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Partial ordering on hyper-power sets

Published in:
Florentin Smarandache & Jean Dezert (Editors)
Advances and Applications of DSmT for Information Fusion
(Collected works), Vol. I
American Research Press, Rehoboth, 2004
ISBN: 1-931233-82-9
Chapter III, pp. 49 - 60

Abstract: *In this chapter, we examine several issues for ordering or partially ordering elements of hyper-power sets involved in the DSmT. We will show the benefit of some of these issues to obtain a nice and interesting structure of matrix representation of belief functions.*

3.1 Introduction to matrix calculus for belief functions

As rightly emphasized recently by Smets in [9], the mathematic of belief functions is often cumbersome because of the many summations symbols and all its subscripts involved in equations. This renders equations very difficult to read and understand at first sight and might discourage potential readers for their complexity. Actually, this is just an appearance because most of the operations encountered in DST with belief functions and basic belief assignments $m(.)$ are just simple linear operations and can be easily represented using matrix notation and be handled by elementary matrix calculus. We just focus here our presentation on the matrix representation of the relationship between a basic belief assignment $m(.)$ and its associated belief function Bel(.). A nice and more complete presentation of matrix calculus for belief functions can be found in [6, 7, 9]. One important aspect for the simplification of matrix representation and calculus in DST, concerns the choice of the order of the elements of the

This chapter is based on a paper [4] presented during the International Conference on Information Fusion, Fusion 2003, Cairns, Australia, in July 2003 and is reproduced here with permission of the International Society of Information Fusion.

power set 2^Θ. The order of elements of 2^Θ can be chosen arbitrarily actually, and it can be easily seen by denoting $\mathbf{m}$ the bba vector of size $2^n \times 1$ and $\mathbf{Bel}$ its corresponding belief vector of same size, that the set of equations $\text{Bel}(A) = \sum_{B\subseteq A} m(B)$ holding for all $A \subseteq \Theta$ is strictly equivalent to the following general matrix equation

$$\mathbf{Bel} = \mathbf{BM} \cdot \mathbf{m} \quad \Leftrightarrow \quad \mathbf{m} = \mathbf{BM}^{-1} \cdot \mathbf{Bel} \tag{3.1}$$

where the internal structure of $\mathbf{BM}$ depends on the choice of the order for enumerating the elements of 2^Θ. But it turns out that the simplest ordering based on the enumeration of integers from 0 to $2^n - 1$ expressed as n-binary strings with the lower bit on the right (LBR) (where $n = |\Theta|$) to characterize all the elements of power set, is the most efficient solution and best encoding method for matrix calculus and for developing efficient algorithms in MatLab[1] or similar programming languages [9]. By choosing the basic increasing binary enumeration (called *bibe system*), one obtains a very nice recursive algorithm on the dimension n of Θ for computing the matrix $\mathbf{BM}$. The computation of $\mathbf{BM}$ for $|\Theta| = n$ is just obtained from the iterations up to $i + 1 = n$ of the recursive relation [9] starting with $\mathbf{BM}_0 \triangleq [1]$ and where $\mathbf{0}_{i+1}$ denotes the zero-matrix of size $(i + 1) \times (i + 1)$,

$$\mathbf{BM}_{i+1} = \begin{bmatrix} \mathbf{BM}_i & \mathbf{0}_{i+1} \\ \mathbf{BM}_i & \mathbf{BM}_i \end{bmatrix} \tag{3.2}$$

$\mathbf{BM}$ is a binary unimodular matrix ($\det(\mathbf{BM}) = \pm 1$). $\mathbf{BM}$ is moreover triangular inferior and symmetrical with respect to its antidiagonal.

