TARBIAT MODARES UNIVERSITY

DOCTORAL THESIS

(Fuzzy and neutrosophic) soft hyper BCK-ideals

Author:
Somayeh KHADEMAN

Supervisor:
Prof. Mohammad Mehdi ZAHEDI

Advisors:
Prof. Ali IRANMANESH and **Prof. Young Bae JUN**

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Declaration of Authorship

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- I have acknowledged all main sources of help.
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TARBIAT MODARES UNIVERSITY

Abstract

Faculty of Mathematical Sciences
Department of Mathematics

Doctor of Philosophy

(Fuzzy and neutrosophic) soft hyper BCK-ideals

by Somayeh KHADEMAN

In this thesis, We have introduced four types of fuzzy soft positive implicative hyper BCK-ideals as types $(\subseteq, \subseteq, \subseteq)$, $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$.

We have also given examples and theorems to examine the relations between them and their relations with fuzzy soft (weak, strong) hyper BCK-ideals.

Then, we have introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal, reflexive neutrosophic hyper BCK-ideal and neutrosophic commutative hyper BCK-ideal of types $(\subseteq, \subseteq)$, $(\subseteq, \ll)$, $(\ll, \subseteq)$ and $(\ll, \ll)$ and indicated some relevant properties and their relations. Finally, we introduce the notions of neutrosophic soft (weak, strong) hyper BCK-ideal and (weak, strong) neutrosophic soft hyper p-ideal and have got some results on them.

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Contents

Dedicated to my Dear family

Chapter 1.

Introduction

1 Abstract

The notion of BCK-algebra was formulated first in 1966 by K. Iseki, Japanese Mathematician [20, 21, 22, 23, 24, 25]. This notion is originated from two different ways. One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations introduced by L. Kantorovic and E. Livenson to make a new set from old sets. These fundamental operations are union, intersection and the set difference. Then, as a generalization of those three operations and properties, we have the notion of Boolean algebra. If we take both of the union and the intersection, then as a general algebra, the notion of distributive lattice is obtained. Moreover, if we consider the notion of union or intersection, we have the notion of an upper semilattice or a lower semilattice. But the notion of set difference was not considered systematically before K. Iseki. Another Motivation is taken from classical and non-classical propositional calcului. There are some systems which contain the only implication functor among the logical functors. These examples are the systems of positive implicational calculus, weak positive implicational calculus by A. Church, and BCI, BCK-systems by C. A .Meredith. We know the following simple relations in set theory:

$$(A - B) - (A - C) \subseteq C - B,$$
$$A - (A - B) \subseteq B.$$

In propositional calcului, these relations are denoted by

$$(p \to q) \to ((q \to r) \to (p \to r)),$$
$$p \to ((p \to q) \to q).$$

From these relationships, K. Iseki introduced a new notion called a *BCK-algebra*.

In ordinary algebras the concept of operation is a fundamental. It can be generalized to multioperation and consequently leads to the emergence of multialgebras. This generalization was already made and as far as is knownthe first to idealize it for group theory

was the French mathematician Frederic Marty, in 1934, with the publication of the paper "Sur une generalisation de la notion de groupe" [54].

An operation is a relation that manipulate elements of a set and returns a value that is in another set. A multioperation (or hyperoperation) is a generalization of an operation when it returns a set of values instead of a single value. The class of structures composed by a set and at least one multioperation is what we call of algebraic hyperstructure. Multialgebras (or hyperalgebras) are a kind of hyperstructures as well as hypergroups, hyperrings, hyperlattices and so on. The hyperstructures theory was studied from many points of view and applied to several areas of Mathematics, Computer Science and Logic. Unfortunately F. Marty died young, during the Second World War, when his airplane was shot down over the Baltic Sea, while he was going on a mission to Finland. In the duration of his short life (1911-1940), F. Marty studied properties and applications of the hypergroups in two more communications [55, 56]. Many mathematicians in several countries contributed to the studies of the hypergroups theory[4, 5, 6, 47, 48, 59, 60, 61, 62, 69]. One of the first books dedicated to hypergroups and a good reference for applications of hyperstructures was written by P. Corsini in 1993 [15, 16].

In [40], Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra. Since then, Jun et al. studied more notions and results in [10, 11, 34, 37, 38] and [39]. Borzooei et al. [13] introduced the concept of the hyper K-algebra which is a generalization of the hyper BCK-algebra, and Zahedi et al. [71] defined the notions of (weak, strong) implicative hyper K-algebras. Borumand Saeid et al. [9] studied (weak) implicative hyper K-ideals in hyper K-algebras.

The concept of fuzzy sets was introduced by Lotfi A. Zadeh in 1965 [70]. Since then the fuzzy sets and fuzzy logic have been applied in many real life problems in uncertain, ambiguous environment. The traditional fuzzy sets is characterised by the membership value or the grade of membership value. Some times it may be very difficult to assign the membership value for a fuzzy sets. Consequently the concept of interval valued fuzzy sets was proposed [68] to capture the uncertainty of grade of membership value. In some real life problems in expert system, belief system, information fusion and so on, we must consider the truth-membership as well as the falsity-membership for proper description of an object in uncertain, ambiguous environment. Neither the fuzzy sets nor the interval valued fuzzy sets is appropriate for such a situation. Intuitionistic fuzzy sets introduced by Atanassov [7] is appropriate for such a situation. The intuitionistic fuzzy sets can only handle the incomplete information considering both the truth-membership (or simply membership) and falsity-membership (or non-membership) values. It does not handle the indeterminate and inconsistent information which exists in belief system. Smarandache [63, 64, 65] introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data [1, 2, 3, 12, 19, 57].

Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social science etc. These problems can not be dealt with by classical methods, because classical methods have inherent difficulties. Molodtsov suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [58] proposed a new approach, which was called soft set theory, for modeling uncertainty. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years.

Maji et al. [49, 50, 51, 52] extended the study of soft sets to fuzzy soft sets and neutrosophic soft sets. They introduced these concepts as a generalization of the standard soft sets, and presented applications of fuzzy soft sets and neutrosophic soft sets in a decision making problem.

Jun et al. applied the notions of fuzzy sets, neutrosophic sets, soft sets, fuzzy soft sets and neutrosophic soft sets to the theory of BCK/BCI-algebras and hyper BCK-algebras and studied ideal theory of BCK/BCI-algebras and and hyper BCK-algebras based on these notions [9, 14, 18, 30, 31, 33, 35, 36, 53, 66, 67, 72].

1.1 BCK-algebras

Definition 1.1 ([32])**.** Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a *BCK-algebra* if it satisfies the following conditions:

BCI-1 $((x * y) * (x * z)) * (z * y) = 0$,
BCI-2 $(x * (x * y)) * y = 0$,
BCI-3 $x * x = 0$,
BCI-4 $x * y = 0$ and $y * x = 0$ imply $x = y$,
BCk-5 $0 * x = 0$,
for all $x, y, z \in X$.

For brevity we also call X a BCK-algebra. In X we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$, for all $x, y \in X$.

Proposition 1.2 ([32])**.** *Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is BCK-algebra if and only if it satisfies:*

BCI-1' $(x * y) * (x * z) \leq z * y$,
BCI-2' $x * (x * y) \leq y$,
BCI-3' $x \leq x$,
BCI-4' $x \leq y$ *and* $y \leq x$ *imply* $x = y$,
BCk-5' $0 \leq x$,
BCI-6' $x \leq y$ *if and only if* $x * y = 0$,
for all $x, y, z \in X$.

From now on, for any BCK-algebra X, $*$ and $\leq$ are called a *BCK-operetion* and *BCK-ordering* on X respectively.

Example 1.3. Let S be a set. Denote 2^S for the power set of S in the sense that 2^S that is the collection of all subsets of S, $\setminus$ for the set difference and $\emptyset$ for the empty set. Then $(2^S, \setminus, \emptyset)$ is a BCK-algebra.

Proposition 1.4 ([32])**.** *In a BCK-algebra X, we have the following properties:*
(*i*) $x \leq y$ *implies* $z * y \leq z * x$,
(*ii*) $x \leq y$ *and* $y \leq z$ *implies* $x \leq z$,
(*iii*) $(x * y) * z = (x * z) * y$,
(*iv*) $x * y \leq z$ *implies* $x * z \leq y$,
(*v*) $(x * z) * (y * z) \leq x * y$,
(*vi*) $x \leq y$ *implies* $x * z \leq y * z$,
(*vii*) $x * y \leq x$,
(*viii*) $x * 0 = x$.

Definition 1.5 ([32])**.** A subset I of a BCK-algebra X is called a *BCK-ideal* of X if it satisfies:

$$0 \in I, \tag{1.1}$$
$$x * y \in I \textit{ and } y \in I \textit{ imply } x \in I \tag{1.2}$$

for all $x, y \in X$.

Definition 1.6 ([32])**.** A subset I of a BCK-algebra X is called a *positive imlicative BCK-ideal* of X if it satisfies (1.1) and

$$(x * y) * z \in I \text{ and } y * z \in I \textit{ imply } x * z \in I \tag{1.3}$$

for all $x, y, z \in X$.

Definition 1.7 ([32])**.** A subset I of a BCK-algebra X is called a *imlicative BCK-ideal* of X if it satisfies (1.1) and

$$(x * (y * z)) * z \in I \text{ and } z \in I \textit{ imply } x \in I \tag{1.4}$$

for all $x, y, z \in X$.

Definition 1.8 ([32])**.** A subset I of a BCK-algebra X is called a *commutative BCK-ideal* of X if it satisfies (1.1) and

$$(x * y) * z \in I \text{ and } z \in I \textit{ imply } x * (y * (y * x)) \in I \tag{1.5}$$

for all $x, y, z \in X$.

Theorem 1.9 ([32])**.** *In a BCK-algebra, every (positive) implicative ideal is an ideal. Also, every commutative ideal is an ideal. but the inverses is not true.*

Theorem 1.10 ([32])**.** *If we are given a BCK-algebra X, then a nonempty subset I of X is an implicative ideal if and only if it is both a commutative ideal and positive imlicative ideal.*

1.2 Hyper BCK-algebras

Let H be a nonempty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. For $A, B \in \mathcal{P}^*(H)$, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

Definition 1.11 ([40])**.** Let H be a nonempty set with a hyper operation "$\circ$" and a constant 0. Then an algebraic hyperstructure $(H, \circ, 0)$ of type $(2, 0)$ is called a hyper BCK-algebra if for all $x, y, z \in H$, it satisfying the following axioms:

(H1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(H3) $x \circ H \ll \{x\}$,

(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, there exists $b \in B$ such that $a \ll b$.

Remark 1.12. In a hyper BCK-algebra H, for all $x, y \in H$, the condition (H3) is equivalent to the condition:

(a1) $x \circ y \ll \{x\}$.

Proposition 1.13 ([40])**.** *In any hyper BCK-algebra H, for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H, the following conditions hold:*

$$x \circ 0 = \{x\}, \ A \circ 0 = A. \tag{1.6}$$
$$0 \circ x = \{0\}, \ 0 \circ A = \{0\}. \tag{1.7}$$
$$x \ll x, \ A \ll A. \tag{1.8}$$
$$(A \circ B) \circ C = (A \circ C) \circ B. \tag{1.9}$$
$$A \circ B \ll A. \tag{1.10}$$
$$A \ll \{0\} \text{ implies } A = \{0\}. \tag{1.11}$$
$$A \subseteq B \text{ implies } A \ll B. \tag{1.12}$$
$$y \ll z \ \text{ implies } \ x \circ z \ll x \circ y. \tag{1.13}$$

Lemma 1.14 ([37])**.** *Every hyper BCK-algebra H satisfies the following condition:*

$$((x \circ z) \circ (y \circ z)) \circ a \ll (x \circ y) \circ a \tag{1.14}$$

for all $x, y, z, a \in H$.

Definition 1.15 ([40])**.** A subset I of a hyper BCK-algebra H is called a *hyper BCK-ideal* of H if it satisfies:

$$0 \in I, \tag{1.15}$$

$$x \circ y \ll I \text{ and } y \in I \text{ imply } x \in I \tag{1.16}$$

for all $x, y \in H$.

Definition 1.16 ([40])**.** A subset I of a hyper BCK-algebra H is called a *weak hyper BCK-ideal* of H if it satisfies (1.15) and

$$x \circ y \subseteq I \text{ and } y \in I \text{ imply } x \in I \tag{1.17}$$

for all $x, y \in H$.

Definition 1.17 ([39])**.** A subset I of a hyper BCK-algebra H is called a *strong hyper BCK-ideal* of H if it satisfies (1.15) and

$$(x \circ y) \cap I \neq \emptyset \text{ and } y \in I \text{ imply } x \in I \tag{1.18}$$

for all $x, y \in H$.

Remark 1.18 ([39])**.** Recall that every strong hyper BCK-ideal is a hyper BCK-ideal. Also, every hyper BCK-ideal is a weak hyper BCK-ideal. But the converses not true in general.

Lemma 1.19 ([37])**.** *Let A be a hyper BCK-ideal of H. Then $I \circ J \subseteq A$ and $J \subseteq A$ imply that $I \subseteq A$ for every nonempty subsets I and J of H.*

Lemma 1.20 ([38])**.** *Let I be a subset of H. If J is a hyper BCK-ideal of H such that $I \ll J$, then I is contained in J.*

Definition 1.21 ([39])**.** A hyper BCK-ideal I of a hyper BCK-algebra H is said to be *reflexive* if $x \circ x \subseteq I$, for all $x, y \in H$.

Remark 1.22 ([39])**.** Every reflexive hyper BCK-ideal is a strong hyper BCK-ideal. But the converse not true in general.

Lemma 1.23 ([39])**.** *Every reflexive hyper BCK-ideal I of H satisfies the following implication.*

$$(x \circ y) \cap I \neq \emptyset \text{ imply } x \circ y \subseteq I \tag{1.19}$$

for all $x, y \in H$.

Definition 1.24 ([11]). A nonempty subset I of a hyper BCK-algebra H is said to be *S-reflexive* if

$$(x \circ y) \cap I \neq \emptyset \text{ imply } x \circ y \subseteq I \tag{1.20}$$

for all $x, y \in H$.

Definition 1.25 ([10]). A subset I of a hyper BCK-algebra H is said to be *closed* if

$$x \ll y \text{ and } y \in I \text{ imply } x \in I \tag{1.21}$$

for all $x, y \in H$.

Definition 1.26 ([37]). Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. Then I is called a *positive imlicative hyper BCK-ideal*

- of type $(\subseteq, \subseteq, \subseteq)$ of H if it satisfies:

$$(x \circ y) \circ z \subseteq I \text{ and } y \circ z \subseteq I \text{ imply } x \circ z \subseteq I, \tag{1.22}$$

  for all $x, y, z \in H$.

- of type $(\subseteq, \ll, \subseteq)$ of H if it satisfies:

$$(x \circ y) \circ z \subseteq I \text{ and } y \circ z \ll I \text{ imply } x \circ z \subseteq I, \tag{1.23}$$

  for all $x, y, z \in H$.

- of type $(\ll, \subseteq, \subseteq)$ of H if it satisfies:

$$(x \circ y) \circ z \ll I \text{ and } y \circ z \subseteq I \text{ imply } x \circ z \subseteq I, \tag{1.24}$$

  for all $x, y, z \in H$.

- of type $(\ll, \ll, \subseteq)$ of H if it satisfies:

$$(x \circ y) \circ z \ll I \text{ and } y \circ z \ll I \text{ imply } x \circ z \subseteq I, \tag{1.25}$$

  for all $x, y, z \in H$.

Theorem 1.27 ([10]). *Let I be a nonempty subset of a hyper BCK-algebra H. Then*

(1) *If I is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ or type $(\subseteq, \ll, \subseteq)$, then I is a positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.*

(2) *If I is a positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$, then I is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$.*

Theorem 1.28 ([11])**.** *Let I be a nonempty closed subset of a hyper BCK-algebra H. If I is a positive imlicative hyper BCK-ideal of type $(\alpha, \beta, \subseteq)$, then I is a positive imlicative hyper BCK-ideal of type $(\alpha, \beta, \subseteq)$, where $\alpha, \beta \in \{\ll, \subseteq\}$.*

Lemma 1.29 ([37])**.** *Every positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ is a hyper BCK-ideal.*

Lemma 1.30 ([10])**.** *Every positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ is a weak hyper BCK-ideal.*

Lemma 1.31 ([37])**.** *For a subset I of H such that $x \circ x \subseteq I$ for all $x \in H$, the following assertions are equivalent.*

(1) *I is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.*

(2) *I is a hyper BCK-ideal of H such that*

$$(x \circ y) \circ z \subseteq I \textit{ imply } (x \circ z) \circ (y \circ z) \subseteq I$$

for all $x, y, z \in H$.

Definition 1.32 ([11])**.** Let I be a subset of a hyper BCK-algebra H with $0 \in I$. Then I is called a *commutative hyper BCK-ideal* of

- type $(\subseteq, \subseteq)$ if it satisfies:

$$((x \circ y) \circ z \subseteq I \text{ and } z \in I \textit{ imply } x \circ (y \circ (y \circ x)) \subseteq I), \tag{1.26}$$

for all $x, y, z \in H$.

- type $(\subseteq, \ll)$ if it satisfies:

$$((x \circ y) \circ z \subseteq I \text{ and } z \in I \textit{ imply } x \circ (y \circ (y \circ x)) \ll I), \tag{1.27}$$

for all $x, y, z \in H$.

- type $(\ll, \subseteq)$ if it satisfies:

$$((x \circ y) \circ z \ll I \text{ and } z \in I \textit{ imply } x \circ (y \circ (y \circ x)) \subseteq I), \tag{1.28}$$

for all $x, y, z \in H$.

- type $(\ll, \ll)$ if it satisfies:

$$((x \circ y) \circ z \ll I \text{ and } z \in I \textit{ imply } x \circ (y \circ (y \circ x)) \ll I), \tag{1.29}$$

for all $x, y, z \in H$.

Theorem 1.33 ([11]). *Let I be a nonempty subset of a hyper BCK-algebra H. Then*

(1) *If I is a commutative hyper BCK-ideal of type $(\subseteq, \subseteq)$ or type $(\ll, \ll)$, then I is a commutative hyper BCK-ideal of type $(\subseteq, \ll)$.*

(2) *If I is a commutative hyper BCK-ideal of type $(\ll, \subseteq)$, then I is a commutative hyper BCK-ideal of type $(\subseteq, \subseteq)$ and $(\ll, \ll)$.*

Definition 1.34 ([53]). Let I be a subset of a hyper BCK-algebra H with $0 \in I$. Then I is called

- a *weak hyper p-ideal* of H if it satisfies:

$$((x \circ z) \circ (y \circ z) \subseteq A \text{ and } y \in A \text{ imply } x \in A), \tag{1.30}$$

  for all $x, y, z \in H$.

- a *hyper p-ideal* of H if it satisfies:

$$((x \circ z) \circ (y \circ z) \ll A \text{ and } y \in A \text{ imply } x \in A), \tag{1.31}$$

  for all $x, y, z \in H$.

- a *strong hyper p-ideal* of H if it satisfies:

$$(((x \circ z) \circ (y \circ z)) \cap A \neq \emptyset \text{ and } y \in A \text{ imply } x \in A), \tag{1.32}$$

  for all $x, y, z \in H$.

Remark 1.35 ([53]). Every (weak, strong) hyper p-ideal is a (weak, strong) hyper BCK-ideal. Also, every strong hyper p-ideal is a hyper p-ideal and every hyper p-ideal is a weak hyper p-ideal. But the converses not true in general.

1.3 Fuzzy sets, neutrosophic sets and soft sets in hyper BCK-algebras

Let X be a nonempty set. By a fuzzy set μ in X we mean a function $\mu : X \to [0, 1]$. For a fuzzy set μ in X and $t \in [0, 1]$, the set $U(\mu; t) := \{x \in X | \mu(x) \geq t\}$ is called a level subset of μ.

Definition 1.36 ([35]). A fuzzy set μ over a hyper BCK-algebra H is called a *fuzzy hyper BCK-ideal* of H, if for all $x, y \in H$ satisfies the following conditions:

$$x \ll y \text{ imply } \mu(x) \geq \mu(y), \tag{1.33}$$

$$\mu(x) \geq \min\{\inf_{a \in x \circ y} \mu(a), \mu(y)\}. \tag{1.34}$$

Theorem 1.37 ([35])**.** *A fuzzy set μ over H is a fuzzy hyper BCK-ideal of H if and only if the set $U(\mu;t)$ is a hyper BCK-ideal of H for all $t \in [0,1]$.*

Definition 1.38 ([63, 64, 65])**.** Let X be a nonempty set. A *neutrosophic set* (NS) in X is a structure of the form:

$$N := \{\langle x; N_T(x), N_I(x), N_F(x)\rangle \mid x \in X\}$$

where $N_T : X \to [0,1]$ is a truth membership function, $N_I : X \to [0,1]$ is an indeterminate membership function, and $N_F : X \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $N = (N_T, N_I, N_F)$ for the neutrosophic set

$$N := \{\langle x; N_T(x), N_I(x), N_F(x)\rangle \mid x \in X\}.$$

Let $N = (N_T, N_I, N_F)$ be a neutrosophic set in X. We define the following sets:

$$\begin{aligned}
U(N_T, \varepsilon_T) &:= \{x \in X \mid N_T(x) \geq \varepsilon_T\},\\
U(N_I, \varepsilon_I) &:= \{x \in X \mid N_I(x) \geq \varepsilon_I\},\\
L(N_F, \varepsilon_F) &:= \{x \in X \mid \tilde{\lambda}_F(x) \leq \varepsilon_F\},
\end{aligned}$$

where ε_T, ε_I, $\varepsilon_F \in [0,1]$.

Definition 1.39 ([58])**.** Let U be an initial universe set and E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and $A \subseteq E$. A pair (λ, A) is called a *soft set* over U, where λ is a mapping given by

$$\lambda : A \to \mathcal{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\lambda(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (λ, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [58].

Definition 1.40 ([50])**.** Let U be an initial universe set and E be a set of parameters. Let $\mathcal{F}(U)$ denote the set of all fuzzy sets in U. Then $(\tilde{\lambda}, A)$ is called a *fuzzy soft set* over U where $A \subseteq E$ and $\tilde{\lambda}$ is a mapping given by $\tilde{\lambda} : A \to \mathcal{F}(U)$.

In general, for every parameter u in A, $\tilde{\lambda}[u]$ is a fuzzy set in U and it is called *fuzzy value set* of parameter u. If for every $u \in A$, $\tilde{\lambda}[u]$ is a crisp subset of U, then $(\tilde{\lambda}, A)$ is degenerated to be the standard soft set. Thus, from the above definition, it is clear that fuzzy soft set is a generalization of standard soft set.

Definition 1.41 ([8])**.** Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H and $t \in [0,1]$. The following set

$$U(\tilde{\lambda}[u]; t) := \left\{x \in H \mid \tilde{\lambda}[u](x) \geq t\right\} \tag{1.35}$$

where u is a parameter in A, is called *level set* of $(\tilde{\lambda}, A)$.

Definition 1.42 ([8]). A fuzzy soft set $(\tilde{\lambda}, A)$ over a hyper BCK-algebra H is called a *fuzzy soft hyper BCK-ideal* based on a paramenter $u \in A$ over H (briefly, u-fuzzy soft hyper BCK-ideal of H) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0,1]$ of u, for all $x, y \in H$ satisfies the following conditions:

$$x \ll y \text{ imply } \tilde{\lambda}[u](x) \geq \tilde{\lambda}[u](y), \tag{1.36}$$

$$\tilde{\lambda}[u](x) \geq \min\{\inf_{a \in x \circ y} \tilde{\lambda}[u](a), \tilde{\lambda}[u](y)\}. \tag{1.37}$$

If $(\tilde{\lambda}, A)$ is a fuzzy soft hyper BCK-ideal based on a paramenter u over H for all $u \in A$, we say that $(\tilde{\lambda}, A)$ is a *fuzzy soft hyper BCK-ideal* of H.

Proposition 1.43 ([8]). *For every fuzzy soft hyper BCK-ideal $(\tilde{\lambda}, A)$ of H, the following assertions are valid.*

(1) $(\tilde{\lambda}, A)$ *satisfies the condition*

$$(\forall x \in H) \left(\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x) \right). \tag{1.38}$$

where u is any parameter in A.

(2) *If $(\tilde{\lambda}, A)$ satisfies the following condition:*

$$(\forall T \subseteq H)(\exists x_0 \in T) \left(\tilde{\lambda}[u](x_0) = \inf_{a \in T} \tilde{\lambda}[u](a) \right) \tag{1.39}$$

where u is any parameter in A, then

$$(\forall x, y \in H)(\exists a \in x \circ y) \left(\tilde{\lambda}[u](x) \geq \min\{\tilde{\lambda}[u](a), \tilde{\lambda}[u](y)\} \right)$$

Definition 1.44 ([8]). A fuzzy soft set $(\tilde{\lambda}, A)$ over H is called a *fuzzy soft weak hyper BCK-ideal* based on a paramenter $u \in A$ over H (briefly, *u-fuzzy soft weak hyper BCK-ideal* of H) if the fuzzy value set

$$\tilde{\lambda}[u] : H \to [0,1]$$

of u satisfies conditions (2.2) and (1.38). If $(\tilde{\lambda}, A)$ is a fuzzy soft weak hyper BCK-ideal based on u over H for all $u \in A$, we say that $(\tilde{\lambda}, A)$ is a *fuzzy soft weak hyper BCK-ideal* of H.

Definition 1.45 ([8]). A fuzzy soft set $(\tilde{\lambda}, A)$ over H is called a *fuzzy soft strong hyper BCK-ideal* over H based on a parameter u in A (briefly, u-fuzzy soft strong hyper BCK-ideal of H) if the fuzzy value set

$$\tilde{\lambda}[u] : H \to [0,1]$$

of u satisfies the following conditions

$$(\forall x, y \in H) \left(\tilde{\lambda}[u](x) \geq \min\{ \sup_{a \in x \circ y} \tilde{\lambda}[u](a), \tilde{\lambda}[u](y)\} \right), \tag{1.40}$$

$$(\forall x \in H) \left(\inf_{a \in x \circ x} \tilde{\lambda}[u](A) \geq \tilde{\lambda}[u](x) \right). \tag{1.41}$$

If $(\tilde{\lambda}, A)$ is a u-fuzzy soft strong hyper BCK-ideal of H for all $u \in A$, we say that $(\tilde{\lambda}, A)$ is a *fuzzy soft strong hyper BCK-ideal* of H.

Lemma 1.46 ([8])**.** *A fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft hyper BCK-ideal of H if and only if the set $U(\tilde{\lambda}[u]; t)$ in* (1.35) *is a hyper BCK-ideal of H for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u]; t) \neq \emptyset$.*

Definition 1.47 ([49])**.** Let U be an initial universe set and E be a set of parameters. Let $\mathcal{NS}(U)$ denote the set of all neutrosophic sets in U. Then a pair $(\widetilde{\mathcal{N}}, A)$ is called a *neutrosophic soft set* over U where $A \subseteq E$ and $\widetilde{\mathcal{N}}$ is a mapping given by $\widetilde{\mathcal{N}} : A \to \mathcal{NS}(U)$.

For every $e \in A$, the image of e under $\widetilde{\mathcal{N}}$, denoted by $\widetilde{\mathcal{N}}^e$, is a neutrosophic set in U:

$$\widetilde{\mathcal{N}}^e = \left\{ \langle x; \widetilde{\mathcal{N}}^e_T(x), \widetilde{\mathcal{N}}^e_I(x), \widetilde{\mathcal{N}}^e_F(x) \rangle \mid x \in U \right\},$$

and it is simply denoted by $\widetilde{\mathcal{N}}^e = (\widetilde{\mathcal{N}}^e_T, \widetilde{\mathcal{N}}^e_I, \widetilde{\mathcal{N}}^e_F)$.

Chapter 2.

