Neutrosophic Hypervector Spaces

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Abstract

The objective of this paper is to study *neutrosophic* hypervector spaces. Some basic definitions and properties of the hypervector spaces are generalized.
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1 Introduction

The theory of fuzzy set introduced by L.A. Zadeh [20] is mainly concerned with the measurement of the degree of membership and non-membership of a given abstract situation. Despite its wide range of real life applications, fuzzy set theory cannot be applied to model an abstract situation where indeterminancy is involved. In his quest to modeling situations involving indeterminates, F. Smarandache introduced the theory of *neutrosophy* in 1995. *Neutrosophic* logic is an extension of the fuzzy logic in which indeterminancy is included. In the *neutrosophic* logic, each proposition is characterized by the degree of truth in the set (T), the degree of falsehood in the set (F) and the degree of indeterminancy in the set (I) where T, F, I are subsets of $]-0, 1+[$. *Neutrosophic* logic has wide applications in science, engineering, IT, law, politics, economics, finance etc. The concept of *neutrosophic* algebraic structures was introduced by F. Smarandache and W.B. Vasantha Kandasamy in 2006. However, for details about *neutrosophy* and *neutrosophic* algebraic structures, the reader should see [1, 2, 3, 11, 16, 17, 18, 19].

The concept of hyperstructures was introduced by F. Marty [10] in 1934 at the 8th Congress of Scandinavian Mathematicians. The concept has been further studied, developed and generalized by many reseachers in hyperstructures. In the development of studies in hyperstructures, M.S. Talini [13] introduced the concept of hypervector spaces in 1990 at the 4th International Congress on Algebraic Hyperstructures and Applications. Since its introduction in 1990, hypervector spaces have been further studied and expanded by other reseachers. For further details about hypervector spaces, the reader should see [5, 6, 7, 8, 9, 12, 14, 15].

The concept of *neutrosophic* vector spaces was studied by A.A.A. Agboola and S.A. Akinleye in [4]. In the present paper, we are concerned with the study of *neutrosophic* hypervector spaces. Some basic definitions and properties of the hypervector spaces are generalized.

2 Preliminaries

In this section, we present some known definitions and results that will be used in the present paper.

Definition 2.1. Let $(G, *)$ be any group and let $G(I) =< G \cup I >$. The couple $(G(I), *)$ is called a *neutrosophic* group generated by G and I under the binary operation $*$. The indeterminancy factor I is such that $I * I = I$. If $*$ is ordinary multiplication, then $I * I * \cdots * I = I^n = I$ and if $*$ is ordinary addition, then $I * I * I * \cdots * I = nI$ for $n \in \mathbb{N}$.

If $a * b = b * a$ for all $a, b \in G(I)$, we say that $G(I)$ is commutative. Otherwise, $G(I)$ is called a non-commutative *neutrosophic* group. $(\mathbb{R}(I), +), (\mathbb{Q}(I), +), (\mathbb{C}(I), +)$ are examples of commutative *neutrosophic* groups while $(A_{m\times n}(I), .)$ is a non-commutative *neutrosophic* group.

Definition 2.2. Let $(K, +, .)$ be any field and let $K(I) =< K \cup I >$ be a *neutrosophic* set generated by K and I. The tripple $(K(I), +, .)$ is called a *neutrosophic* field. The zero element $0 \in K$ is represented by $0 + 0I$ in $K(I)$ and $1 \in K$ is represented by $1 + 0I$ in $K(I)$. Examples of *neutrosophic* field include $(\mathbb{Q}(I), .), (\mathbb{R}(I), .)$ and $(\mathbb{C}(I), .)$.

Definition 2.3. Let $K(I)$ be a *neutrosophic* field and let $F(I)$ be a nonempty subset of $K(I)$. $F(I)$ is called a *neutrosophic* subfield of $K(I)$ if $F(I)$ is itself a *neutrosophic* field. It is essential that $F(I)$ contains a proper subset which is a field. $(\mathbb{Q}(I), .)$ is a *neutrosophic* subfield of $(\mathbb{R}(I), .)$ and $(\mathbb{R}(I), .)$ is a *neutrosophic* subfield of $(\mathbb{C}(I), .)$.

Definition 2.4. [4] Let $(V, +, .)$ be any vector space over a field K and let $V(I) =< V \cup I >$ be a *neutrosophic* set generated by V and I. The tripple $(V(I), +, .)$ is called a weak *neutrosophic* vector space over a field K. If $V(I)$ is a *neutrosophic* vector space over a *neutrosophic* field $K(I)$, then $V(I)$ is called a strong *neutrosophic* vector space. The elements of $V(I)$ are called *neutrosophic* vectors and the elements of $K(I)$ are called *neutrosophic* scalars.

If $u = a + bI, v = c + dI \in V(I)$ where a, b, c and d are vectors in V and $\alpha = k + mI \in K(I)$ where k and m are scalars in K, we define:

$$\begin{aligned} u + v &= (a + bI) + (c + dI) = (a + c) + (b + d)I, \text{ and} \\ \alpha.u &= (k + mI).(a + bI) = k.a + (k.b + m.a + m.b)I. \end{aligned}$$

Theorem 2.5. *[4] Every strong neutrosophic vector space is a weak neutrosophic vector space.*

Theorem 2.6. *[4] Every weak (strong) neutrosophic vector space is a vector space.*

Example 1.
(1) $\mathbb{R}(I)$ is a weak *neutrosophic* vector space over a field $\mathbb{Q}$ and it is a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{Q}(I)$.