Example for $\Theta = \{\theta_1, \theta_2, \theta_3\}$

The *bibe system* gives us the following order for elements of $2^\Theta = \{\alpha_0, \ldots, \alpha_7\}$:

$$\begin{array}{llll} \alpha_0 \equiv 000 \equiv \emptyset & \alpha_1 \equiv 001 \equiv \theta_1 & \alpha_2 \equiv 010 \equiv \theta_2 & \alpha_3 \equiv 011 \equiv \theta_1 \cup \theta_2 \\ \alpha_4 \equiv 100 \equiv \theta_3 & \alpha_5 \equiv 101 \equiv \theta_1 \cup \theta_3 & \alpha_6 \equiv 110 \equiv \theta_2 \cup \theta_3 & \alpha_7 \equiv 111 \equiv \theta_1 \cup \theta_2 \cup \theta_3 \equiv \Theta \end{array}$$

Each element α_i of 2^Θ is a 3-bits string. With this bibe system, one has $\mathbf{m} = [m(\alpha_0), \ldots, m(\alpha_7)]'$ and $\mathbf{Bel} = [\text{Bel}(\alpha_0), \ldots, \text{Bel}(\alpha_7)]'$. The expressions of the matrix $\mathbf{BM}_3$ and its inverse $\mathbf{BM}_3{}^{-1}$ are given by

$$\mathbf{BM}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

[1]Matlab is a trademark of The MathWorks, Inc.

$$\mathbf{BM}_3{}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

3.2 Ordering elements of hyper-power set for matrix calculus

As within the DST framework, the order of the elements of D^Θ can be arbitrarily chosen. We denote the Dedekind number or order n as $d(n) \triangleq |D^\Theta|$ for $n = |\Theta|$. We denote also $\mathbf{m}$ the gbba vector of size $d(n) \times 1$ and **Bel** its corresponding belief vector of the same size. The set of equations $\text{Bel}(A) = \sum_{B \in D^\Theta, B \subseteq A} m(B)$ holding for all $A \in D^\Theta$ is then strictly equivalent to the following general matrix equation

$$\mathbf{Bel} = \mathbf{BM} \cdot \mathbf{m} \quad \Leftrightarrow \quad \mathbf{m} = \mathbf{BM}^{-1} \cdot \mathbf{Bel} \tag{3.3}$$

Note the similarity between these relations with the previous ones (3.1). The only difference resides in the size of vectors **Bel** and **m** and the size of matrix **BM** and their components. We explore in the following sections the possible choices for ordering (or partially ordering) the elements of hyper-power set D^Θ, to obtain an interesting matrix structure of **BM** matrix. Only three issues are examined and briefly presented in the sequel. The first method is based on the direct enumeration of elements of D^Θ according to their recursive generation via the algorithm for generating all isotone Boolean functions presented in the previous chapter and in [3]. The second (partial) ordering method is based on the notion of DSm cardinality which will be introduced in section 3.2.2. The last and most interesting solution proposed for partial ordering over D^Θ is obtained by introducing the notion of intrinsic informational strength $s(.)$ associated to each element of hyper-power set.

3.2.1 Order based on the enumeration of isotone Boolean functions

We have presented in chapter 2 a recursive algorithm based on isotone Boolean functions for generating D^Θ with didactic examples. Here is briefly the principle of the method. Let's consider $\Theta = \{\theta_1, \ldots, \theta_n\}$ satisfying the DSm model and the DSm order $\mathbf{u}_n$ of Smarandache's codification of parts of Venn diagram Θ with n partially overlapping elements θ_i, $i = 1, \ldots, n$. All the elements α_i of D^Θ can then be obtained by the very simple linear equation $\mathbf{d}_n = \mathbf{D}_n \cdot \mathbf{u}_n$ where $\mathbf{d}_n \equiv [\alpha_0 \equiv \emptyset, \alpha_1, \ldots, \alpha_{d(n)-1}]'$ is the vector of elements of D^Θ, $\mathbf{u}_n$ is the proper codification vector and D_n a particular binary matrix. The final result $\mathbf{d}_n$ is obtained from the previous *matrix product* after identifying $(+, \cdot)$ with $(\cup, \cap)$ operators, $0 \cdot x$ with $\emptyset$

and $1 \cdot x$ with x. D_n is actually a binary matrix corresponding to isotone (i.e. non-decreasing) Boolean functions obtained by applying recursively the steps (starting with $\mathbf{D}_0^c = [0\,1]'$)

- $\mathbf{D}_n^c$ is built from $\mathbf{D}_{n-1}^c$ by adjoining to each row $\mathbf{r}_i$ of $\mathbf{D}_{n-1}^c$ any row $\mathbf{r}_j$ of $\mathbf{D}_{n-1}^c$ such that $\mathbf{r}_i \cup \mathbf{r}_j = \mathbf{r}_j$. Then $\mathbf{D}_n$ is obtained by removing the first column and the last line of $\mathbf{D}_n^c$.