Fuzzy soft positive imlicative hyper BCK-ideals of several types

2 Abstract

Fuzzy soft positive implicative hyper BCK-ideal of types $(\subseteq, \subseteq, \subseteq)$, $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$ are introduced, and their relations are investigated. Relations between fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ and fuzzy soft hyper BCK-ideal is considered. Also, relations between fuzzy soft strong hyper BCK-ideal and fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$ are discussed. Characterizations of fuzzy soft positive implicative hyper BCK-ideals are provided and we proved that the level set of fuzzy soft positive implicative hyper BCK-ideal of types $(\subseteq, \subseteq, \subseteq)$, $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ are positive implicative hyper BCK-ideal of types $(\subseteq, \subseteq, \subseteq)$, $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, respectively. Using the notion of positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, a fuzzy soft weak (strong) hyper BCK-ideal is established. Conditions for a fuzzy soft set to be a fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, respectively, are founded, and conditions for a fuzzy soft set to be a fuzzy soft weak hyper BCK-ideal are considered.

In what follows, let H and E be a hyper BCK-algebra and a set of parameters, respectively, and A be a subset of E unless otherwise specified.

2.1 Fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$

In the first chapter, we define the fuzzy soft hyper BCK-ideal based on a paramenter $u \in A$ over H, as follows:

Definition 2.1 ([8])**.** A fuzzy soft set $(\tilde{\lambda}, A)$ over a hyper BCK-algebra H is called a *fuzzy soft hyper BCK-ideal* based on a paramenter $u \in A$ over H (briefly, u-fuzzy soft hyper BCK-ideal of H) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0, 1]$ of u, for all $x, y \in H$

satisfies the following conditions:

$$x \ll y \ \Rightarrow\ \tilde{\lambda}[u](x) \geq \tilde{\lambda}[u](y), \tag{2.1}$$

$$\tilde{\lambda}[u](x) \geq \min\{\inf_{a \in x \circ y} \tilde{\lambda}[u](a), \tilde{\lambda}[u](y)\}. \tag{2.2}$$

If $(\tilde{\lambda}, A)$ is a fuzzy soft hyper BCK-ideal based on a paramenter u over H for all $u \in A$, we say that $(\tilde{\lambda}, A)$ is a *fuzzy soft hyper BCK-ideal* of H.

Now, we introduce the notion of fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ based on u over H.

Definition 2.2. Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H. Given a parameter $u \in A$, we say that $(\tilde{\lambda}, A)$ is a *fuzzy soft positive imlicative hyper BCK-ideal of type* $(\subseteq, \subseteq, \subseteq)$ based on u over H (briefly, *u-fuzzy soft positive imlicative hyper BCK-ideal of type* $(\subseteq, \subseteq, \subseteq)$) if the fuzzy value set

$$\tilde{\lambda}[u] : H \to [0, 1]$$

of u satisfies (2.1) and

$$\inf_{a \in x \circ z} \tilde{\lambda}[u](a) \geq \min\left\{\inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\right\}. \tag{2.3}$$

for all $x, y, z \in H$. If $(\tilde{\lambda}, A)$ is a u-fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ for all $u \in A$, we say that $(\tilde{\lambda}, A)$ is a *fuzzy soft positive imlicative hyper BCK-ideal of type* $(\subseteq, \subseteq, \subseteq)$.

Example 2.3. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" in Table 1.

Table 1: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Given a set $A = \{x, y, z\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 2. Then, $\tilde{\lambda}[x]$ and $\tilde{\lambda}[z]$ satisfies conditions (2.1) and (2.3). Hence, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ based on x and z. But $\tilde{\lambda}[y]$ does not satisfy the condition (2.1) since $a \ll b$ and $\tilde{\lambda}[y](a) < \tilde{\lambda}[y](b)$, and does not satisfy the condition (2.3) because of

$$\inf_{e \in a \circ 0} \tilde{\lambda}[y](e) < \min\left\{\inf_{f \in (a \circ b) \circ 0} \tilde{\lambda}[y](f), \inf_{g \in b \circ 0} \tilde{\lambda}[y](g)\right\}.$$

Table 2: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.3
y	0.8	0.4	0.6
z	0.7	0.7	0.4

Thus, $(\tilde{\lambda}, A)$ is not a y-fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ over H.

Example 2.4. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" in Table 3.

Table 3: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Given a set $A = \{x, y, z\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 4. Then, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.

Table 4: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.8	0.7	0.6
y	0.5	0.3	0.2
z	0.9	0.6	0.1

Lemma 2.5. *In every fuzzy soft positive imlicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \subseteq, \subseteq)$, the assertion* (1.38) *is valid.*

Proof. It is clear that the condition (1.38) is induced from the condition (2.1). $\square$

Table 5: Tabular representation of the binary operation $\circ$

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Example 2.6. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given in Table 5. Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 6. It is clear that $\tilde{\lambda}[y](0) \geq \tilde{\lambda}[y](z)$ for all $z \in H$. But $a \ll b$ and

Table 6: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.3
y	0.8	0.4	0.6

$\tilde{\lambda}[y](a) = 0.4 < 0.6 = \tilde{\lambda}[y](b)$. Hence, $(\tilde{\lambda}, A)$ is not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.

Theorem 2.7. *Every fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ is a fuzzy soft hyper BCK-ideal.*

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ and u be any parameter in A. Taking $z = 0$ in (2.3) and using (1.6) imply that

$$\begin{aligned}
\tilde{\lambda}[u](x) &= \inf_{a \in x \circ 0} \tilde{\lambda}[u](a) \\
&\geq \min\{\inf_{b \in (x \circ y) \circ 0} \tilde{\lambda}[u](b), \inf_{c \in y \circ 0} \tilde{\lambda}[u](c)\} \\
&= \min\{\inf_{d \in x \circ y} \tilde{\lambda}[u](d), \tilde{\lambda}[u](y)\}.
\end{aligned}$$

Therefore, $(\tilde{\lambda}, A)$ is a fuzzy soft hyper BCK-ideal of H. □

The converse of Theorem 2.7 is not true as seen in the following example.

Example 2.8. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" in Table 7. Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 8.

Table 7: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0, a\}$

Table 8: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.3
y	0.5	0.4	0.4

Then, $(\tilde{\lambda}, A)$ is a x-fuzzy soft hyper BCK-ideal over H. But it is not a y-fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$, since

$$\inf_{c \in b \circ b} \tilde{\lambda}[y](c) = \tilde{\lambda}[y](a) = 0.4$$

and

$$\begin{aligned} &\min\left\{ \inf_{d \in (b \circ a) \circ b} \tilde{\lambda}[y](d), \inf_{e \in a \circ b} \tilde{\lambda}[y](e) \right\} \\ &= \tilde{\lambda}[y](0) = 0.5. \end{aligned}$$

Therefore, any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.

Theorem 2.9. *A fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ if and only if the set $U(\tilde{\lambda}[u]; t)$ in (1.35) is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u]; t) \neq \emptyset$.*

Proof. Assume that $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$. Let u be a parameter in A and $t \in [0, 1]$ be such that $U(\tilde{\lambda}[u]; t) \neq \emptyset$. Since $0 \ll x$ for all $x \in H$, it follows from (2.1) that $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in H$. Hence,

$$\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$$

for all $x \in U(\tilde{\lambda}[u]; t)$, and so $\tilde{\lambda}[u](0) \geq t$. Thus, $0 \in U(\tilde{\lambda}[u]; t)$. Let $x, y, z \in H$ be such that

$$(x \circ y) \circ z \ll U(\tilde{\lambda}[u]; t)$$

and $y \circ z \subseteq U(\tilde{\lambda}[u];t)$. For every $a \in (x \circ y) \circ z$, there exists $b \in U(\tilde{\lambda}[u];t)$ such that $a \ll b$. Hence, $\tilde{\lambda}[u](a) \geq \tilde{\lambda}[u](b)$ by (2.1), and thus, $\tilde{\lambda}[u](a) \geq t$ for all $a \in (x \circ y) \circ z$. Let $c \in x \circ z$. Using (2.3), we have

$$\begin{aligned}\tilde{\lambda}[u](c) &\geq \inf_{e \in x \circ z} \tilde{\lambda}[u](e)\\ &\geq \min\left\{\inf_{f \in (x \circ y) \circ z} \tilde{\lambda}[u](f), \inf_{g \in y \circ z} \tilde{\lambda}[u](g)\right\}\\ &\geq t.\end{aligned}$$

Thus, $c \in U(\tilde{\lambda}[u];t)$, and so $x \circ z \subseteq U(\tilde{\lambda}[u];t)$. Therefore, $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ for all $t \in [0,1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t) \neq \emptyset$.

Conversely, assume that $U(\tilde{\lambda}[u];t) \neq \emptyset$ for $t \in [0,1]$ and any parameter u in A. Suppose that $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Then, $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H by Lemma 1.29. It follows from Lemma 1.46 that $\tilde{\lambda}[u]$ is a hyper BCK-ideal of H. Thus, the condition (2.1) is valid. Let $t = \min\left\{\inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\right\}$. Then,

$$\tilde{\lambda}[u](b) \geq \inf_{p \in (x \circ y) \circ z} \tilde{\lambda}[u](p) \geq t$$

and

$$\tilde{\lambda}[u](c) \geq \inf_{q \in y \circ z} \tilde{\lambda}[u](q) \geq t$$

for all $b \in (x \circ y) \circ z$ and $c \in y \circ z$. Hence, $b, c \in U(\tilde{\lambda}[u];t)$. Therefore, $(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u];t)$ and $y \circ z \subseteq U(\tilde{\lambda}[u];t)$. Using (1.12) and (1.24), we have $x \circ z \subseteq U(\tilde{\lambda}[u];t)$, and so

$$\inf_{d \in x \circ z} \tilde{\lambda}[u](a) \geq t = \min\left\{\inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\right\}.$$

Consequently, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq$). □

Corollary 2.10. *If a fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$, then $\bigcap_{u \in A} U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ for $t \in [0,1]$.*

Question. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$. For any parameter u in A, if $U(\tilde{\lambda}[u];t)$ is reflexive for all $t \in Im(\tilde{\lambda}[u])$, then is the following inequality valid?*

$$\inf_{a \in x \circ y} \tilde{\lambda}[u](a) \geq \inf_{b \in (x \circ y) \circ y} \tilde{\lambda}[u](b) \tag{2.4}$$

for all $x, y \in H$.

The answer to the above question is negative. For example, note that $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ based on x and z in Example 2.3. Also $U(\tilde{\lambda}[x]; t)$ and $U(\tilde{\lambda}[z]; t)$ are reflexive for all $t \in Im(\tilde{\lambda}[u])$. But

$$\inf_{e \in b \circ 0} \tilde{\lambda}[x](e) = 0.3 < 0.9 = \inf_{f \in (x \circ 0) \circ 0} \tilde{\lambda}[x](f).$$

Hence, (2.4) is not true.

Proposition 2.11. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft hyper BCK-ideal of H. For any parameter u in A, if $(\tilde{\lambda}, A)$ satisfies the condition* (2.4)*, then it satisfies the following condition.*

$$\inf_{a \in (x \circ z) \circ (y \circ z)} \tilde{\lambda}[u](a) \geq \inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b) \tag{2.5}$$

for all $x, y, z \in H$. Moreover if the nonempty level set $U(\tilde{\lambda}[u]; t)$ of $(\tilde{\lambda}, A)$ is reflexive for all $t \in [0, 1]$, then

$$\inf_{c \in x \circ y} \tilde{\lambda}[u](c) \geq \min \left\{ \tilde{\lambda}[u](z), \inf_{d \in ((x \circ y) \circ y) \circ z} \tilde{\lambda}[u](d) \right\} \tag{2.6}$$

for all $x, y, z \in H$.

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft hyper BCK-ideal of H which satisfies the condition (2.4) for any parameter u in A. Let $t = \inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b)$. Then,

$$(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u]; t).$$

Using (1.7) and Lemma 1.14 induces

$$((x \circ (y \circ z)) \circ z) \circ z = ((x \circ z) \circ (y \circ z)) \circ z \ll (x \circ y) \circ z, \tag{2.7}$$

and so $((x \circ (y \circ z)) \circ z) \circ z \ll U(\tilde{\lambda}[u]; t)$. It follows from Lemma 1.20 that

$$((x \circ (y \circ z)) \circ z) \circ z \subseteq U(\tilde{\lambda}[u]; t)$$

and so that $(q \circ z) \circ z \subseteq U(\tilde{\lambda}[u]; t)$ for all $q \in x \circ (y \circ z)$. Using the condition (2.4), we get

$$\inf_{r \in q \circ z} \tilde{\lambda}[u](r) \geq \inf_{s \in (q \circ z) \circ z} \tilde{\lambda}[u](s) \geq t,$$

and so $q \circ z \subseteq U(\tilde{\lambda}[u]; t)$ for all $q \in x \circ (y \circ z)$. Hence,

$$(x \circ z) \circ (y \circ z) = (x \circ (y \circ z)) \circ z = \bigcup_{q \in x \circ (y \circ z)} q \circ z \subseteq U(\tilde{\lambda}[u]; t),$$

and therefore,

$$\inf_{a \in (x \circ z) \circ (y \circ z)} \tilde{\lambda}[u]() \geq t = \inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b).$$

This proves (2.5). Suppose that $U(\tilde{\lambda}[u];t)$ of $(\tilde{\lambda},A)$ is reflexive for all $t\in[0,1]$. For any $x,y,z\in H$, put

$$s=\min\left\{\tilde{\lambda}[u](z),\inf_{d\in((x\circ y)\circ y)\circ z}\tilde{\lambda}[u](d)\right\}.$$

Then, $z\in U(\tilde{\lambda}[u];s)$ and $((x\circ z)\circ y)\circ y=((x\circ y)\circ y)\circ z\subseteq U(\tilde{\lambda}[u];s)$. Thus, $(q\circ y)\circ y\subseteq U(\tilde{\lambda}[u];s)$, which implies from Lemma 1.31 that $(q\circ y)\circ(y\circ y)\subseteq U(\tilde{\lambda}[u];s)$ for all $q\in x\circ z$. Thus,

$$((x\circ z)\circ y)\circ(y\circ y)\subseteq U(\tilde{\lambda}[u];s),$$

and so $(x\circ y)\circ z=(x\circ z)\circ y\subseteq U(\tilde{\lambda}[u];s)$ by Lemma 1.19 and (H2). Since $z\in U(\tilde{\lambda}[u];s)$, we have $x\circ y\subseteq U(\tilde{\lambda}[u];s)$ by Lemma 1.19. Hence,

$$\inf_{c\in x\circ y}\tilde{\lambda}[u](c)\geq s=\min\left\{\tilde{\lambda}[u](z),\inf_{d\in((x\circ y)\circ y)\circ z}\tilde{\lambda}[u](d)\right\}$$

for all $x,y,z\in H$. This completes the proof. □

Using the notion of positive imlicative hyper BCK-ideal of H, we establish a fuzzy soft weak hyper BCK-ideal.

Theorem 2.12. *Let I be a positive imlicative hyper BCK-ideal of type $(\ll,\subseteq,\subseteq)$ and let $z\in H$. For a fuzzy soft set $(\tilde{\lambda},A)$ over H and any parameter u in A, if we define the fuzzy value set $\tilde{\lambda}[u]$ by*

$$\tilde{\lambda}[u]:H\to[0,1],\ x\mapsto\begin{cases}t & \text{if } x\in I_z,\\ s & \text{otherwise},\end{cases}\tag{2.8}$$

where $t>s$ in $[0,1]$ and $I_z:=\{y\in H\mid y\circ z\subseteq I\}$, then $(\tilde{\lambda},A)$ is a u-fuzzy soft weak hyper BCK-ideal of H.

Proof. It is clear that $\tilde{\lambda}[u](0)\geq\tilde{\lambda}[u](x)$ for all $x\in H$. Let $x,y\in H$. If $y\notin I_z$, then $\tilde{\lambda}[u](y)=s$ and so

$$\tilde{\lambda}[u](x)\geq s=\min\left\{\tilde{\lambda}[u](y),\inf_{e\in x\circ y}\tilde{\lambda}[u](e)\right\}.\tag{2.9}$$

If $x\circ y\nsubseteq I_z$, then there exists $a\in x\circ y\setminus I_z$, and thus, $\tilde{\lambda}[u](a)=s$. Hence,

$$\min\left\{\tilde{\lambda}[u](y),\inf_{e\in x\circ y}\tilde{\lambda}[u](e)\right\}=s\leq\tilde{\lambda}[u](x).\tag{2.10}$$

Assume that $x\circ y\subseteq I_z$ and $y\in I_z$. Then, $(x\circ y)\circ z\subseteq I$ and $y\circ z\subseteq I$, which imply that $(x\circ y)\circ z\ll I$ and $y\circ z\subseteq I$. It follows from (1.24) that $x\circ z\subseteq I$, i.e., $x\in I_z$. Thus,

$$\tilde{\lambda}[u](x)=t\geq\min\left\{\tilde{\lambda}[u](y),\inf_{e\in x\circ y}\tilde{\lambda}[u](e)\right\}.$$

Therefore, $(\tilde{\lambda},A)$ is a fuzzy soft weak hyper BCK-ideal of H. □

Theorem 2.13. *If $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ in which the nonempty level set $U(\tilde{\lambda}[u];t)$ of $(\tilde{\lambda}, A)$ is reflexive for all $t \in Im(\tilde{\lambda}[u])$, then the set*

$$\tilde{\lambda}[u]_z = \{x \in H \mid x \circ z \subseteq U(\tilde{\lambda}[u];t)\} \tag{2.11}$$

is a hyper BCK-ideal of H for all $z \in H$.

Proof. Obviously $0 \in \tilde{\lambda}[u]_z$. Let $x, y \in H$ be such that $x \circ y \subseteq \tilde{\lambda}[u]_z$ and $y \in \tilde{\lambda}[u]_z$. Then,

$$(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u];t)$$

and $y \circ z \subseteq U(\tilde{\lambda}[u];t)$ for all $t \in Im(\tilde{\lambda}[u])$. Using (1.12), we know that $(x \circ y) \circ z \ll U(\tilde{\lambda}[u];t)$. Since $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, it follows from (1.24) that $x \circ z \subseteq U(\tilde{\lambda}[u];t)$, that is, $x \in \tilde{\lambda}[u]_z$. This shows that $\tilde{\lambda}[u]_z$ is a weak hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \circ y \ll \tilde{\lambda}[u]_z$ and $y \in \tilde{\lambda}[u]_z$, and let $a \in x \circ y$. Then there exists $b \in \tilde{\lambda}[u]_z$ such that $a \ll b$, that is, $0 \in a \circ b$. Thus, $(a \circ b) \cap U(\tilde{\lambda}[u];t) \neq \emptyset$. Since $U(\tilde{\lambda}[u];t)$ is a reflexive hyper BCK-ideal of H, it follows from (H1) and Lemma 1.23 that

$$(a \circ z) \circ (b \circ z) \ll a \circ b \subseteq U(\tilde{\lambda}[u];t)$$

and so that $a \circ z \subseteq U(\tilde{\lambda}[u];t)$ since $b \circ z \subseteq U(\tilde{\lambda}[u];t)$. Hence, $a \in \tilde{\lambda}[u]_z$, and so $x \circ y \subseteq \tilde{\lambda}[u]_z$. Since $\tilde{\lambda}[u]_z$ is a weak hyper BCK-ideal of H, we get $x \in \tilde{\lambda}[u]_z$. Consequently $\tilde{\lambda}[u]_z$ is a hyper BCK-ideal of H. $\square$

The following example shows that any positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ is neither S-reflexive nor a strong hyper BCK-ideal.

Example 2.14. Consider the hyper BCK-algebra $H = \{0, a, b\}$ in Example 2.3. Then, the set $I := \{0, a\}$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. But it is not S-reflexive since $(b \circ a) \cap I \neq \emptyset$ but $b \circ a \not\subseteq I$. Also, I is not a strong hyper BCK-ideal of H since $(b \circ a) \cap I \neq \emptyset$ and $a \in I$, but $b \notin I$.

Using the notion of positive imlicative hyper BCK-ideal of H, we establish a fuzzy soft strong hyper BCK-ideal.

Lemma 2.15. *Every S-reflexive positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ is a strong hyper BCK-ideal.*

Proof. Let I be an S-reflexive positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and let $x, y \in H$ be such that $(x \circ y) \cap I \neq \emptyset$ and $y \in I$. Then, $(x \circ y) \circ 0 = x \circ y \subseteq I$ since I is S-reflexive and $A \circ 0 = A$ for any subset A of H. It follows from (1.12) that $(x \circ y) \circ 0 \ll I$. Since $y \circ 0 \subseteq I$, we have $\{x\} = x \circ 0 \subseteq I$ and so $x \in I$. Therefore, I is a strong hyper BCK-ideal of H. $\square$

Table 9: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{a\}$	$\{0\}$	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0\}$

The following example shows that the converse of Lemma 2.15 is not true in general.

Example 2.16. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation "$\circ$" in Table 9. Then, $I := \{0, c\}$ is a strong hyper BCK-ideal and S-reflexive. But it is not a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, since $(b \circ a) \circ a \ll I$ and $a \circ a \subseteq I$ but $b \circ a \nsubseteq I$.

Lemma 2.17 ([8])**.** *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H such that*

$$(\forall T \subseteq H)(\exists x_0 \in T)\left(\tilde{\lambda}[u](x_0) = \sup_{a\in T}\tilde{\lambda}[u](a)\right) \tag{2.12}$$

where u is any parameter in A. If the set $U(\tilde{\lambda}[u]; t)$ in (1.35) *is a strong hyper BCK-ideal of H for all $t \in [0, 1]$ with $U(\tilde{\lambda}[u]; t) \neq \emptyset$, then $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H.*

Using Lemmas 2.15 and 2.17, we have the following theorem.

Theorem 2.18. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H satisfying the condition* (2.12)*. If the set $U(\tilde{\lambda}[u]; t)$ in* (1.35) *is an S-reflexive positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ for all $t \in [0, 1]$ with $U(\tilde{\lambda}[u]; t) \neq \emptyset$, then $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H.*

The following example shows that the converse of Theorem 2.18 is not true in general.

Example 2.19. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" in Table 10.
Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 11.
It is routine to verify that $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H. If $t > 0.6$, then the set $U(\tilde{\lambda}[x]; t) = \{0\}$ is not a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, since $(0 \circ b) \circ b = \{0\} \ll U(\tilde{\lambda}[x]; t)$, $b \circ b = \{0, b\} \nsubseteq U(\tilde{\lambda}[x]; t)$ and

$$0 \circ b = \{0\} \subseteq U(\tilde{\lambda}[x]; t).$$

Table 10: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Table 11: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.3
y	0.8	0.6	0.6

2.2 Fuzzy soft positive imlicative hyper BCK-ideals of types $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$

Definition 2.20. Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H. Then, $(\tilde{\lambda}, A)$ is called

- a *fuzzy soft positive imlicative hyper BCK-ideal* of type $(\subseteq, \ll, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0, 1]$ of u satisfies (2.1) and

$$(\forall x, y, z \in H)\ (\inf_{a \in x \circ z} \tilde{\lambda}[u](a) \geq \min\{\inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \sup_{c \in y \circ z} \tilde{\lambda}[u](c)\}). \quad (2.13)$$

- a *fuzzy soft positive imlicative hyper BCK-ideal* of type $(\ll, \subseteq, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0, 1]$ of u satisfies (2.1) and

$$(\forall x, y, z \in H)\ (\inf_{a \in x \circ z} \tilde{\lambda}[u](a) \geq \min\{\sup_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\}). \quad (2.14)$$

- a *fuzzy soft positive imlicative hyper BCK-ideal* of type $(\ll, \ll, \subseteq)$ based on a parameter $u \in A$ over H (briefly, u-fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$) if the fuzzy value set $\tilde{\lambda}[u] : H \to [0, 1]$ of u satisfies (2.1) and

$$(\forall x, y, z \in H)\ (\inf_{a \in x \circ z} \tilde{\lambda}[u](a) \geq \min\{\sup_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\}). \quad (2.15)$$

Theorem 2.21. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H.*

(1) *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive imlicative hyper* BCK*-ideal of type* $(\ll, \subseteq, \subseteq)$ *or type* $(\subseteq, \ll, \subseteq)$*, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive imlicative hyper* BCK*-ideal of type* $(\subseteq, \subseteq, \subseteq)$.

(2) *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive imlicative hyper* BCK*-ideal of type* $(\ll, \ll, \subseteq)$*, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive imlicative hyper* BCK*-ideal of type* $(\ll, \subseteq, \subseteq)$ *and* $(\subseteq, \ll, \subseteq)$.

Proof. (1) Assume that $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ or type $(\subseteq, \ll, \subseteq)$. Then,

$$\begin{aligned} \inf_{a \in x \circ z} \tilde{\lambda}[u](a) &\geq \min\{ \sup_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\} \\ &\geq \min\{ \inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\} \end{aligned}$$

or

$$\begin{aligned} \inf_{a \in x \circ z} \tilde{\lambda}[u](a) &\geq \min\{ \inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\} \\ &\geq \min\{ \inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\}, \end{aligned}$$

respectively. Thus, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$.

(2) Suppose that $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Then,

$$\begin{aligned} \inf_{a \in x \circ z} \tilde{\lambda}[u](a) &\geq \min\{ \sup_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\} \\ &\geq \min\{ \sup_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\} \end{aligned}$$

and

$$\begin{aligned} \inf_{a \in x \circ z} \tilde{\lambda}[u](a) &\geq \min\{ \sup_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\} \\ &\geq \min\{ \inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\}. \end{aligned}$$

Therefore, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$. $\square$

Corollary 2.22. *If* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive imlicative hyper* BCK*-ideal of type* $(\ll, \ll, \subseteq)$ *, then* $(\tilde{\lambda}, A)$ *is a fuzzy soft positive imlicative hyper* BCK*-ideal of type* $(\subseteq, \subseteq, \subseteq)$.

The following example shows that any fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ is not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Table 12: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, b, c\}$

Table 13: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b	c
x	0.9	0.8	0.5	0.3
y	0.9	0.7	0.6	0.4

Example 2.23. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation "$\circ$" in Table 12. Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 13. Then, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$. Since

$$\inf_{r \in c \circ 0} \tilde{\lambda}[x](r) = 0.3 < 0.5 = \min\left\{ \sup_{s \in (c \circ b) \circ 0} \tilde{\lambda}[x](s), \inf_{t \in b \circ 0} \tilde{\lambda}[x](t) \right\},$$

it is not an x-fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, and thus, it is not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Question. *Is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$?*

The following example shows that any fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ is not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ or $(\ll, \ll, \subseteq)$.

Example 2.24. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" in Table 14. Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 15. Then, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. Since

$$\inf_{r \in b \circ b} \tilde{\lambda}[x](r) = 0.3 < 0.9 = \min\left\{ \sup_{s \in (b \circ a) \circ b} \tilde{\lambda}[x](s), \sup_{t \in a \circ b} \tilde{\lambda}[x](t) \right\},$$

Table 14: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Table 15: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.3
y	0.8	0.7	0.1

it is not an x-fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and so not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Also, since

$$\inf_{r \in b \circ b} \tilde{\lambda}[y](r) = 0.1 < 0.8 = \min \left\{ \sup_{s \in (b \circ 0) \circ b} \tilde{\lambda}[y](s), \sup_{t \in 0 \circ b} \tilde{\lambda}[y](t) \right\},$$

it is not a y-fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$ and so not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$.

Question. *Is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ or $(\ll, \ll, \subseteq)$?*

Using Theorems 2.21 and 2.7, we have the following corollary.

