), \tag{12.8}$$

$$(\psi^\infty \rightarrow_{INL} \varphi^\infty) \leftrightarrow_{INL} (\psi^\infty \leftrightarrow_{INL} (\psi^\infty \wedge_{INL} \varphi^\infty)) \leftrightarrow_{INL} (\varphi^\infty \rightarrow_{INL} (\psi^\infty \vee_{INL} \varphi^\infty)). \tag{12.9}$$

The only inference rule of **INL** is modus ponens.

We can take also the non-Archimedean case of axiom schemata (4.1) – (4.4) for the axiomatization of **INL**, because **INL** is a generalization of non-Archimedean valued Łukasiewicz's logic (see the previous section). This means that we can also set **INL** as generalization of non-Archimedean valued Gödel's, Product or Post's logics.

Chapter 13

Conclusion

The informal sense of ARCHIMEDES' axiom is that anything can be measured by a ruler. The negation of this axiom allows to postulate infinitesimals and infinitely large integers and, as a result, to consider non-wellfounded and neutrality phenomena. In this book we examine the non-Archimedean fuzziness, i.e. fuzziness that runs over the non-Archimedean number systems. We show that this fuzziness is constructed in the framework of the t-norm based approach. We consider two cases of the non-Archimedean fuzziness: one with many-validity in the interval $[0, 1]$ of hypernumbers and one with many-validity in the ring $\mathbf{Z}_p$ of p-adic integers. This fuzziness has a lot of practical applications, e.g. it can be used in non-Kolmogorovian approaches to probability theory.

Non-Archimedean logic is constructed on the base of infinite DSm models. Its instances are the following multi-valued logics:

- hyperrational valued ŁUKASIEWICZ's, GÖDEL's, and Product logics,
- hyperreal valued ŁUKASIEWICZ's, GÖDEL's, and Product logics,
- p-adic valued ŁUKASIEWICZ's, GÖDEL's, and POST's logics.

These systems can be used in probabilistic and fuzzy reasoning.

Hyper-valued (resp. p-adic valued) interval neutrosophic logic **INL** by which we can describe neutrality phenomena is an extension of non-Archimedean valued fuzzy logic that is obtained by adding a truth triple $\langle t, i, f\rangle \in ({}^*[0, 1])^3$ (resp. $\langle t, i, f\rangle \in (\mathbf{Z}_p)^3$) instead of one truth value $t \in {}^*[0, 1]$ (resp. $t \in \mathbf{Z}_p$) to the formula valuation, where t is a truth-degree, i is an indeterminacy-degree, and f is a falsity-degree.

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In this book, we consider various many-valued logics: standard, linear, hyperbolic, parabolic, non-Archimedean, p-adic, interval, neutrosophic, etc. We survey also results which show the tree different proof-theoretic frameworks for many-valued logics, e.g. frameworks of the following deductive calculi: Hilbert's style, sequent, and hypersequent. Recall that hypersequents are a natural generalization of Gentzen's style sequents that was introduced independently by Avron and Pottinger. In particular, we consider Hilbert's style, sequent, and hypersequent calculi for infinite-valued logics based on the three fundamental continuous t-norms: Łukasiewicz's, Gödel's, and Product logics.

We present a general way that allows to construct systematically analytic calculi for a large family of non-Archimedean many-valued logics: hyperrational-valued, hyperreal-valued, and p-adic valued logics characterized by a special format of semantics with an appropriate rejection of Archimedes' axiom. These logics are built as different extensions of standard many-valued logics (namely, Łukasiewicz's, Gödel's, Product, and Post's logics).

The informal sense of Archimedes' axiom is that anything can be measured by a ruler. Also logical multiple-validity without Archimedes' axiom consists in that the set of truth values is infinite and it is not well-founded and well-ordered.

We consider two cases of non-Archimedean multi-valued logics: the first with many-validity in the interval [0,1] of hypernumbers and the second with many-validity in the ring of p-adic integers. Notice that in the second case we set discrete infinite-valued logics. The following logics are investigated:

1. hyperrational valued Łukasiewicz's, Gödel's, and Product logics,
2. hyperreal valued Łukasiewicz's, Gödel's, and Product logics,
3. p-adic valued Łukasiewicz's, Gödel's, and Post's logics.

Hajek proposes basic fuzzy logic BL which has validity in all logics based on continuous t-norms. In this book, for the first time we survey hypervalued and p-adic valued extensions of basic fuzzy logic BL.

On the base of non-Archimedean valued logics, we construct non-Archimedean valued interval neutrosophic logic INL by which we can describe neutrality phenomena. This logic is obtained by adding to the truth valuation a truth triple t, i, f instead of one truth value t, where t is a truth-degree, i is an indeterminacy-degree, and f is a falsity-degree. Each parameter of this triple runs either the unit interval [0,1] of hypernumbers or the ring of p-adic integers.

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