We denote $r^{iso}(\alpha_i)$ the position of α_i into the column vector $\mathbf{d}_n$ obtained from the previous enumeration/generation system. Such a system provides a total order over D^Θ defined $\forall \alpha_i, \alpha_j \in D^\Theta$ as $\alpha_i \prec \alpha_j$ (α_i precedes α_j) if and only if $r^{iso}(\alpha_i) < r^{iso}(\alpha_j)$. Based on this order, the **BM** matrix involved in (3.3) presents unfortunately no particular interesting structure. We have thus to look for better solutions for ordering the elements of hyper-power sets.

3.2.2 Ordering based on the DSm cardinality

A second possibility for ordering the elements of D^Θ is to (partially) order them by their increasing *DSm cardinality*.

Definition of the DSm cardinality

The *DSm cardinality* of any element $A \in D^\Theta$, denoted $\mathcal{C}_{\mathcal{M}}(A)$, corresponds to the number of parts of A in the Venn diagram of the problem (model $\mathcal{M}$) taking into account the set of integrity constraints (if any), i.e. all the possible intersections due to the nature of the elements θ_i. This *intrinsic cardinality* depends on the model $\mathcal{M}$. $\mathcal{M}$ is the model that contains A which depends on the dimension of Venn diagram, (i.e. the number of sets $n = |\Theta|$ under consideration), and on the number of non-empty intersections in this diagram. $\mathcal{C}_{\mathcal{M}}(A)$ must not be confused with the classical cardinality $|A|$ of a given set A (i.e. the number of its distinct elements) - that's why a new notation is necessary here.

Some properties of the DSm cardinality

First note that one has always $1 \leq \mathcal{C}_{\mathcal{M}}(A) \leq 2^n - 1$. In the (general) case of the free-model $\mathcal{M}^f$ (i.e. the DSm model) where all conjunctions are non-empty, one has for intersections:

$\mathcal{C}_{\mathcal{M}^f}(\theta_1) = \ldots = \mathcal{C}_{\mathcal{M}^f}(\theta_n) = 2^{n-1}$

$\mathcal{C}_{\mathcal{M}^f}(\theta_i \cap \theta_j) = 2^{n-2}$ for $n \geq 2$

$\mathcal{C}_{\mathcal{M}^f}(\theta_i \cap \theta_j \cap \theta_k) = 2^{n-3}$ for $n \geq 3$

It can be proven by induction that for $1 \leq m \leq n$, one has $\mathcal{C}_{\mathcal{M}^f}(\theta_{i_1} \cap \theta_{i_2} \cap \ldots \cap \theta_{i_m}) = 2^{n-m}$. For the cases $n = 1, 2, 3, 4$, this formula can be checked on the corresponding Venn diagrams. Let's consider this formula true for n sets, and prove it for $n+1$ sets (when all intersections/conjunctions are considered non-empty). From the Venn diagram of n sets, we can get a Venn diagram with $n+1$ sets if one draws a closed curve that cuts each of the $2^n - 1$ parts of the previous diagram (and, as a consequence, divides each part into two disjoint subparts). Therefore, the number of parts of each intersection is doubling when passing from a diagram of dimension n to a diagram of dimension $n+1$. Q.e.d.

