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Probabilized logics related to DSmT and Bayes inference

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Abstract: *This work proposes a logical interpretation of the non hybrid Dezert Smarandache Theory (DSmT). As probability is deeply related to a classical semantic, it appears that DSmT relies on an alternative semantic of decision. This semantic is characterized as a probabilized multi-modal logic. It is noteworthy that this interpretation justifies clearly some hypotheses usually made about the fusion rule (*ie. *the independence between the sensors). At last, a conclusion arises: there could be many possible fusion rules, depending on the chosen semantic of decision; and the choice of a semantic depends on how the actual problem is managed. Illustrating this fact, a logical interpretation of the Bayesian inference is proposed as a conclusion to this chapter.*

8.1 Introduction

When a non deterministic problem appears to be too badly shaped, it becomes difficult to make a coherent use of the probabilistic models. A particular difficulty, often neglected, comes from the interpretation of the raw data. The raw data could have a good probabilistic modelling, but in general such informations are useless: an interpretation is necessary. Determining the model of interpretation, and its probabilistic law, is the true issue. Due to the *forgotten/unknown case* syndrome, it is possible that such model cannot be entirely constructed. In some cases, only a rather weak approximation of the model is possible. Such approximated model of interpretation may produce paradoxical results. This is particularly true in information fusion problems.

Several new theories have been proposed for managing these difficulties. Dempster Shafer Theory of evidence [1, 5] is one of them. In this paper, we are interested in the Dezert Smarandache Theory (DSmT) [3], a closely related theory. These theories, and particularly the DSmT, are able to manipulate the model contradictions. But a difficulty remains: it seems uneasy to link these various theories. In particular, their relation with the theory of probability seems unclear. Such a relation is perhaps not possible, as could claim some authors, but it is necessary: it is sometimes needed to combine methods and algorithms based on different theories. This paper intends to establish such relations. A probabilized multi-modal logic is constructed. This probabilized logic, intended for the information fusion, induces the same conjunctive fusion operator as DSmT (*ie.* operator $\oplus$). By the way, the necessity of independent sources for applying the operator $\oplus$ is clarified and confirmed. Moreover, this logical interpretation induces a possible semantic of the DSmT, and somehow enlightens the intuitions behind this theory. Near the end, the paper keeps going by giving a similar interpretation of the Bayes inference. Although the Bayes inference is not related to the DSmT, this last result suggests that probabilized logics could be a possible common frame for several non deterministic theories.

Section 8.2 is beginning by a general discussion about probability. It is shown that probabilistic modellings are sometimes questionable. Following this preliminary discussion, two versions of the theory of evidence are introduced: the historical Dempster Shafer Theory and the Transferable Belief Model of Smets [8]. Section 8.3 makes a concise presentation of the Dezert Smarandache Theory. The short section 8.4 establishes some definitions about probability (and partial probability) over a set of logical propositions. These general definitions are needed in the following sections. Section 8.5 gives a logical interpretation of the DSmT on a small example. This section does not enter the theory too deeply: the modal logic associated to this interpretation is described with *practical words*, not with formulae! Section 8.6 generalizes the results to any cases. This section is much more theoretic. The modal logic is defined mathematically. Section 8.7 proposes a similar logical interpretation of the Bayesian inference. Section 8.8 concludes.

8.2 Belief Theory Models

8.2.1 Preliminary: about probability

This subsection argues about the difficulty to modelize "everything" with probability. Given a measurable universe of abstract events (or propositions) $\Omega = \{\omega_i, i \in I\}$, a probability P could be defined as a bounded and normalized measure over Ω. In this paper, we are interested in finite models (I is finite).

A probability P could also be defined from the probabilities $\rho(\omega)$ of the elementary events $\omega \in \Omega$. The density of probability ρ should verify (finite case):

$$\rho : \Omega \mapsto \mathbb{R}^+,$$

and:

$$\sum_{\omega \in \Omega} \rho(\omega) = 1 \,.$$

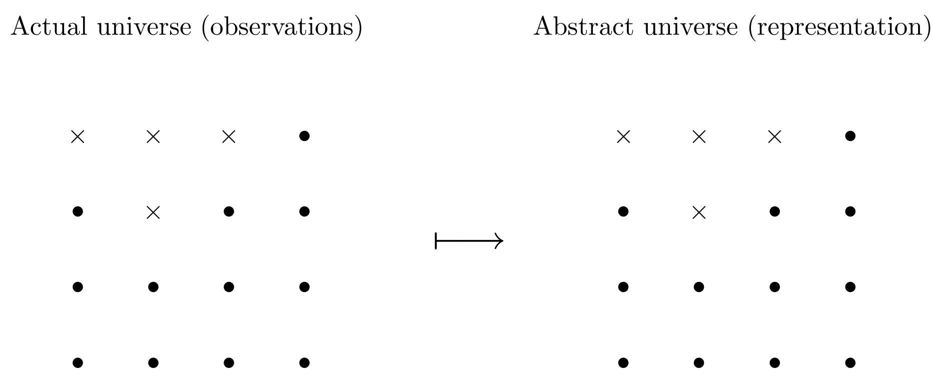
The probability P is recovered by means of the additivity property:

$$\forall A \subset \Omega,\ P(A) = \sum_{\omega \in A} \rho(\omega).$$

It is important to remember how such abstract definitions are related to a concrete notion of "chance" in the actual universe. Behind the formalism, behind the abstract events, there are *actual events*. The formalism introduced by the abstract universe Ω is just a modelling of the actual universe. Such a model is expected to be more suitable to mathematical manipulations and reasoning. But there is no reason that these actual events are compatible with the abstract events. Probability theory assumes this compatibility. More precisely, probability assumes that either the abstract and actual events are the same, either *there is a mapping from the actual events to the abstract events* (figure 8.1). When this mapping hypothesis is made, the density function makes sense then, in regard to the observation. Indeed, a practical construction of ρ becomes possible with a frequentist taste:

1. Set $\rho(\omega) = 0$ for all $\omega \in \Omega$,
2. Make N tossing of an actual event. For each tossed event, a, do:
   (a) Select the $\omega \in \Omega$ such that a maps to ω,
   (b) Set $\rho(\omega) = \rho(\omega) + 1$,
3. Set $\rho(\omega) \simeq \frac{1}{N}\rho(\omega)$ for all $\omega \in \Omega$.