(2) $\mathbb{R}^n(I)$ is a weak *neutrosophic* vector space over a field $\mathbb{R}$ and it is a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{R}(I)$.

(3) $M_{m\times n}(I) = \{[a_{ij}] : a_{ij} \in \mathbb{Q}(I)\}$ is a weak *neutrosophic* vector space over a field $\mathbb{Q}$ and it is a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{Q}(I)$.

Definition 2.7. [4] Let $V(I)$ be a strong *neutrosophic* vector space over a *neutrosophic* field $K(I)$ and let $W(I)$ be a nonempty subset of $V(I)$. $W(I)$ is called a strong *neutrosophic* subspace of $V(I)$ if $W(I)$ is itself a strong *neutrosophic* vector space over $K(I)$. It is essential that $W(I)$ contains a proper subset which is a vector space.

Example 2. Let $V(I) = \mathbb{R}^3(I)$ be a strong *neutrosophic* vector space over a *neutrosophic* field $\mathbb{R}(I)$ and let

$$W(I) = \{(u = a + bI, v = c + dI, 0 = 0 + 0I) \in V(I) : a, b, c, d \in V\}.$$

Then $W(I)$ is a strong *neutrosophic* subspace of $V(I)$.

Definition 2.8. [4] Let $W(I)$ be a strong *neutrosophic* subspace of a strong *neutrosophic* vector space $V(I)$ over a *neutrosophic* field $K(I)$. The quotient $V(I)/W(I)$ is defined by the set

$$\{v + W(I) : v \in V(I)\}.$$

$V(I)/W(I)$ can be made a strong *neutrosophic* vector space over a *neutrosophic* field $K(I)$ if addition and multiplication are defined for all $u+W(I), v+W(I) \in V(I)/W(I)$ and $\alpha \in K(I)$ as follows:

$$\begin{aligned} (u + W(I)) + (v + W(I)) &= (u + v) + W(I), \\ \alpha(u + W(I)) &= \alpha u + W(I). \end{aligned}$$

The strong *neutrosophic* vector space $(V(I)/W(I), +, .)$ over a *neutrosophic* field $K(I)$ is called a strong *neutrosophic* quotient space.

Definition 2.9. [13] Let $P(V)$ be the power set of a set V, $P^*(V) = P(V)\backslash\{\emptyset\}$ and let K be a field. The quadruple $(V, +, \bullet, K)$ is called a hypervector space over a field K if:

(1) $(V, +)$ is an abelian group.

(2) $\bullet : K \times V \to P^*(V)$ is a hyperoperation such that for all $k, m \in K$ and $u, v \in V$, the following conditions hold:

(i) $(k+m)\bullet u \subseteq (k\bullet u)+(m\bullet u)$,

(ii) $k\bullet(u+v) \subseteq (k\bullet u)+(k\bullet v)$,

(iii) $k\bullet(m\bullet u) = (km)\bullet u$, where $k\bullet(m\bullet u) = \{k\bullet v : v\in m\bullet u\}$,

(iv) $(-k)\bullet u = k\bullet(-u)$,

(v) $u\in 1\bullet u$.

A hypervector space is said to be strongly left distributive (resp. strongly right distributive) if equality holds in (i) (resp. in (ii)). $(V,+,\bullet,K)$ is called a strongly distributive hypervector space if it is both strongly left and strongly right distributive.

3 Neutrosophic Hypervector Spaces and Neutrosophic Subhypervector Spaces

In this section, we develop the concept of *neutrosophic* hypervector spaces and present some of their basic properties.

Definition 3.1. Let $(V,+,\bullet,K)$ be any strongly distributive hypervector space over a field K and let

$$V(I) = < V\cup I > = \{u=(a,bI) : a,b\in V\}$$

be a set generated by V and I. The quadruple $(V(I),+,\bullet,K)$ is called a weak *neutrosophic* strongly distributive hypervector space over a field K.

For every $u=(a,bI), v=(c,dI)\in V(I)$ and $k\in K$, we define

$$\begin{aligned} u+v &= (a+c,(b+d)I)\in V(I),\\ k\bullet u &= \{(x,yI) : x\in k\bullet a, y\in k\bullet b\}. \end{aligned}$$

If K is a *neutrosophic* field, that is, $K=K(I)$, then the quadruple $(V(I),+,\bullet,K(I))$ is called a strong *neutrosophic* strongly distributive hypervector space over a *neutrosophic* field $K(I)$.

For every $u=(a,bI), v=(c,dI)\in V(I)$ and $\alpha=(k,mI)\in K(I)$, we define

$$\begin{aligned} u+v &= (a+c,(b+d)I)\in V(I),\\ \alpha\bullet u &= \{(x,yI) : x\in k\bullet a, y\in k\bullet b\cup m\bullet a\cup m\bullet b\}. \end{aligned}$$

The elements of $V(I)$ are called *neutrosophic* vectors and the elements of $K(I)$ are called *neutrosophic* scalars. The zero *neutrosophic* vector of $V(I)$, $(0,0I)$, is denoted by θ, the zero element $0\in K$ is represented by $(0,0I)$ in $K(I)$ and $1\in K$ is represented by $(1,0I)$ in $K(I)$.

Example 3. (1) Let $V(I)=\mathbb{R}(I)$ and let $K=\mathbb{R}$. For all $u=(a,bI), v=(c,dI)\in V(I)$ and $k\in K$, define:

$$\begin{aligned} u+v &= (a+c,(b+d)I),\\ k\bullet u &= \{(x,yI) : x\in k\bullet a, y\in k\bullet b\}. \end{aligned}$$

Then $(V(I), +, \bullet, K)$ is a weak *neutrosophic* strongly distributive hypervector space over the field K.