Corollary 2.25. *Every fuzzy soft positive imlicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of types $(\ll, \subseteq, \subseteq)$, $(\subseteq, \ll, \subseteq)$ or $(\ll, \ll, \subseteq)$ is a fuzzy soft hyper BCK-ideal.*

We can check that the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 2.23 is a fuzzy soft hyper BCK-ideal of H, but it is not a fuzzy soft positive imlicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$. This shows that any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive imlicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$. Also, we know that the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 2.24 is a fuzzy soft hyper BCK-ideal of H, but it is a fuzzy soft hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Thus, any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Let $(\tilde{\lambda}, A)$ be a fuzzy soft hyper BCK-ideal of H. If $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \ll, \subseteq)$, then, it is a fuzzy soft positive imlicative hyper BCK-ideal

$(\tilde{\lambda}, A)$ of type $(\subseteq, \subseteq, \subseteq)$ by Theorem 2.21(1). Hence, every fuzzy soft hyper BCK-ideal of H is a fuzzy soft positive imlicative hyper BCK-ideal $(\tilde{\lambda}, A)$ of type $(\subseteq, \subseteq, \subseteq)$. But this is contradictory to Example 2.8. Therefore, we know that any fuzzy soft hyper BCK-ideal may not be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$.

We consider relation between a fuzzy soft positive imlicative hyper BCK-ideal of any type and a fuzzy soft strong hyper BCK-ideal.

Theorem 2.26. *Every fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ is a fuzzy soft strong hyper BCK-ideal of H.*

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and let u be any parameter in A. Since $x \circ x \ll \{x\}$ for all $x \in H$, it follows from (2.1) that

$$\inf_{a \in x \circ x} \tilde{\lambda}[u](a) \geq \inf_{a \in \{x\}} \tilde{\lambda}[u](a) = \tilde{\lambda}[u](x).$$

Taking $z = 0$ in (2.14) and using (1.6) imply that

$$\begin{aligned} \tilde{\lambda}[u](x) &= \inf_{a \in x \circ 0} \tilde{\lambda}[u](a) \\ &\geq \min\{\sup_{b \in (x \circ y) \circ 0} \tilde{\lambda}[u](b), \inf_{c \in y \circ 0} \tilde{\lambda}[u](c)\} \\ &= \min\{\sup_{b \in x \circ y} \tilde{\lambda}[u](b), \tilde{\lambda}[u](y)\}. \end{aligned}$$

Therefore, $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H. □

Corollary 2.27. *Every fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$ is a fuzzy soft strong hyper BCK-ideal of H.*

The following example shows that the converse of Theorem 2.26 and Corollary 2.27 is not true in general.

Example 2.28. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given in Table 16. Given a set $A = \{x, y\}$ of parameters, we define a fuzzy

Table 16: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Table 17: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.1	0.5
y	0.7	0.2	0.6

soft set $(\tilde{\lambda}, A)$ by Table 17. Then, $(\tilde{\lambda}, A)$ is a fuzzy soft strong hyper BCK-ideal of H. Since

$$\inf_{r\in b\circ b}\tilde{\lambda}[x](r) = 0.5 < 0.9 = \min\left\{\sup_{s\in(b\circ 0)\circ b}\tilde{\lambda}[x](s), \inf_{t\in 0\circ b}\tilde{\lambda}[x](t)\right\},$$

we know that $(\tilde{\lambda}, A)$ is not an x-fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ and so it is not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Also

$$\inf_{r\in b\circ b}\tilde{\lambda}[y](r) = 0.6 < 0.7 = \min\left\{\sup_{s\in(b\circ b)\circ b}\tilde{\lambda}[y](s), \sup_{t\in b\circ b}\tilde{\lambda}[y](t)\right\},$$

and so $(\tilde{\lambda}, A)$ it is not a y-fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Thus, it is not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \ll, \subseteq)$. Therefore, any fuzzy soft strong hyper BCK-ideal of H may not be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ or $(\ll, \ll, \subseteq)$.

Consider the hyper BCK-algebra $H = \{0, a, b, c\}$ in Example 2.23 and a set $A = \{x, y\}$ of parameters. We define a fuzzy soft set $(\tilde{\lambda}, A)$ by Table 13 in Example 2.23. Then, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$. But $(\tilde{\lambda}, A)$ is not a fuzzy soft strong hyper BCK-ideal of H since

$$\tilde{\lambda}[y](c) = 0.4 < 0.6 = \min\left\{\sup_{r\in c\circ b}\tilde{\lambda}[y](r), \tilde{\lambda}[y](b)\right\}.$$

Hence, we know that any fuzzy soft positive imlicative hyper BCK-ideal of types $(\subseteq, \subseteq, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ is not a fuzzy soft strong hyper BCK-ideal of H.

Lemma 2.29. *If a fuzzy soft set $(\tilde{\lambda}, A)$ over H satisfies the condition* (2.1)*, then $0 \in U(\tilde{\lambda}[u]; t)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u]; t) \neq \emptyset$.*

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H which satisfies the condition (2.1). For any $t \in [0, 1]$ and any parameter u in A, assume that $U(\tilde{\lambda}[u]; t) \neq \emptyset$. Since $0 \ll x$ for all $x \in H$, it follows from (2.1) that $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in H$. Hence, $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in U(\tilde{\lambda}[u]; t)$, and so $\tilde{\lambda}[u](0) \geq t$. Thus, $0 \in U(\tilde{\lambda}[u]; t)$. $\square$

Theorem 2.30. *If a fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$, then the set $U(\tilde{\lambda}[u]; t)$ is a positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u]; t) \neq \emptyset$.*

Proof. Assume that a fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. Then, $0 \in U(\tilde{\lambda}[u]; t)$ by Lemma 2.29. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u]; t)$ and $y \circ z \ll U(\tilde{\lambda}[u]; t)$. Then,

$$\tilde{\lambda}[u](a) \geq t \text{ for all } a \in (x \circ y) \circ z \tag{2.16}$$

and

$$(\forall b \in y \circ z)(\exists c \in U(\tilde{\lambda}[u]; t))(b \ll c). \tag{2.17}$$

The condition (2.16) implies $\inf\limits_{a \in (x \circ y) \circ z} \tilde{\lambda}[u](a) \geq t$, and the condition (2.17) implies from (2.1) that $\tilde{\lambda}[u](b) \geq \tilde{\lambda}[u](c) \geq t$ for all $b \in y \circ z$. Let $d \in x \circ z$. Using (2.13), we have

$$\tilde{\lambda}[u](d) \geq \inf_{d \in x \circ z} \tilde{\lambda}[u](d) \geq \min\left\{ \inf_{a \in (x \circ y) \circ z} \tilde{\lambda}[u](a), \sup_{b \in y \circ z} \tilde{\lambda}[u](b) \right\} \geq t.$$

Thus, $d \in U(\tilde{\lambda}[u]; t)$, and so $x \circ z \subseteq U(\tilde{\lambda}[u]; t)$. Therefore, $U(\tilde{\lambda}[u]; t)$ is a positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. □

The following example shows that the converse of Theorem 2.30 is not true in general.

Example 2.31. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" in Table 18. Given a set $A = \{x, y\}$ of parameters, we define a fuzzy soft set $(\tilde{\lambda}, A)$

Table 18: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

by Table 19. Then,

$$U(\tilde{\lambda}[x]; t) = \begin{cases} \emptyset & \text{if } t \in (0.9, 1], \\ \{0\} & \text{if } t \in (0.8, 0.9], \\ \{0, b\} & \text{if } t \in (0.5, 0.8], \\ H & \text{if } t \in [0, 0.5] \end{cases}$$

Table 19: Tabular representation of $(\tilde{\lambda}, A)$

$\tilde{\lambda}$	0	a	b
x	0.9	0.5	0.8
y	0.8	0.3	0.6

and

$$U(\tilde{\lambda}[y];t) = \begin{cases} \emptyset & \text{if } t \in (0.8, 1], \\ \{0\} & \text{if } t \in (0.6, 0.8], \\ \{0, b\} & \text{if } t \in (0.3, 0.6], \\ H & \text{if } t \in [0, 0.3], \end{cases}$$

which are positive imlicative hyper BCK-ideals of type $(\subseteq, \ll, \subseteq)$. Note that $a \ll b$ and $\tilde{\lambda}[u](a) < \tilde{\lambda}[u](b)$ for all $u \in A$. Thus, $(\tilde{\lambda}, A)$ is not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$.

Lemma 2.32. *If any subset I of H is closed and satisfies the condition* (1.17)*, then the condition* (1.16) *is valid.*

Proof. Assume that $x \circ y \ll I$ and $y \in I$ for all $x, y \in H$. Let $a \in x \circ y$. Then there exists $b \in I$ such that $a \ll b$. Since I is closed, we have $a \in I$ and thus, $x \circ y \subseteq I$. It follows from (1.17) that $x \in I$. $\square$

Theorem 2.33. *Let A be a fuzzy soft set over H satisfying the condition* (2.1) *and*

$$(\forall T \in \mathcal{P}(H))(\exists x_0 \in T)\left(\tilde{\lambda}[u](x_0) = \sup_{r \in T} \tilde{\lambda}[u](r)\right). \tag{2.18}$$

If the set $U(\tilde{\lambda}[u];t)$ is a reflexive positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ for all $t \in [0,1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t) \neq \emptyset$, then $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$.

Proof. For any $x, y, z \in H$ let

$$t := \min\{\inf_{a \in (x \circ y) \circ z} \tilde{\lambda}[u](a), \inf_{b \in y \circ z} \tilde{\lambda}[u](b)\}.$$

Then, $\inf_{a \in (x \circ y) \circ z} \tilde{\lambda}[u](a) \geq t$ and so $\tilde{\lambda}[u](a) \geq t$ for all $a \in (x \circ y) \circ z$. Since $\inf_{b \in y \circ z} \tilde{\lambda}[u](b) \geq t$, it follows from (2.18) that $\tilde{\lambda}[u](b_0) = \inf_{b \in y \circ z} \tilde{\lambda}[u](b) \geq t$ for some $b_0 \in y \circ z$. Hence, $b_0 \in U(\tilde{\lambda}[u];t)$, and thus, $U(\tilde{\lambda}[u];t) \cap (y \circ z) \neq \emptyset$. Since $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ and Hence, of type $(\subseteq, \subseteq, \subseteq)$, $U(\tilde{\lambda}[u];t)$ is a weak hyper BCK-ideal of H by Lemma 1.30. Let $x \in H$ be such that $x \ll y$. If $y \in U(\tilde{\lambda}[u];t)$, then $\tilde{\lambda}[u](x) \geq \tilde{\lambda}[u](y) \geq t$ by (2.1) and so $x \in U(\tilde{\lambda}[u];t)$, that is, $U(\tilde{\lambda}[u];t)$ is closed. Hence, $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H by Lemma 2.32. Since $U(\tilde{\lambda}[u];t)$ is reflexive, it follows from Lemma 1.23 that $y \circ z \ll U(\tilde{\lambda}[u];t)$. Hence, $x \circ z \subseteq U(\tilde{\lambda}[u];t)$ since $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. Hence,

$$\tilde{\lambda}[u](a) \geq t = \min\{\inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\}$$

for all $a \in x \circ z$, and thus,

$$\inf_{a \in x \circ z} \tilde{\lambda}[u](a) \geq \min\{\inf_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c)\}$$

for all $x, y, z \in H$. Therefore, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$. □

Corollary 2.34. *Let A be a fuzzy soft set over H satisfying the condition* (2.1) *and* (2.18). *For any $t \in [0, 1]$ and any parameter u in A, assume that $U(\tilde{\lambda}[u];t)$ is nonempty and reflexive. Then, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$ if and only if $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\subseteq, \ll, \subseteq)$.*

Theorem 2.35. *If a fuzzy soft set $(\tilde{\lambda}, A)$ over H is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, then the set $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t) \neq \emptyset$.*

Proof. Let $(\tilde{\lambda}, A)$ be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Then, $0 \in U(\tilde{\lambda}[u];t)$ by Lemma 2.29. Let $x, y, z \in H$ be such that $(x \circ y) \circ z \ll U(\tilde{\lambda}[u];t)$ and $y \circ z \subseteq U(\tilde{\lambda}[u];t)$. Then, for all $a \in (x \circ y) \circ z$, there exists $b \in U(\tilde{\lambda}[u];t)$ such that $a \ll b$, which implies from (2.1) that $\tilde{\lambda}[u](a) \geq \tilde{\lambda}[u](b)$ for all $a \in (x \circ y) \circ z$. Since $y \circ z \subseteq U(\tilde{\lambda}[u];t)$, we have $\tilde{\lambda}[u](a) \geq t$ for all $a \in y \circ z$. Let $c \in x \circ z$. Then,

$$\tilde{\lambda}[u](c) \geq \inf_{c \in x \circ z} \tilde{\lambda}[u](c) \geq \min\{\sup_{a \in (x \circ y) \circ z} \tilde{\lambda}[u](a), \inf_{b \in y \circ z} \tilde{\lambda}[u](b)\} \geq t$$

for all $x, y, z \in H$ by (2.14), and thus, $c \in U(\tilde{\lambda}[u];t)$. Hence, $x \circ z \subseteq U(\tilde{\lambda}[u];t)$. Therefore, $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. □

The converse of Theorem 2.35 is not true as seen in the following example.

Example 2.36. Consider the hyper BCK-algebra $H = \{0, a, b\}$ and the fuzzy soft set $(\tilde{\lambda}, A)$ in Example 2.24. Then,

$$U(\tilde{\lambda}[x];t) = \begin{cases} \emptyset & \text{if } t \in (0.9, 1], \\ \{0\} & \text{if } t \in (0.5, 0.9], \\ \{0, a\} & \text{if } t \in (0.3, 0.5], \\ H & \text{if } t \in [0, 0.3] \end{cases}$$

and

$$U(\tilde{\lambda}[y];t) = \begin{cases} \emptyset & \text{if } t \in (0.8, 1], \\ \{0\} & \text{if } t \in (0.7, 0.8], \\ \{0, a\} & \text{if } t \in (0.1, 0.7], \\ H & \text{if } t \in [0, 0.1], \end{cases}$$

which are positive imlicative hyper BCK-ideals of type $(\ll, \subseteq, \subseteq)$. But we know $(\tilde{\lambda}, A)$ is not a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

We provide conditions for a fuzzy soft set to be a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Theorem 2.37. *Let A be a fuzzy soft set over H satisfying the condition* (2.18)*. If the set $U(\tilde{\lambda}[u];t)$ is a reflexive positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ for all $t \in [0, 1]$ and any parameter u in A with $U(\tilde{\lambda}[u];t) \neq \emptyset$, then $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.*

Proof. Assume that $U(\tilde{\lambda}[u];t) \neq \emptyset$ for all $t \in [0, 1]$ and any parameter u in A. Suppose that $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. Then, $U(\tilde{\lambda}[u];t)$ is a hyper BCK-ideal of H by Lemma 1.29. It follows from Lemma (1.46) that $(\tilde{\lambda}, A)$ is a fuzzy soft hyper BCK-ideal of H. Thus, the condition (2.1) is valid. Now, let $t = \min\left\{ \sup\limits_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf\limits_{c \in y \circ z} \tilde{\lambda}[u](c) \right\}$ for $x, y, z \in H$. Since $(\tilde{\lambda}, A)$ satisfies the condition (2.18), there exists $x_0 \in (x \circ y) \circ z$ such that $\tilde{\lambda}[u](x_0) = \sup\limits_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b) \geq t$ and so $x_0 \in U(\tilde{\lambda}[u];t)$. Hence, $((x \circ y) \circ z) \cap U(\tilde{\lambda}[u];t) \neq \emptyset$ and so $(x \circ y) \circ z \ll U(\tilde{\lambda}[u];t)$ by Lemma 1.23 and (1.12). Moreover $\tilde{\lambda}[u](c) \geq \inf\limits_{c \in y \circ z} \tilde{\lambda}[u](c) \geq t$ for all $c \in y \circ z$, and Hence, $c \in U(\tilde{\lambda}[u];t)$ which shows that $y \circ z \subseteq U(\tilde{\lambda}[u];t)$. Since $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$, it follows that $x \circ z \subseteq U(\tilde{\lambda}[u];t)$. Thus, $\tilde{\lambda}[u](a) \geq t$ for all $a \in x \circ z$, and so

$$\inf_{a \in x \circ z} \tilde{\lambda}[u](a) \geq t = \min\left\{ \sup_{b \in (x \circ y) \circ z} \tilde{\lambda}[u](b), \inf_{c \in y \circ z} \tilde{\lambda}[u](c) \right\}.$$

Consequently, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$. □

Corollary 2.38. *Let A be a fuzzy soft set over H satisfying the condition* (2.18)*. For any $t \in [0, 1]$ and any parameter u in A, assume that $U(\tilde{\lambda}[u];t)$ is nonempty and reflexive. Then, $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$ if and only if $U(\tilde{\lambda}[u];t)$ is a positive imlicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.*

Using a positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ (resp., $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$), we establish a fuzzy soft weak hyper BCK-ideal.

Theorem 2.39. *Let I be a positive imlicative hyper BCK-ideal of type $(\subseteq, \subsetneq, \subseteq)$ (resp., $(\subseteq, \ll, \subseteq)$, $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$) and let $z \in H$. For a fuzzy soft set $(\tilde{\lambda}, A)$ over H and any parameter u in A, if we define the fuzzy value set $\tilde{\lambda}[u]$ by*

$$\tilde{\lambda}[u] : H \to [0,1],\ x \mapsto \begin{cases} t & \text{if } x \in I_z, \\ s & \text{otherwise,} \end{cases} \tag{2.19}$$

where $t > s$ in $[0,1]$ and $I_z := \{y \in H \mid y \circ z \subseteq I\}$, then $(\tilde{\lambda}, A)$ is a u-fuzzy soft weak hyper BCK-ideal of H.

Proof. It is clear that $\tilde{\lambda}[u](0) \geq \tilde{\lambda}[u](x)$ for all $x \in H$. Let $x, y \in H$. If $y \notin I_z$, then $\tilde{\lambda}[u](y) = s$ and so

$$\tilde{\lambda}[u](x) \geq s = \min\left\{\tilde{\lambda}[u](y), \inf_{a \in x \circ y} \tilde{\lambda}[u](a)\right\}. \tag{2.20}$$

If $x \circ y \nsubseteq I_z$, then there exists $a \in x \circ y \setminus I_z$, and thus, $\tilde{\lambda}[u](a) = s$. Hence,

$$\tilde{\lambda}[u](x) \geq s = \min\left\{\tilde{\lambda}[u](y), \inf_{a \in x \circ y} \tilde{\lambda}[u](a)\right\}. \tag{2.21}$$

Assume that $x \circ y \subseteq I_z$ and $y \in I_z$. Then,

$$(x \circ y) \circ z \subseteq I \text{ and } y \circ z \subseteq I. \tag{2.22}$$

If I is of type $(\subseteq, \subseteq, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Thus,

$$\tilde{\lambda}[u](x) = t \geq \min\left\{\tilde{\lambda}[u](y), \inf_{a \in x \circ y} \tilde{\lambda}[u](a)\right\}. \tag{2.23}$$

The condition (2.22) implies that $(x \circ y) \circ z \ll I$ and $y \circ z \ll I$ by (1.12). Hence, if I is of type $(\ll, \ll, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore, we have (2.23). From the condition (2.22), we have $(x \circ y) \circ z \subseteq I$ and $y \circ z \ll I$. If I is of type $(\subseteq, \ll, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore, we have (2.23). From the condition (2.22), we have $(x \circ y) \circ z \ll I$ and $y \circ z \subseteq I$. If I is of type $(\ll, \subseteq, \subseteq)$, then $x \circ z \subseteq I$, i.e., $x \in I_z$. Therefore, we have (2.23). Therefore, $(\tilde{\lambda}, A)$ is a fuzzy soft weak hyper BCK-ideal of H. □

Theorem 2.40. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H in which the nonempty level set $U(\tilde{\lambda}[u]; t)$ of $(\tilde{\lambda}, A)$ is reflexive for all $t \in [0,1]$. If $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of H of type $(\ll, \subseteq, \subseteq)$, then the set*

$$\tilde{\lambda}[u]_z := \{x \in H \mid x \circ z \subseteq U(\tilde{\lambda}[u]; t)\} \tag{2.24}$$

is a (weak) hyper BCK-ideal of H for all $z \in H$.

Proof. Assume that $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of H of type $(\ll, \subseteq, \subseteq)$. Obviously $0 \in \tilde{\lambda}[u]_z$. Then, $(\tilde{\lambda}, A)$ is a fuzzy soft hyper BCK-ideal of H, and so $U(\tilde{\lambda}[u]; t)$ is a hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \circ y \subseteq \tilde{\lambda}[u]_z$ and $y \in \tilde{\lambda}[u]_z$. Then, $(x \circ y) \circ z \subseteq U(\tilde{\lambda}[u]; t)$ and $y \circ z \subseteq U(\tilde{\lambda}[u]; t)$ for all $t \in [0, 1]$. Using (1.12), we know that $(x \circ y) \circ z \ll U(\tilde{\lambda}[u]; t)$. Since $U(\tilde{\lambda}[u]; t)$ is a positive imlicative hyper BCK-ideal of H of type $(\ll, \subseteq, \subseteq)$, it follows from (1.24) that $x \circ z \subseteq U(\tilde{\lambda}[u]; t)$, that is, $x \in \tilde{\lambda}[u]_z$. This shows that $\tilde{\lambda}[u]_z$ is a weak hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \circ y \ll \tilde{\lambda}[u]_z$ and $y \in \tilde{\lambda}[u]_z$, and let $a \in x \circ y$. Then there exists $b \in \tilde{\lambda}[u]_z$ such that $a \ll b$, that is, $0 \in a \circ b$. Thus, $(a \circ b) \cap U(\tilde{\lambda}[u]; t) \neq \emptyset$. Since $U(\tilde{\lambda}[u]; t)$ is a reflexive hyper BCK-ideal of H, it follows from (H1) and Lemma 1.23 that $(a \circ z) \circ (b \circ z) \ll a \circ b \subseteq U(\tilde{\lambda}[u]; t)$ and so that $a \circ z \subseteq U(\tilde{\lambda}[u]; t)$ since $b \circ z \subseteq U(\tilde{\lambda}[u]; t)$. Hence, $a \in \tilde{\lambda}[u]_z$, and so $x \circ y \subseteq \tilde{\lambda}[u]_z$. Since $\tilde{\lambda}[u]_z$ is a weak hyper BCK-ideal of H, we get $x \in \tilde{\lambda}[u]_z$. Consequently $\tilde{\lambda}[u]_z$ is a hyper BCK-ideal of H. □

Corollary 2.41. *Let $(\tilde{\lambda}, A)$ be a fuzzy soft set over H in which the nonempty level set $U(\tilde{\lambda}[u]; t)$ of $(\tilde{\lambda}, A)$ is reflexive for all $t \in [0, 1]$. If $(\tilde{\lambda}, A)$ is a fuzzy soft positive imlicative hyper BCK-ideal of H of type $(\ll, \ll, \subseteq)$, then the set*

$$\tilde{\lambda}[u]_z := \{x \in H \mid x \circ z \subseteq U(\tilde{\lambda}[u]; t)\} \tag{2.25}$$

is a (weak) hyper BCK-ideal of H for all $z \in H$.

Chapter 3.

Neutrosophic hyper BCK-ideals of several types

3 Abstract

In this chapter, we introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal and reflexive neutrosophic hyper BCK-ideal. Some relevant properties and their relations are indicated. Characterization of neutrosophic (weak) hyper BCK-ideal is considered. Conditions for a neutrosophic set to be a (reflexive) neutrosophic hyper BCK-ideal and a neutrosophic strong hyper BCK-ideal are discussed. Some conditions for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal, and conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal are provided.

Also, we introduced the notions of neutrosophic commutative hyper BCK-ideals of types $(\subseteq, \subseteq)$, $(\subseteq, \ll)$, $(\ll, \subseteq)$ and $(\ll, \ll)$. Some relevant properties and their relations are indicated. Relations between commutative neutrosophic hyper BCK-ideal of types $(\subseteq, \subseteq)$, $(\ll, \subseteq)$, neutrosophic weak hyper BCK-ideal and neutrosophic strong hyper BCK-ideal are discussed. We provide a condition for a neutrosophic weak hyper BCK-ideal to be a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$. A condition for a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ to be a neutrosophic s-weak hyper BCK-ideal is discussed. Characterization of a commutative neutrosophic hyper BCK-ideal of types $(\subseteq, \subseteq)$, $(\subseteq, \ll)$, $(\ll, \subseteq)$ and $(\ll, \ll)$ are considered. Fainally, relations between commutative neutrosophic hyper BCK-ideal of types $(\subseteq, \subseteq)$, $(\subseteq, \ll)$, $(\ll, \subseteq)$ and $(\ll, \ll)$ and a spesial subset of H are discussed.

In what follows, let H denote a hyper BCK-algebra unless otherwise specified.

3.1 Neutrosophic (strong, weak, s-weak) hyper BCK-ideals

Definition 3.1. A neutrosophic set $N = (N_T, N_I, N_F)$ in H is called a *neutrosophic hyper BCK-ideal* of H if it satisfies the following assertions.

$$(\forall x, y \in H)\left(x \ll y \Rightarrow \begin{cases} N_T(x) \geq N_T(y) \\ N_I(x) \geq N_I(y) \\ N_F(x) \leq N_F(y) \end{cases} \right), \tag{3.1}$$

$$(\forall x, y \in H)\left(\begin{array}{l} N_T(x) \geq \min\left\{ \inf\limits_{a \in x \circ y} N_T(a), N_T(y) \right\} \\ N_I(x) \geq \min\left\{ \inf\limits_{b \in x \circ y} N_I(b), N_I(y) \right\} \\ N_F(x) \leq \max\left\{ \sup\limits_{c \in x \circ y} N_F(c), N_F(y) \right\} \end{array} \right). \tag{3.2}$$

Example 3.2. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 20.

Table 20: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Let $N = (N_T, N_I, N_F)$ be a neutrosophic set in H which is described in Table 27.

Table 21: Tabular representation of $N = (N_T, N_I, N_F)$

H	$N_T(x)$	$N_I(x)$	$N_F(x)$
0	0.77	0.65	0.08
a	0.55	0.47	0.57
b	0.11	0.27	0.69

It is easy to verify that $N = (N_T, N_I, N_F)$ is a neutrosophic hyper BCK-ideal of H.