The next paragraph explains why the mapping from the actual events to the abstract events is not always possible and how to overcome this difficulty.



An abstract event is a connected component; in this example,
the ×-ed observations map to the unique ×-ed component

Figure 8.1: Event mapping: probabilist case

8.2.1.1 The impossible quest of the perfect universe

It is always possible to assume that there is a perfect universe, where all problems could be modeled, but we are not able to construct it or to manipulate it practically. However, we are able to think with it. Let $\mathcal{A}$ be the actual universe, let Ω be the abstract universe, and let $\mathcal{Z}$ be this perfect universe.

The structure of Ω is well known; it describes our modelling of the actual world. *This is how we interpret the observations.* Practically, such interpretation is almost always necessary, while the raw observation may be useless. But Ω is only an hypothesis: our knowledge about the observation is generally insufficient for a *true interpretation.*

The universe $\mathcal{A}$ is observed, but like $\mathcal{Z}$ its structure is not really known: although an observation is possible, it is not necessary possible to know the meaning, the *true interpretation*, of this observation. For example, what is the meaning of an observation for a situation never seen before?

The universe $\mathcal{Z}$ is perfect, which means that it *contains* the two other, and is unknown. The word *contains* has a logical signification here, *ie.* the events/propositions of $\mathcal{A}$ or Ω are macro-events/macro-propositions of $\mathcal{Z}$ (figure 8.2):

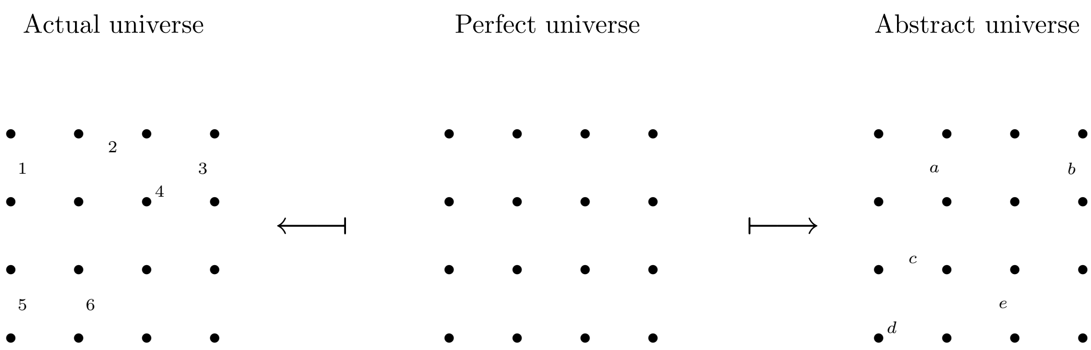
$$\mathcal{A} \subset \mathcal{P}(\mathcal{Z}) \quad \text{and} \quad \Omega \subset \mathcal{P}(\mathcal{Z}) \ ,$$

with the following exhaustiveness (**x**) and coherence (**c**) hypotheses for $\mathcal{A}$ and Ω:

x. $\mathcal{Z} = \bigcup_{a \in \mathcal{A}} a = \bigcup_{\omega \in \Omega} \omega \,,$

c1. $[a_1, a_2 \in \mathcal{A}\,,\ a_1 \neq a_2] \Rightarrow a_1 \cap a_2 = \emptyset\,,$

c2. $[\omega_1, \omega_2 \in \Omega\,,\ \omega_1 \neq \omega_2] \Rightarrow \omega_1 \cap \omega_2 = \emptyset\,.$



An abstract event (*ie.* a, b, c, d, e) is a $-$ connected component

An actual event (*ie.* $1, 2, 3, 4, 5, 6$) is a $=$ connected component

Figure 8.2: Event mapping: general case

The exhaustiveness and coherence hypotheses are questionable; it will be seen that these hypotheses induce contradictions when fusing informations.

Of course, the abstract universe Ω is a coherent interpretation of the observations, when any actual event $a \in \mathcal{A}$ is a subevent of an abstract event $\omega \in \Omega$. But since the interpretation of $\mathcal{A}$ is necessarily partial and subjective, this property does not hold in general. The figure 8.2 gives an example of erroneous interpretation of the observations: the actual event 5 intersects both the abstract event d and the abstract event c. More precisely, if an actual event $a \in \mathcal{A}$ happens, there is a perfect event $z \in a$ which has happened. Since $\mathcal{Z}$ contains (*ie.* maps to) Ω, there is an unique abstract event, $\omega \in \Omega$, which checks z, *ie.* $z \in \omega$. As a conclusion, when a given actual event a happens, any abstract event $\omega \in \Omega$ such that $\omega \cap a \neq \emptyset$ is likely to happen. Practically, such situation is easy to decide, since it just happens when a doubt appears in a measure classification. The table 8.1, refering to the example of figure 8.2, gives the possible abstract events related to each tossed observation.