(2) Let $V(I) = \mathbb{R}^2(I)$ and let $K = \mathbb{R}(I)$. For all $u = ((a, bI), (c, dI)), v = ((e, fI), (g, hI)) \in V(I)$ and $\alpha = (k, mI) \in K(I)$, define:

$$\begin{aligned} u + v &= ((a + e, (b + f)I), (c + g, (d + h)I)), \\ \alpha \bullet u &= \{((x, yI), (w, zI)) : x \in k \bullet a, y \in k \bullet b \cup m \bullet a \cup m \bullet b, \\ &\quad w \in k \bullet c, z \in k \bullet d \cup m \bullet c \cup m \bullet d\}. \end{aligned}$$

Then $(V(I), +, \bullet, K(I))$ is a strong *neutrosophic* strongly distributive hypervector space over the *neutrosophic* field $K(I)$.

From now on, every weak(strong) *neutrosophic* strongly distributive hypervector space will simply be called a weak(strong) *neutrosophic* hypervector space.

Lemma 3.2. *Let $V(I)$ be a weak neutrosophic hypervector space over a field K. Then for all $k \in K$ and $u = (a, bI) \in V(I)$, we have*

(1) $k \bullet \theta = \{\theta\}$.

(2) $k \bullet u = \{\theta\}$ *implies that* $k = 0$ *or* $u = \theta$.

(3) $-u \in (-1) \bullet u$.

Theorem 3.3. *Every strong neutrosophic hypervector space is a weak neutrosophic hypervector space.*

Proof. Obvious since $K \subseteq K(I)$. □

Theorem 3.4. *Every weak neutrosophic hypervector space is a strongly distributive hypervector space.*

Proof. Suppose that $V(I)$ is a weak *neutrosophic* hypervector space over a field K. Obviously, $(V(I), +)$ is an abelian group. Let $u = (a, bI), v = (c, dI) \in V(I)$ and $k, m \in K$ be arbitrary. Then
(1)

$$\begin{aligned} k \bullet u + m \bullet u &= \{(p, qI) : p \in k \bullet a, q \in k \bullet b\} + \{(r, sI) : r \in m \bullet a, s \in m \bullet b\} \\ &= \{(p + r, (q + s)I) : p + r \in k \bullet a + m \bullet a, q + s \in k \bullet b + m \bullet b\}. \end{aligned}$$

Also,

$$\begin{aligned} (k + m) \bullet u &= \{(x, yI) : x \in (k + m) \bullet a, y \in (k + m) \bullet b\} \\ &= \{(x, yI) : x \in k \bullet a + m \bullet a, y \in k \bullet b + m \bullet b\} \\ &= k \bullet u + m \bullet u. \end{aligned}$$

(2)

$$\begin{aligned} k \bullet u + k \bullet v &= \{(p, qI) : p \in k \bullet a, q \in k \bullet b\} + \{(r, sI) : r \in k \bullet c, s \in k \bullet d\} \\ &= \{(p + r, (q + s)I) : p + r \in k \bullet a + k \bullet c, q + s \in k \bullet b + k \bullet d\}. \end{aligned}$$

Also,

$$\begin{aligned} k \bullet (u + v) &= k \bullet (a + c, (b + d)I) \\ &= \{(x, yI) : x \in k \bullet (a + c), y \in k \bullet (b + d)\} \\ &= \{(x, yI) : x \in k \bullet a + k \bullet c, y \in k \bullet b + k \bullet d\} \\ &= k \bullet u + k \bullet v. \end{aligned}$$

(3)

$$\begin{aligned} k \bullet (m \bullet u) &= k \bullet \{(x, yI) : x \in m \bullet a, y \in m \bullet b\} \\ &= \{(p, qI) : p \in k \bullet x, q \in k \bullet y\} \\ &= \{(p, qI) : p \in k \bullet (m \bullet a), q \in k \bullet (m \bullet b)\} \\ &= \{(p, qI) : p \in (km) \bullet a, q \in (km) \bullet b\} \\ &= (km) \bullet (a, bI) \\ &= (km) \bullet u. \end{aligned}$$

(4)

$$\begin{aligned} (-k) \bullet u &= \{(x, yI) : x \in (-k) \bullet a, y \in (-k) \bullet b\} \\ &= \{(x, yI) : x \in k \bullet (-a), y \in k \bullet (-b)\} \\ &= k \bullet (-a, -bI) \\ &= k \bullet (-u). \end{aligned}$$

(5)

$$\begin{aligned} 1 \bullet u &= \{(x, yI) : x \in 1 \bullet a, y \in 1 \bullet b\} \\ &= \{(a, bI) : a \in 1 \bullet a, b \in 1 \bullet b\} \end{aligned}$$

showing that $u \in 1 \bullet u$. Accordingly, $V(I)$ is a strongly distributive hypervector space. □

Theorem 3.5. *Let $V(I)$ be a strong neutrosophic hypervector space over a neutrosophic field $K(I)$. Then*

(1) $V(I)$ generally is not a strongly distributive hypervector space.

(2) $V(I)$ always contain a strongly distributive hypervector space.