Proposition 3.3. *For every neutrosophic hyper BCK-ideal $N = (N_T,\ N_I,\ N_F)$ of H, the following assertions are valid.*

(1) *$N = (N_T, N_I, N_F)$ satisfies*

$$(\forall x \in H) \begin{pmatrix} N_T(0) \geq N_T(x) \\ N_I(0) \geq N_I(x) \\ N_F(0) \leq N_F(x) \end{pmatrix}. \tag{3.3}$$

(2) *If $N = (N_T, N_I, N_F)$ satisfies*

$$(\forall S \subseteq H)(\exists\, a, b, c \in S) \begin{pmatrix} N_T(a) = \inf\limits_{x \in S} N_T(x) \\ N_I(b) = \inf\limits_{x \in S} N_I(x) \\ N_F(c) = \sup\limits_{x \in S} N_F(x) \end{pmatrix}, \tag{3.4}$$

then, the following assertion is valid.

$$(\forall x, y \in H)(\exists\, a, b, c \in x \circ y) \begin{pmatrix} N_T(x) \geq \min\{N_T(a), N_T(y)\} \\ N_I(x) \geq \min\{N_I(b), N_I(y)\} \\ N_F(x) \leq \max\{N_F(c), N_F(y)\} \end{pmatrix}. \tag{3.5}$$

Proof. Since $0 \ll x$ for all $x \in H$, it follows from (3.1) that

$$N_T(0) \geq N_T(x),\ N_I(0) \geq N_I(x) \text{ and } N_F(0) \leq N_F(x).$$

Assume that $N = (N_T, N_I, N_F)$ satisfies the condition (3.4). For any $x, y \in H$, there exists $a_0, b_0, c_0 \in x \circ y$ such that

$$N_T(a_0) = \inf_{a \in x \circ y} N_T(a),\ N_I(b_0) = \inf_{b \in x \circ y} N_I(b) \text{ and } N_F(c_0) = \sup_{c \in x \circ y} N_F(c).$$

It follows from (3.2) that

$$\begin{aligned} N_T(x) &\geq \min\left\{\inf_{a \in x \circ y} N_T(a), N_T(y)\right\} = \min\{N_T(a_0), N_T(y)\} \\ N_I(x) &\geq \min\left\{\inf_{b \in x \circ y} N_I(b), N_I(y)\right\} = \min\{N_I(b_0), N_I(y)\} \\ N_F(x) &\leq \max\left\{\sup_{c \in x \circ y} N_F(c), N_F(y)\right\} = \max\{N_F(c_0), N_F(y)\}. \end{aligned}$$

This completes the proof. □

Theorem 3.4. *A neutrosophic set $N = (N_T, N_I, N_F)$ is a neutrosophic hyper BCK-ideal of H if and only if the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$.*

Proof. Assume that $N = (N_T, N_I, N_F)$ is a neutrosophic hyper BCK-ideal of H and suppose that $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are nonempty for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$. It is clear that $0 \in U(N_T; \varepsilon_T)$, $0 \in U(N_I; \varepsilon_I)$ and $0 \in L(N_F; \varepsilon_F)$. Let $x, y \in H$ be such that $x \circ y \ll U(N_T; \varepsilon_T)$ and $y \in U(N_T; \varepsilon_T)$. Then, $N_T(y) \geq \varepsilon_T$ and for any $a \in x \circ y$ there exists $a_0 \in U(N_T; \varepsilon_T)$ such that $a \ll a_0$. It follows from (3.1) that $N_T(a) \geq N_T(a_0) \geq \varepsilon_T$ for all $a \in x \circ y$. Hence, $\inf\limits_{a \in x \circ y} N_T(a) \geq \varepsilon_T$, and so

$$N_T(x) \geq \min \left\{ \inf_{a \in x \circ y} N_T(a), N_T(y) \right\} \geq \varepsilon_T,$$

that is, $x \in U(N_T; \varepsilon_T)$. Similarly, we show that if $x \circ y \ll U(N_I; \varepsilon_I)$ and $y \in U(N_I; \varepsilon_I)$, then, $x \in U(N_I; \varepsilon_I)$. Hence, $U(N_T; \varepsilon_T)$ and $U(N_I; \varepsilon_I)$ are hyper BCK-ideals of H. Let $x, y \in H$ be such that $x \circ y \ll L(N_F; \varepsilon_F)$ and $y \in L(N_F; \varepsilon_F)$. Then, $N_F(y) \leq \varepsilon_F$. Let $b \in x \circ y$. Then, there exists $b_0 \in L(N_F; \varepsilon_F)$ such that $b \ll b_0$, which implies from (3.1) that $N_F(b) \leq N_F(b_0) \leq \varepsilon_F$. Thus, $\sup\limits_{b \in x \circ y} N_F(b) \leq \varepsilon_F$, and so

$$N_F(x) \leq \max \left\{ \sup_{b \in x \circ y} N_F(b), N_F(y) \right\} \leq \varepsilon_F.$$

Hence, $x \in L(N_F; \varepsilon_F)$, and therefore $L(N_F; \varepsilon_F)$ is a hyper BCK-ideal of H.

Conversely, suppose that the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$. Let $x, y \in H$ be such that $x \ll y$. Then

$$y \in U(N_T; N_T(y)) \cap U(N_I; N_I(y)) \cap L(N_F; N_F(y)),$$

and thus, $x \ll U(N_T; N_T(y))$, $x \ll U(N_I; N_I(y))$ and $x \ll L(N_F; N_F(y))$. It follows from Lemma 1.20 that $x \in U(N_T; N_T(y))$, $x \in U(N_I; N_I(y))$ and $x \in L(N_F; N_F(y))$ which imply that $N_T(x) \geq N_T(y)$, $N_I(x) \geq N_I(y)$ and $N_F(x) \leq N_F(y)$. For any $x, y \in H$, let $\varepsilon_T := \min \left\{ \inf\limits_{a \in x \circ y} N_T(a), N_T(y) \right\}$, $\varepsilon_I := \min \left\{ \inf\limits_{b \in x \circ y} N_I(b), N_I(y) \right\}$ and $\varepsilon_F := \max \left\{ \sup\limits_{c \in x \circ y} N_F(c), N_F(y) \right\}$. Then,

$$y \in U(N_T; \varepsilon_T) \cap U(N_I; \varepsilon_I) \cap L(N_F; \varepsilon_F),$$

and for each $a_T, b_I, c_F \in x \circ y$ we have

$$N_T(a_T) \geq \inf_{a \in x \circ y} N_T(a) \geq \min \left\{ \inf_{a \in x \circ y} N_T(a), N_T(y) \right\} = \varepsilon_T,$$

$$N_I(b_I) \geq \inf_{b \in x \circ y} N_I(b) \geq \min \left\{ \inf_{b \in x \circ y} N_I(b), N_I(y) \right\} = \varepsilon_I$$

and

$$N_F(c_F) \leq \sup_{c \in x \circ y} N_F(c) \leq \max\left\{\sup_{c \in x \circ y} N_F(c), N_F(y)\right\} = \varepsilon_F.$$

Hence, $a_T \in U(N_T; \varepsilon_T)$, $b_I \in U(N_I; \varepsilon_I)$ and $c_F \in L(N_F; \varepsilon_F)$, which imply that $x \circ y \subseteq U(N_T; \varepsilon_T)$, $x \circ y \subseteq U(N_I; \varepsilon_I)$ and $x \circ y \subseteq L(N_F; \varepsilon_F)$. Using (1.12), we have $x \circ y \ll U(N_T; \varepsilon_T)$, $x \circ y \ll U(N_I; \varepsilon_I)$ and $x \circ y \ll L(N_F; \varepsilon_F)$. It follows from (1.16) that

$$x \in U(N_T; \varepsilon_T) \cap U(N_I; \varepsilon_I) \cap L(N_F; \varepsilon_F).$$

Hence,

$$N_T(x) \geq \varepsilon_T = \min\left\{\inf_{a \in x \circ y} N_T(a), N_T(y)\right\},$$

$$N_I(x) \geq \varepsilon_I = \min\left\{\inf_{b \in x \circ y} N_I(b), N_I(y)\right\}$$

and

$$N_F(x) \leq \varepsilon_F = \max\left\{\sup_{c \in x \circ y} N_F(c), N_F(y)\right\}.$$

Therefore, $N = (N_T, N_I, N_F)$ is a neutrosophic hyper BCK-ideal of H. □

Theorem 3.5. *If $N = (N_T, N_I, N_F)$ is a neutrosophic hyper BCK-ideal of H, then, the set*

$$J := \{x \in H \mid N_T(x) = N_T(0), N_I(x) = N_I(0), N_F(x) = N_F(0)\} \tag{3.6}$$

is a hyper BCK-ideal of H.

Proof. It is clear that $0 \in J$. Let $x, y \in H$ be such that $x \circ y \ll J$ and $y \in J$. Then, $N_T(y) = N_T(0)$, $N_I(y) = N_I(0)$ and $N_F(y) = N_F(0)$. Let $a \in x \circ y$. Then, there exists $a_0 \in J$ such that $a \ll a_0$, and thus, $N_T(a) \geq N_T(a_0) = N_T(0)$, $N_I(a) \geq N_I(a_0) = N_I(0)$ and $N_F(a) \leq N_F(a_0) = N_F(0)$ by (3.1). It follows from (3.2) that

$$N_T(x) \geq \min\left\{\inf_{a \in x \circ y} N_T(a), N_T(y)\right\} \geq N_T(0),$$

$$N_I(x) \geq \min\left\{\inf_{a \in x \circ y} N_I(a), N_I(y)\right\} \geq N_I(0)$$

and

$$N_F(x) \leq \max\left\{\sup_{a \in x \circ y} N_F(a), N_F(y)\right\} \leq N_F(0).$$

Hence, $N_T(x) = N_T(0)$, $N_I(x) = N_I(0)$ and $N_F(x) = N_F(0)$, that is, $x \in J$. Therefore, J is a hyper BCK-ideal of H. □

We provide conditions for a neutrosophic set $N = (N_T, N_I, N_F)$ to be a neutrosophic hyper BCK-ideal of H.

Theorem 3.6. *Let H satisfy $|x \circ y| < \infty$ for all $x, y \in H$, and let $\{J_t \mid t \in \Lambda \subseteq [0, 0.5]\}$ be a collection of hyper BCK-ideals of H such that*

$$H = \bigcup_{t \in \Lambda} J_t, \tag{3.7}$$

$$(\forall s, t \in \Lambda)(s > t \;\Leftrightarrow\; J_s \subset J_t). \tag{3.8}$$

Then, a neutrosophic set $N = (N_T, N_I, N_F)$ in H defined by

$$N_T : H \to [0, 1],\ x \mapsto \sup\{t \in \Lambda \mid x \in J_t\},$$
$$N_I : H \to [0, 1],\ x \mapsto \sup\{t \in \Lambda \mid x \in J_t\},$$
$$N_F : H \to [0, 1],\ x \mapsto \inf\{t \in \Lambda \mid x \in J_t\}$$

is a neutrosophic hyper BCK-ideal of H.

Proof. We first shows that

$$q \in [0, 1] \Rightarrow \bigcup_{p \in \Lambda, p \geq q} J_p \text{ is a hyper } BCK\text{-ideal of } H. \tag{3.9}$$

It is clear that $0 \in \bigcup_{p \in \Lambda, p \geq q} J_p$ for all $q \in [0, 1]$. Let $x, y \in H$ be such that $x \circ y = \{a_1, a_2, \cdots, a_n\}$, $x \circ y \ll \bigcup_{p \in \Lambda, p \geq q} J_p$ and $y \in \bigcup_{p \in \Lambda, p \geq q} J_p$. Then, $y \in J_r$ for some $r \in \Lambda$ with $q \leq r$, and for every $a_i \in x \circ y$ there exists $b_i \in \bigcup_{p \in \Lambda, p \geq q} J_p$, and so $b_i \in J_{t_i}$ for some $t_i \in \Lambda$ with $q \leq t_i$, such that $a_i \ll b_i$. If we let $t := \min\{t_i \mid i \in \{1, 2, \cdots, n\}\}$, then, $J_{t_i} \subset J_t$ for all $i \in \{1, 2, \cdots, n\}$ and so $x \circ y \ll J_t$ with $q \leq t$. We may assume that $r > t$ without loss of generality, and so $J_r \subset J_t$. Using (1.16), we have $x \in J_t \subset \bigcup_{p \in \Lambda, p \geq q} J_p$. Hence, $\bigcup_{p \in \Lambda, p \geq q} J_p$ is a hyper BCK-ideal of H. Next, we consider the following two cases:

$$\text{(i) } t = \sup\{q \in \Lambda \mid q < t\}, \ \text{ (ii) } t \neq \sup\{q \in \Lambda \mid q < t\}. \tag{3.10}$$

If the first case is valid, then,

$$x \in U(N_T, t) \;\Leftrightarrow\; x \in J_q \text{ for all } q < t \;\Leftrightarrow\; x \in \bigcap_{q < t} J_q,$$

and so $U(N_T, t) = \bigcap_{q < t} J_q$ which is a hyper BCK-ideal of H. Similarly, we know that $U(N_I, t)$ is a hyper BCK-ideal of H. For the second case, we will show that $U(N_T, t) = \bigcup_{q \geq t} J_q$. If $x \in \bigcup_{q \geq t} J_q$, then, $x \in J_q$ for some $q \geq t$. Thus, $N_T(x) \geq q \geq t$, and so $x \in U(N_T, t)$

which shows that $\bigcup_{q\geq t} J_q \subseteq U(N_T, t)$. Assume that $x \notin \bigcup_{q\geq t} J_q$. Then, $x \notin J_q$ for all $q \geq t$, and so there exist $\delta > 0$ such that $(t-\delta, t) \cap \Lambda = \emptyset$. Thus, $x \notin J_q$ for all $q > t-\delta$, that is, if $x \in J_q$ then $q \leq t-\delta < t$. Hence, $x \notin U(N_T, t)$. This shows that $U(N_T, t) = \bigcup_{q\geq t} J_q$ which is a hyper BCK-ideal of H by (3.9). Similarly we can prove that $U(N_I, t)$ is a hyper BCK-ideal of H. Now we consider the following two cases:

$$s = \inf\{r \in \Lambda \mid s < r\} \text{ and } s \neq \inf\{r \in \Lambda \mid s < r\}. \tag{3.11}$$

The first case implies that

$$x \in L(N_F, s) \ \Leftrightarrow\ x \in J_r \text{ for all } s < r \ \Leftrightarrow\ x \in \bigcap_{s<r} J_r,$$

and so $L(N_F, s) = \bigcap_{s<r} J_r$ which is a hyper BCK-ideal of H. For the second case, there exists $\delta > 0$ such that $(s, s+\delta) \cap \Lambda = \emptyset$. If $x \in \bigcup_{s\geq r} J_r$, then, $x \in J_r$ for some $s \geq r$. Thus, $N_F(x) \leq r \leq s$, that is, $x \in L(N_F, s)$. Hence, $\bigcup_{s\geq r} J_r \subseteq L(N_F, s)$. If $x \notin \bigcup_{s\geq r} J_r$, then, $x \notin J_r$ for all $r \leq s$ and thus, $x \notin J_r$ for all $r < s+\delta$. This shows that if $x \in J_r$ then $r \geq s+\delta$. Hence, $N_F(x) \geq s+\delta > s$, i.e., $x \notin L(N_F, s)$. Therefore, $L(N_F, s) \subseteq \bigcup_{s\geq r} J_r$. Consequently, $L(N_F, s) = \bigcup_{s\geq r} J_r$ which is a hyper BCK-ideal of H by (3.9). It follows from Theorem 3.4 that $N = (N_T, N_I, N_F)$ is a neutrosophic hyper BCK-ideal of H. □

Definition 3.7. A neutrosophic set $N = (N_T, N_I, N_F)$ in H is called a *neutrosophic strong hyper BCK-ideal* of H if it satisfies the following assertions.

$$\begin{aligned}
&\inf_{a\in x\circ x} N_T(a) \geq N_T(x) \geq \min\left\{\sup_{a_0\in x\circ y} N_T(a_0), N_T(y)\right\},\\
&\inf_{b\in x\circ x} N_I(b) \geq N_I(x) \geq \min\left\{\sup_{b_0\in x\circ y} N_I(b_0), N_I(y)\right\},\\
&\sup_{c\in x\circ x} N_F(c) \leq N_F(x) \leq \max\left\{\inf_{c_0\in x\circ y} N_F(c_0), N_F(y)\right\}
\end{aligned} \tag{3.12}$$

for all $x, y \in H$.

Example 3.8. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 22. Let $N = (N_T, N_I, N_F)$ be a neutrosophic set in H which is described in Table 23. It is routine to verify that $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H.

Theorem 3.9. *For every neutrosophic strong hyper BCK-ideal $N = (N_T, N_I, N_F)$ of H, the following assertions are valid.*

Table 22: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Table 23: Tabular representation of $N = (N_T, N_I, N_F)$

H	$N_T(x)$	$N_I(x)$	$N_F(x)$
0	0.86	0.75	0.09
a	0.65	0.57	0.17
b	0.31	0.37	0.29

(1) *$N = (N_T, N_I, N_F)$ satisfies the conditions* (3.1) *and* (3.3).

(2) *$N = (N_T, N_I, N_F)$ satisfies*

$$(\forall x, y \in H)(\forall a, b, c \in x \circ y) \begin{pmatrix} N_T(x) \geq \min\{N_T(a), N_T(y)\} \\ N_I(x) \geq \min\{N_I(b), N_I(y)\} \\ N_F(x) \leq \max\{N_F(c), N_F(y)\} \end{pmatrix}. \tag{3.13}$$

Proof. (1) Since $x \ll x$, i.e., $0 \in x \circ x$ for all $x \in H$, we get

$$\begin{aligned} N_T(0) &\geq \inf_{a \in x \circ x} N_T(a) \geq N_T(x), \\ N_I(0) &\geq \inf_{b \in x \circ x} N_I(b) \geq N_I(x), \\ N_F(0) &\leq \sup_{c \in x \circ x} N_F(c) \leq N_F(x), \end{aligned}$$

which shows that (3.3) is valid. Let $x, y \in H$ be such that $x \ll y$. Then, $0 \in x \circ y$, and so

$$\sup_{c \in x \circ y} N_T(c) \geq N_T(0), \ \sup_{b \in x \circ y} N_I(b) \geq N_I(0) \text{ and } \inf_{a \in x \circ y} N_F(a) \leq N_F(0).$$

It follows from (3.3) that

$$N_T(x) \geq \min\left\{\sup_{c\in x\circ y} N_T(c), N_T(y)\right\} \geq \min\{N_T(0), N_T(y)\} = N_T(y),$$

$$N_I(x) \geq \min\left\{\sup_{b\in x\circ y} N_I(b), N_I(y)\right\} \geq \min\{N_I(0), N_I(y)\} = N_I(y),$$

$$N_F(x) \leq \max\left\{\inf_{a\in x\circ y} N_F(a), N_F(y)\right\} \leq \max\{N_F(0), N_F(y)\} = N_F(y).$$

Hence, $N = (N_T, N_I, N_F)$ satisfies the condition (3.1).

(2) Let $x, y, a, b, c \in H$ be such that $a, b, c \in x \circ y$. Then,

$$N_T(x) \geq \min\left\{\sup_{a_0\in x\circ y} N_T(a_0), N_T(y)\right\} \geq \min\{N_T(a), N_T(y)\},$$

$$N_I(x) \geq \min\left\{\sup_{b_0\in x\circ y} N_I(b_0), N_I(y)\right\} \geq \min\{N_I(b), N_I(y)\},$$

$$N_F(x) \leq \max\left\{\inf_{c_0\in x\circ y} N_F(c_0), N_F(y)\right\} \leq \max\{N_F(c), N_F(y)\}.$$

This completes the proof. □

Theorem 3.10. *If a neutrosophic set $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H, then, the nonempty sets $U(N_T;\varepsilon_T)$, $U(N_I;\varepsilon_I)$ and $L(N_F;\varepsilon_F)$ are strong hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0,1]$.*

Proof. Let $N = (N_T, N_I, N_F)$ be a neutrosophic strong hyper BCK-ideal of H. Then, $N = (N_T, N_I, N_F)$ is a neutrosophic hyper BCK-ideal of H. Assume that $U(N_T;\varepsilon_T)$, $U(N_I;\varepsilon_I)$ and $L(N_F;\varepsilon_F)$ are nonempty for all ε_T, ε_I, $\varepsilon_F \in [0,1]$. Then, there exist $a \in U(N_T;\varepsilon_T)$, $b \in U(N_I;\varepsilon_I)$ and $c \in L(N_F;\varepsilon_F)$, that is, $N_T(a) \geq \varepsilon_T$, $N_I(b) \geq \varepsilon_I$ and $N_F(c) \leq \varepsilon_F$. It follows from (3.3) that $N_T(0) \geq N_T(a) \geq \varepsilon_T$, $N_I(0) \geq N_I(b) \geq \varepsilon_I$ and $N_F(0) \leq N_F(c) \leq \varepsilon_F$. Hence,

$$0 \in U(N_T;\varepsilon_T) \cap U(N_I;\varepsilon_I) \cap L(N_F;\varepsilon_F).$$

Let $x, y, a, b, u, v \in H$ be such that $(x \circ y) \cap U(N_T;\varepsilon_T) \neq \emptyset$, $y \in U(N_T;\varepsilon_T)$, $(a \circ b) \cap U(N_I;\varepsilon_I) \neq \emptyset$, $b \in U(N_I;\varepsilon_I)$, $(u \circ v) \cap L(N_F;\varepsilon_F) \neq \emptyset$ and $v \in L(N_F;\varepsilon_F)$. Then, there exist $x_0 \in (x \circ y) \cap U(N_T;\varepsilon_T)$, $a_0 \in (a \circ b) \cap U(N_I;\varepsilon_I)$ and $u_0 \in (u \circ v) \cap L(N_F;\varepsilon_F)$. It follows that

$$N_T(x) \geq \min\left\{\sup_{c\in x\circ y} N_T(c), N_T(y)\right\} \geq \min\{N_T(x_0), N_T(y)\} \geq \varepsilon_T,$$

$$N_I(a) \geq \min\left\{\sup_{d\in a\circ b} N_I(d), N_I(b)\right\} \geq \min\{N_I(a_0), N_I(b)\} \geq \varepsilon_I$$

and

$$N_F(u) \leq \max\left\{\inf_{e\in u\circ v} N_F(e), N_F(v)\right\} \leq \max\{N_F(u_0), N_F(v)\} \leq \varepsilon_F.$$

Hence, $x \in U(N_T;\varepsilon_T)$, $a \in U(N_I;\varepsilon_I)$ and $u \in L(N_F;\varepsilon_F)$. Therefore, $U(N_T;\varepsilon_T)$, $U(N_I;\varepsilon_I)$ and $L(N_F;\varepsilon_F)$ are strong hyper BCK-ideals of H. □

Theorem 3.11. *For every neutrosophic set $N = (N_T, N_I, N_F)$ in H satisfying the condition*

$$(\forall S \subseteq H)(\exists\, a, b, c \in S)\left(\begin{array}{l} N_T(a) = \sup\limits_{x\in S} N_T(x) \\ N_I(b) = \sup\limits_{x\in S} N_I(x) \\ N_F(c) = \inf\limits_{x\in S} N_F(x) \end{array}\right), \tag{3.14}$$

if the nonempty sets $U(N_T;\varepsilon_T)$, $U(N_I;\varepsilon_I)$ and $L(N_F;\varepsilon_F)$ are strong hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0,1]$, then, $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H.

Proof. Assume that $U(N_T;\varepsilon_T)$, $U(N_I;\varepsilon_I)$ and $L(N_F;\varepsilon_F)$ are nonempty and strong hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0,1]$. For any $x, y, z \in H$, we have $x \in U(N_T;N_T(x))$, $y \in U(N_I;N_I(y))$ and $z \in L(N_F;N_F(z))$. Since $x \circ x \ll x$, $y \circ y \ll y$ and $z \circ z \ll z$ by (a1), we have $x \circ x \ll U(N_T;N_T(x))$, $y \circ y \ll U(N_I;N_I(y))$ and $z \circ z \ll L(N_F;N_F(z))$. It follows from Lemma 1.20 that $x \circ x \subseteq U(N_T;N_T(x))$, $y \circ y \subseteq U(N_I;N_I(y))$ and $z \circ z \subseteq L(N_F;N_F(z))$. Hence, $a \in U(N_T;N_T(x))$, $b \in U(N_I;N_I(y))$ and $c \in L(N_F;N_F(z))$ for all $a \in x \circ x$, $b \in y \circ y$ and $c \in z \circ z$. Therefore, $\inf\limits_{a\in x\circ x} N_T(a) \geq N_T(x)$, $\inf\limits_{b\in y\circ y} N_I(b) \geq N_I(y)$ and $\sup\limits_{c\in z\circ z} N_F(c) \leq N_F(z)$. Now, let $\varepsilon_T := \min\left\{\sup\limits_{a\in x\circ y} N_T(a), N_T(y)\right\}$, $\varepsilon_I := \min\left\{\sup\limits_{b\in x\circ y} N_I(b), N_I(y)\right\}$ and $\varepsilon_F := \max\left\{\inf\limits_{c\in x\circ y} N_F(c), N_F(y)\right\}$. Using (3.14), we have

$$N_T(a_0) = \sup_{a\in x\circ y} N_T(a) \geq \min\left\{\sup_{a\in x\circ y} N_T(a), N_T(y)\right\} = \varepsilon_T,$$

$$N_I(b_0) = \sup_{b\in x\circ y} N_I(b) \geq \min\left\{\sup_{b\in x\circ y} N_I(b), N_I(y)\right\} = \varepsilon_I$$

and

$$N_F(c_0) = \inf_{c\in x\circ y} N_F(c) \leq \max\left\{\inf_{c\in x\circ y} N_F(c), N_F(y)\right\} = \varepsilon_F$$

for some $a_0, b_0, c_0 \in x \circ y$. Hence, $a_0 \in U(N_T; \varepsilon_T)$, $b_0 \in U(N_I; \varepsilon_I)$ and $c_0 \in L(N_F; \varepsilon_F)$ which imply that $(x \circ y) \cap U(N_T; \varepsilon_T)$, $(x \circ y) \cap U(N_I; \varepsilon_I)$ and $(x \circ y) \cap L(N_F; \varepsilon_F)$ are nonempty. Since $y \in U(N_T; \varepsilon_T) \cap U(N_I; \varepsilon_I) \cap L(N_F; \varepsilon_F)$, it follows from (1.18) that $x \in U(N_T; \varepsilon_T) \cap U(N_I; \varepsilon_I) \cap L(N_F; \varepsilon_F)$. Thus,

$$N_T(x) \geq \varepsilon_T = \min\left\{\sup_{a \in x \circ y} N_T(a), N_T(y)\right\},$$

$$N_I(x) \geq \varepsilon_I = \min\left\{\sup_{b \in x \circ y} N_I(b), N_I(y)\right\}$$

and

$$N_F(x) \leq \varepsilon_F = \max\left\{\inf_{c \in x \circ y} N_F(c), N_F(y)\right\}.$$

Consequently, $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H. $\square$

Since every neutrosophic set $N = (N_T, N_I, N_F)$ satisfies the condition (3.14) in a finite hyper BCK-algebra, we have the following corollary.

Corollary 3.12. *Let $N = (N_T, N_I, N_F)$ be a neutrosophic set in a finite hyper BCK-algebra H. Then, $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H if and only if the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are strong hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$.*

Definition 3.13. A neutrosophic set $N = (N_T, N_I, N_F)$ in H is called a *neutrosophic weak hyper BCK-ideal* of H if it satisfies the following assertions.

$$\begin{aligned} N_T(0) \geq N_T(x) &\geq \min\left\{\inf_{a \in x \circ y} N_T(a), N_T(y)\right\}, \\ N_I(0) \geq N_I(x) &\geq \min\left\{\inf_{b \in x \circ y} N_I(b), N_I(y)\right\}, \\ N_F(0) \leq N_F(x) &\leq \max\left\{\sup_{c \in x \circ y} N_F(c), N_F(y)\right\} \end{aligned} \tag{3.15}$$

for all $x, y \in H$.

Definition 3.14. A neutrosophic set $N = (N_T, N_I, N_F)$ in H is called a *neutrosophic s-weak hyper BCK-ideal* of H if it satisfies the conditions (3.3) and (3.5).

Example 3.15. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation "$\circ$" which is given by Table 24. Let $N = (N_T, N_I, N_F)$ be a neutrosophic set in H which is described in Table 25. It is routine to verify that $N = (N_T, N_I, N_F)$ is a neutrosophic weak hyper BCK-ideal of H.

Table 24: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, b, c\}$

Table 25: Tabular representation of $N = (N_T, N_I, N_F)$

H	$N_T(x)$	$N_I(x)$	$N_F(x)$
0	0.98	0.85	0.02
a	0.81	0.69	0.19
b	0.56	0.43	0.32
c	0.34	0.21	0.44

Theorem 3.16. *Every neutrosophic s-weak hyper BCK-ideal is a neutrosophic weak hyper BCK-ideal.*

Proof. Let $N = (N_T, N_I, N_F)$ be a neutrosophic s-weak hyper BCK-ideal of H and let $x, y \in H$. Then, there exist $a, b, c \in x \circ y$ such that

$$N_T(x) \geq \min\{N_T(a), N_T(y)\} \geq \min\left\{\inf_{a_0 \in x \circ y} N_T(a_0), N_T(y)\right\},$$

$$N_I(x) \geq \min\{N_I(b), N_I(y)\} \geq \min\left\{\inf_{b_0 \in x \circ y} N_I(b_0), N_I(y)\right\},$$

$$N_F(x) \leq \max\{N_F(c), N_F(y)\} \leq \max\left\{\sup_{c_0 \in x \circ y} N_F(c_0), N_F(y)\right\}.$$

Hence, $N = (N_T, N_I, N_F)$ is a neutrosophic weak hyper BCK-ideal of H. □

We can conjecture that the converse of Theorem 3.16 is not true. But it is not easy to find an example of a neutrosophic weak hyper BCK-ideal which is not a neutrosophic s-weak hyper BCK-ideal.