Finally, it does not seem possible to define a density of probability for unique abstract events from partially decidable observations. But it is possible to define a density function for multiple events. Again, a construction of such function, still denoted ρ, is possible in a frequentist manner:

1. Set $\rho(\phi) = 0$ for all $\phi \subset \Omega$,
2. Make N tossing of an actual event. For each tossed event, a, do:
   (a) Define the set $\phi(a) = \{\omega \in \Omega / \omega \cap a \neq \emptyset\}$,
   (b) Set $\rho(\phi(a)) = \rho(\phi(a)) + 1$,
3. Set $\rho(\phi) \simeq \frac{1}{N}\rho(\phi)$ for all $\phi \subset \Omega$.

Tossed observation Possible abstract events

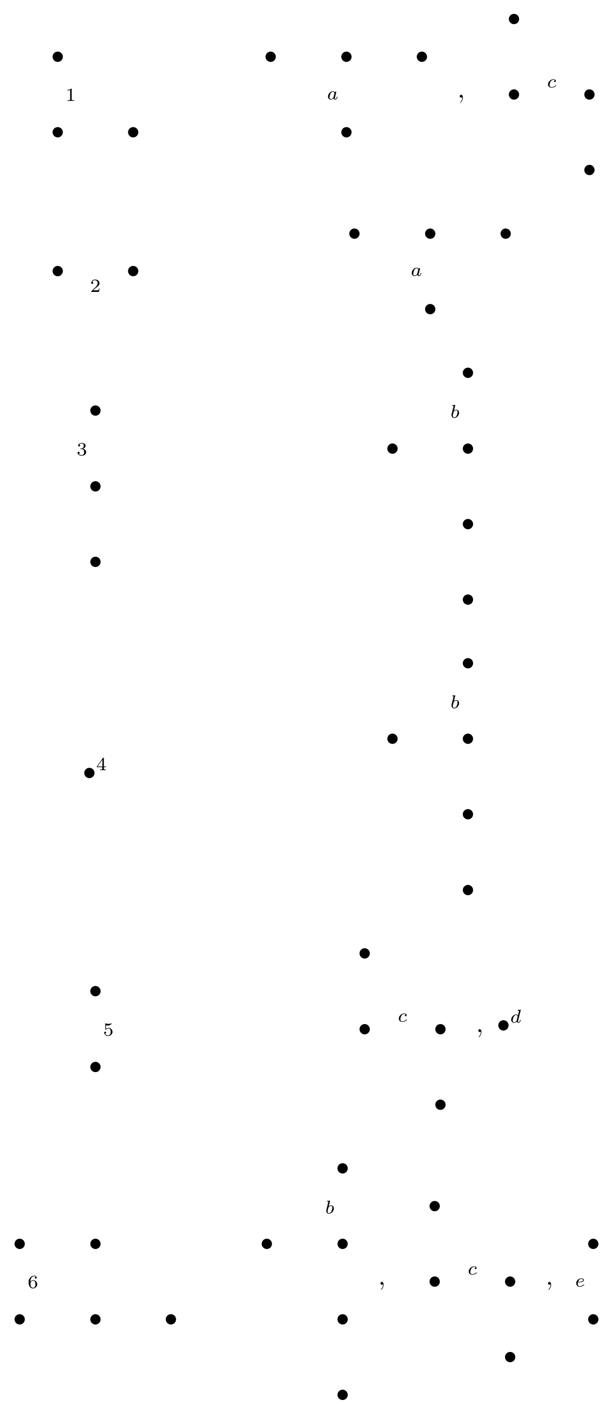


Table 8.1: Event *multi-mapping* for figure 8.2

In particular, $\rho(\emptyset) = 0$.

In the particular case of table 8.1, this density is related to the probability of observation by:

$$\rho\{a,c\} = p(1)\,,\ \rho\{a\} = p(2)\,,\ \rho\{b\} = p(3) + p(4)\,,\ \rho\{c,d\} = p(5)\,,\ \rho\{b,c,e\} = p(6)\,.$$

The previous discussion has shown that the definition of a density of probability for the abstract events does not make sense, when the interpretations of the observations are approximative. However, it is possible to construct a density for multiple abstract events. Such a density looks quite similarly to the *Basic Belief Assignment* of DST, defined in the next section.

8.2.2 Dempster Shafer Theory

8.2.2.1 Definition

A Dempster Shafer model [1, 2, 5] is characterized by a pair (Ω, m), where Ω is a set of abstract events and the basic belief assignment (bba) m is a non negatively valued function defined over $\mathcal{P}(\Omega)$, the set of subsets of Ω, such that:

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{\phi \subset \Omega} m(\phi) = 1 \,.$$

A DSm (Ω, m) could be seen as a non deterministic interpretation of the actuality. Typically, it is a tool providing informations from a sensor.

8.2.2.2 Belief of a proposition

Let $\phi \subset \Omega$ be a proposition. Assume a basic belief assignment m. The *degree of belief of* ϕ, $\mathrm{Bel}(\phi)$, and the *plausibility* of ϕ, $\mathrm{Pl}(\phi)$, are defined by:

$$\mathrm{Bel}(\phi) = \sum_{\psi \subset \phi} m(\psi) \quad \text{and} \quad \mathrm{Pl}(\phi) = \sum_{\psi \cap \phi \neq \emptyset} m(\psi) \,.$$

Bel and Pl do not satisfy the additivity property of probability. $\mathrm{Bel}(\phi)$ and $\mathrm{Pl}(\phi)$ are the lower and upper measures of the "credibility" of the proposition ϕ. These measures are sometimes considered as the lower bound and the upper bound of the probability of ϕ:

$$\mathrm{Bel}(\phi) \leq P(\phi) \leq \mathrm{Pl}(\phi) \,.$$

This interpretation is dangerous, since it is generally admitted that probability and DST are quite different theories.