In the case of the free-model $\mathcal{M}^f$, one has for unions:

$\mathcal{C}_{\mathcal{M}^f}(\theta_i \cup \theta_j) = 3(2^{n-2})$ for $n \geq 2$

$\mathcal{C}_{\mathcal{M}^f}(\theta_i \cup \theta_j \cup \theta_k) = 7(2^{n-3})$ for $n \geq 3$

It can be proven also by induction that for $1 \leq m \leq n$, one has $\mathcal{C}_{\mathcal{M}^f}(\theta_{i_1} \cup \theta_{i_2} \cup \ldots \cup \theta_{i_m}) = (2^m - 1)(2^{n-m})$. The proof is similar to the previous one, and keeping in mind that passing from a Venn diagram of dimension n to a dimension $n+1$, each part that forms the union $\theta_i \cap \theta_j \cap \theta_k$ will be split into two disjoint parts, hence the number of parts is doubling.

For other elements A in D^Θ, formed by unions and intersections, the closed form for $\mathcal{C}_{\mathcal{M}^f}(A)$ seems more complicated to obtain. But from the generation algorithm of D^Θ, DSm cardinal of a set A from D^Θ is exactly equal to the sum of its coefficients in the $\mathbf{u}_n$ basis, i.e. the sum of its row elements in the $\mathbf{D}_n$ matrix, which is actually very easy to compute by programming. The *DSm cardinality* plays in important role in the definition of the Generalized Pignistic Transformation (GPT) for the construction of subjective/pignistic probabilities of elements of D^Θ for decision-making at the pignistic level as explained in chapter 7 and in [5]. If one imposes a constraint that a set B from D^Θ is empty, then one suppresses the columns corresponding to the parts which compose B in the $\mathbf{D}_n$ matrix and the row of B and the rows of all elements of D^Θ which are subsets of B, getting a new matrix $\mathbf{D'}_n$ which represents a new model $\mathcal{M}'$. In the $\mathbf{u}_n$ basis, one similarly suppresses the parts that form B, and now this basis has the dimension $2^n - 1 - \mathcal{C}_{\mathcal{M}}(B)$.

Example of DSm cardinals on $\mathcal{M}^f$

Consider the 3D case $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with the free-model $\mathcal{M}^f$ corresponding to the following Venn diagram (where $< i >$ denotes the part which belongs to θ_i only, $< ij >$ denotes the part which belongs to θ_i and θ_j only, etc; this is Smarandache's codification (see the previous chapter).

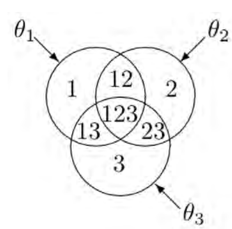


Figure 3.1: Venn Diagram for $\mathcal{M}^f$

The corresponding *partial ordering* for elements of D^Θ is then summarized in the following table:

$A \in D^\Theta$	$\mathcal{C}_{\mathcal{M}^f}(A)$
$\alpha_0 \triangleq \emptyset$	0
$\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3$	1
$\alpha_2 \triangleq \theta_1 \cap \theta_2$	2
$\alpha_3 \triangleq \theta_1 \cap \theta_3$	2
$\alpha_4 \triangleq \theta_2 \cap \theta_3$	2
$\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3$	3
$\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2$	3
$\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1$	3
$\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$	4
$\alpha_9 \triangleq \theta_1$	4
$\alpha_{10} \triangleq \theta_2$	4
$\alpha_{11} \triangleq \theta_3$	4
$\alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3$	5
$\alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2$	5
$\alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1$	5
$\alpha_{15} \triangleq \theta_1 \cup \theta_2$	6
$\alpha_{16} \triangleq \theta_1 \cup \theta_3$	6
$\alpha_{17} \triangleq \theta_2 \cup \theta_3$	6
$\alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3$	7

Table 3.1: $\mathcal{C}_{\mathcal{M}^f}(A)$ for free DSm model $\mathcal{M}^f$

Note that this partial ordering doesn't properly catch the intrinsic informational structure/strength of elements since by example $(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$ and θ_1 have the same DSm cardinal although they don't look similar because the part $< 1 >$ in θ_1 belongs only to θ_1 but none of the parts of $(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$ belongs to only one part of some θ_i. A better ordering function is then necessary to catch the intrinsic informational structure of elements of D^Θ. This is the purpose of the next section.