Now we provide a condition for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal.

Theorem 3.17. *If $N = (N_T, N_I, N_F)$ is a neutrosophic weak hyper BCK-ideal of H which satisfies the condition* (3.4)*, then, $N = (N_T, N_I, N_F)$ is a neutrosophic s-weak hyper BCK-ideal of H.*

Proof. Let $N = (N_T, N_I, N_F)$ be a neutrosophic weak hyper BCK-ideal of H in which the condition (3.4) is true. Then, there exist $a_0, b_0, c_0 \in x \circ y$ such that $N_T(a_0) = \inf_{a \in x \circ y} N_T(a)$, $N_I(b_0) = \inf_{b \in x \circ y} N_I(b)$ and $N_F(c_0) = \sup_{c \in x \circ y} N_F(c)$. Hence,

$$
\begin{aligned}
N_T(x) &\geq \min\left\{ \inf_{a \in x \circ y} N_T(a), N_T(y) \right\} = \min\{N_T(a_0), N_T(y)\}, \\
N_I(x) &\geq \min\left\{ \inf_{b \in x \circ y} N_I(b), N_I(y) \right\} = \min\{N_I(b_0), N_I(y)\}, \\
N_F(x) &\leq \max\left\{ \sup_{c \in x \circ y} N_F(c), N_F(y) \right\} = \max\{N_F(c_0), N_F(y)\}.
\end{aligned}
$$

Therefore, $N = (N_T, N_I, N_F)$ is a neutrosophic s-weak hyper BCK-ideal of H. □

Remark 3.18. In a finite hyper BCK-algebra, every neutrosophic set satisfies the condition (3.4). Hence, the concept of neutrosophic s-weak hyper BCK-ideal and neutrosophic weak hyper BCK-ideal coincide in a finite hyper BCK-algebra.

Theorem 3.19. *A neutrosophic set $N = (N_T, N_I, N_F)$ is a neutrosophic weak hyper BCK-ideal of H if and only if the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are weak hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$.*

Proof. The proof is similar to the proof of Theorem 3.4. □

Definition 3.20. A neutrosophic set $N = (N_T, N_I, N_F)$ in H is called a *reflexive neutrosophic hyper BCK-ideal* of H if it satisfies

$$
(\forall x, y \in H) \left(\begin{array}{c} \inf_{a \in x \circ x} N_T(a) \geq N_T(y) \\ \inf_{b \in x \circ x} N_I(b) \geq N_I(y) \\ \sup_{c \in x \circ x} N_F(c) \leq N_F(y) \end{array} \right), \tag{3.16}
$$

and

$$
(\forall x, y \in H) \left(\begin{array}{c} N_T(x) \geq \min\left\{ \sup_{a \in x \circ y} N_T(a), N_T(y) \right\} \\ N_I(x) \geq \min\left\{ \sup_{b \in x \circ y} N_I(b), N_I(y) \right\} \\ N_F(x) \leq \max\left\{ \inf_{c \in x \circ y} N_F(c), N_F(y) \right\} \end{array} \right). \tag{3.17}
$$

Theorem 3.21. *Every reflexive neutrosophic hyper BCK-ideal is a neutrosophic strong hyper BCK-ideal.*

Proof. Straightforward. □

Theorem 3.22. *If $N = (N_T, N_I, N_F)$ is a reflexive neutrosophic hyper BCK-ideal of H, then, the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are reflexive hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$.*

Proof. Assume that $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are nonempty for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$. Let $a \in U(N_T; \varepsilon_T)$, $b \in U(N_I; \varepsilon_I)$ and $c \in L(N_F; \varepsilon_F)$. If $N = (N_T, N_I, N_F)$ is a reflexive neutrosophic hyper BCK-ideal of H, then, $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H by Theorem 3.21, and so it is a neutrosophic hyper BCK-ideal of H. It follows from Theorem 3.4 that $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are hyper BCK-ideals of H. For each $x \in H$, let $a_0, b_0, c_0 \in x \circ x$. Then,

$$\begin{aligned} N_T(a_0) &\geq \inf_{u \in x \circ x} N_T(u) \geq N_T(a) \geq \varepsilon_T, \\ N_I(b_0) &\geq \inf_{v \in x \circ x} N_I(v) \geq N_I(b) \geq \varepsilon_I, \\ N_F(c_0) &\leq \sup_{w \in x \circ x} N_F(w) \leq N_F(c) \leq \varepsilon_F, \end{aligned}$$

and so $a_0 \in U(N_T; \varepsilon_T)$, $b_0 \in U(N_I; \varepsilon_I)$ and $c_0 \in L(N_F; \varepsilon_F)$. Hence, $x \circ x \subseteq U(N_T; \varepsilon_T)$, $x \circ x \subseteq U(N_I; \varepsilon_I)$ and $x \circ x \subseteq L(N_F; \varepsilon_F)$. Therefore, $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are reflexive hyper BCK-ideals of H. $\square$

We consider the converse of Theorem 3.22 by adding a condition.

Theorem 3.23. *Let $N = (N_T, N_I, N_F)$ be a neutrosophic set in H satisfying the condition* (3.14). *If the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are reflexive hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$, then, $N = (N_T, N_I, N_F)$ is a reflexive neutrosophic hyper BCK-ideal of H.*

Proof. If the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are reflexive hyper BCK-ideals of H, then, they are strong hyper BCK-ideals of H by Lemma 1.20. It follows from Theorem 3.11 that $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H. Hence, the condition (3.17) is valid. Let $x, y \in H$. Then, the sets $U(N_T; N_T(y))$, $U(N_I; N_I(y))$ and $L(N_F; N_F(y))$ are reflexive hyper BCK-ideals of H, and so $x \circ x \subseteq U(N_T; N_T(y))$, $x \circ x \subseteq U(N_I; N_I(y))$ and $x \circ x \subseteq L(N_F; N_F(y))$. Hence, $N_T(a) \geq N_T(y)$, $N_I(b) \geq N_I(y)$ and $N_F(c) \leq N_F(y)$ for all $a, b, c \in x \circ x$. It follows that $\inf\limits_{a \in x \circ x} N_T(a) \geq N_T(y)$, $\inf\limits_{b \in x \circ x} N_I(b) \geq N_I(y)$ and $\sup\limits_{c \in x \circ x} N_F(c) \leq N_F(y)$. Therefore, $N = (N_T, N_I, N_F)$ is a reflexive neutrosophic hyper BCK-ideal of H. $\square$

We provide conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal.

Theorem 3.24. *Let $N = (N_T, N_I, N_F)$ be a neutrosophic strong hyper BCK-ideal of H which satisfies the condition* (3.14). *Then, $N = (N_T, N_I, N_F)$ is a reflexive neutrosophic* *hyper BCK-ideal of H if and only if the following assertion is valid.*

$$(\forall x \in H)\left(\begin{array}{c} \inf_{a\in x\circ x} N_T(a) \geq N_T(0) \\ \inf_{b\in x\circ x} N_I(b) \geq N_I(0) \\ \sup_{c\in x\circ x} N_F(c) \leq N_F(0) \end{array}\right). \tag{3.18}$$

Proof. It is clear that if $N = (N_T, N_I, N_F)$ is a reflexive neutrosophic hyper BCK-ideal of H, then, the condition (3.18) is valid.

Conversely, assume that $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H which satisfies the conditions (3.14) and (3.18). Then, $N_T(0) \geq N_T(y)$, $N_I(0) \geq N_I(y)$ and $N_F(0) \leq N_F(y)$ for all $y \in H$. Hence,

$$\inf_{a\in x\circ x} N_T(a) \geq N_T(y), \ \inf_{b\in x\circ x} N_I(b) \geq N_I(y) \text{ and } \sup_{c\in x\circ x} N_F(c) \leq N_F(y).$$

For any $x, y \in H$, let

$$\begin{aligned} \varepsilon_T &:= \min\left\{\sup_{a\in x\circ y} N_T(a), N_T(y)\right\}, \\ \varepsilon_I &:= \min\left\{\sup_{b\in x\circ y} N_I(b), N_I(y)\right\}, \\ \varepsilon_F &:= \max\left\{\inf_{c\in x\circ y} N_F(c), N_F(y)\right\}. \end{aligned}$$

Then, $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are strong hyper BCK-ideals of H by Theorem 3.10. Since $N = (N_T, N_I, N_F)$ satisfies the condition (3.14), there exist $a_0, b_0, c_0 \in x \circ y$ such that

$$N_T(a_0) = \sup_{a\in x\circ y} N_T(a), \ N_I(b_0) = \sup_{b\in x\circ y} N_I(b), \ N_F(c_0) = \inf_{c\in x\circ y} N_F(c).$$

Hence, $N_T(a_0) \geq \varepsilon_T$, $N_I(b_0) \geq \varepsilon_I$ and $N_F(c_0) \leq \varepsilon_F$, that is, $a_0 \in U(N_T; \varepsilon_T)$, $b_0 \in U(N_I; \varepsilon_I)$ and $c_0 \in L(N_F; \varepsilon_F)$. Hence, $(x \circ y) \cap U(N_T; \varepsilon_T) \neq \emptyset$, $(x \circ y) \cap U(N_I; \varepsilon_I) \neq \emptyset$ and $(x \circ y) \cap L(N_F; \varepsilon_F) \neq \emptyset$. Since $y \in U(N_T; \varepsilon_T) \cap U(N_I; \varepsilon_I) \cap L(N_F; \varepsilon_F)$, it follows from (1.18) that $x \in U(N_T; \varepsilon_T) \cap U(N_I; \varepsilon_I) \cap L(N_F; \varepsilon_F)$. Thus,

$$\begin{aligned} N_T(x) &\geq \varepsilon_T = \min\left\{\sup_{a\in x\circ y} N_T(a), N_T(y)\right\}, \\ N_I(x) &\geq \varepsilon_I = \min\left\{\sup_{b\in x\circ y} N_I(b), N_I(y)\right\}, \\ N_F(x) &\leq \varepsilon_F = \max\left\{\inf_{c\in x\circ y} N_F(c), N_F(y)\right\}. \end{aligned}$$

Therefore, $N = (N_T, N_I, N_F)$ is a reflexive neutrosophic hyper BCK-ideal of H. $\square$

3.2 Commutative neutrosophic hyper BCK-ideals

Definition 3.25. Let $N = (N_T, N_I, N_F)$ be a neutrosophic set $N = (N_T, N_I, N_F)$ is called a *commutative neutrosophic hyper BCK-ideal*

- of *type* $(\subseteq, \subseteq)$ over H if for all $x, y, z \in H$ and for all $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$,

$$\left(\begin{array}{l} N_T(\alpha_T) \geq \min\left\{ \inf\limits_{a \in (x \circ y) \circ z} N_T(a), N_T(z) \right\} \\ N_I(\alpha_I) \geq \min\left\{ \inf\limits_{b \in (x \circ y) \circ z} N_I(b), N_I(z) \right\} \\ N_F(\alpha_F) \leq \max\left\{ \sup\limits_{c \in (x \circ y) \circ z} N_F(c), N_F(z) \right\} \end{array} \right), \tag{3.19}$$

- of *type* $(\subseteq, \ll)$ over H if for all $x, y, z \in H$, there exist $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$ such that

$$\left(\begin{array}{l} N_T(\alpha_T) \geq \min\left\{ \inf\limits_{a \in (x \circ y) \circ z} N_T(a), N_T(z) \right\} \\ N_I(\alpha_I) \geq \min\left\{ \inf\limits_{b \in (x \circ y) \circ z} N_I(b), N_I(z) \right\} \\ N_F(\alpha_F) \leq \max\left\{ \sup\limits_{c \in (x \circ y) \circ z} N_F(c), N_F(z) \right\} \end{array} \right), \tag{3.20}$$

- of *type* $(\ll, \subseteq)$ over H if for all $x, y, z \in H$ and for all $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$,

$$\left(\begin{array}{l} N_T(\alpha_T) \geq \min\left\{ \sup\limits_{a \in (x \circ y) \circ z} N_T(a), N_T(z) \right\} \\ N_I(\alpha_I) \geq \min\left\{ \sup\limits_{b \in (x \circ y) \circ z} N_I(b), N_I(z) \right\} \\ N_F(\alpha_F) \leq \max\left\{ \inf\limits_{c \in (x \circ y) \circ z} N_F(c), N_F(z) \right\} \end{array} \right), \tag{3.21}$$

- of *type* $(\ll, \ll)$ over H if for all $x, y, z \in H$, there exist $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$ such that

$$\left(\begin{array}{l} N_T(\alpha_T) \geq \min\left\{ \sup\limits_{a \in (x \circ y) \circ z} N_T(a), N_T(z) \right\} \\ N_I(\alpha_I) \geq \min\left\{ \sup\limits_{b \in (x \circ y) \circ z} N_I(b), N_I(z) \right\} \\ N_F(\alpha_F) \leq \max\left\{ \inf\limits_{c \in (x \circ y) \circ z} N_F(c), N_F(z) \right\} \end{array} \right). \tag{3.22}$$

It is clear that every commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ (resp., type $(\ll, \subseteq)$) is of type $(\subseteq, \ll)$ (resp., type $(\ll, \ll)$), and every commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ (resp., type $(\ll, \ll)$) is of type $(\subseteq, \subseteq)$ (resp., type $(\subseteq, \ll)$).
The following example shows that there is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ (resp., type $(\subseteq, \ll)$) which is not of type $(\ll, \subseteq)$ (resp., type $(\ll, \ll)$).

Example 3.26. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given in Table 26. We define a neutrosophic set $N = (N_T, N_I, N_F)$ on H

Table 26: Tabular representation of the binary operation $\circ$

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

by Table 27. Then, $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal

Table 27: Tabular representation of $N = (N_T, N_I, N_F)$

H	$N_T(x)$	$N_I(x)$	$N_F(x)$
0	0.82	0.68	0.08
a	0.51	0.45	0.57
b	0.16	0.33	0.69

of type $(\subseteq, \subseteq)$ and $(\subseteq, \ll)$. But if we take $x = b$, $y = a$ and $z = 0$, then, it is not a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$, since $b \in b \circ (a \circ (a \circ b))$ and

$$N_T(b) \leq N_T(a) = \min\left\{ \sup_{a_0 \in (b \circ a) \circ 0} N_T(a_0), N_T(0) \right\},$$

$$N_I(b) \leq N_I(a) = \min\left\{ \sup_{b_0 \in (b \circ a) \circ 0} N_I(b_0), N_I(0) \right\}$$

and

$$N_F(b) \geq N_F(a) = \max\left\{ \inf_{c_0 \in (b \circ a) \circ 0} N_F(c_0), N_F(0) \right\}.$$

Also if we take $x = b$, $y = 0$ and $z = a$ then $N = (N_T, N_I, N_F)$ is not a commutative neutrosophic hyper BCK-ideal of type $(\ll, \ll)$, since $b \in b \circ (0 \circ (0 \circ b))$ and

$$N_T(b) \leq N_T(a) = \min\left\{\sup_{a_0 \in (b \circ 0) \circ a} N_T(a_0), N_T(a)\right\},$$

$$N_I(b) \leq N_I(a) = \min\left\{\sup_{b_0 \in (b \circ 0) \circ a} N_I(b_0), N_I(a)\right\}$$

and

$$N_F(b) \geq N_F(a) = \max\left\{\inf_{c_0 \in (b \circ 0) \circ a} N_F(c_0), N_F(a)\right\}.$$

Theorem 3.27. *Every commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ is a neutrosophic weak hyper BCK-ideal.*

Proof. Let $N = (N_T, N_I, N_F)$ be a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ over H. For any $x, y \in H$, we have $x \in x \circ (0 \circ (0 \circ x))$. It follows from (3.19) that

$$\begin{pmatrix} N_T(x) \geq \min\left\{\inf_{a \in (x \circ 0) \circ y} N_T(a), N_T(y)\right\} \\ N_I(x) \geq \min\left\{\inf_{b \in (x \circ 0) \circ y} N_I(b), N_I(y)\right\} \\ N_F(x) \leq \max\left\{\sup_{c \in (x \circ 0) \circ y} N_F(c), N_F(y)\right\} \end{pmatrix}. \tag{3.23}$$

Combining (3.3) and (3.23) induce (3.15). Therefore, $N = (N_T, N_I, N_F)$ is a neutrosophic weak hyper BCK-ideal of H. □

The converse of Theorem 3.27 is not true in general as seen in the following example.

Example 3.28. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given in Table 28. We define a neutrosophic set $N = (N_T, N_I, N_F)$ on H by

Table 28: Tabular representation of the binary operation $\circ$

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0, a\}$

Table 29.

Table 29: Tabular representation of $N = (N_T, N_I, N_F)$

H	$N_T(x)$	$N_I(x)$	$N_F(x)$
0	0.93	0.88	0.18
a	0.62	0.78	0.41
b	0.36	0.45	0.72

Then, $N = (N_T, N_I, N_F)$ is a neutrosophic weak hyper BCK-ideal. But if we take $x = b$, $y = 0$ and $z = a$ then it is not a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$, since $b \in b \circ (0 \circ (0 \circ b))$ and

$$N_T(b) \leq N_T(a) = \min\left\{ \inf_{a_0 \in (b \circ 0) \circ a} N_T(a_0), N_T(a) \right\},$$

$$N_I(b) \leq N_I(a) = \min\left\{ \inf_{b_0 \in (b \circ 0) \circ a} N_I(b_0), N_I(a) \right\}$$

and

$$N_F(b) \geq N_F(a) = \max\left\{ \sup_{c_0 \in (b \circ 0) \circ a} N_F(c_0), N_F(a) \right\}.$$

Now we provide a condition for a neutrosophic weak hyper BCK-ideal to be a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$.

Theorem 3.29. *If $N = (N_T, N_I, N_F)$ is a neutrosophic weak hyper BCK-ideal of H which satisfies the following condition*

$$(\forall x, y \in H) \begin{pmatrix} \inf\limits_{a \in x \circ (y \circ (y \circ x))} N_T(a) \geq \inf\limits_{b \in x \circ y} N_T(b), \\ \inf\limits_{a \in x \circ (y \circ (y \circ x))} N_I(a) \geq \inf\limits_{b \in x \circ y} N_I(b), \\ \sup\limits_{a \in x \circ (y \circ (y \circ x))} N_F(a) \leq \sup\limits_{b \in x \circ y} N_F(b) \end{pmatrix}, \tag{3.24}$$

Then, $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$.

Proof. Let $x, y, z \in H$ and $d \in x \circ y$. By (3.15) we have

$$N_T(d) \geq \min\left\{ \inf_{a \in d \circ z} N_T(a), N_T(z) \right\} \geq \min\left\{ \inf_{a \in (x \circ y) \circ z} N_T(a), N_T(z) \right\},$$

$$N_I(d) \geq \min\left\{ \inf_{b \in d \circ z} N_I(b), N_I(z) \right\} \geq \min\left\{ \inf_{b \in (x \circ y) \circ z} N_I(b), N_I(z) \right\}$$

and

$$N_F(d) \leq \max\left\{\sup_{c\in d\circ z} N_F(c), N_F(z)\right\} \leq \max\left\{\sup_{c\in (x\circ y)\circ z} N_F(c), N_F(z)\right\}.$$

Then, (3.24) implies that for all $\alpha_T, \alpha_I, \alpha_F \in x\circ(y\circ(y\circ x))$

$$N_T(\alpha_T) \geq \inf_{d\in x\circ y} N_T(d) \geq \min\left\{\inf_{a\in (x\circ y)\circ z} N_T(a), N_T(z)\right\},$$

$$N_I(\alpha_I) \geq \inf_{d\in x\circ y} N_I(d) \geq \min\left\{\inf_{a\in (x\circ y)\circ z} N_I(a), N_I(z)\right\}$$

and

$$N_F(\alpha_F) \leq \sup_{d\in x\circ y} N_F(d) \leq \max\left\{\sup_{a\in (x\circ y)\circ z} N_F(a), N_F(z)\right\}.$$

Therefore, $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$. □

Proposition 3.30. *Every commutative neutrosophic hyper BCK-ideal $N = (N_T, N_I, N_F)$ of type $(\ll, \subseteq)$ over H satisfies* (3.1) *and*

$$\left(\begin{array}{l} N_T(x) \geq \min\left\{\sup_{a\in x\circ y} N_T(a), N_T(y)\right\} \\ N_I(x) \geq \min\left\{\sup_{b\in x\circ y} N_I(b), N_I(y)\right\} \\ N_F(x) \leq \max\left\{\inf_{c\in x\circ y} N_F(c), N_F(y)\right\} \end{array}\right). \tag{3.25}$$

Proof. Let $N = (N_T, N_I, N_F)$ be a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H. For any $x, y \in H$, we have $x \in x\circ(0\circ(0\circ x))$. It follows from (3.21) that

$$N_T(x) \geq \min\left\{\sup_{a\in (x\circ 0)\circ y} N_T(a), N_T(y)\right\} = \min\left\{\sup_{a\in x\circ y} N_T(a), N_T(y)\right\},$$

$$N_I(x) \geq \min\left\{\sup_{b\in (x\circ 0)\circ y} N_I(b), N_I(y)\right\} = \min\left\{\sup_{b\in x\circ y} N_I(b), N_I(y)\right\}$$

and

$$N_F(x) \leq \max\left\{\inf_{c\in (x\circ 0)\circ y} N_F(c), N_F(y)\right\} = \max\left\{\inf_{c\in x\circ y} N_F(c), N_F(y)\right\}.$$

Hence, (3.25) is valid. Let $x, y \in H$ such that $x \ll y$. Then, $0 \in x \circ y$. Thus, by (3.25) and (3.3), we have,

$$N_T(x) \geq \min\left\{\sup_{a \in x \circ y} N_T(a), N_T(y)\right\} \geq \min\{N_T(0), N_T(y)\} = N_T(y),$$

$$N_I(x) \geq \min\left\{\sup_{b \in x \circ y} N_I(b), N_I(y)\right\} \geq \min\{N_I(0), N_I(y)\} = N_I(y)$$

and

$$N_F(x) \leq \max\left\{\inf_{c \in x \circ y} N_F(c), N_F(y)\right\} \leq \max\{N_F(0), N_F(y)\} = N_F(y).$$

□

Theorem 3.31. *Every commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ is a neutrosophic strong hyper BCK-ideal.*

Proof. Let $N = (N_T, N_I, N_F)$ be a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H. For any $x \in H$, let $a \in x \circ x$. Then, $a \ll x$, and so by (3.1), $N_T(a) \geq N_T(x)$, $N_I(a) \geq N_I(x)$ and $N_F(a) \leq N_F(x)$. Hence, $\inf\limits_{a \in x \circ x} N_T(a) \geq N_T(x)$, $\inf\limits_{b \in x \circ x} N_I(b) \geq N_I(x)$ and $\sup\limits_{c \in x \circ x} N_F(c) \leq N_F(x)$. Therefore, $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H. □

In the following example, we show that the converse of Theorem 3.31 may not be true, in general.

Example 3.32. Let $N = (N_T, N_I, N_F)$ be the neutrosophic set as in Example 3.28. Then, it is easy to see that $N = (N_T, N_I, N_F)$ is a neutrosophic strong hyper BCK-ideal of H. But if we take $x = b$, $y = b$ and $z = 0$, then, it is not a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$, since $a \in b \circ (b \circ (b \circ b))$ and

$$N_T(a) \leq N_T(0) = \min\left\{\sup_{a_0 \in (b \circ b) \circ 0} N_T(a_0), N_T(0)\right\},$$

$$N_I(a) \leq N_I(0) = \min\left\{\sup_{b_0 \in (b \circ b) \circ 0} N_I(b_0), N_I(0)\right\}$$

and/or

$$N_F(a) \geq N_F(0) = \max\left\{\inf_{c_0 \in (b \circ b) \circ 0} N_F(c_0), N_F(0)\right\}.$$

Theorem 3.33. *If $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H which satisfies the following condition*

$$(\forall K \subseteq H)(\exists x_0, y_0, z_0 \in K) \begin{pmatrix} N_T(x_0) = \inf\limits_{x\in K} N_T(x) \\ N_I(y_0) = \inf\limits_{y\in K} N_I(y) \\ N_F(z_0) = \sup\limits_{z\in K} N_F(z) \end{pmatrix}, \tag{3.26}$$

Then, $N = (N_T, N_I, N_F)$ is a neutrosophic s-weak hyper BCK-ideal of H.

Proof. Let $N = (N_T, N_I, N_F)$ be a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H satisfying the condition (3.26). Then, by Proposition 3.30, we have

$$N_T(x) \geq \min\left\{\sup_{a\in x\circ y} N_T(a), N_T(y)\right\} \geq \min\left\{\inf_{a\in x\circ y} N_T(a), N_T(y)\right\},$$

$$N_I(x) \geq \min\left\{\sup_{b\in x\circ y} N_I(b), N_I(y)\right\} \geq \min\left\{\inf_{b\in x\circ y} N_I(b), N_I(y)\right\}$$

and

$$N_F(x) \leq \max\left\{\inf_{c\in x\circ y} N_F(c), N_F(y)\right\} \leq \max\left\{\sup_{c\in x\circ y} N_F(c), N_F(y)\right\}.$$

Now, by (3.26), for every $x, y \in H$, there exist $a_0, b_0, c_0 \in x \circ y$ such that

$$\begin{pmatrix} N_T(a_0) = \inf\limits_{a\in x\circ y} N_T(a) \\ N_I(b_0) = \inf\limits_{b\in x\circ y} N_I(b) \\ N_F(c_0) = \sup\limits_{c\in x\circ y} N_F(c) \end{pmatrix}.$$

Then,

$$(\forall x, y \in H)(\exists\, a_0, b_0, c_0 \in x \circ y) \begin{pmatrix} N_T(x) \geq \min\{N_T(a_0), N_T(y)\} \\ N_I(x) \geq \min\{N_I(b_0), N_I(y)\} \\ N_F(x) \leq \max\{N_F(c_0), N_F(y)\} \end{pmatrix}.$$

Therefore, $N = (N_T, N_I, N_F)$ is a neutrosophic s-weak hyper BCK-ideal of H. $\square$

The following example shows that there exists a neutrosophic s-weak hyper BCK-ideal which is not a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H.

Example 3.34. The neutrosophic set $N = (N_T, N_I, N_F)$ in Example 3.26 is a neutrosophic s-weak hyper BCK-ideal of H by Remark 3.18. But it is not a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H.

Theorem 3.35. *A neutrosophic set $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ over H if and only if for all ε_T, ε_I, $\varepsilon_F \in [0,1]$, the nonempty sets $U(N_T;\varepsilon_T)$, $U(N_I;\varepsilon_I)$ and $L(N_F;\varepsilon_F)$ are commutative hyper BCK-ideals of type $(\subseteq, \subseteq)$.*

Proof. Assume that $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ over H. Let $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0,1]$ such that $U(N_T;\varepsilon_T)$, $U(N_I;\varepsilon_I)$ and $L(N_F;\varepsilon_F)$ are nonempty subsets of H. Obviously, $0 \in U(N_T;\varepsilon_T) \cap U(N_I;\varepsilon_I) \cap L(N_F;\varepsilon_F)$. Let $x, y, z \in H$ such that $(x \circ y) \circ z \subseteq U(N_T;\varepsilon_T)$ and $z \in U(N_T;\varepsilon_T)$. Then, for all $a \in (x \circ y) \circ z$, $N_T(z) \geq \varepsilon_T$ and $N_T(a) \geq \varepsilon_T$. Thus, by (3.19), for any $\alpha_T \in x \circ (y \circ (y \circ x))$ we obtain

$$N_T(\alpha_T) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} N_T(a), N_T(z)\right\} \geq \varepsilon_T.$$

Hence, $\alpha_T \in U(N_T;\varepsilon_T)$, and so $x \circ (y \circ (y \circ x)) \subseteq U(N_T;\varepsilon_T)$. Therefore, for all $\varepsilon_T \in [0,1]$, $U(N_T;\varepsilon_T)$ is a commutative hyper BCK-ideals of type $(\subseteq, \subseteq)$.