8.2.2.3 Fusion rule

Assume two bba m_1 and m_2, defined on the same universe Ω, obtained from two different sources. It is generally admitted that the sources are independent. Then, the bba $m_1 \oplus m_2$ is defined by:

$$\begin{cases} m_1 \oplus m_2(\emptyset) = 0 \,, \\ m_1 \oplus m_2(\phi) = \dfrac{1}{Z} \displaystyle\sum_{\psi_1 \cap \psi_2 = \phi} m_1(\psi_1) m_2(\psi_2) \,, \quad \text{where } Z = 1 - \displaystyle\sum_{\psi_1 \cap \psi_2 = \emptyset} m_1(\psi_1) m_2(\psi_2) \,. \end{cases}$$

The operator $\oplus$ describes the (conjunctive) information fusion between two bba.

The normalizer Z is needed since the bba is zeroed for the empty set $\emptyset$. Except some specific cases, it is indeed possible that:

$$m_1(\psi_1)m_2(\psi_2) > 0 \ , \tag{8.1}$$

$$\psi_1 \cap \psi_2 = \emptyset \ . \tag{8.2}$$

In particular, the property (8.2) is related to an implied coherence hypothesis; more precisely, since the universe Ω is defined as a *set* of events, the intersection of distinct singletons is empty:

$$\forall \{\omega_1\}, \{\omega_2\} \subset \Omega \, , \ \{\omega_1\} \neq \{\omega_2\} \Rightarrow \{\omega_1\} \cap \{\omega_2\} = \emptyset \ .$$

Notice that this hypothesis is quite similar to the hypothesis **c2.** of section 8.2.1 . *The coherence hypothesis seems to be the source of the contradictions in the abstract model, when fusing informations.* Finally, $Z < 1$ means that our abstract universe Ω has been incorrectly defined and is thus unable to fit the both sensors. Z measures the error in our model of interpretation. This ability of the rule $\oplus$ is really new in comparison with probabilistic rules.

8.2.3 Transferable Belief Model

Smets has made an extensive explanation of TBM [8]. This section focuses on a minimal and somewhat simplified description of the model.

8.2.3.1 Definition

A Transferable Belief Model is characterized by a pair (Ω, m), where Ω is a set of abstract events and the basic belief assignment m is a non negatively valued function defined over $\mathcal{P}(\Omega)$ such that:

$$\sum_{\phi \subset \Omega} m(\phi) = 1 \ .$$

In this definition, *the hypothesis* $m(\emptyset) = 0$ *does not hold anymore.*

8.2.3.2 Fusion rule

Smets' rule looks like a refinement of Dempster and Shafer's rule:

$$m_1 \oplus m_2(\phi) = \sum_{\psi_1 \cap \psi_2 = \phi} m_1(\psi_1)m_2(\psi_2) \ .$$

Notice that the normalizer does not exist anymore. The measure of contradiction has been moved into $m(\emptyset)$. *This theory has been justified from an axiomatization of the fusion rule.*

8.2.3.3 TBM generalizes DST

First, notice that any bba for DST is a valid bba for TBM, but the converse is false because of $\emptyset$. Now, for any bba m_T of TBM such that $m_T(\emptyset) < 1$, construct the bba $\Delta(m_T)$ of DST defined by:

$$\Delta(m_T)(\emptyset) = 0 \text{ and } \forall\phi \subset \Omega : \phi \neq \emptyset\,,\ \Delta(m_T)(\phi) = \frac{m_T(\phi)}{1 - m_T(\emptyset)}\ .$$

Δ is an onto mapping. Any bba m_D of DST is a bba of TBM, and $\Delta(m_D) = m_D$.

Δ is a morphism for $\oplus$. *IE.* $\Delta(m_{T,1} \oplus m_{T,2}) = \Delta(m_{T,1}) \oplus \Delta(m_{T,2})$.

Proof. By definition, it is clear that:

$$\Delta(m_{T,1}) \oplus \Delta(m_{T,2})(\emptyset) = 0 = \Delta(m_{T,1} \oplus m_{T,2})(\emptyset)\ .$$

Now, for any $\phi \subset \Omega$, such that $\phi \neq \emptyset$:

$$\Delta(m_{T,1}) \oplus \Delta(m_{T,2})(\phi) = \frac{\sum_{\psi_1\cap\psi_2=\phi} \Delta(m_{T,1})(\psi_1)\Delta(m_{T,2})(\psi_2)}{\sum_{\phi\neq\emptyset}\sum_{\psi_1\cap\psi_2=\phi} \Delta(m_{T,1})(\psi_1)\Delta(m_{T,2})(\psi_2)}$$

$$= \frac{\sum_{\psi_1\cap\psi_2=\phi} \frac{m_{T,1}(\psi_1)}{1-m_{T,1}(\emptyset)} \times \frac{m_{T,2}(\psi_2)}{1-m_{T,2}(\emptyset)}}{\sum_{\phi\neq\emptyset}\sum_{\psi_1\cap\psi_2=\phi} \frac{m_{T,1}(\psi_1)}{1-m_{T,1}(\emptyset)} \times \frac{m_{T,2}(\psi_2)}{1-m_{T,2}(\emptyset)}} = \frac{\sum_{\psi_1\cap\psi_2=\phi} m_{T,1}(\psi_1)m_{T,2}(\psi_2)}{\sum_{\phi\neq\emptyset}\sum_{\psi_1\cap\psi_2=\phi} m_{T,1}(\psi_1)m_{T,2}(\psi_2)}$$

$$= \frac{m_{T,1} \oplus m_{T,2}(\phi)}{\sum_{\phi\neq\emptyset} m_{T,1} \oplus m_{T,2}(\phi)} = \frac{m_{T,1} \oplus m_{T,2}(\phi)}{1 - m_{T,1} \oplus m_{T,2}(\emptyset)} = \Delta(m_{T,1} \oplus m_{T,2})(\phi)\ .$$

□□□

Since Δ is an onto morphism, TBM is a generalization of DST. More precisely, a bba of TBM contains more information than a bba of DST, *ie.* the measure of contradiction $m(\emptyset)$, but this complementary information remains compatible with the fusion rule of DST.