Example of DSm cardinals on an hybrid DSm model $\mathcal{M}$

Consider now the same 3D case with the hybrid DSm model $\mathcal{M} \neq \mathcal{M}^f$ in which we force all possible conjunctions to be empty, but $\theta_1 \cap \theta_2$ according to the following Venn diagram.

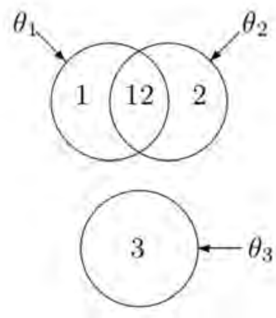


Figure 3.2: Venn Diagram for $\mathcal{M}$

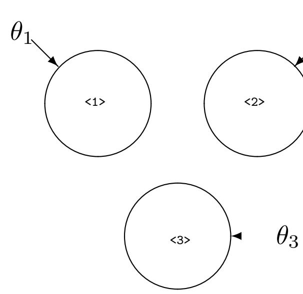
The corresponding *partial ordering* for elements of D^Θ is then summarized in the following table:

$A \in D^\Theta$	$\mathcal{C}_\mathcal{M}(A)$
$\alpha_0 \triangleq \emptyset$	0
$\alpha_1 \triangleq \theta_1 \cap \theta_2$	1
$\alpha_2 \triangleq \theta_3$	1
$\alpha_3 \triangleq \theta_1$	2
$\alpha_4 \triangleq \theta_2$	2
$\alpha_5 \triangleq \theta_1 \cup \theta_2$	3
$\alpha_6 \triangleq \theta_1 \cup \theta_3$	3
$\alpha_7 \triangleq \theta_2 \cup \theta_3$	3
$\alpha_8 \triangleq \theta_1 \cup \theta_2 \cup \theta_3$	4

Table 3.2: $\mathcal{C}_\mathcal{M}(A)$ for hybrid DSm model $\mathcal{M}$

Another example based on Shafer's model

Consider now the same 3D case but including all exclusivity constraints on θ_i, $i = 1, 2, 3$. This corresponds to the 3D Shafer's model $\mathcal{M}^0$ presented in the following Venn diagram.



Then, one gets the following list of elements (with their DSm cardinal) for the restricted D^Θ, which coincides naturally with the classical power set 2^Θ:

$A \in (D^\Theta \equiv 2^\Theta)$	$\mathcal{C}_{\mathcal{M}^0}(A)$
$\alpha_0 \triangleq \emptyset$	0
$\alpha_1 \triangleq \theta_1$	1
$\alpha_2 \triangleq \theta_2$	1
$\alpha_3 \triangleq \theta_3$	1
$\alpha_4 \triangleq \theta_1 \cup \theta_2$	2
$\alpha_5 \triangleq \theta_1 \cup \theta_3$	2
$\alpha_6 \triangleq \theta_2 \cup \theta_3$	2
$\alpha_7 \triangleq \theta_1 \cup \theta_2 \cup \theta_3$	3

Table 3.3: $\mathcal{C}_{\mathcal{M}^0}(A)$ for Shafer's model $\mathcal{M}^0$

The partial ordering of D^Θ based on DSm cardinality does not provide in general an efficient solution to get an interesting structure for the **BM** matrix involved in (3.3), contrary to the structure obtained by Smets in the DST framework as in section 3.1. The partial ordering presented in the sequel will however allow us to get such a nice structure for the matrix calculus of belief functions.

3.2.3 Ordering based on the intrinsic informational content

As already pointed out, the DSm cardinality is insufficient to catch the intrinsic informational content of each element d_i of D^Θ. A better approach to obtain this, is based on the following new function $s(.)$, which describes the *intrinsic information strength* of any $d_i \in D^\Theta$. A previous, but cumbersome, definition of $s(.)$ had been proposed in our previous works [1, 2] but it was difficult to handle and questionable with respect to the formal equivalent (dual) representation of elements belonging to D^Θ.