Similarly, we can verify that $U(N_I;\varepsilon_I)$ is a commutative hyper BCK-ideals of type $(\subseteq, \subseteq)$ for all $\varepsilon_I \in [0,1]$. Let $x, y, z \in H$ such that $(x \circ y) \circ z \subseteq L(N_F;\varepsilon_F)$ and $z \in L(N_F;\varepsilon_F)$. Then, $N_F(z) \leq \varepsilon_F$ and $N_F(c) \leq \varepsilon_F$ for all $c \in (x \circ y) \circ z$. It follows from (3.19) that

$$N_F(\alpha_F) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} N_F(c), N_F(z)\right\} \leq \varepsilon_F$$

for all $\alpha_F \in x \circ (y \circ (y \circ x))$. Hence, $\alpha_F \in L(N_F;\varepsilon_F)$, and so $x \circ (y \circ (y \circ x)) \subseteq L(N_F;\varepsilon_F)$. Consequently, $L(N_F;\varepsilon_F)$ is a commutative hyper BCK-ideals of type $(\subseteq, \subseteq)$ for all $\varepsilon_F \in [0,1]$.

Conversely, suppose that for all ε_T, ε_I, $\varepsilon_F \in [0,1]$, the nonempty sets $U(N_T;\varepsilon_T)$, $U(N_I;\varepsilon_I)$ and $L(N_F;\varepsilon_F)$ are commutative hyper BCK-ideals of type $(\subseteq, \subseteq)$. Let $x, y, z \in H$. If we put

$$\delta_T = \min\left\{\inf_{a \in (x \circ y) \circ z} N_T(a), N_T(z)\right\},$$

$$\delta_I = \min\left\{\inf_{b \in (x \circ y) \circ z} N_I(b), N_I(z)\right\}$$

and

$$\delta_F = \max\left\{\sup_{c \in (x \circ y) \circ z} N_F(a), N_F(z)\right\},$$

then $z \in U(N_T;\delta_T) \cap U(N_I;\delta_I) \cap L(N_F;\delta_F)$, $a \in U(N_T;\delta_T)$, $b \in U(N_I;\delta_I)$ and $c \in L(N_F;\delta_F)$ for all $a, b, c \in (x \circ y) \circ z$. Hence, $(x \circ y) \circ z \subseteq U(N_T;\delta_T)$, $(x \circ y) \circ z \subseteq U(N_I;\delta_I)$ and $(x \circ y) \circ z \subseteq L(N_F;\delta_F)$. Thus, $x \circ (y \circ (y \circ x)) \subseteq U(N_T;\delta_T)$, $x \circ (y \circ (y \circ x)) \subseteq U(N_I;\delta_I)$ and $x \circ (y \circ (y \circ x)) \subseteq L(N_F;\delta_f T)$. It follows that for all $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$,

$$N_T(\alpha_T) \geq \delta_T = \min\left\{\inf_{a \in (x \circ y) \circ z} N_T(a), N_T(z)\right\},$$

$$N_I(\alpha_I) \geq \delta_I = \min\left\{ \inf_{b \in (x \circ y) \circ z} N_I(b), N_I(z) \right\}$$

and

$$N_F(\alpha_F) \leq \delta_F = \max\left\{ \sup_{c \in (x \circ y) \circ z} N_F(a), N_F(z) \right\}.$$

Obviously, $N = (N_T, N_I, N_F)$ satisfies the condition (3.3). Therefore, $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ over H. $\square$

Corollary 3.36. *If a neutrosophic set $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H, then, for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$ the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are commutative hyper BCK-ideals of type $(\subseteq, \subseteq)$.*

Corollary 3.37. *If a neutrosophic set $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H, then, for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$ the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are commutative hyper BCK-ideals of type $(\subseteq, \ll)$.*

Theorem 3.38. *If a neutrosophic set $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H, then, for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$ the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are commutative hyper BCK-ideals of type $(\ll, \subseteq)$.*

Proof. Assume that $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H. Let ε_T, ε_I, $\varepsilon_F \in [0, 1]$ such that $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are nonempty sets. Obviously, $0 \in U(N_T; \varepsilon_T) \cap U(N_I; \varepsilon_I) \cap L(N_F; \varepsilon_F)$. Let $x, y, z \in H$ such that $(x \circ y) \circ z \ll U(N_T; \varepsilon_T)$ and $z \in U(N_T; \varepsilon_T)$. Then, $N_T(z) \geq \varepsilon_T$ and for all $a \in (x \circ y) \circ z$ there exists $b \in U(N_T; \varepsilon_T)$ such that $a \ll b$. It follows from (3.1) that $N_T(a) \geq N_T(b) \geq \varepsilon_T$, and so for all $\alpha_T \in x \circ (y \circ (y \circ x))$ by (3.21),

$$N_T(\alpha_T) \geq \min\left\{ \sup_{a \in (x \circ y) \circ z} N_T(a), N_T(z) \right\} \geq \varepsilon_T.$$

Hence, $\alpha_T \in U(N_T; \varepsilon_T)$, and so $x \circ (y \circ (y \circ x)) \subseteq U(N_T; \varepsilon_T)$. Consequently, $U(N_T; \varepsilon_T)$ is a commutative hyper BCK-ideals of type $(\ll, \subseteq)$ for all $\varepsilon_T \in [0, 1]$. Similarly, we can verify that $U(N_I; \varepsilon_I)$ is a commutative hyper BCK-ideals of type $(\ll, \subseteq)$ for all $\varepsilon_I \in [0, 1]$. Let $x, y, z \in H$ such that $(x \circ y) \circ z \ll L(N_F; \varepsilon_F)$ and $z \in L(N_F; \varepsilon_F)$. Then, $N_F(z) \leq \varepsilon_F$ and for all $c \in (x \circ y) \circ z$ there exist $b \in L(N_F; \varepsilon_F)$ such that $c \ll b$. Hence, $N_F(c) \leq N_F(b) \leq \varepsilon_F$ by (3.1). Then, $c \in L(N_F; \varepsilon_F)$ for all $c \in (x \circ y) \circ z$ and $\inf_{a \in (x \circ y) \circ z} N_F(c) \leq \varepsilon_F$. It follows from (3.21) that

$$N_F(\alpha_F) \leq \max\left\{ \inf_{c \in (x \circ y) \circ z} N_F(c), N_F(z) \right\} \leq \varepsilon_F$$

for all $\alpha_F \in x \circ (y \circ (y \circ x))$. Hence, $\alpha_F \in L(N_F; \varepsilon_F)$, and so $x \circ (y \circ (y \circ x)) \subseteq L(N_F; \varepsilon_F)$. Therefore, $L(N_F; \varepsilon_F)$ is a commutative hyper BCK-ideals of type $(\ll, \subseteq)$ for all $\varepsilon_F \in [0, 1]$. $\square$

In the following example, we show that the converse of Theorem 3.38 may not be true, in general.

Example 3.39. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given in Table 30. We define a neutrosophic set $N = (N_T, N_I, N_F)$ on H by

Table 30: Tabular representation of the binary operation $\circ$

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a\}$
b	$\{b\}$	$\{0, b\}$	$\{0, b\}$

Table 31. Then, $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are commutative hyper BCK-ideal

Table 31: Tabular representation of $N = (N_T, N_I, N_F)$

H	$N_T(x)$	$N_I(x)$	$N_F(x)$
0	0.88	0.91	0.12
a	0.43	0.45	0.68
b	0.76	0.53	0.37

of type $(\ll, \subseteq)$ for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$. But if we take $x = a$, $y = a$ and $z = 0$, then, it is not a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$, since $b \in a \circ (a \circ (a \circ a))$ and

$$N_T(b) \leq N_T(0) = \min \left\{ \sup_{a_0 \in (a \circ a) \circ 0} N_T(a_0), N_T(0) \right\},$$

$$N_I(b) \leq N_T(0) = \min \left\{ \sup_{b_0 \in (a \circ a) \circ 0} N_T(b_0), N_T(0) \right\}$$

and

$$N_F(b) \geq N_T(0) = \max \left\{ \sup_{c_0 \in (a \circ a) \circ 0} N_T(c_0), N_T(0) \right\}.$$

We present the following open problem.

Open problem. Let $N = (N_T, N_I, N_F)$ be a neutrosophic set of H such that the nonempty sets $U(N_T; \varepsilon_T)$, $U(N_I; \varepsilon_I)$ and $L(N_F; \varepsilon_F)$ are commutative hyper BCK-ideals of type $(\ll, \subseteq)$ for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$. Then, by what condition $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H?

Given a nonempty subset K of H, let $N(K) = (N(K)_T,\ N(K)_I,\ N(K)_F)$ be a neutrosophic set in H defined by

$$N(K)_T : H \to [0,1],\ x \mapsto \begin{cases} \varepsilon_T & \text{if } x \in K, \\ \delta_T & \text{otherwise,} \end{cases}$$

$$N(K)_I : H \to [0,1],\ x \mapsto \begin{cases} \varepsilon_I & \text{if } x \in K, \\ \delta_I & \text{otherwise,} \end{cases}$$

$$N(K)_F : H \to [0,1],\ x \mapsto \begin{cases} \varepsilon_F & \text{if } x \in K, \\ \delta_F & \text{otherwise,} \end{cases}$$

where ε_T, ε_I, ε_F, δ_T, δ_I, $\delta_F \in [0, 1]$ with $\varepsilon_T > \delta_T$, $\varepsilon_I > \delta_I$ and $\varepsilon_F < \delta_F$.

Theorem 3.40. *Let (α, β) be any one of $(\subseteq, \subseteq)$, $(\subseteq, \ll)$, $(\ll, \subseteq)$ and $(\ll, \ll)$. A nonempty subset K of H is a commutative hyper BCK-ideal of type (α, β) if and only if the neutrosophic set $N(K) = (N(K)_T,\ N(K)_I,\ N(K)_F)$ is a commutative neutrosophic hyper BCK-ideal of type (α, β) over H.*

Proof. Let a nonempty subset K of H be a commutative hyper BCK-ideal of type $(\subseteq, \subseteq)$. Let $x, y, z \in H$ and $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$.

(1) If $(x \circ y) \circ z \subseteq K$ and $z \in K$, then, $x \circ (y \circ (y \circ x)) \subseteq K$ by (1.26). Hence, for all $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$,

$$\begin{aligned} N(K)_T(\alpha_T) &= \varepsilon_T \geq \min\left\{ \inf_{a \in (x \circ y) \circ z} N(K)_T(a), N(K)_T(z) \right\} = \varepsilon_T, \\ N(K)_I(\alpha_I) &= \varepsilon_I \geq \min\left\{ \inf_{b \in (x \circ y) \circ z} N(K)_I(b), N(K)_I(z) \right\} = \varepsilon_I, \\ N(K)_F(\alpha_F) &= \varepsilon_F \leq \max\left\{ \sup_{c \in (x \circ y) \circ z} N(K)_F(c), N(K)_F(z) \right\} \leq \delta_F \end{aligned}$$

and the neutrosophic set $N(K) = (N(K)_T,\ N(K)_I,\ N(K)_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$.

(2) If $(x \circ y) \circ z \nsubseteq K$ and $z \in K$, then, there exist $a_0, b_0, c_0 \in (x \circ y) \circ z$ such that $N(K)_T(a_0) = \delta_T$, $N(K)_I(b_0) = \delta_I$ and $N(K)_F(c_0) = \delta_F$. Hence, for all $\alpha_T, \alpha_I, \alpha_F \in$

$x \circ (y \circ (y \circ x))$,

$$N(K)_T(\alpha_T) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} N(K)_T(a), N(K)_T(z)\right\} = \min\{\delta_T, \varepsilon_T\} = \delta_T,$$
$$N(K)_I(\alpha_I) \geq \min\left\{\inf_{b \in (x \circ y) \circ z} N(K)_I(b), N(K)_I(z)\right\} = \min\{\delta_I, \varepsilon_I\} = \delta_I,$$
$$N(K)_F(\alpha_F) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} N(K)_F(c), N(K)_F(z)\right\} = \max\{\varepsilon_F, \varepsilon_F\} = \varepsilon_F$$

and the neutrosophic set $N(K) = (N(K)_T,\ N(K)_I,\ N(K)_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$.

(3) If $(x \circ y) \circ z \subseteq K$ and $z \notin K$, then, $N(K)_T(z) = \delta_T$, $N(K)_I(z) = \delta_I$ and $N(K)_F(z) = \delta_F$. Hence, for all $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$, we get that

$$N(K)_T(\alpha_T) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} N(K)_T(a), N(K)_T(z)\right\} = \min\{\varepsilon_T, \delta_T\} = \delta_T,$$
$$N(K)_I(\alpha_I) \geq \min\left\{\inf_{b \in (x \circ y) \circ z} N(K)_I(b), N(K)_I(z)\right\} = \min\{\varepsilon_I, \delta_I\} = \delta_I,$$
$$N(K)_F(\alpha_F) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} N(K)_F(c), N(K)_F(z)\right\} = \max\{\varepsilon_F, \delta_F\} = \varepsilon_F$$

and the neutrosophic set $N(K) = (N(K)_T,\ N(K)_I,\ N(K)_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$.

(4) If $(x \circ y) \circ z \nsubseteq K$ and $z \notin K$, then, there exist $a_0, b_0, c_0 \in (x \circ y) \circ z$ such that $N(K)_T(a_0) = \delta_T$, $N(K)_I(b_0) = \delta_I$ and $N(K)_F(c_0) = \delta_F$. Also $N(K)_T(z) = \delta_T$, $N(K)_I(z) = \delta_I$ and $N(K)_F(z) = \delta_F$. Hence, for all $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$,

$$N(K)_T(\alpha_T) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} N(K)_T(a), N(K)_T(z)\right\} = \min\{\delta_T, \delta_T\} = \delta_T,$$
$$N(K)_I(\alpha_I) \geq \min\left\{\inf_{b \in (x \circ y) \circ z} N(K)_I(b), N(K)_I(z)\right\} = \min\{\delta_I, \delta_I\} = \delta_I,$$
$$N(K)_F(\alpha_F) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} N(K)_F(c), N(K)_F(z)\right\} = \max\{\varepsilon_F, \delta_F\} = \varepsilon_F$$

and the neutrosophic set $N(K) = (N(K)_T,\ N(K)_I,\ N(K)_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$.

Conversely, suppose that the neutrosophic set $N(K) = (N(K)_T,\ N(K)_I,\ N(K)_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$. It follows from (3.3) that

$0 \in K$. Let $(x \circ y) \circ z \subseteq K$ and $z \in K$. Hence, for all $\alpha_T, \alpha_I, \alpha_F \in x \circ (y \circ (y \circ x))$,

$$N(K)_T(\alpha_T) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} N(K)_T(a), N(K)_T(z)\right\} = \min\{\varepsilon_T, \varepsilon_T\} = \varepsilon_T,$$
$$N(K)_I(\alpha_I) \geq \min\left\{\inf_{b \in (x \circ y) \circ z} N(K)_I(b), N(K)_I(z)\right\} = \min\{\varepsilon_I, \varepsilon_I\} = \varepsilon_I,$$
$$N(K)_F(\alpha_F) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} N(K)_F(c), N(K)_F(z)\right\} = \max\{\varepsilon_F, \varepsilon_F\} = \varepsilon_F.$$

Therefore, $x \circ (y \circ (y \circ x)) \subseteq K$ and K is a commutative hyper BCK-ideal of type $(\subseteq, \subseteq)$ over H.
The proof of the other types are similar with some modifications. $\square$

Theorem 3.41. *If $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ over H, then, the set*

$$K := \{x \in H \mid N_T(x) = N_T(0), N_I(x) = N_I(0), N_F(x) = N_F(0)\} \tag{3.27}$$

is a commutative hyper BCK-ideal of type $(\subseteq, \subseteq)$.

Proof. It is clear that $0 \in K$. Assume that $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ over H. Let $x, y, z \in H$ such that $(x \circ y) \circ z \subseteq K$ and $z \in K$. Then, $N_T(z) = N_T(0)$, $N_I(z) = N_I(0)$, $N_F(z) = N_F(0)$, $N_T(a) = N_T(0)$, $N_I(a) = N_I(0)$ and $N_F(a) = N_F(0)$ for all $a \in (x \circ y) \circ z$. Let $b \in x \circ (y \circ (y \circ x))$. Then,

$$N_T(b) \geq \min\left\{\inf_{a \in (x \circ y) \circ z} N_T(a), N_T(z)\right\} = N_T(0),$$

$$N_I(b) \geq \min\left\{\inf_{b \in (x \circ y) \circ z} N_I(b), N_I(z)\right\} = N_I(0),$$

$$N_F(b) \leq \max\left\{\sup_{c \in (x \circ y) \circ z} N_F(c), N_F(z)\right\} = N_F(0)$$

and so $N_T(b) = N_T(0)$, $N_I(b) = N_I(0)$ and $N_F(b) = N_F(0)$. Hence, $b \in K$, and thus, $x \circ (y \circ (y \circ x)) \subseteq K$. Therefore, K is a commutative hyper BCK-ideal of type $(\subseteq, \subseteq)$. $\square$

Corollary 3.42. *If $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H, then, the set K in (3.27) is a commutative hyper BCK-ideal of type $(\subseteq, \subseteq)$.*

Corollary 3.43. *If $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ over H, then, the set K in (3.27) is a commutative hyper BCK-ideal of type $(\subseteq, \ll)$.*

Corollary 3.44. *If $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H, then, the set K in* (3.27) *is a commutative hyper BCK-ideal of type $(\subseteq, \ll)$.*

Lemma 3.45. *Every commutative neutrosophic hyper BCK-ideal $N = (N_T, N_I, N_F)$ of type $(\ll, \subseteq)$ over H satisfies the condition* (3.1)*.*

Proof. By using Theorems 3.31 and 3.9, the proof is clear. □

Theorem 3.46. *If $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H, then, the set K in* (3.27) *is a commutative hyper BCK-ideal of type $(\ll, \subseteq)$.*

Proof. It is clear that $0 \in K$. Assume that $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H. Let $x, y, z \in H$ such that $(x \circ y) \circ z \ll K$ and $z \in K$. Then, for all $a \in (x \circ y) \circ z$, there exists $c \in K$ such that $a \ll c$ and Lemma 3.45 implies that $N_T(a) = N_T(0)$, $N_I(a) = N_I(0)$ and $N_F(a) = N_F(0)$. Suppose $b \in x \circ (y \circ (y \circ x))$. Then,

$$N_T(b) \geq \min\left\{\sup_{a \in (x \circ y) \circ z} N_T(a), N_T(z)\right\} = N_T(0),$$

$$N_I(b) \geq \min\left\{\sup_{b \in (x \circ y) \circ z} N_I(b), N_I(z)\right\} = N_I(0)$$

and

$$N_F(b) \leq \max\left\{\inf_{c \in (x \circ y) \circ z} N_F(c), N_F(z)\right\} = N_F(0).$$

Hence, $b \in K$, and so $x \circ (y \circ (y \circ x)) \subseteq K$. Therefore, K is a commutative hyper BCK-ideal of type $(\ll, \subseteq)$. □

Corollary 3.47. *If $N = (N_T, N_I, N_F)$ is a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ over H, then, the set K in* (3.27) *is a commutative hyper BCK-ideal of type $(\ll, \ll)$.*

Chapter 4.

Neutrosophic soft hyper BCK-ideals

4 Abstract

The aim of this chapter is to apply neutrosophic soft set for dealing with several kinds of theories in hyper BCK-algebras. The notions of neutrosophic soft hyper BCK-ideal, neutrosophic soft weak hyper BCK-ideal and neutrosophic soft strong hyper BCK-ideal are introduced. Some relevant properties and their relations are indicated. Also, the notion of (strong, weak) neutrosophic soft hyper p-ideal is introduced, and their relations are investigated. Relations between (strong, weak) neutrosophic soft hyper BCK-ideal and (strong, weak) neutrosophic soft hyper p-ideal are discussed. Characterizations of neutrosophic soft hyper BCK-ideal and neutrosophic soft hyper p-ideal are considered.

In what follows, let H and E be a hyper BCK-algebra and a set of parameters, respectively, and A be a subset of E unless otherwise specified.

4.1 Neutrosophic soft hyper BCK-ideals

Definition 4.1. Let $e \in A$ be a parameter. A neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over H is called a *neutrosophic soft hyper BCK-ideal* of H based on e if the following assertions are valid.

$$\widetilde{\mathcal{N}}_T^e(x) \geq \widetilde{\mathcal{N}}_T^e(y),\ \widetilde{\mathcal{N}}_I^e(x) \geq \widetilde{\mathcal{N}}_I^e(y),\ \widetilde{\mathcal{N}}_F^e(x) \leq \widetilde{\mathcal{N}}_F^e(y) \tag{4.1}$$

for all $x, y \in H$, such that $x \ll y$, and

$$(\forall x, y \in H) \begin{pmatrix} \widetilde{\mathcal{N}}_T^e(x) \geq \min\left\{\inf\limits_{a \in x \circ y} \widetilde{\mathcal{N}}_T^e(a), \widetilde{\mathcal{N}}_T^e(y)\right\} \\ \widetilde{\mathcal{N}}_I^e(x) \geq \min\left\{\inf\limits_{b \in x \circ y} \widetilde{\mathcal{N}}_I^e(b), \widetilde{\mathcal{N}}_I^e(y)\right\} \\ \widetilde{\mathcal{N}}_F^e(x) \leq \max\left\{\sup\limits_{c \in x \circ y} \widetilde{\mathcal{N}}_F^e(c), \widetilde{\mathcal{N}}_F^e(y)\right\} \end{pmatrix}. \tag{4.2}$$

Table 32: Cayley table for the binary operation “$\circ$”

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$	$\{0,a\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0,a\}$

Example 4.2. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation “$\circ$” which is given by Table 32. Let $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set in H which is described in Table 33. It is routine to verify that $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper

Table 33: Tabular representation of $(\widetilde{\mathcal{N}}, A)$

H	$\widetilde{\mathcal{N}}^e_T(x)$	$\widetilde{\mathcal{N}}^e_I(x)$	$\widetilde{\mathcal{N}}^e_F(x)$
0	0.98	0.85	0.02
a	0.98	0.85	0.02
b	0.56	0.43	0.32
c	0.34	0.21	0.44

BCK-ideal of H.

Proposition 4.3. *Every neutrosophic soft hyper BCK-ideal $(\widetilde{\mathcal{N}}, A)$ of H satisfies:*

$$\widetilde{\mathcal{N}}^e_T(0) \geq \widetilde{\mathcal{N}}^e_T(x),\ \widetilde{\mathcal{N}}^e_I(0) \geq \widetilde{\mathcal{N}}^e_I(x),\ \widetilde{\mathcal{N}}^e_F(0) \leq \widetilde{\mathcal{N}}^e_F(x) \tag{4.3}$$

for all $x \in H$ and $e \in A$.

Proof. Straightforward. □

Given a neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over H and a parameter $e \in A$, we consider the following sets:

$$\begin{aligned} U(\widetilde{\mathcal{N}}^e_T; e_T) &:= \{x \in H \mid \widetilde{\mathcal{N}}^e_T(x) \geq e_T\}, \\ U(\widetilde{\mathcal{N}}^e_I; e_I) &:= \{x \in H \mid \widetilde{\mathcal{N}}^e_I(x) \geq e_I\}, \\ L(\widetilde{\mathcal{N}}^e_F; e_F) &:= \{x \in H \mid \widetilde{\mathcal{N}}^e_F(x) \leq e_F\}, \end{aligned}$$

which are called *neutrosophic soft level sets* based on e of $(\widetilde{\mathcal{N}}, A)$ where $e_T, e_I, e_F \in [0, 1]$ which are related to the parameter e. In this case, we say that e_T, e_I and e_F are *parameter e-numbers.*

Theorem 4.4. *Let H be a hyper BCK-algebra and $e \in A$ be a parameter. If a neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over H is a neutrosophic soft hyper BCK-ideal of H based on e, then the non-empty neutrosophic soft level sets of $(\widetilde{\mathcal{N}}, A)$ based on e are hyper BCK-ideal of H for all parameter e-numbers.*

Proof. Assume that $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper BCK-ideal of H, $e \in A$ be a parameter and $e_T, e_I, e_F \in [0,1]$ be such that $U(\widetilde{\mathcal{N}}^e_T; e_T)$, $U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $L(\widetilde{\mathcal{N}}^e_F; e_F)$ are nonempty. It is easy to see that $0 \in U(\widetilde{\mathcal{N}}^e_T; e_T)$, $0 \in U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $0 \in L(\widetilde{\mathcal{N}}^e_F; e_F)$. Let $x, y \in H$ such that $x \circ y \ll U(\widetilde{\mathcal{N}}^e_T; e_T)$ and $y \in U(\widetilde{\mathcal{N}}^e_T; e_T)$. Then, for any $a \in x \circ y$, there exists $a_0 \in U(\widetilde{\mathcal{N}}^e_T; e_T)$ such that $a \ll a_0$ and $\widetilde{\mathcal{N}}^e_T(y) \geq e_T$. We conclude from (4.1) that $\widetilde{\mathcal{N}}^e_T(a) \geq \widetilde{\mathcal{N}}^e_T(a_0) \geq e_T$ for all $a \in x \circ y$. Hence, $\inf\limits_{a \in x \circ y} \widetilde{\mathcal{N}}^e_T(a) \geq e_T$, and so

$$\widetilde{\mathcal{N}}^e_T(x) \geq \min\left\{ \inf_{a \in x \circ y} \widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y) \right\} \geq e_T,$$

that is, $x \in U(\widetilde{\mathcal{N}}^e_T; e_T)$. Similarly, we can prove that if $x \circ y \ll U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $y \in U(\widetilde{\mathcal{N}}^e_I; e_I)$, then $x \in U(\widetilde{\mathcal{N}}^e_I; e_I)$. Hence, $U(\widetilde{\mathcal{N}}^e_T; e_T)$ and $U(\widetilde{\mathcal{N}}^e_I; e_I)$ based on e are hyper BCK-ideals of H for all parameter e-numbers. Let $x, y \in H$ such that $x \circ y \ll L(\widetilde{\mathcal{N}}^e_F; e_F)$ and $y \in L(\widetilde{\mathcal{N}}^e_F; e_F)$. Then, $\widetilde{\mathcal{N}}^e_F(y) \leq e_F$. Let $b \in x \circ y$. Then, there exists $b_0 \in L(\widetilde{\mathcal{N}}^e_F; e_F)$ such that $b \ll b_0$, thus, by (4.1), $\widetilde{\mathcal{N}}^e_F(b) \leq \widetilde{\mathcal{N}}^e_F(b_0) \leq e_F$. Hence, $\sup\limits_{b \in x \circ y} \widetilde{\mathcal{N}}^e_F(b) \leq e_F$, and so

$$\widetilde{\mathcal{N}}^e_F(x) \leq \max\left\{ \sup_{b \in x \circ y} \widetilde{\mathcal{N}}^e_F(b), \widetilde{\mathcal{N}}^e_F(y) \right\} \leq e_F.$$

Then, $x \in L(\widetilde{\mathcal{N}}^e_F; e_F)$. Therefore, $L(\widetilde{\mathcal{N}}^e_F; e_F)$ based on e is a hyper BCK-ideal of H, for all parameter e-numbers. $\square$

Theorem 4.5. *Given a hyper BCK-algebra H and a parameter $e \in A$. Let $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set over H such that the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are hyper BCK-ideal of H for all parameter e-numbers. Then, $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper BCK-ideal of H based on e.*