Theorem 3.6. *Let $(V_1(I), +_1, \bullet_1, K(I))$ and $(V_2(I), +_2', \bullet_2', K(I))$ be two strong neutrosophic hypervector spaces over a neutrosophic field $K(I)$. Let*

$$V_1(I) \times V_2(I) = \{((a_1, b_1I), (a_2, b_2I)) : (a_1, b_1I) \in V_1(I), (a_2, b_2I) \in V_2(I)\}$$

and for all $u = ((a_1, b_1I), (a_2, b_2I)), v = ((a_1', b_1'I), (a_2', b_2'I)) \in V_1(I) \times V_2(I)$ and $\alpha = (k, mI) \in K(I)$, define:

$$\begin{aligned} u + v &= ((a_1 + a_1', (b_1 + b_1')I), (a_2 + a_2', (b_2 + b_2')I)), \\ \alpha \bullet u &= \{((x, yI), (p, qI)) : x \in k \bullet a_1, y \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1, \\ &\quad p \in k \bullet a_2, q \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2\}. \end{aligned}$$

Then $(V_1(I) \times V_2(I), +, \bullet, K(I))$ is a strong neutrosophic hypervector space.

Definition 3.7. Let $(V(I), +, \bullet, K(I))$ be a strong *neutrosophic* hypervector space over a *neutrosophic* field $K(I)$ and let $W[I]$ be a nonempty subset of $V(I)$. $W[I]$ is said to be a subhypervector space of $V(I)$ if $(W[I], +, \bullet, K(I))$ is also a *neutrosophic* hypervector space over the *neutrosophic* field $K(I)$. It is essential that $W[I]$ contains a proper subset which is a hypervector space over a field K.

Theorem 3.8. *Let $W[I]$ be a subset of a strong neutrosophic hypervector space $(V(I), +, \bullet, K(I))$ over a neutrosophic field $K(I)$. Then $W[I]$ is a neutrosophic subhypervector space of $V(I)$ if and only if for all $u = (a, bI), v = (c, dI) \in V(I)$ and $\alpha = (k, mI) \in K(I)$, the following conditions hold:*

(1) $W[I] \neq \emptyset$,

(2) $u + v \in W[I]$,

(3) $\alpha \bullet u \subseteq W[I]$,

(4) $W[I]$ *contains a proper subset which is a hypervector space over* K.

Corollary 3.9. Let $W[I]$ be a subset of a strong *neutrosophic* hypervector space $(V(I), +, \bullet, K(I))$ over a *neutrosophic* field $K(I)$. Then $W[I]$ is a *neutrosophic* subhypervector space of $V(I)$ if and only if for all $u = (a, bI), v = (c, dI) \in V(I)$ and $\alpha = (k, mI), \beta = (r, sI) \in K(I)$, the following conditions hold:

(1) $W[I] \neq \emptyset$,

(2) $\alpha \bullet u + \beta \bullet v \subseteq W[I]$,

(3) $W[I]$ contains a proper subset which is a hypervector space over K.

Theorem 3.10. *Let $W_1[I], W_2[I], \cdots, W_n[I]$ be neutrosophic subhypervector spaces of a strong neutrosophic hypervector space $(V(I), +, \bullet, K(I))$ over a neutrosophic field $K(I)$. Then $\bigcap_{i=1}^{n} W_i[I]$ is a neutrosophic subhypervector space of $V(I)$.*

Remark 1. If $W_1[I]$ and $W_2[I]$ are *neutrosophic* subhypervector spaces of a strong *neutrosophic* hypervector space $V(I)$ over a *neutrosophic* field $K(I)$, then generally, $W_1[I] \cup W_2[I]$ is not a *neutrosophic* subhypervector space of $V(I)$ except if $W_1[I] \subseteq W_2[I]$ or $W_2[I] \subseteq W_1[I]$. However, $W_1[I] \cup W_2[I]$ is a *neutrosophic* bihypervector space over $K(I)$.

Example 4. Let $V(I)$ be the strong *neutrosophic* hypervector space of Example 3(2) and let

$$W[I] = \{((a, bI), \theta) \in V(I) : a, b \in \mathbb{R}\}.$$

Then $W[I]$ is a *neutrosophic* subhypervector space of $V(I)$.

Definition 3.11. Let $W[I]$ and $X[I]$ be two *neutrosophic* subhypervector spaces of a strong *neutrosophic* hypervector space $(V(I), +, \bullet, K(I))$ over a *neutrosophic* field $K(I)$. The sum of $W[I]$ and $X[I]$ denoted by $W[I] + X[I]$ is defined by the set

$$\bigcup \{w + x : w = (a, bI) \in W[I], x = (c, dI) \in X[I]\}.$$

If $W[I] \cap X[I] = \{\theta\}$, then the sum of $W[I]$ and $X[I]$ is denoted by $W[I] \oplus X[I]$ and it is called the direct sum of $W[I]$ and $X[I]$.

Theorem 3.12. *Let $W[I]$ and $X[I]$ be two neutrosophic subhypervector spaces of a strong neutrosophic hypervector space $(V(I), +, \bullet, K(I))$ over a neutrosophic field $K(I)$.*

(1) $W[I] + X[I]$ is a neutrosophic subhypervector space of $V(I)$.

(2) $W[I] + X[I]$ is the least neutrosophic subhypervector space of $V(I)$ containing $W[I]$ and $X[I]$.

Definition 3.13. Let $W[I]$ and $X[I]$ be two *neutrosophic* subhypervector spaces of a strong *neutrosophic* hypervector space $(V(I), +, \bullet, K(I))$ over a *neutrosophic* field $K(I)$. $V(I)$ is said to be the direct sum of $W[I]$ and $X[I]$ written $V(I) = W[I] \oplus X[I]$ if every element $v \in V(I)$ can be written uniquely as $v = w + x$ where $w \in W[I]$ and $x \in X[I]$.