The Dezert Smarandache Theory is introduced in the next section. This theory shares many common points with TBM. But there is a main and fundamental contribution of this theory. It does not make the coherence hypothesis anymore and the contradictions are managed differently: the abstract model is more flexible to the interpretation and it is not needed to rebuild the model in case of contradicting sensors.

8.3 Dezert Smarandache Theory (DSmT)

Both in DST and in TBM, the difficulty in the model definition appears when dealing with the contradictions between the informations. But contradictions are unavoidable, when dealing with imprecise informations. This assertion is illustrated by the following example.

B&W example. Assume a sensor s_1 which tells us if an object is white (W) or not (NW), and gives no answer (NA_1) in litigious cases. The actual universe for this sensor is $\mathcal{A}_1 = \{\mathrm{W}, \mathrm{NW}, \mathrm{NA}_1\}$. Assume a sensor s_2 which tells us if an object is black (B) or not (NB), and gives no answer (NA_2) in litigious cases. The actual universe for this sensor is $\mathcal{A}_2 = \{\mathrm{B}, \mathrm{NB}, \mathrm{NA}_2\}$. These characteristics are not known, but the sensors have been tested with black or white objects. For this reason, it is natural to model our world by $\Omega = \{\text{black}, \text{white}\}$. When a litigious case happens, its interpretation will just be the pair $\{\text{black}, \text{white}\}$. Otherwise the good answer is expected. The following properties are then verified:

$$\mathrm{B}, \mathrm{NW} \subset \text{black} \quad \text{and} \quad \mathrm{W}, \mathrm{NB} \subset \text{white}\,.$$

The coherence hypothesis is assumed, that is $\text{black} \cap \text{white} = \emptyset$. The event $\text{black} \cap \text{white}$ is impossible. This model works well, as long as the sensors work separately or the objects are still black or white. Now, in a true universe there are many objects which are neither white and neither black, and this without any litigation. For example: gray objects. Assume that the two sensors are activated. Then, the fused sensors will answer $\mathrm{NW} \cap \mathrm{NB}$, which will be interpreted by $\text{black} \cap \text{white}$. This contradicts the coherence hypothesis.

Conclusion. This example is a sketch of what generally happens, when constructing a system of decision. Several sources of information are available (two sensors here). These sources have different discrimination abilities. In fact, these discrimination abilities are not really known, but by running these sources on several test samples (black and white objects here), a model of theses abilities is obtained (here it is learned within Ω that our sensors distinguish between black and white objects). Of course, it is never sure that this model is complete. It is still possible actually that some new unknown cases could be discriminated by the information sources. In the example, the combination of two sensors made it possible to discriminate a new class of objects: the neither black, neither white objects. But when fusing these sensors, the new cases will become contradictions regarding the coherence hypothesis. Not only the coherence hypothesis makes our model contradictory, but it also prevents us from discovering new cases. The coherence hypothesis should be removed! Dezert and Smarandache proposed a model without the coherence hypothesis.

8.3.1 Dezert Smarandache model

In DST and TBM, the coherence hypothesis was implied by the use of a set, Ω, to represent the abstract universe. Moreover, the set operators $\cap$, $\cup$ and c (*ie.* set complement) were used to explain the interactions between the propositions $\phi \subset \Omega$. In fact, the notion of propositions is related to the notion of Boolean Algebra. Sets together with set operators are particular models of Boolean Algebra. Since DSmT does not make the coherence hypothesis, DSmT cannot rely on the set formalism. However, some boolean relations are needed to explain the relations between propositions. Another fundamental Boolean

Algebra is the propositional logic. This model should be used for the representation of the propositions of DSmT. Nevertheless, the negation operator will be removed from our logic, since it implies itself some coherence hypotheses, *eg.* $\phi \wedge \neg\phi \equiv \bot$! By identifying the equivalent propositions of the resulting logic, an *hyper-power set* of propositions is obtained. *Hyper-power sets are used as models of universe for the DSmT.*

8.3.1.1 Hyper-power set

Let $\Phi = \{\phi_i / i \in I\}$ be a set of propositions. The hyper-power set $< \Phi >$ is the free boolean pre-algebra generated by Φ and the boolean operators $\wedge$ and $\vee$:

$$\Phi, < \Phi > \wedge < \Phi >, < \Phi > \vee < \Phi > \subset < \Phi >$$

and $\wedge, \vee$ verify the properties:

Commutative. $\phi \wedge \psi \equiv \psi \wedge \phi$ and $\phi \vee \psi \equiv \psi \vee \phi$,

Associative. $\phi \wedge (\psi \wedge \eta) \equiv (\phi \wedge \psi) \wedge \eta$ and $\phi \vee (\psi \vee \eta) \equiv (\phi \vee \psi) \vee \eta$,

Distributive. $\phi \wedge (\psi \vee \eta) \equiv (\phi \wedge \psi) \vee (\phi \wedge \eta)$ and $\phi \vee (\psi \wedge \eta) \equiv (\phi \vee \psi) \wedge (\phi \vee \eta)$,

Idempotent. $\phi \wedge \phi \equiv \phi$ and $\phi \vee \phi \equiv \phi$,

Neutral sup/sub-elements. $\phi \wedge (\phi \vee \psi) \equiv \phi$ and $\phi \vee (\phi \wedge \psi) \equiv \phi$,

for any $\phi, \psi, \eta \in < \Phi >$.