Definition of the $s(.)$ function

We propose here a better choice for $s(.)$, based on a very simple and natural geometrical interpretation of the relationships between the parts of the Venn diagram belonging to each $d_i \in D^\Theta$. All the values of the $s(.)$ function (stored into a vector $\mathbf{s}$) over D^Θ are defined by the following equation:

$$\mathbf{s} = \mathbf{D}_n \cdot \mathbf{w}_n \tag{3.4}$$

with $\mathbf{s} \triangleq [s(d_0) \, \dots \, s(d_p)]'$ where p is the cardinal of D^Θ for the model $\mathcal{M}$ under consideration. p is equal to Dedekind's number $d(n) - 1$ if the free-model $\mathcal{M}^f$ is chosen for $\Theta = \{\theta_1, \dots, \theta_n\}$. $\mathbf{D}_n$ is the *hyper-power set generating matrix*. The components w_i of vector $\mathbf{w}_n$ are obtained from the components of the *DSm encoding basis* vector $\mathbf{u}_n$ as follows (see previous chapter for details about $\mathbf{D}_n$ and $\mathbf{u}_n$) :

$$w_i \triangleq 1/l(u_i) \tag{3.5}$$

where $l(u_i)$ is the length of Smarandache's codification u_i of the part of the Venn diagram of the model $\mathcal{M}$, i.e the number of symbols involved in the codification.

For example, if $u_i =< 123 >$, then $l(u_i) = 3$ just because only three symbols 1, 2, and 3 enter in the codification u_i, thus $w_i = 1/3$.

From this new DSm ordering function s(.) we can partially order all the elements $d_i \in D^\Theta$ by the increasing values of $s(.)$.

Example of ordering on $D^{\Theta=\{\theta_1,\theta_2\}}$ with $\mathcal{M}^f$

In this simple case, the DSm ordering of D^Θ is given by

$\alpha_i \in D^\Theta$	$s(\alpha_i)$
$\alpha_0 = \emptyset$	$s(\alpha_0) = 0$
$\alpha_1 = \theta_1 \cap \theta_2$	$s(\alpha_1) = 1/2$
$\alpha_2 = \theta_1$	$s(\alpha_2) = 1 + 1/2$
$\alpha_3 = \theta_2$	$s(\alpha_3) = 1 + 1/2$
$\alpha_4 = \theta_1 \cup \theta_2$	$s(\alpha_4) = 1 + 1 + 1/2$

Based on this ordering, it can be easily verified that the matrix calculus of the beliefs **Bel** from **m** by equation (3.3), is equivalent to

$$\underbrace{\begin{bmatrix} \text{Bel}(\emptyset) \\ \text{Bel}(\theta_1 \cap \theta_2) \\ \text{Bel}(\theta_1) \\ \text{Bel}(\theta_2) \\ \text{Bel}(\theta_1 \cup \theta_2) \end{bmatrix}}_{\mathbf{Bel}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{\mathbf{BM}_2} \underbrace{\begin{bmatrix} m(\emptyset) \\ m(\theta_1 \cap \theta_2) \\ m(\theta_1) \\ m(\theta_2) \\ m(\theta_1 \cup \theta_2) \end{bmatrix}}_{\mathbf{m}}$$

where the $\mathbf{BM}_2$ matrix has a interesting structure (triangular inferior and unimodular properties, $\det(\mathbf{BM}_2) = \det(\mathbf{BM}_2^{-1}) = 1$). Conversely, the calculus of the generalized basic belief assignment **m** from beliefs **Bel** will be obtained by the inversion of the previous linear system of equations

$$\underbrace{\begin{bmatrix} m(\emptyset) \\ m(\theta_1 \cap \theta_2) \\ m(\theta_1) \\ m(\theta_2) \\ m(\theta_1 \cup \theta_2) \end{bmatrix}}_{\mathbf{m}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \end{bmatrix}}_{\mathbf{MB}_2 = \mathbf{BM}_2^{-1}} \underbrace{\begin{bmatrix} \mathrm{Bel}(\emptyset) \\ \mathrm{Bel}(\theta_1 \cap \theta_2) \\ \mathrm{Bel}(\theta_1) \\ \mathrm{Bel}(\theta_2) \\ \mathrm{Bel}(\theta_1 \cup \theta_2) \end{bmatrix}}_{\mathbf{Bel}}$$