Theorem 3.14. *Let $W[I]$ and $X[I]$ be two neutrosophic subhypervector spaces of a strong neutrosophic hypervector space $V(I), +, \bullet, K(I))$ over a neutrosophic field $K(I)$. $V(I) = W[I] \oplus X[I]$ if and only if the following conditions hold:*

(1) $V(I) = W[I] + X[I]$.

(2) $W[I] \cap X[I] = \{\theta\}$.

Lemma 3.15. *Let $W[I]$ be a neutrosophic subhypervector space of a strong neutrosophic hypervector space $(V(I), +, \bullet, K(I))$ over a neutrosophic field $K(I)$. Then:*

(1) $W[I] + W[I] = W[I]$.

(2) $w + W[I] = W[I]$ *for all* $w \in W[I]$.

Definition 3.16. Let $W[I]$ be a *neutrosophic* subhypervector space of a strong *neutrosophic* hypervector space $(V(I), +, \bullet, K(I))$ over a *neutrosophic* field $K(I)$. The quotient $V(I)/W[I]$ is defined by the set

$$\{[v] = v + W[I] : v \in V(I)\}.$$

If for every $[u], [v] \in V(I)/W[I]$ and $\alpha \in K(I)$, we define:

$$\begin{aligned} [u] \oplus [v] &= (u + v) + W[I] \text{ and} \\ \alpha \odot [u] &= [\alpha \bullet u] = \{[x] : x \in \alpha \bullet u\}, \end{aligned}$$

it can be shown that $(V(I)/W[I], \oplus, \odot, K(I))$ is a strong *neutrosophic* hypervector over a *neutrosophic* field $K(I)$ called a strong *neutrosophic* quotient hypervector space.

4 Bases and Dimensions of Neutrosophic Hypervector Spaces

Theorem 4.1. *Let* $(V(I), +, \bullet, K(I))$ *be a strong neutrosophic hypervector space over a neutrosophic field* $K(I)$ *and let* $u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots, u_n = (a_n, b_nI) \in V(I)$, $\alpha_1 = (k_1, m_1I), \alpha_2 = (k_2, m_2I), \cdots, \alpha_1 = (k_n, m_nI) \in K(I)$. *If*

$$W(I) = \bigcup\{\alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n : u_i \in V(I), \alpha_i \in K(I)\},$$

then:

(1) $(W(I), +, \bullet, K(I))$ *is a neutrosophic subhypervector space of* $V(I)$.

(2) $W(I)$ *is the smallest neutrosophic subhypervector space of* $V(I)$ *containing* $u_1, u_2, \cdots, u_n$.

Remark 2. The *neutrosophic* subhypervector space $W(I)$ of the strong *neutrosophic* hypervector space $V(I)$ over a *neutrosophic* field $K(I)$ of Theorem 4.1 is said to be generated or spanned by the *neutrosophic* vectors $u_1, u_2, \cdots, u_n$ and we write $W(I) = span\{u_1, u_2, \cdots, u_n\}$.

Definition 4.2. Let $(V(I), +, \bullet, K(I))$ be a strong *neutrosophic* hypervector space over a *neutrosophic* field $K(I)$ and let $B(I) = \{u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots, u_n = (a_n, b_nI)\}$ be a subset of $V(I)$. $B(I)$ is said to generate or span $V(I)$ if $V(I) = span(B(I))$.

Example 5. Let $V(I) = \mathbb{R}^3(I)$ be a strong *neutrosophic* hypervector space over a *neutrosophic* field $\mathbb{R}(I)$ and let $B(I) = \{u_1 = ((1, 0I), (0, 0I), (0, 0I)), u_2 = ((0, 0I), (1, 0I), (0, 0I)), u_3 = ((0, 0I), (0, 0I), (1, 0I))\}$. Then $B(I)$ spans $V(I)$.

Example 6. Let $V(I) = \mathbb{R}^2(I)$ be a weak *neutrosophic* hypervector space over a field $\mathbb{R}$ and let $B(I) = \{u_1 = ((1, 0I), (0, 0I)), u_2 = ((0, 0I), (1, 0I)), u_3 = ((0, I), (0, 0I)), u_4 = ((0, 0I), (0, I))\}$. Then $B(I)$ spans $V(I)$.

Definition 4.3. Let $(V(I), +, \bullet, K(I))$ be a strong *neutrosophic* hypervector space over a *neutrosophic* field $K(I)$. The *neutrosophic* vector $u = (a, bI) \in V(I)$ is said to be a linear combination of the *neutrosophic* vectors $u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots, u_n = (a_n, b_nI) \in V(I)$ if there exists *neutrosophic* scalars $\alpha_1 = (k_1, m_1I), \alpha_2 = (k_2, m_2I), \cdots, \alpha_1 = (k_n, m_nI) \in K(I)$ such that

$$u \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n.$$

Definition 4.4. Let $(V(I), +, \bullet, K(I))$ be a strong *neutrosophic* hypervector space over a *neutrosophic* field $K(I)$ and let $B(I) = \{u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots, u_n = (a_n, b_nI)\}$ be a subset of $V(I)$.