Unless more specifications about the free pre-algebra are made, this definition forbids the propositions to be exclusive (no coherence assumption) or to be exhaustive. In particular, the negation operator, $\neg$, and the *never happen/always happen*, $\bot/\top$, are excluded from the formalism. Indeed, the negation is related to the coherence hypothesis, since $\top$ is related to the exhaustiveness hypothesis.

Property. It is easily proved from the definition that:

$$\forall \phi, \psi \in < \Phi >, \; \phi \wedge \psi \equiv \phi \iff \phi \vee \psi \equiv \psi \, .$$

The order $\leq$ is a meta-operator defined over $< \Phi >$ by:

$$\phi \leq \psi \iff \phi \wedge \psi \equiv \phi \iff \phi \vee \psi \equiv \psi \, .$$

The order $<$ is a meta-operator defined over $< \Phi >$ by:

$$\phi < \psi \iff \left[\phi \leq \psi \text{ and } \phi \not\equiv \psi \right] .$$

The hyper-power set order $\leq$ is the analogue of the set order $\subset$.

8.3.1.2 Dezert Smarandache Model

A Dezert Smarandache model (DSmm) is a pair (Φ, m), where the (abstract) universe Φ is a set of propositions and the basic belief assignment m is a non negatively valued function defined over $< \Phi >$ such that:

$$\sum_{\phi \in <\Phi>} m(\phi) = 1 \,.$$

8.3.1.3 Belief Function

The belief function Bel is defined by:

$$\forall \phi \in < \Phi >, \, \mathrm{Bel}(\phi) = \sum_{\psi \in <\Phi> : \psi \leq \phi} m(\psi) \,. \tag{8.3}$$

Since propositions are never exclusive within $< \Phi >$, the (classical) plausibility function is just equal to 1. The equation (8.3) is invertible:

$$\forall \phi \in < \Phi >, \, m(\phi) = \mathrm{Bel}(\phi) - \sum_{\psi \in <\Phi> : \psi < \phi} m(\psi) \,.$$

8.3.2 Fusion rule

For a given universe Φ , and two basic belief assignments m_1 and m_2, associated to different sensors, the fused basic belief assignment is $m_1 \oplus m_2$, defined by:

$$m_1 \oplus m_2(\phi) = \sum_{\psi_1 \wedge \psi_2 \equiv \phi} m_1(\psi_1) m_2(\psi_2) \,. \tag{8.4}$$

8.3.2.1 Dezert & Smarandache's example

Assume a thief (45 years old) witnessed by a granddad and a grandson. The witnesses answer the question: *is the thief young or old?* The universe is then $\Phi = \{\text{young}, \text{old}\}$. The granddad answers that the thief is rather young. Its testimony is described by the bba:

$$m_1(\text{young}) = 0.9 \quad \text{and} \quad m_1(\text{young} \vee \text{old}) = 0.1 \text{ (slight unknown)} \,.$$

Of course, the grandson thinks he is rather old:

$$m_2(\text{old}) = 0.9 \quad \text{and} \quad m_2(\text{young} \vee \text{old}) = 0.1 \text{ (slight unknown)} \,.$$

How to interpret the testimonies? The fusion rule says:

$$\begin{cases} m_1 \oplus m_2(\text{young} \wedge \text{old}) = 0.9801 \quad \text{(highly contradicts} \rightarrow \text{third case)} \\ m_1 \oplus m_2(\text{young}) = m_1 \oplus m_2(\text{old}) = 0.0099 \\ m_1 \oplus m_2(\text{young} \vee \text{old}) = 0.0001 \end{cases}$$

Our hypotheses contradict. There were a third case: the thief is middle aged.

8.3.2.2 Comments

In DSmT, there is not a clear distinction between the notion of conjunction, $\wedge$, the notion of *third case* and the notion of contradiction. The model does not decide for that and leaves this distinction to our last interpretation. It is our interpretation of the model which will make the distinction. Thus, the DSm model avoids any *over-abstraction* of the actual universe. Consequently, it never fails although we could fail in the last instance by interpreting it. Another good consequence is that *DSmT specifies any contradiction/third case*: the contradiction $\phi \wedge \psi$ is not just a contradiction, it is the contradiction between ϕ and ψ.

8.4 Probability over logical propositions

Probabilities are classically defined over measurable sets. However, this is only a manner to modelize the notion of probability, which is essentially a measure of the belief of logical propositions. Probability could be defined without reference to the measure theory, at least when the number of propositions is finite. In this section, the notion of probability is explained within a strict logical formalism. This formalism is of constant use in the sequel.

Intuitively, a probability over a set of logical propositions is a measure of belief which is additive (disjoint propositions are adding their chances) and increasing with the proposition (weak propositions are more probable). This measure should be zeroed for the impossible propositions and full for the ever-true propositions. Moreover, *a probability is a multiplicative measure for independent propositions.* The independence of propositions is a meta-relation between propositions, which generally depends on the problem setting.

These intuitions are formalized now. It is assumed that the reader is used with some logical notions.

8.4.1 Definition

Let L be at least an extension of the classical logic of propositions, that is L contains the operators $\wedge$, $\vee$, $\neg$ (and, or, negation) and the propositions $\bot$, $\top$ (always false, always true) . Assume moreover that some propositions pairs of L are *recognized* as *independent propositions* (this is a meta-relation not necessarily related to the logic itself). A probability p over L is a $\mathbb{R}^+$ valued function such that for any proposition ϕ and ψ of L :

Additivity. $p(\phi \wedge \psi) + p(\phi \vee \psi) = p(\phi) + p(\psi)$,

Coherence. $p(\bot) = 0$,

Finiteness. $p(\top) = 1$,

Multiplicativity. When ϕ and ψ are *independent* propositions, then $p(\phi \wedge \psi) = p(\phi)p(\psi)$.