Proof. Suppose that the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are hyper BCK-ideal of H for all parameter e-numbers. Let $x, y \in H$ such that $x \ll y$. Then,

$$y \in U(\widetilde{\mathcal{N}}^e_T; \widetilde{\mathcal{N}}^e_T(y)) \cap U(\widetilde{\mathcal{N}}^e_I; \widetilde{\mathcal{N}}^e_I(y)) \cap L(\widetilde{\mathcal{N}}^e_F; \widetilde{\mathcal{N}}^e_F(y)),$$

and so $\{x\} \ll U(\widetilde{\mathcal{N}}^e_T; \widetilde{\mathcal{N}}^e_T(y))$, $\{x\} \ll U(\widetilde{\mathcal{N}}^e_I; \widetilde{\mathcal{N}}^e_I(y))$ and $\{x\} \ll L(\widetilde{\mathcal{N}}^e_F; \widetilde{\mathcal{N}}^e_F(y))$. By Lemma 1.20, $x \in U(\widetilde{\mathcal{N}}^e_T; \widetilde{\mathcal{N}}^e_T(y))$, $x \in U(\widetilde{\mathcal{N}}^e_I; \widetilde{\mathcal{N}}^e_I(y))$ and $x \in L(\widetilde{\mathcal{N}}^e_F; \widetilde{\mathcal{N}}^e_F(y))$. Hence, $\widetilde{\mathcal{N}}^e_T(x) \geq \widetilde{\mathcal{N}}^e_T(y)$, $\widetilde{\mathcal{N}}^e_I(x) \geq \widetilde{\mathcal{N}}^e_I(y)$ and $\widetilde{\mathcal{N}}^e_F(x) \leq \widetilde{\mathcal{N}}^e_F(y)$. Now, for any $x, y \in H$, let $e_T := \min\left\{ \inf\limits_{a_T \in x \circ y} \widetilde{\mathcal{N}}^e_T(a_T), \widetilde{\mathcal{N}}^e_T(y) \right\}$, $e_I := \min\left\{ \inf\limits_{b_I \in x \circ y} \widetilde{\mathcal{N}}^e_I(b_I), \widetilde{\mathcal{N}}^e_I(y) \right\}$ and $e_F :=$

$\max\left\{\sup_{c_F\in x\circ y}\widetilde{\mathcal{N}}_F^e(c_F), \widetilde{\mathcal{N}}_F^e(y)\right\}$. Then, $y \in U(\widetilde{\mathcal{N}}_T^e; e_T) \cap U(\widetilde{\mathcal{N}}_I^e; e_I) \cap L(\widetilde{\mathcal{N}}_F^e; e_F)$, and for any $a_T, b_I, c_F \in x \circ y$ we have,

$$\widetilde{\mathcal{N}}_T^e(a_T) \geq \inf_{a_T\in x\circ y}\widetilde{\mathcal{N}}_T^e(a_T) \geq \min\left\{\inf_{a_T\in x\circ y}\widetilde{\mathcal{N}}_T^e(a_T), \widetilde{\mathcal{N}}_T^e(y)\right\} = e_T,$$

$$\widetilde{\mathcal{N}}_I^e(b_I) \geq \inf_{b_I\in x\circ y}\widetilde{\mathcal{N}}_I^e(b_I) \geq \min\left\{\inf_{b_I\in x\circ y}\widetilde{\mathcal{N}}_I^e(b_I), \widetilde{\mathcal{N}}_I^e(y)\right\} = e_I$$

and

$$\widetilde{\mathcal{N}}_F^e(c_F) \leq \sup_{c_F\in x\circ y}\widetilde{\mathcal{N}}_F^e(c_F) \leq \max\left\{\sup_{c_F\in x\circ y}\widetilde{\mathcal{N}}_F^e(c_F), \widetilde{\mathcal{N}}_F^e(y)\right\} = e_F.$$

Hence, $a_T \in U(\widetilde{\mathcal{N}}_T^e; e_T)$, $b_I \in U(\widetilde{\mathcal{N}}_I^e; e_I)$ and $c_F \in L(\widetilde{\mathcal{N}}_F^e; e_F)$, and so $x \circ y \subseteq U(\widetilde{\mathcal{N}}_T^e; e_T)$, $x \circ y \subseteq U(\widetilde{\mathcal{N}}_I^e; e_I)$ and $x \circ y \subseteq L(\widetilde{\mathcal{N}}_F^e; e_F)$. By (1.12), we have $x \circ y \ll U(\widetilde{\mathcal{N}}_T^e; e_T)$, $x \circ y \ll U(\widetilde{\mathcal{N}}_I^e; e_I)$ and $x \circ y \ll L(\widetilde{\mathcal{N}}_F^e; e_F)$. However, by (1.16), we get that

$$x \in U(\widetilde{\mathcal{N}}_T^e; e_T) \cap U(\widetilde{\mathcal{N}}_I^e; e_I) \cap L(\widetilde{\mathcal{N}}_F^e; e_F).$$

Then,

$$\widetilde{\mathcal{N}}_T^e(x) \geq e_T = \min\left\{\inf_{a_T\in x\circ y}\widetilde{\mathcal{N}}_T^e(a_T), \widetilde{\mathcal{N}}_T^e(y)\right\},$$

$$\widetilde{\mathcal{N}}_I^e(x) \geq e_I = \min\left\{\inf_{b_I\in x\circ y}\widetilde{\mathcal{N}}_I^e(b_I), \widetilde{\mathcal{N}}_I^e(y)\right\}$$

and

$$\widetilde{\mathcal{N}}_F^e(x) \leq e_F = \max\left\{\sup_{c_F\in x\circ y}\widetilde{\mathcal{N}}_F^e(c_F), \widetilde{\mathcal{N}}_F^e(y)\right\}.$$

Therefore, $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper BCK-ideal of H based on e. □

Definition 4.6. Let $e \in A$ be a parameter. A neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over a hyper BCK-algebra H is called

- a *weak neutrosophic soft hyper BCK-ideal* of H based on e if it satisfies:

$$(\forall x, y \in H)\left(\begin{array}{l} \widetilde{\mathcal{N}}_T^e(0) \geq \widetilde{\mathcal{N}}_T^e(x) \geq \min\left\{\inf_{a\in x\circ y}\widetilde{\mathcal{N}}_T^e(a), \widetilde{\mathcal{N}}_T^e(y)\right\} \\ \widetilde{\mathcal{N}}_I^e(0) \geq \widetilde{\mathcal{N}}_I^e(x) \geq \min\left\{\inf_{b\in x\circ y}\widetilde{\mathcal{N}}_I^e(b), \widetilde{\mathcal{N}}_I^e(y)\right\} \\ \widetilde{\mathcal{N}}_F^e(0) \leq \widetilde{\mathcal{N}}_F^e(x) \leq \max\left\{\sup_{c\in x\circ y}\widetilde{\mathcal{N}}_F^e(c), \widetilde{\mathcal{N}}_F^e(y)\right\} \end{array}\right). \tag{4.4}$$

- a *strong neutrosophic soft hyper BCK-ideal* of H based on e if it satisfies:

$$(\forall x, y \in H) \left(\begin{array}{l} \inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}_T^e(a) \geq \widetilde{\mathcal{N}}_T^e(x) \geq \min\left\{ \sup\limits_{b \in x \circ y} \widetilde{\mathcal{N}}_T^e(b), \widetilde{\mathcal{N}}_T^e(y) \right\} \\ \inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}_I^e(a) \geq \widetilde{\mathcal{N}}_I^e(x) \geq \min\left\{ \sup\limits_{b \in x \circ y} \widetilde{\mathcal{N}}_I^e(b), \widetilde{\mathcal{N}}_I^e(y) \right\} \\ \sup\limits_{a \in x \circ x} \widetilde{\mathcal{N}}_I^e(a) \leq \widetilde{\mathcal{N}}_F^e(x) \leq \max\left\{ \inf\limits_{c \in x \circ y} \widetilde{\mathcal{N}}_F^e(c), \widetilde{\mathcal{N}}_F^e(y) \right\} \end{array} \right). \tag{4.5}$$

If a neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over H is a (weak, strong) neutrosophic soft hyper BCK-ideal of H based on all parameters, then $(\widetilde{\mathcal{N}}, A)$ is called a (weak, strong) neutrosophic soft hyper BCK-ideal of H.

Example 4.7. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 34. Let $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set in H which is

Table 34: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a\}$	$\{0, a\}$

described in Table 35. It is easy to check that $(\widetilde{\mathcal{N}}, A)$ is a weak neutrosophic soft hyper

Table 35: Tabular representation of $(\widetilde{\mathcal{N}}, A)$

H	$\widetilde{\mathcal{N}}_T^e(x)$	$\widetilde{\mathcal{N}}_I^e(x)$	$\widetilde{\mathcal{N}}_F^e(x)$
0	0.98	0.85	0.12
a	0.48	0.35	0.82
b	0.67	0.48	0.32

BCK-ideal of H. But it is not strong neutrosophic soft hyper BCK-ideal of H, since $\inf\limits_{x \in b \circ b} \widetilde{\mathcal{N}}_T^e(x) < \widetilde{\mathcal{N}}_T^e(b)$, $\inf\limits_{y \in b \circ b} \widetilde{\mathcal{N}}_I^e(y) < \widetilde{\mathcal{N}}_I^e(b)$ and $\sup\limits_{z \in b \circ b} \widetilde{\mathcal{N}}_I^e(z) > \widetilde{\mathcal{N}}_F^e(b)$.

Proposition 4.8. *Every strong neutrosophic soft hyper BCK-ideal of H based on a parameter e is a neutrosophic soft hyper BCK-ideal of H. Also, every neutrosophic soft hyper BCK-ideal of H based on a parameter e is a weak neutrosophic soft hyper BCK-ideal of H.*

Proof. The proof is straightforward. □

Example 4.9. Consider a hyper BCK-algebra $H = \{0, a, b\}$ and the soft hyper BCK-ideal $(\widetilde{\mathcal{N}}, A)$ in Example 4.2. Then, it is a strong neutrosophic soft hyper BCK-ideal of H.

Theorem 4.10. *Let H be a hyper BCK-algebra, $e \in A$ be a parameter and $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set over H. Then, $(\widetilde{\mathcal{N}}, A)$ is a weak neutrosophic soft hyper BCK-ideal of H based on e if and only if the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are weak hyper BCK-ideal of H for all parameter e-numbers.*

Proof. The proof is similar to the proof of Theorems 4.4 and 4.5. □

Theorem 4.11. *Let H be a hyper BCK-algebra and $e \in A$ be a parameter. If a neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over H is a strong neutrosophic soft hyper BCK-ideal of H based on e, then the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are strong hyper BCK-ideal of H for all parameter e-numbers.*

Proof. Let $(\widetilde{\mathcal{N}}, A)$ over H is a strong neutrosophic soft hyper BCK-ideal of H based on e. Then, $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper BCK-ideal of H. Assume that the neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are non-empty for all e_T, e_I, $e_F \in [0, 1]$. Then, there exist $a \in U(\widetilde{\mathcal{N}}^e_T; e_T)$, $b \in U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $c \in L(\widetilde{\mathcal{N}}^e_F; e_F)$, such that $\widetilde{\mathcal{N}}^e_T(a) \geq e_T$, $\widetilde{\mathcal{N}}^e_I(b) \geq e_I$ and $\widetilde{\mathcal{N}}^e_F(c) \leq e_F$. It follows from (4.3) that $\widetilde{\mathcal{N}}^e_T(0) \geq \widetilde{\mathcal{N}}^e_T(a) \geq e_T$, $\widetilde{\mathcal{N}}^e_I(0) \geq \widetilde{\mathcal{N}}^e_I(b) \geq e_I$ and $\widetilde{\mathcal{N}}^e_F(0) \leq \widetilde{\mathcal{N}}^e_F(c) \leq e_F$. Hence,

$$0 \in U(\widetilde{\mathcal{N}}^e_T; e_T) \cap U(\widetilde{\mathcal{N}}^e_I; e_I) \cap L(\widetilde{\mathcal{N}}^e_F; e_F).$$

Let $x, y, a, b, u, v \in H$ such that $(x \circ y) \cap U(\widetilde{\mathcal{N}}^e_T; e_T) \neq \emptyset$, $y \in U(\widetilde{\mathcal{N}}^e_T; e_T)$, $(a \circ b) \cap U(\widetilde{\mathcal{N}}^e_I; e_I) \neq \emptyset$, $b \in U(\widetilde{\mathcal{N}}^e_I; e_I)$, $(u \circ v) \cap L(\widetilde{\mathcal{N}}^e_F; e_F) \neq \emptyset$ and $v \in L(\widetilde{\mathcal{N}}^e_F; e_F)$. Then, there exist $x_0 \in (x \circ y) \cap U(\widetilde{\mathcal{N}}^e_T; e_T)$, $a_0 \in (a \circ b) \cap U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $u_0 \in (u \circ v) \cap L(\widetilde{\mathcal{N}}^e_F; e_F)$, and so

$$\widetilde{\mathcal{N}}^e_T(x) \geq \min\left\{\sup_{d \in x \circ y} \widetilde{\mathcal{N}}^e_T(d), \widetilde{\mathcal{N}}^e_T(y)\right\} \geq \min\{\widetilde{\mathcal{N}}^e_T(x_0), \widetilde{\mathcal{N}}^e_T(y)\} \geq e_T,$$

$$\widetilde{\mathcal{N}}^e_I(a) \geq \min\left\{\sup_{f \in x \circ y} \widetilde{\mathcal{N}}^e_I(f), \widetilde{\mathcal{N}}^e_I(b)\right\} \geq \min\{\widetilde{\mathcal{N}}^e_I(a_0), \widetilde{\mathcal{N}}^e_I(b)\} \geq e_I$$

and

$$\widetilde{\mathcal{N}}^e_F(u) \leq \max\left\{\inf_{g \in x \circ y} \widetilde{\mathcal{N}}^e_F(g), \widetilde{\mathcal{N}}^e_F(v)\right\} \leq \max\{\widetilde{\mathcal{N}}^e_F(u_0), \widetilde{\mathcal{N}}^e_F(v)\} \leq e_F.$$

Hence, $x \in U(\widetilde{\mathcal{N}}^e_T; e_T)$, $a \in U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $u \in L(\widetilde{\mathcal{N}}^e_F; e_F)$. Therefore, $U(\widetilde{\mathcal{N}}^e_T; e_T)$, $U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $L(\widetilde{\mathcal{N}}^e_F; e_F)$ are strong hyper BCK-ideals of H. □

We consider the converse of Theorem 4.11.

Theorem 4.12. *Let H be a finite hyper BCK-algebra, $e \in A$ be a parameter and $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set over H such that the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are strong hyper BCK-ideal of H for all parameter e-numbers. Then, $(\widetilde{\mathcal{N}}, A)$ is a strong neutrosophic soft hyper BCK-ideal of H based on e.*

Proof. Assume that $U(\widetilde{\mathcal{N}}^e_T; e_T)$, $U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $L(\widetilde{\mathcal{N}}^e_F; e_F)$ are nonempty and strong hyper BCK-ideals of H for all $e_T,\ e_I,\ e_F \in [0, 1]$. For any $x, y, z \in H$, we get that $x \in U(\widetilde{\mathcal{N}}^e_T; \widetilde{\mathcal{N}}^e_T(x))$, $y \in U(\widetilde{\mathcal{N}}^e_I; \widetilde{\mathcal{N}}^e_I(y))$ and $z \in L(\widetilde{\mathcal{N}}^e_F; \widetilde{\mathcal{N}}^e_F(z))$. By (a1), $x \circ x \ll \{x\}$, $y \circ y \ll \{y\}$ and $z \circ z \ll \{z\}$, and so $x \circ x \ll U(\widetilde{\mathcal{N}}^e_T; \widetilde{\mathcal{N}}^e_T(x))$, $y \circ y \ll U(\widetilde{\mathcal{N}}^e_I; \widetilde{\mathcal{N}}^e_I(y))$ and $z \circ z \ll L(\widetilde{\mathcal{N}}^e_F; \widetilde{\mathcal{N}}^e_F(z))$. By Lemma 1.20, $x \circ x \subseteq U(\widetilde{\mathcal{N}}^e_T; \widetilde{\mathcal{N}}^e_T(x))$, $y \circ y \subseteq U(\widetilde{\mathcal{N}}^e_I; \widetilde{\mathcal{N}}^e_I(y))$ and $z \circ z \subseteq L(\widetilde{\mathcal{N}}^e_F; \widetilde{\mathcal{N}}^e_F(z))$. Hence, $a \in U(\widetilde{\mathcal{N}}^e_T; \widetilde{\mathcal{N}}^e_T(x))$, $b \in U(\widetilde{\mathcal{N}}^e_I; \widetilde{\mathcal{N}}^e_I(y))$ and $c \in L(\widetilde{\mathcal{N}}^e_F; \widetilde{\mathcal{N}}^e_F(z))$ for all $a \in x \circ x$, $b \in y \circ y$ and $c \in z \circ z$. Therefore, $\inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_T(a) \geq \widetilde{\mathcal{N}}^e_T(x)$, $\inf\limits_{b \in y \circ y} \widetilde{\mathcal{N}}^e_I(b) \geq \widetilde{\mathcal{N}}^e_I(y)$ and $\sup\limits_{c \in z \circ z} \widetilde{\mathcal{N}}^e_F(c) \leq \widetilde{\mathcal{N}}^e_F(z)$. Let $e_T := \min\left\{\sup\limits_{p \in x \circ y} \widetilde{\mathcal{N}}^e_T(p), \widetilde{\mathcal{N}}^e_T(y)\right\}$, $e_I := \min\left\{\sup\limits_{q \in x \circ y} \widetilde{\mathcal{N}}^e_I(q), \widetilde{\mathcal{N}}^e_I(y)\right\}$ and $e_F := \max\left\{\inf\limits_{r \in x \circ y} \widetilde{\mathcal{N}}^e_F(r), \widetilde{\mathcal{N}}^e_F(y)\right\}$. Then, $y \in U(\widetilde{\mathcal{N}}^e_T; e_T) \cap U(\widetilde{\mathcal{N}}^e_I; e_I) \cap L(\widetilde{\mathcal{N}}^e_F; e_F)$. Since H is a finite hyper BCK-algebra, then for all $x, y \in H$, there exist $a_0, b_0, c_0 \in x \circ y$ such that

$$\widetilde{\mathcal{N}}^e_T(a_0) = \sup_{p \in x \circ y} \widetilde{\mathcal{N}}^e_T(p) \geq \min\left\{\sup_{p \in x \circ y} \widetilde{\mathcal{N}}^e_T(p), \widetilde{\mathcal{N}}^e_T(y)\right\} = e_T,$$

$$\widetilde{\mathcal{N}}^e_I(b_0) = \sup_{q \in x \circ y} \widetilde{\mathcal{N}}^e_I(q) \geq \min\left\{\sup_{q \in x \circ y} \widetilde{\mathcal{N}}^e_I(q), \widetilde{\mathcal{N}}^e_I(y)\right\} = e_I$$

and

$$\widetilde{\mathcal{N}}^e_F(c_0) = \inf_{r \in x \circ y} \widetilde{\mathcal{N}}^e_F(r) \leq \max\left\{\inf_{r \in x \circ y} \widetilde{\mathcal{N}}^e_F(r), \widetilde{\mathcal{N}}^e_F(y)\right\} = e_F.$$

Thus, $a_0 \in U(\widetilde{\mathcal{N}}^e_T; e_T)$, $b_0 \in U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $c_0 \in L(\widetilde{\mathcal{N}}^e_F; e_F)$, and so $(x \circ y) \cap U(\widetilde{\mathcal{N}}^e_T; e_T)$, $(x \circ y) \cap U(\widetilde{\mathcal{N}}^e_I; e_I)$ and $(x \circ y) \cap L(\widetilde{\mathcal{N}}^e_F; e_F)$ are nonempty. Then, $x \in U(\widetilde{\mathcal{N}}^e_T; e_T) \cap U(\widetilde{\mathcal{N}}^e_I; e_I) \cap L(\widetilde{\mathcal{N}}^e_F; e_F)$ by (1.18). Hence,

$$\widetilde{\mathcal{N}}^e_T(x) \geq e_T = \min\left\{\sup_{p \in x \circ y} \widetilde{\mathcal{N}}^e_T(p), \widetilde{\mathcal{N}}^e_T(y)\right\},$$

$$\widetilde{\mathcal{N}}^e_I(x) \geq e_I = \min\left\{\sup_{q \in x \circ y} \widetilde{\mathcal{N}}^e_I(q), \widetilde{\mathcal{N}}^e_I(y)\right\}$$

and

$$\widetilde{\mathcal{N}}^e_F(x) \leq e_F = \max\left\{\inf_{r\in x\circ y}\widetilde{\mathcal{N}}^e_F(r), \widetilde{\mathcal{N}}^e_F(y)\right\}.$$

Consequently, $(\widetilde{\mathcal{N}}, A)$ is a strong neutrosophic soft hyper BCK-ideal of H based on e. □

Corollary 4.13. *Let H be a finite hyper BCK-algebra, $e \in A$ be a parameter and $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set over H. Then, $(\widetilde{\mathcal{N}}, A)$ is a strong neutrosophic soft hyper BCK-ideal of H based on e if and only if the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are strong hyper BCK-ideal of H for all parameter e-numbers.*

Theorem 4.14. *Let H be a hyper BCK-algebra and let $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set over H in which*

$$\widetilde{\mathcal{N}}^e_T(x) = \begin{cases} \varepsilon_T & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases} \quad \widetilde{\mathcal{N}}^e_I(x) = \begin{cases} \varepsilon_I & \text{if } x \in F, \\ 0 & \text{otherwise,} \end{cases} \quad \widetilde{\mathcal{N}}^e_F(x) = \begin{cases} \varepsilon_F & \text{if } x \in F, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x \in H$ where F is a subset of H, $\varepsilon_T, \varepsilon_I \in (0, 1]$ and $\varepsilon_F \in [0, 1)$. Then, $(\widetilde{\mathcal{N}}, A)$ is a (weak, strong) neutrosophic soft hyper BCK-ideal of H if and only if F is a (weak, strong) hyper BCK-ideal of H.

Proof. Let $(\widetilde{\mathcal{N}}, A)$ be a weak neutrosophic soft hyper BCK-ideal of H. Then, for any $x \in H$ and for all $\varepsilon_T, \varepsilon_I \in (0, 1]$ and $\varepsilon_F \in [0, 1)$, we get that $U(\widetilde{\mathcal{N}}^e_T; \varepsilon_T) = U(\widetilde{\mathcal{N}}^e_I; \varepsilon_I) = L(\widetilde{\mathcal{N}}^e_F; \varepsilon_F) = F$ and so, F is a weak hyper BCK-ideal of H, by Theorem 4.10.
Conversely, let F be a weak hyper BCK-ideal of H. Then, $0 \in F$ and for any $x \in H$, we get that $\widetilde{\mathcal{N}}^e_T(0) \geq \widetilde{\mathcal{N}}^e_T(x)$, $\widetilde{\mathcal{N}}^e_I(0) \geq \widetilde{\mathcal{N}}^e_I(x)$ and $\widetilde{\mathcal{N}}^e_F(0) \leq \widetilde{\mathcal{N}}^e_F(x)$, for all $\varepsilon_T, \varepsilon_I \in (0, 1]$ and $\varepsilon_F \in [0, 1)$. Now, let $x, y \in H$. If $x \circ y \subseteq F$ and $y \in F$, then by (1.17), we have $x \in F$ and so,

$$\begin{aligned}
\widetilde{\mathcal{N}}^e_T(x) = \varepsilon_T &\geq \min\left\{\inf_{a\in x\circ y}\widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y)\right\} = \min\{\varepsilon_T, \varepsilon_T\} = \varepsilon_T, \\
\widetilde{\mathcal{N}}^e_I(x) = \varepsilon_I &\geq \min\left\{\inf_{b\in x\circ y}\widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y)\right\} = \min\{\varepsilon_I, \varepsilon_I\} = \varepsilon_I, \\
\widetilde{\mathcal{N}}^e_F(x) = \varepsilon_F &\leq \max\left\{\sup_{c\in x\circ y}\widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y)\right\} = \max\{\varepsilon_F, \varepsilon_F\} = \varepsilon_F.
\end{aligned}$$

Also, in the other cases, for all $x, y \in H$, we have

$$\begin{aligned}
\widetilde{\mathcal{N}}^e_T(x) &\geq \min\left\{\inf_{a\in x\circ y}\widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y)\right\} = 0, \\
\widetilde{\mathcal{N}}^e_I(x) &\geq \min\left\{\inf_{b\in x\circ y}\widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y)\right\} = 0, \\
\widetilde{\mathcal{N}}^e_F(x) &\leq \max\left\{\sup_{c\in x\circ y}\widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y)\right\} = 1.
\end{aligned}$$

Hence, for all $x, y \in H$ and for any $\varepsilon_T, \varepsilon_I \in (0, 1]$ and $\varepsilon_F \in [0, 1)$, the condition (4.4) holds. Therefore, $(\widetilde{\mathcal{N}}, A)$ is a weak neutrosophic soft hyper BCK-ideal of H. □

4.2 Neutrosophic soft hyper p-ideals

Definition 4.15. Let $e \in A$ be a parameter. A neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over H is called

- a *neutrosophic soft hyper p-ideal* of H based on e if it satisfies (4.1) and

$$(\forall x, y, z \in H) \left(\begin{array}{l} \widetilde{\mathcal{N}}^e_T(x) \geq \min\left\{ \inf\limits_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y) \right\} \\ \widetilde{\mathcal{N}}^e_I(x) \geq \min\left\{ \inf\limits_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y) \right\} \\ \widetilde{\mathcal{N}}^e_F(x) \leq \max\left\{ \sup\limits_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y) \right\} \end{array} \right). \tag{4.6}$$

- a *weak neutrosophic soft hyper p-ideal* of H based on e if it satisfies:

$$(\forall x, y, z \in H) \left(\begin{array}{l} \widetilde{\mathcal{N}}^e_T(0) \geq \widetilde{\mathcal{N}}^e_T(x) \geq \min\left\{ \inf\limits_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y) \right\} \\ \widetilde{\mathcal{N}}^e_I(0) \geq \widetilde{\mathcal{N}}^e_I(x) \geq \min\left\{ \inf\limits_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y) \right\} \\ \widetilde{\mathcal{N}}^e_F(0) \leq \widetilde{\mathcal{N}}^e_F(x) \leq \max\left\{ \sup\limits_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y) \right\} \end{array} \right). \tag{4.7}$$

- a *strong neutrosophic soft hyper p-ideal* of H based on e if it satisfies:

$$(\forall x, y, z \in H) \left(\begin{array}{l} \inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_T(a) \geq \widetilde{\mathcal{N}}^e_T(x) \geq \min\left\{ \sup\limits_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_T(b), \widetilde{\mathcal{N}}^e_T(y) \right\} \\ \inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_I(a) \geq \widetilde{\mathcal{N}}^e_I(x) \geq \min\left\{ \sup\limits_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y) \right\} \\ \sup\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_I(a) \leq \widetilde{\mathcal{N}}^e_F(x) \leq \max\left\{ \inf\limits_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y) \right\} \end{array} \right). \tag{4.8}$$

If a neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over H is a (weak, strong) neutrosophic soft hyper p-ideal of H based on all parameters, we say that $(\widetilde{\mathcal{N}}, A)$ is a (weak, strong) neutrosophic soft hyper p-ideal of H.

Example 4.16. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 36. Let $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set in H which is described in Table 37. It is easy to check that $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper p-ideal of H.