(1) $B(I)$ is called a linearly dependent set if there exists *neutrosophic* scalars $\alpha_1 = (k_1, m_1I), \alpha_2 = (k_2, m_2I), \cdots, \alpha_1 = (k_n, m_nI)$ (not all zero) such that

$$\theta \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n.$$

(2) $B(I)$ is called a linearly independent set if

$$\theta \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n \text{ implies that } \alpha_1 = \alpha_2 = \cdots = \alpha_n = (0, 0I).$$

Theorem 4.5. *Let $(V(I), +, \bullet, K)$ be a weak neutrosophic hypervector space over a field K and let $\theta \neq u = (a, bI) \in V(I)$. Then $B(I) = \{u\}$ is a linearly independent set.*

Proof. Suppose that $\theta \neq u = (a, bI) \in V(I)$. Let $\theta \in k \bullet u$ and suppose that $0 \neq k \in K$. Then $k^{-1} \in K$ and therefore, $k^{-1} \bullet \theta \subseteq k^{-1} \bullet (k \bullet u)$ so that

$$\begin{aligned}
\theta &\in (k^{-1}k) \bullet u \\
&= 1 \bullet u \\
&= \{(x, yI) : x \in 1 \bullet a, y \in 1 \bullet b\} \\
&= \{(x, yI) : x \in \{a\}, y \in \{b\}\} \\
&= \{(a, bI)\} \\
&= \{u\}.
\end{aligned}$$

This shows that $u = \theta$ which is a contradiction. Hence, $k = 0$ and thus, $B = \{u\}$ is a linearly independent set. □

Theorem 4.6. *Let $(V(I), +, \bullet, K)$ be a weak neutrosophic hypervector space over a field K and let $B(I) = \{u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots, u_n = (a_n, b_nI)\}$ be a subset of $V(I)$. Then $B(I)$ is a linearly independent set if and only if at least one element of $B(I)$ can be expressed as a linear combination of the remaining elements of $B(I)$.*

Proof. Suppose that $B(I)$ is a linearly dependent set. Then there exists scalars $k_1, k_2, \cdots, k_n$ not all zero in K such that

$$\theta \in k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n.$$

Suppose that $k_1 \neq 0$. Then $k_1^{-1} \in K$ and therefore

$$\begin{aligned} k_1^{-1} \bullet \theta &\subseteq k_1^{-1} \bullet (k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n) \\ &= (k_1^{-1}k_1) \bullet u_1 + (k_1^{-1}k_2) \bullet u_2 + \cdots + (k_n^{-1}k_n) \bullet u_n \\ &= 1 \bullet u_1 + (k_1^{-1}k_2) \bullet u_2 + \cdots + (k_n^{-1}k_n) \bullet u_n \end{aligned}$$

so that

$$\theta \in 1 \bullet u_1 + \{u\}$$

where $u = (a, bI) \in (k_1^{-1}k_2) \bullet u_2 + \cdots + (k_n^{-1}k_n) \bullet u_n$. Thus $\theta \in \{(a, a_1, (b + b_1)I)\}$ from which we obtain $u_1 = (a_1, b_1 I) = -u = -(a, bI)$ so that

$$\begin{aligned} u_1 &\in (-1) \bullet u \\ &\subseteq (-1) \bullet ((k_1^{-1}k_2) \bullet u_2 + \cdots + (k_n^{-1}k_n) \bullet u_n) \\ &\subseteq (-k_1^{-1}k_2) \bullet u_2 + (-k_1^{-1}k_3) \bullet u_3 + \cdots + (-k_1^{-1}k_n) \bullet u_n. \end{aligned}$$

This shows that $u_1 \in span\{u_2, u_3, \cdots, u_n\}$.

Conversely, suppose that $u_1 \in span\{u_2, u_3, \cdots, u_n\}$ and suppose that $0 \neq -1 \in K$. Then there exists $k_2, k_3, \cdots, k_n \in K$ such that

$$u_1 \in k_2 \bullet u_2 + k_3 \bullet u_3 + \cdots + k_n \bullet u_n.$$

and we have

$$u_1 + (-u_1) \in (-1) \bullet u_1 + \in k_2 \bullet u_2 + k_3 \bullet u_3 + \cdots + k_n \bullet u_n.$$

from which we have

$$\theta \in (-1) \bullet u_1 + \in k_2 \bullet u_2 + k_3 \bullet u_3 + \cdots + k_n \bullet u_n.$$

Since $-1 \neq 0$ in K, it follows that $B(I)$ is a linearly dependent set. □

Corollary 4.7. Let $(V(I), +, \bullet, K)$ be a weak *neutrosophic* hypervector space over a field K and let $B(I) = \{u_1, u_2, \cdots, u_n\}$ be a subset of $V(I)$. Then $B(I)$ is a linearly independent set if and only if $u_i \in B(I)$ can be expressed as a linear combination of $\{u_1, u_2, \cdots, u_{i-1}\}$.

Theorem 4.8. *Let $(V(I), +, \bullet, K(I))$ be a strong neutrosophic hypervector space over a neutrosophic field $K(I)$ and let $B_1(I)$ and $B_2(I)$ be subsets of $V(I)$ such that $B_1(I) \subseteq B_2(I)$. If $B_1(I)$ is linearly dependent, then $B_2(I)$ is linearly dependent.*

Proof. Obvious. □

Theorem 4.9. *Let $(V(I), +, \bullet, K(I))$ be a strong neutrosophic hypervector space over a neutrosophic field $K(I)$ and let $B_1(I)$ and $B_2(I)$ be subsets of $V(I)$ such that $B_1(I) \subseteq B_2(I)$. If $B_2(I)$ is linearly indedependent, then $B_1(I)$ is linearly independent.*

Proof. Obvious. □

Definition 4.10. Let $(V(I), +, \bullet, K(I))$ be a strong *neutrosophic* hypervector space over a *neutrosophic* field $K(I)$ and let $B(I) = \{u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots\}$ be a subset of $V(I)$. $B(I)$ is said to be a basis for $V(I)$ if the following conditions hold:

(1) $B(I)$ is a linearly independent set.