8.4.2 Property

The coherence and additivity implies the increaseness of p:

Increaseness. $p(\phi \wedge \psi) \leq p(\phi)$.

Proof. Since $\phi \equiv (\phi \wedge \psi) \vee (\phi \wedge \neg\psi)$ and $(\phi \wedge \psi) \wedge (\phi \wedge \neg\psi) \equiv \bot$, it follows from the additivity:

$$p(\phi) + p(\bot) = p(\phi \wedge \psi) + p(\phi \wedge \neg\psi) \ .$$

From the coherence $p(\bot) = 0$, it is deduced $p(\phi) = p(\phi \wedge \psi) + p(\phi \wedge \neg\psi)$. Since p is non negatively valued, $p(\phi) \geq p(\phi \wedge \psi)$.

□□□

8.4.3 Partially defined probability

In the sequel, knowledges are alternately described by partially known probabilities over a logical system. Typically, the probability p will be known only for a subset of propositions $\ell \subset L$.

Partial probabilities have been investigated by other works [9], for the representation of partial knowledge. In these works, the probabilities are characterized by constraints. It is believed that this area has been insufficiently investigated. And although our presentation is essentially focused on the logical aspect of the knowledge representation, it should be noticed that it is quite related to this notion of partial probability. In particular, the knowledge of the probability for a subset of propositions implies the definition of constraints for the probability over the whole logical system. For example, the knowledge of $\pi = p(\phi \wedge \psi)$ implies a lower bound for $p(\phi)$ and $p(\psi)$: $p(\phi) \geq \pi$ and $p(\psi) \geq \pi$.

The next section introduces, on a small example, a new interpretation of DSmT by means of probabilized logic.

8.5 Logical interpretation of DSmT: an example

A bipropositional DSm model $\Delta = (\{\phi_1, \phi_2\}, m)$ is considered. This section proposes an interpretation of this DSm model by means of probabilized modal propositions.

8.5.1 A possible modal interpretation

Consider the following *modal* propositions:

U **.** Unable to decide between the ϕ_i's ,

α_i **.** Proposition ϕ_i is sure ; No Other Information (NOI) ,

I **.** Contradiction between the ϕ_i's .

It is noticeable that these propositions are exclusive:

$$\forall\, a, b \in \{U, \alpha_1, \alpha_2, I\}\,,\ a \not\equiv b \Rightarrow a \wedge b \equiv \bot\ . \tag{8.5}$$

These propositions are clearly related to the propositions ϕ_i :

$$\begin{cases} I \leq \phi_1 \wedge \phi_2\,,\ \phi_1\,,\ \phi_2\,,\ \phi_1 \vee \phi_2 & \text{[the \textit{contradiction} } I \text{ implies everything]} \\ \alpha_i \leq \phi_i\,,\ \phi_1 \vee \phi_2\,,\ \ \text{for } i = 1, 2 & [\alpha_i \text{ implies } \phi_i \text{ and } \phi_1 \vee \phi_2] \\ U \leq \phi_1 \vee \phi_2 & [U \text{ only implies } \phi_1 \vee \phi_2] \end{cases} \tag{8.6}$$

These propositions are also exhaustive; *ie.* in the universe Φ , either one of the propositions I, α_1, α_2, U should be verified:

$$I \vee \alpha_1 \vee \alpha_2 \vee U \equiv \phi_1 \vee \phi_2\ . \tag{8.7}$$

Since the propositions α_i, U, I are characterizing the knowledge about ϕ_i (with NOI), the doubt or the contradiction, it seems natural to associate to these propositions a belief equivalent to $m(\phi_i)$, $m(\phi_1 \vee \phi_2)$ and $m(\phi_1 \wedge \phi_2)$. These beliefs will be interpreted as probabilities over I, U and α_i :

$$p(I) = m(\phi_1 \wedge \phi_2)\ , \qquad p(U) = m(\phi_1 \vee \phi_2)\ . \qquad p(\alpha_i) = m(\phi_i)\ ,\ \ \text{for } i = 1, 2\ . \tag{8.8}$$

Such probabilistic interpretation is natural but questionable: it mixes probabilities together with bba. Since the propositions ϕ_i are not directly manipulated, this interpretation is not forbidden however. In fact, it will be shown next that this interpretation *implies* the fusion rule $\oplus$ and this will be a posterior justification of such hypothesis .

8.5.2 Deriving a fusion rule

In this section, a fusion rule is deduced from the previous *probabilized modal interpretation.* This rule happens to be the (conjunctive) fusion rule of DSmT.
Let $\Delta_j = (\{\phi_1, \phi_2\}, m_j)$ be the DSm models associated to sensors $j = 1, 2$ working beside the same abstract universe $\{\phi_1, \phi_2\}$. Define the set of modal propositions $S_j = \{I_j, \alpha_{j1}, \alpha_{j2}, U_j\}$:

U_j **.** Unable to decide between the ϕ_i's, according to sensor j ,

α_{ji} **.** Proposition ϕ_i is sure and NOI, according to sensor j ,

I_j **.** Contradiction between the ϕ_i's, according to sensor j .