Example of ordering on $D^{\Theta=\{\theta_1,\theta_2,\theta_3\}}$ with $\mathcal{M}^f$

In this more complicated case, the DSm ordering of D^Θ is now given by

$\alpha_i \in D^\Theta, i = 0, ..., 18$	$s(\alpha_i)$
$\emptyset$	0
$\theta_1 \cap \theta_2 \cap \theta_3$	$1/3$
$\theta_1 \cap \theta_2$	$1/3 + 1/2$
$\theta_1 \cap \theta_3$	$1/3 + 1/2$
$\theta_2 \cap \theta_3$	$1/3 + 1/2$
$(\theta_1 \cup \theta_2) \cap \theta_3$	$1/3 + 1/2 + 1/2$
$(\theta_1 \cup \theta_3) \cap \theta_2$	$1/3 + 1/2 + 1/2$
$(\theta_2 \cup \theta_3) \cap \theta_1$	$1/3 + 1/2 + 1/2$
$(\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$	$1/3 + 1/2 + 1/2 + 1/2$
θ_1	$1/3 + 1/2 + 1/2 + 1$
θ_2	$1/3 + 1/2 + 1/2 + 1$
θ_3	$1/3 + 1/2 + 1/2 + 1$
$(\theta_1 \cap \theta_2) \cup \theta_3$	$1/3 + 1/2 + 1/2 + 1 + 1/2$
$(\theta_1 \cap \theta_3) \cup \theta_2$	$1/3 + 1/2 + 1/2 + 1 + 1/2$
$(\theta_2 \cap \theta_3) \cup \theta_1$	$1/3 + 1/2 + 1/2 + 1 + 1/2$
$\theta_1 \cup \theta_2$	$1/3 + 1/2 + 1/2 + 1 + 1/2 + 1$
$\theta_1 \cup \theta_3$	$1/3 + 1/2 + 1/2 + 1 + 1/2 + 1$
$\theta_2 \cup \theta_3$	$1/3 + 1/2 + 1/2 + 1 + 1/2 + 1$
$\theta_1 \cup \theta_2 \cup \theta_3$	$1/3 + 1/2 + 1/2 + 1 + 1/2 + 1 + 1$

The order for elements generating the same value of $s(.)$ can be chosen arbitrarily and doesn't change the structure of the matrix $\mathbf{BM}_3$ given right after. That's why only a partial order is possible from $s(.)$. It can be verified that $\mathbf{BM}_3$ holds also the same previous interesting matrix structure properties and that $\det(\mathbf{BM}_3) = \det(\mathbf{BM}_3^{-1}) = 1$. Similar structure can be shown for problems of higher dimensions ($n > 3$).

Although a nice structure for matrix calculus of belief functions has been obtained in this work, and conversely to the recursive construction of $\mathbf{BM}_n$ in DST framework, a recursive algorithm (on dimension n) for the construction of $\mathbf{BM}_n$ from $\mathbf{BM}_{n-1}$ has not yet be found (if such recursive algorithm exists ...) and is still an open difficult problem for further research.

$$\mathbf{BM}_3 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

3.3 Conclusion

In this chapter, one has analyzed several issues to obtain an interesting matrix representation of the belief functions defined in the DSmT. For ordering the elements of hyper-power set D^Θ we propose three such orderings: first, using the direct enumeration of isotone Boolean functions, second, based on the DSm cardinality, and third, and maybe the most interesting, by introducing the intrinsic informational strength function $s(.)$ constructed from the DSm encoding basis. The third order permits to get a nice internal structure of the transition matrix **BM** in order to compute directly and easily by programming the belief vector **Bel** from the basic belief mass vector **m** and conversely by inversion of matrix **BM**.

3.4 References

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