(2) $V(I) = span(B(I))$.

If $B(I)$ is finite and its cardinality is n, then $V(I)$ is called an n-dimensional strong *neutrosophic* hypervector space and we write $dim_s(V(I)) = n$. If $B(I)$ is not finite, then $V(I)$ is called an infinite-dimensional strong *neutrosophic* hypervector space.

Definition 4.11. Let $(V(I), +, \bullet, K)$ be a weak *neutrosophic* hypervector space over a field K and let $B(I) = \{u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots\}$ be a subset of $V(I)$. $B(I)$ is said to be a basis for $V(I)$ if the following conditions hold:

(1) $B(I)$ is a linearly independent set.

(2) $V(I) = span(B(I))$.

If $B(I)$ is finite and its cardinality is n, then $V(I)$ is called an n-dimensional weak *neutrosophic* hypervector space and we write $dim_w(V(I)) = n$. If $B(I)$ is not finite, then $V(I)$ is called an infinite-dimensional weak *neutrosophic* hypervector space.

Example 7. (1) In Example 5, $B(I)$ is a basis for $V(I)$ and $dim_s(V(I)) = 3$.

(2) In Example 6, $B(I)$ is a basis for $V(I)$ and $dim_w(V(I)) = 4$.

Theorem 4.12. *Let $(V(I), +, \bullet, K(I))$ be a strong neutrosophic hypervector space over a neutrosophic field $K(I)$ and let $B(I) = \{u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots, u_n = (a_n, b_nI)\}$ be a subset of $V(I)$. Then $B(I)$ is a basis for $V(I)$ if and only if each neutrosophic vector $u = (a, bI) \in V(I)$ can be expressed uniquely as a linear combination of the elements of $B(I)$.*

Proof. Suppose that each *neutrosophic* vector $u = (a, bI) \in V(I)$ can be expressed uniquely as a linear combination of the elements of $B(I)$. Then $u \in span(B(I)) = V(I)$. Since such a representation is unique, it follows that $B(I)$ is a linearly independent set and since $u \in V(I)$ is arbitrary, it follows that $B(I)$ is a basis for $V(I)$.

Conversely, suppose that $B(I)$ is a basis for $V(I)$. Then $V(I) = span(B(I))$ and $B(I)$ is linearly independent. We show that $u = (a, bI) \in V(I)$ can be expressed uniquely as a linear combination of the elements of $B(I)$. To this end, for $\alpha_1 = (k_1, m_1I), \alpha_2 = (k_2, m_2I), \cdots, \alpha_n = (k_n, m_nI), \beta_1 = (r_1, s_1I), \beta_2 = (r_2, s_2I), \cdots, \beta_n = (r_n, s_nI) \in K(I)$, let us express u in two ways as follows:

$$u \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n, \quad (1)$$

$$u \in \beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \cdots + \beta_n \bullet u_n. \quad (2)$$

From equation(2), we have

$$\begin{aligned} -u &\in (-1)\bullet u \subseteq (-1)\bullet(\beta_1\bullet u_1+\beta_2\bullet u_2+\cdots+\beta_n\bullet u_n)\\ &= ((-1)\beta_1)\bullet u_1+((-1)\beta_2)\bullet u_2+\cdots+((-1)\beta_n)\bullet u_n\\ &= (-\beta_1)\bullet u_1+(-\beta_2)\bullet u_2+\cdots+(-1\beta_n)\bullet u_n. \end{aligned} \tag{3}$$

From equations (1) and (3), we have

$$\begin{aligned} u+(-u) &\in (\alpha_1+(-\beta_1))\bullet u_1+(\alpha_2+(-\beta_2))\bullet u_2+\cdots+(\alpha_n+(-\beta_n))\bullet u_n\\ \Rightarrow\theta &\in (\alpha_1-\beta_1)\bullet u_1+(\alpha_2-\beta_2)\bullet u_2+\cdots+(\alpha_n-\beta_n)\bullet u_n. \end{aligned}$$

Since $B(I)$ is linearly independent, it follows that $\alpha_1-\beta_1=\alpha_2-\beta_2=\cdots=\alpha_n-\beta_n=(0,0I)$ and therefore, $\alpha_1=\beta_1,\alpha_2=\beta_2,\cdots,\alpha_n=\beta_n$. This shows that u has been be expressed uniquely as a linear combination of the elements of $B(I)$. The proof is complete. □

Theorem 4.13. *Let $(V(I),+,\bullet,K)$ be a weak neutrosophic hypervector space over a field K and let $B_1(I)=\{u_1=(a_1,b_1I),u_2=(a_2,b_2I),\cdots,u_n=(a_n,b_nI)\}$ be a linearly independent subset of $V(I)$. If $u\in V(I)\backslash B(I)$ is arbitrary, then $B_2(I)=\{u_1=(a_1,b_1I),u_2=(a_2,b_2I),\cdots,u_n=(a_n,b_nI),u\}$ is a linearly dependent set if and only if $u\in span((B(I))$.*

Proof. Suppose that $B_2(I)$ is a linearly dependent set. Then there exists scalars $k_1,k_2,\cdots,k_n,k$ not all zero such that

$$\theta\in k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\bullet u_n+k\bullet u. \tag{4}$$