Table 36: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 37: Tabular representation of $(\widetilde{\mathcal{N}}, A)$

H	$\widetilde{\mathcal{N}}^e_T(x)$	$\widetilde{\mathcal{N}}^e_I(x)$	$\widetilde{\mathcal{N}}^e_F(x)$
0	0.97	0.85	0.09
a	0.77	0.65	0.43
b	0.61	0.48	0.76

Example 4.17. Consider a hyper BCK-algebra $H = \{0, 1, 2, ...\}$ with the following hyper operation:

$$x \circ y = \begin{cases} \{0, x\} & x \leq y \\ \{x\} & \text{otherwise} \end{cases}$$

Let $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set in H which is described by

$$\begin{aligned} \widetilde{\mathcal{N}}^e_T &: H \to [0,1],\ x \mapsto \frac{1}{2x+1}, \\ \widetilde{\mathcal{N}}^e_I &: H \to [0,1],\ x \mapsto \frac{1}{x+r}, \\ \widetilde{\mathcal{N}}^e_F &: H \to [0,1],\ x \mapsto \frac{-k}{2x+r}, \end{aligned}$$

where $k, r \in \mathbb{N}$. If $x \ll y$, then $0 \in x \circ y$, that is, $x \leq y$. Thus, $\widetilde{\mathcal{N}}^e_T(x) \geq \widetilde{\mathcal{N}}^e_T(y)$, $\widetilde{\mathcal{N}}^e_I(x) \geq \widetilde{\mathcal{N}}^e_I(y)$ and $\widetilde{\mathcal{N}}^e_F(x) \leq \widetilde{\mathcal{N}}^e_F(y)$. Hence, $(\widetilde{\mathcal{N}}, A)$ satisfies (4.1). In order to check that $(\widetilde{\mathcal{N}}, A)$ satisfies (4.6), we consider the following cases:

$$\begin{array}{lll} (1)\ 0 \leq x \leq y \leq z, & (2)\ 0 \leq x \leq z \leq y, & (3)\ 0 \leq y \leq x \leq z, \\ (4)\ 0 \leq y \leq z \leq x, & (5)\ 0 \leq z \leq x \leq y, & (6)\ 0 \leq z \leq y \leq x. \end{array}$$

For the first case, we have $(x \circ z) \circ (y \circ z) = \{0, x, y\}$. Then,

$$\widetilde{\mathcal{N}}^e_T(x) \geq \min\left\{\inf_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y)\right\} = \min\{\widetilde{\mathcal{N}}^e_T(y), \widetilde{\mathcal{N}}^e_T(y)\} = \widetilde{\mathcal{N}}^e_T(y),$$
$$\widetilde{\mathcal{N}}^e_I(x) \geq \min\left\{\inf_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_I(a), \widetilde{\mathcal{N}}^e_I(y)\right\} = \min\{\widetilde{\mathcal{N}}^e_I(y), \widetilde{\mathcal{N}}^e_I(y)\} = \widetilde{\mathcal{N}}^e_I(y),$$
$$\widetilde{\mathcal{N}}^e_F(x) \leq \max\left\{\sup_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_F(a), \widetilde{\mathcal{N}}^e_F(y)\right\} = \max\{\widetilde{\mathcal{N}}^e_F(y, \widetilde{\mathcal{N}}^e_F(y)\} = \widetilde{\mathcal{N}}^e_F(y).$$

Similarly, we can verify that $(\widetilde{\mathcal{N}}, A)$ satisfies (4.6) for other cases. Therefore, $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper p-ideal of H.

Theorem 4.18. *Every (weak, strong) neutrosophic soft hyper p-ideal is a (weak, strong) neutrosophic soft hyper BCK-ideal.*

Proof. Let $e \in A$ be a parameter and $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft hyper p-ideal of H base on e. By taking $z = 0$ in (4.6), for all $x, y, z \in H$, we have

$$\widetilde{\mathcal{N}}^e_T(x) \geq \min\left\{\inf_{a \in (x \circ 0) \circ (y \circ 0)} \widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y)\right\} = \min\left\{\inf_{a \in x \circ y} \widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y)\right\},$$
$$\widetilde{\mathcal{N}}^e_I(x) \geq \min\left\{\inf_{b \in (x \circ 0) \circ (y \circ 0)} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y)\right\} = \min\left\{\inf_{b \in x \circ y} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y)\right\},$$
$$\widetilde{\mathcal{N}}^e_F(x) \leq \max\left\{\sup_{c \in (x \circ 0) \circ (y \circ 0)} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y)\right\} = \max\left\{\sup_{c \in x \circ y} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y)\right\}.$$

Therefore, $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper BCK-ideal base on a parameter e. The proof of other cases is similar. □

The converse of Theorem 4.18 is not true as seen in the following example.

Example 4.19. In Example 4.2, it is easy to check that $(\widetilde{\mathcal{N}}, A)$ is a weak neutrosophic soft hyper BCK-ideal of H. But it is not a weak neutrosophic soft hyper p-ideal of H. Because if we take $x = b$, $y = a$ and $z = b$, then

$$\widetilde{\mathcal{N}}^e_T(b) \leq \min\left\{\inf_{a_0 \in (b \circ b) \circ (a \circ b)} \widetilde{\mathcal{N}}^e_T(a_0), \widetilde{\mathcal{N}}^e_T(a)\right\} = \min\left\{\widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(a)\right\} = \widetilde{\mathcal{N}}^e_T(a)$$
$$\widetilde{\mathcal{N}}^e_I(b) \leq \min\left\{\inf_{b_0 \in (b \circ b) \circ (a \circ b)} \widetilde{\mathcal{N}}^e_I(b_0), \widetilde{\mathcal{N}}^e_I(a)\right\} = \min\left\{\widetilde{\mathcal{N}}^e_I(a), \widetilde{\mathcal{N}}^e_I(a)\right\} = \widetilde{\mathcal{N}}^e_I(a)$$
$$\widetilde{\mathcal{N}}^e_F(b) \leq \max\left\{\sup_{c_0 \in (b \circ b) \circ (a \circ b)} \widetilde{\mathcal{N}}^e_F(c_0), \widetilde{\mathcal{N}}^e_F(a)\right\} = \max\left\{\widetilde{\mathcal{N}}^e_F(0), \widetilde{\mathcal{N}}^e_F(a)\right\} = \widetilde{\mathcal{N}}^e_F(0)$$

Theorem 4.20. *Every strong neutrosophic soft hyper p-ideal is a neutrosophic soft hyper p-ideal, and every neutrosophic soft hyper p-ideal is a weak neutrosophic soft hyper p-ideal.*

Proof. Let $(\widetilde{\mathcal{N}}, A)$ be a strong neutrosophic soft hyper p-ideal of H based on e. By Theorem 4.18 and Proposition 4.8, imply that the condition (4.1) is valid. Also, for all $x, y, z \in H$, we have

$$\begin{aligned}
\widetilde{\mathcal{N}}^e_T(x) &\geq \min\left\{\sup_{b\in(x\circ z)\circ(y\circ z)} \widetilde{\mathcal{N}}^e_T(b), \widetilde{\mathcal{N}}^e_T(y)\right\} \geq \min\left\{\inf_{b\in(x\circ z)\circ(y\circ z)} \widetilde{\mathcal{N}}^e_T(b), \widetilde{\mathcal{N}}^e_T(y)\right\},\\
\widetilde{\mathcal{N}}^e_I(x) &\geq \min\left\{\sup_{b\in(x\circ z)\circ(y\circ z)} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y)\right\} \geq \min\left\{\inf_{b\in(x\circ z)\circ(y\circ z)} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y)\right\},\\
\widetilde{\mathcal{N}}^e_F(x) &\leq \max\left\{\inf_{c\in(x\circ z)\circ(y\circ z)} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y)\right\} \leq \max\left\{\sup_{c\in(x\circ z)\circ(y\circ z)} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y)\right\}.
\end{aligned}$$

Therefore, $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper p-ideal of H based on e. Now, let $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft hyper p-ideal of H based on e. In every hyper BCK-algebra H, for all $x \in H$ we have $0 \ll x$. Then, by combining (4.1) and (4.6) we can conclude that the condition (4.7) holds. Therefore, $(\widetilde{\mathcal{N}}, A)$ is a weak neutrosophic soft hyper p-ideal of H based on e. □

In the following example, we show that the converse of Theorem 4.20 may not be true, in general.

Example 4.21. In Example 4.17, $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper p-ideal of H. But it is not a strong neutrosophic soft hyper p-ideal of H. Because if we take $x = b$, $y = a$ and $z = b$, then

$$\begin{aligned}
\widetilde{\mathcal{N}}^e_T(b) &\leq \min\left\{\sup_{a_0\in(b\circ b)\circ(a\circ b)} \widetilde{\mathcal{N}}^e_T(a_0), \widetilde{\mathcal{N}}^e_T(a)\right\} = \min\left\{\widetilde{\mathcal{N}}^e_T(0), \widetilde{\mathcal{N}}^e_T(a)\right\} = \widetilde{\mathcal{N}}^e_T(a),\\
\widetilde{\mathcal{N}}^e_I(b) &\leq \min\left\{\sup_{b_0\in(b\circ b)\circ(a\circ b)} \widetilde{\mathcal{N}}^e_I(b_0), \widetilde{\mathcal{N}}^e_I(a)\right\} = \min\left\{\widetilde{\mathcal{N}}^e_I(0), \widetilde{\mathcal{N}}^e_I(a)\right\} = \widetilde{\mathcal{N}}^e_I(a),\\
\widetilde{\mathcal{N}}^e_F(b) &\leq \max\left\{\inf_{c_0\in(b\circ b)\circ(a\circ b)} \widetilde{\mathcal{N}}^e_F(c_0), \widetilde{\mathcal{N}}^e_F(a)\right\} = \max\left\{\widetilde{\mathcal{N}}^e_F(b), \widetilde{\mathcal{N}}^e_F(a)\right\} = \widetilde{\mathcal{N}}^e_F(a).
\end{aligned}$$

Lemma 4.22. *Every (weak, strong) hyper p-ideal of H is a (weak, strong) hyper BCK-ideal of H.*

Proof. Let I be a hyper p-ideal of H. Taking $z = 0$ in (1.31). Then,

$$(x \circ 0) \circ (y \circ 0) = x \circ y \ll I,\ y \in I \ \Rightarrow\ x \in I,$$

for all $x, y \in H$. Therefore, I is a hyper BCK-ideal of H. □

Theorem 4.23. *Let H be a hyper BCK-algebra, $e \in A$ be a parameter and $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set over H. Then, $(\widetilde{\mathcal{N}}, A)$ is a (weak) neutrosophic soft hyper p-ideal of H based on e if and only if the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are (weak) hyper p-ideal of H for all parameter e-numbers.*

Proof. Let $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft hyper p-ideal of H based on e and $e_T, e_I, e_F \in [0,1]$ such that $U(\widetilde{\mathcal{N}}_T^e; e_T)$, $U(\widetilde{\mathcal{N}}_I^e; e_I)$ and $L(\widetilde{\mathcal{N}}_F^e; e_F)$ are nonempty. Obviously, $0 \in U(\widetilde{\mathcal{N}}_T^e; e_T) \cap U(\widetilde{\mathcal{N}}_I^e; e_I) \cap L(\widetilde{\mathcal{N}}_F^e; e_F)$. Let $x, y, z \in H$ such that $(x \circ z) \circ (y \circ z) \ll U(\widetilde{\mathcal{N}}_T^e; e_T)$ and $y \in U(\widetilde{\mathcal{N}}_T^e; e_T)$. Then, for any $a \in (x \circ z) \circ (y \circ z)$ there exists $a_0 \in U(\widetilde{\mathcal{N}}_T^e; e_T)$ such that $a \ll a_0$ and $\widetilde{\mathcal{N}}_T^e(y) \geq e_T$. Now, by Theorem 4.18, we conclude that $\widetilde{\mathcal{N}}_T^e(a) \geq \widetilde{\mathcal{N}}_T^e(a_0) \geq e_T$ for all $a \in (x \circ z) \circ (y \circ z)$. Hence,

$$\widetilde{\mathcal{N}}_T^e(x) \geq \min\left\{ \inf_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_T^e(a), \widetilde{\mathcal{N}}_T^e(y) \right\} \geq e_T.$$

So, $x \in U(\widetilde{\mathcal{N}}_T^e; e_T)$. By the similar way, we can prove that if $(x \circ z) \circ (y \circ z) \ll U(\widetilde{\mathcal{N}}_I^e; e_I)$ and $y \in U(\widetilde{\mathcal{N}}_I^e; e_I)$, then $x \in U(\widetilde{\mathcal{N}}_I^e; e_I)$. Thus, $U(\widetilde{\mathcal{N}}_T^e; e_T)$ and $U(\widetilde{\mathcal{N}}_I^e; e_I)$ based on e are hyper p-ideal of H for all parameter e-numbers. Let $x, y, z \in H$ such that $(x \circ z) \circ (y \circ z) \ll L(\widetilde{\mathcal{N}}_F^e; e_F)$ and $y \in L(\widetilde{\mathcal{N}}_F^e; e_F)$. Then, $\widetilde{\mathcal{N}}_F^e(y) \leq e_F$. Suppose $b \in (x \circ z) \circ (y \circ z)$. Then, there exists $b_0 \in L(\widetilde{\mathcal{N}}_F^e; e_F)$ such that $b \ll b_0$, which implies from (4.1) that $\widetilde{\mathcal{N}}_F^e(b) \leq \widetilde{\mathcal{N}}_F^e(b_0) \leq e_F$. Thus,

$$\widetilde{\mathcal{N}}_F^e(x) \leq \max\left\{ \sup_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_F^e(b), \widetilde{\mathcal{N}}_F^e(y) \right\} \leq e_F.$$

Hence, $x \in L(\widetilde{\mathcal{N}}_F^e; e_F)$. Therefore, $L(\widetilde{\mathcal{N}}_F^e; e_F)$ based on e is a hyper p-ideal of H for all parameter e-numbers.

Conversely, suppose that the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are hyper p-ideal of H for all parameter e-numbers. Let $x, y \in H$ be such that $x \ll y$. Then,

$$y \in U(\widetilde{\mathcal{N}}_T^e; \widetilde{\mathcal{N}}_T^e(y)) \cap U(\widetilde{\mathcal{N}}_I^e; \widetilde{\mathcal{N}}_I^e(y)) \cap L(\widetilde{\mathcal{N}}_F^e; \widetilde{\mathcal{N}}_F^e(y)),$$

and Thus, $\{x\} \ll U(\widetilde{\mathcal{N}}_T^e; \widetilde{\mathcal{N}}_T^e(y))$, $\{x\} \ll U(\widetilde{\mathcal{N}}_I^e; \widetilde{\mathcal{N}}_I^e(y))$ and $\{x\} \ll L(\widetilde{\mathcal{N}}_F^e; \widetilde{\mathcal{N}}_F^e(y))$. By Lemmas 4.22 and 1.20, we have $x \in U(\widetilde{\mathcal{N}}_T^e; \widetilde{\mathcal{N}}_T^e(y))$, $x \in U(\widetilde{\mathcal{N}}_I^e; \widetilde{\mathcal{N}}_I^e(y))$ and $x \in L(\widetilde{\mathcal{N}}_F^e; \widetilde{\mathcal{N}}_F^e(y))$. Hence, $\widetilde{\mathcal{N}}_T^e(x) \geq \widetilde{\mathcal{N}}_T^e(y)$, $\widetilde{\mathcal{N}}_I^e(x) \geq \widetilde{\mathcal{N}}_I^e(y)$ and $\widetilde{\mathcal{N}}_F^e(x) \leq \widetilde{\mathcal{N}}_F^e(y)$. Now, for any $x, y, z \in H$, let

$$e_T := \min\left\{ \inf_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_T^e(a), \widetilde{\mathcal{N}}_T^e(y) \right\},$$

$$e_I := \min\left\{ \inf_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_I^e(b), \widetilde{\mathcal{N}}_I^e(y) \right\}$$

and

$$e_F := \max\left\{ \sup_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_F^e(c), \widetilde{\mathcal{N}}_F^e(y) \right\}.$$

Then,

$$y \in U(\widetilde{\mathcal{N}}_T^e; e_T) \cap U(\widetilde{\mathcal{N}}_I^e; e_I) \cap L(\widetilde{\mathcal{N}}_F^e; e_F),$$

and for any $a, b, c \in (x \circ z) \circ (y \circ z)$ we have

$$\widetilde{\mathcal{N}}_T^e(a) \geq \inf_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_T^e(a) \geq \min\left\{ \inf_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_T^e(a), \widetilde{\mathcal{N}}_T^e(y) \right\} = e_T,$$

$$\widetilde{\mathcal{N}}_I^e(b) \geq \inf_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_I^e(b) \geq \min\left\{ \inf_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_I^e(b), \widetilde{\mathcal{N}}_I^e(y) \right\} = e_I$$

and

$$\widetilde{\mathcal{N}}_F^e(c) \leq \sup_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_F^e(c) \leq \max\left\{ \sup_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_F^e(c), \widetilde{\mathcal{N}}_F^e(y) \right\} = e_F.$$

Hence, $a \in U(\widetilde{\mathcal{N}}_T^e; e_T)$, $b \in U(\widetilde{\mathcal{N}}_I^e; e_I)$ and $c \in L(\widetilde{\mathcal{N}}_F^e; e_F)$, and so $(x \circ z) \circ (y \circ z) \subseteq U(\widetilde{\mathcal{N}}_T^e; e_T) \cap U(\widetilde{\mathcal{N}}_I^e; e_I) \cap L(\widetilde{\mathcal{N}}_F^e; e_F)$. Then, (1.12) implies that $(x \circ z) \circ (y \circ z) \ll U(\widetilde{\mathcal{N}}_T^e; e_T) \cap U(\widetilde{\mathcal{N}}_I^e; e_I) \cap L(\widetilde{\mathcal{N}}_F^e; e_F)$. It follows from (1.31) that $x \in U(\widetilde{\mathcal{N}}_T^e; e_T) \cap U(\widetilde{\mathcal{N}}_I^e; e_I) \cap L(\widetilde{\mathcal{N}}_F^e; e_F)$. Thus,

$$\widetilde{\mathcal{N}}_T^e(x) \geq e_T = \min\left\{ \inf_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_T^e(a), \widetilde{\mathcal{N}}_T^e(y) \right\},$$

$$\widetilde{\mathcal{N}}_I^e(x) \geq e_I = \min\left\{ \inf_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_I^e(b), \widetilde{\mathcal{N}}_I^e(y) \right\}$$

and

$$\widetilde{\mathcal{N}}_F^e(x) \leq e_F = \max\left\{ \sup_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}_F^e(c), \widetilde{\mathcal{N}}_F^e(y) \right\}.$$

Therefore, $(\widetilde{\mathcal{N}}, A)$ is a neutrosophic soft hyper BCK-ideal of H based on e. □

Theorem 4.24. *Let H be a finite hyper BCK-algebra, $e \in A$ be a parameter and $(\widetilde{\mathcal{N}}, A)$ be a neutrosophic soft set over H. Then, $(\widetilde{\mathcal{N}}, A)$ is a strong neutrosophic soft hyper p-ideal of H based on e if and only if the non-empty neutrosophic soft level sets based on e of $(\widetilde{\mathcal{N}}, A)$ are strong hyper p-ideal of H for all parameter e-numbers.*

Proof. The proof is similar to the proof of Theorem 4.12 with some modification. □

Theorem 4.25. *Let H be a hyper BCK-algebra and consider the neutrosophic soft set $(\widetilde{\mathcal{N}}, A)$ over H in Theorem 4.14. Then, $(\widetilde{\mathcal{N}}, A)$ is a (weak, strong) neutrosophic hyper p-ideal of H if and only if F is a (weak, strong) hyper p-ideal of H.*

Proof. Let $(\widetilde{\mathcal{N}}, A)$ be a strong neutrosophic soft hyper p-ideal of H. Then, for any $x \in H$ and for all $\varepsilon_T, \varepsilon_I \in (0,1]$ and $\varepsilon_F \in [0,1)$, we get that $U(\widetilde{\mathcal{N}}^e_T; \varepsilon_T) = U(\widetilde{\mathcal{N}}^e_I; \varepsilon_I) = L(\widetilde{\mathcal{N}}^e_F; \varepsilon_F) = F$ and so, by Theorem 4.24, F is a strong hyper p-ideal of H.

Conversely, let F be a strong hyper p-ideal of H and $x \in F$. By (a1), $x \circ x \ll \{x\}$ and so $x \circ x \ll F$. Then, by Lemmas 4.22 and 1.20, $x \circ x \subseteq F$. Thus, for all $a \in x \circ x$, $a \in F$, and so, $\inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_T(a) = \widetilde{\mathcal{N}}^e_T(x) = \varepsilon_T$, $\inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_I(b) = \widetilde{\mathcal{N}}^e_I(x) = \varepsilon_I$ and $\sup\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_F(c) = \widetilde{\mathcal{N}}^e_F(x) = \varepsilon_F$. Also, if $x \notin F$, then for all $a \in x \circ x$, we have $\inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_T(a) \geq \widetilde{\mathcal{N}}^e_T(x) = 0$, $\inf\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_I(a) \geq \widetilde{\mathcal{N}}^e_I(x) = 0$ and $\sup\limits_{a \in x \circ x} \widetilde{\mathcal{N}}^e_F(a) \leq \widetilde{\mathcal{N}}^e_F(x) = 1$. Now, for any $x, y, z \in H$, we consider the following cases:

If $(x \circ z) \circ (y \circ z) \cap F \neq \emptyset$ and $y \in F$, then by (1.32), we get that $x \in F$ and so

$$\begin{aligned}
\widetilde{\mathcal{N}}^e_T(x) = \varepsilon_T &\geq \min\left\{\sup_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y)\right\} = \min\{\varepsilon_T, \varepsilon_T\} = \varepsilon_T, \\
\widetilde{\mathcal{N}}^e_I(x) = \varepsilon_I &\geq \min\left\{\sup_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y)\right\} = \min\{\varepsilon_I, \varepsilon_I\} = \varepsilon_I, \\
\widetilde{\mathcal{N}}^e_F(x) = \varepsilon_F &\leq \max\left\{\inf_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y)\right\} = \max\{\varepsilon_F, \varepsilon_F\} = \varepsilon_F.
\end{aligned}$$

In the other cases, for all $x, y \in H$, we get that

$$\begin{aligned}
\widetilde{\mathcal{N}}^e_T(x) &\geq \min\left\{\sup_{a \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_T(a), \widetilde{\mathcal{N}}^e_T(y)\right\} = \min\{0, 0\} = 0, \\
\widetilde{\mathcal{N}}^e_I(x) &\geq \min\left\{\sup_{b \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_I(b), \widetilde{\mathcal{N}}^e_I(y)\right\} = \min\{0, 0\} = 0, \\
\widetilde{\mathcal{N}}^e_F(x) &\leq \max\left\{\inf_{c \in (x \circ z) \circ (y \circ z)} \widetilde{\mathcal{N}}^e_F(c), \widetilde{\mathcal{N}}^e_F(y)\right\} = \max\{1, 1\} = 1.
\end{aligned}$$

Therefore, $(\widetilde{\mathcal{N}}, A)$ is a strong neutrosophic hyper p-ideal of H. $\square$

Chapter 5.

Conclusion

In the paper [50], Maji introduced the concept of fuzzy soft sets and presented some definitions, oprations and properties of this concept. In Chapter 2, we have applied the notion of fuzzy soft sets to the theory of hyper BCK-algebras. We have introduced the notion of fuzzy soft positive imlicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$, and have investigated several properties. We have discussed the relation between fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$ and fuzzy soft hyper BCK-ideal, and have provided characterizations of fuzzy soft positive implicative hyper BCK-ideal of type $(\subseteq, \subseteq, \subseteq)$. We have established a fuzzy soft weak (strong) hyper BCK-ideal by using the notion of positive implicative hyper BCK-ideal of type $(\ll, \subseteq, \subseteq)$.

Also, we have introduced the notions of fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, and have investigated their relations. We have discussed the relations among fuzzy soft strong hyper BCK-ideal and fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$ and $(\ll, \ll, \subseteq)$. We have proved that the level set of fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$ are positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, respectively. We have given conditions for a fuzzy soft set to be a fuzzy soft positive implicative hyper BCK-ideal of types $(\ll, \subseteq, \subseteq)$, $(\ll, \ll, \subseteq)$ and $(\subseteq, \ll, \subseteq)$, respectively, and have provided conditions for a fuzzy soft set to be a fuzzy soft weak hyper BCK-ideal.

Additionally, in Chapter 3, we have introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal and reflexive neutrosophic hyper BCK-ideal. We have considered their relations and related properties. We have discussed characterizations of neutrosophic (weak) hyper BCK-ideal, and have given conditions for a neutrosophic set to be a (reflexive) neutrosophic hyper BCK-ideal and a neutrosophic strong hyper BCK-ideal. We have provided conditions for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal, and have provided conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal.

Moreover, we have introduced the notions of neutrosophic commutative hyper BCK-ideal of types $(\subseteq, \subseteq)$, $(\subseteq, \ll)$, $(\ll, \subseteq)$ and $(\ll, \ll)$ and have indicated some relevant properties and their relations. We have discussed relations among commutative neutrosophic hyper BCK-ideal of types $(\subseteq, \subseteq)$, $(\ll, \subseteq)$, neutrosophic weak hyper BCK-ideal and neutrosophic strong hyper BCK-ideal. We have provided a condition for a neutrosophic weak hyper BCK-ideal to be a commutative neutrosophic hyper BCK-ideal of type $(\subseteq, \subseteq)$ and a condition for a commutative neutrosophic hyper BCK-ideal of type $(\ll, \subseteq)$ to be a neutrosophic s-weak hyper BCK-ideal. We have considered characterization of a commutative neutrosophic hyper BCK-ideal of types $(\subseteq, \subseteq)$, $(\subseteq, \ll)$, $(\ll, \subseteq)$ and $(\ll, \ll)$. Fainally, we have discussed relations among commutative neutrosophic hyper BCK-ideal of types $(\subseteq, \subseteq)$, $(\subseteq, \ll)$, $(\ll, \subseteq)$ and $(\ll, \ll)$ and a spesial subset of H.

In the paper [49], Maji introduced the concept of neutrosophic soft set and presented some definitions, oprations and properties of this concept. The aim of Chapter 4, was to apply neutrosophic soft set for dealing with several kinds of theories in hyper BCK-algebras. We have introduced the notions of neutrosophic soft hyper BCK-ideal, neutrosophic soft weak hyper BCK-ideal and neutrosophic soft strong hyper BCK-ideal and have indicated some relevant properties and their relations. Also, we have introduced the notion of (strong, weak) neutrosophic soft hyper p-ideal and have investigated their relations and relations among (strong, weak) neutrosophic soft hyper BCK-ideal and (strong, weak) neutrosophic soft hyper p-ideal. We have considered characterizations of neutrosophic soft hyper BCK-ideal and neutrosophic soft hyper p-ideal.

The papers derived from the thesis

The papers which are used in thesis, respectively:

(1) S. Khademan, M. M. Zahedi and A. Iranmanesh, Commutative neutrosophic hyper BCK-ideals, (submitted).

(2) S. Khademan, M. M. Zahedi, R. A. Borzooei and Y. B. Jun, Fuzzy soft positive implicative hyper BCK-ideals of several types, (submitted).

(3) S. Khademan, M. M. Zahedi, R. A. Borzooei and Y. B. Jun, Neutrosophic hyper BCK-ideals, *Neutrosophic Sets and Systems*, **27** (2019), 201–217.

(4) S. Khademan, M. M. Zahedi, Y. B. Jun and R. A. Borzooei, Fuzzy soft positive implicative hyper BCK-ideals in hyper BCK-algebras, *Journal of Intelligent and Fuzzy Systems*, **36** (2019), 2605–2613.

(5) S. Khademan, M. M. Zahedi, Y. B. Jun and A. Iranmanesh, Neutrosophic soft hyper BCK-ideals, (submitted).

The conference which I had presentation related to thesis:

(1) S. Khademan, M. M. Zahedi and Y. B. Jun, Result on fuzzy soft hyper BCK-ideals, 7th Iranian Joint Congress on Fuzzy and Intelligent Systems, Bojnord (2019), 167–171.

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