Suppose that $k=0$, then there exists at least one of the k_is say $k_1\neq 0$ and equation (4) becomes

$$\theta\in k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\bullet u_n \tag{5}$$

from which it follows that the set $B_1(I)=\{u_1=(a_1,b_1I),u_2=(a_2,b_2I),\cdots,u_n=(a_n,b_nI)\}$ is linearly dependent. This contradicts the hypothesis that $B_1(I)$ is linearly independent. Hence $k\neq 0$ and therefore $k^{-1}\in K$. From equation (4), we have

$$\begin{aligned} k^{-1}\bullet\theta &\subseteq k^{-1}\bullet(k_1\bullet u_1+k_2\bullet u_2+\cdots+k_n\bullet u_n+k\bullet u)\\ \Rightarrow\theta &\in (k^{-1}k_1)\bullet u_1+(k^{-1}k_2)\bullet u_2+\cdots+(k^{-1}k_n)\bullet u_n+(k^{-1}k)\bullet u\\ \Rightarrow\theta &= v+u(\text{where } (k^{-1}k_1)\bullet u_1+(k^{-1}k_2)\bullet u_2+\cdots+(k^{-1}k_n)\bullet u_n)\\ \Rightarrow u &= -v\in(-1)\bullet v\\ \Rightarrow u &\in (-1)\bullet((k^{-1}k_1)\bullet u_1+(k^{-1}k_2)\bullet u_2+\cdots+(k^{-1}k_n)\bullet u_n)\\ \Rightarrow u &\in (-k^{-1}k_1)\bullet u_1+(-k^{-1}k_2)\bullet u_2+\cdots+(-k^{-1}k_n)\bullet u_n\\ \Rightarrow u &\in span(B_1(I)). \end{aligned}$$

Conversely, suppose that $u \in span(B_1))$. Then there exists $k_1, k_2, \cdots, k_n \in K$ such that

$$\begin{aligned} u &\in k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n \\ \Rightarrow u + (-u) &\in k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n + (-1) \bullet u \\ \Rightarrow \theta &\in k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n + (-1) \bullet u. \end{aligned}$$

Since $u \notin B_1(I)$ and $B_1(I)$ is linearly independent, it follows that $\{u_1, u_2, \cdots, u_n, u\} = B_2(I)$ is a linearly dependent set. The proof is complete. □

Definition 4.14. Let $(V(I), +, \bullet, K(I))$ and $W(I), +', \bullet', K(I))$ be two strong *neutrosophic* hypervector spaces over a *neutrosophic* field $K(I)$. A mapping $\phi : V(I) \to W(I)$ is called a strong *neutrosophic* hypervector space homomorphism if the following conditions hold:

(1) ϕ is a strong hypervector space homomorphism.

(2) $\phi((0, I)) = (0, I)$.

If in addition ϕ is a bijection, we say that $V(I)$ is isomorphic to $W(I)$ and we write $V(I) \cong W(I)$.

Theorem 4.15. *Let $(V(I), +, \bullet, K(I))$ and $W(I), +', \bullet', K(I))$ be two strong neutrosophic hypervector spaces over a neutrosophic field $K(I)$ and let $\phi : V(I) \to W(I)$ be a bijective strong neutrosophic hypervector space homomorphism. If $B(I) = \{u_1 = (a_1, b_1I), u_2 = (a_2, b_2I), \cdots, u_n = (a_n, b_nI)\}$ is a basis for $V(I)$, then $B'(I) = \phi(B(I)) = \{\phi(u_1, \phi(u_2, \cdots, \phi(u_n\}$ is a basis for $W(I)$.*

Proof. Suppose that $B(I)$ is a basis for $V(I)$. Then for an arbitrary $u = (a, bI) \in V(I)$, there exists *neutrosophic* scalars $\alpha_1 = (k_1, m_1I), \alpha_2 = (k_2, m_2I), \cdots, \alpha_n = (k_n, m_nI) \in K(I)$ such that

$$\begin{aligned} u &\in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n \\ \Rightarrow \phi(u) &\in \phi(\alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n) \\ &= \alpha_1 \bullet' \phi(u_1) +' \alpha_2 \bullet' \phi(u_2) +' \cdots +' \alpha_n \bullet' \phi(u_n). \end{aligned}$$

Since ϕ is surjective, it follows that $\phi(u), \phi(u_1, \phi(u_2, \cdots, \phi(u_n \in W(I)$ and therefore $\phi(u) \in span(B'(I))$. To complete the proof, we must show that $B'(I)$ is linearly independent. To this end, suppose that

$$\phi(\theta) \in \beta_1 \bullet' \phi(u_1) +' \beta_2 \bullet' \phi(u_2) +' \cdots +' \beta_n \bullet' \phi(u_n)$$

where $\beta_1 = (r_1, s_1I), \beta_2 = (r_2, s_2I), \cdots, \beta_n = (r_n, s_nI) \in K(I)$, then

$$\begin{aligned} \phi(\theta) &\in \phi(\beta_1 \bullet u_1) +' \phi(\beta_2 \bullet u_2) +' \cdots +' \phi(\beta_n \bullet u_n) \\ &= \phi(\beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \cdots + \beta_n \bullet u_n) \end{aligned}$$

Since ϕ is injective, we must have

$$\theta \in \beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \cdots + \beta_n \bullet u_n$$

Also, since $B(I)$ is linearly independent, we must have $\beta_1 = \beta_2 = \cdots = \beta_n = (0, I)$. Hence $B'(I) = \{\phi(u_1), \phi(u_2), \cdots, \phi(u_n)\}$ is linearly independent and therefore a basis for $W(I)$. □

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