The propositions of S_j verify of course the properties (8.5), (8.6), (8.7) and (8.8), the subscript $_j$ being added when needed. Define:

$$S = S_1 \wedge S_2 = \{a_1 \wedge a_2 / a_1 \in S_1 \text{ and } a_2 \in S_2\} \,.$$

Consider $a \equiv a_1 \wedge a_2$ and $b \equiv b_1 \wedge b_2$, two distinct elements of S. Then, either $a_1 \not\equiv b_1$ or $a_2 \not\equiv b_2$. Since S_j verifies (8.5), it follows $a_1 \wedge b_1 \equiv \bot$ or $a_2 \wedge b_2 \equiv \bot$, thus yielding:

$$(a_1 \wedge a_2) \wedge (b_1 \wedge b_2) \equiv (a_1 \wedge b_1) \wedge (a_2 \wedge b_2) \equiv \bot \,.$$

S is made of exclusive elements. It is also known from (8.7) that $\phi_1 \vee \phi_2 \equiv \bigvee_{a_j \in S_j} a_j$; S_j is exhaustive. It follows:

$$\phi_1 \vee \phi_2 \equiv (\phi_1 \vee \phi_2) \wedge (\phi_1 \vee \phi_2) \equiv \bigwedge_{j=1}^{2} \bigvee_{a_j \in S_j} a_j \equiv \bigvee_{a \in S} a \,.$$

S is exhaustive. In fact, S enumerates all the possible cases of observation. It is thus reasonable to think that the fused knowledge of these sensors could be constructed from S. The question then arising is: what is the signification of a proposition $a_1 \wedge a_2 \in S$? It is remembered that a proposition of S_j just tells *what is known for sure according to sensor j*. But the semantic for combining sure or unsure propositions is quite natural:[1]

- unsure + unsure = unsure
- unsure + sure = sure
- sure + sure = sure OR contradiction
- anything + contradiction = contradiction

In particular contradiction arises, when two informations are sure *and* these informations are known contradictory. This conduces to a general interpretation of S:

$\wedge$	I_2	α_{21}	α_{22}	U_2
I_1	Contradiction	Contradiction	Contradiction	Contradiction
α_{11}	Contradiction	ϕ_1 is sure	Contradiction	ϕ_1 is sure
α_{12}	Contradiction	Contradiction	ϕ_2 is sure	ϕ_2 is sure
U_1	Contradiction	ϕ_1 is sure	ϕ_2 is sure	Unsure

At last, any proposition of S is a sub-event of a proposition I, α_1, α_2 or U, defined by:

U . The sensors are unable to decide between the ϕ_i's,

α_i . The sensors are sure of the proposition ϕ_i, but do not know anything else,

I . The sensors contradict.

[1] In fact, the independence of the sensors is implicitly hypothesized in such combining rule (refer to next section).

Since S is exhaustive, the propositions U, α_i, I are entirely determined by S:

- $I \equiv (I_1 \wedge I_2) \vee (I_1 \wedge \alpha_{21}) \vee (I_1 \wedge \alpha_{22}) \vee (I_1 \wedge U_2) \vee (\alpha_{11} \wedge I_2) \vee$
  $(\alpha_{12} \wedge I_2) \vee (U_1 \wedge I_2) \vee (\alpha_{12} \wedge \alpha_{21}) \vee (\alpha_{11} \wedge \alpha_{22})\,,$
- $\alpha_i \equiv (\alpha_{1i} \wedge \alpha_{2i}) \vee (U_1 \wedge \alpha_{2i}) \vee (\alpha_{1i} \wedge U_2)\,,$
- $U \equiv U_1 \wedge U_2\,.$

The propositions I, α_i, U are thus entirely specified and since S is made of exclusive elements, their probabilities are given by:

- $p(I) = p(I_1 \wedge I_2) + p(I_1 \wedge \alpha_{21}) + p(I_1 \wedge \alpha_{22}) + p(I_1 \wedge U_2) + \cdots + p(\alpha_{11} \wedge \alpha_{22})\,,$
- $p(\alpha_i) = p(\alpha_{1i} \wedge \alpha_{2i}) + p(U_1 \wedge \alpha_{2i}) + p(\alpha_{1i} \wedge U_2)\,,$
- $p(U) = p(U_1 \wedge U_2)\,.$

At this point, *the independence of the sensors is needed.* The hypothesis implies $p(a_1 \wedge a_2) = p(a_1)p(a_2)$. The constraints (8.8) for each sensor j then yield:

- $p(I) = m_1(\phi_1 \wedge \phi_2)m_2(\phi_1 \wedge \phi_2) + m_1(\phi_1 \wedge \phi_2)m_2(\phi_1) + \cdots + m_1(\phi_1)m_2(\phi_2)\,,$
- $p(\alpha_i) = m_1(\phi_i)m_2(\phi_i) + m_1(\phi_1 \vee \phi_2)m_2(\phi_i) + m_1(\phi_i)m_2(\phi_1 \vee \phi_2)\,,$
- $p(U) = m_1(\phi_1 \vee \phi_2)m_2(\phi_1 \vee \phi_2)\,.$

The definition of $m_1 \oplus m_2$ implies finally:

$$p(I) = m_1 \oplus m_2(\phi_1 \wedge \phi_2)\,,\qquad p(\alpha_i) = m_1 \oplus m_2(\phi_i)\,,\qquad \text{and}\qquad p(U) = m_1 \oplus m_2(\phi_1 \vee \phi_2)\,.$$

Our interpretation of DSmT by means of probabilized modal propositions has implied the fusion rule $\oplus$.

This result is investigated rigorously and generally in the next section.

8.6 Multi-modal logic and information fusion

This section generalizes the results of the previous section. The presentation is more formalized. In particular, a multi-modal logic for the information fusion is constructed. This presentation is not fully detailed and it is assumed that the reader is acquainted with some logical notions.

8.6.1 Modal logic

In this introductory section, we are just interested in modal logic, and particularly in the T-system. There is no need to argue about a better system, since we are only interested in manipulating the modalities

$\Box$, $\neg\Box$, $\diamond$ and $\neg\diamond$.

Being given Φ a set of atomic propositions, the set of classical propositions, $C(\Phi)$ more simply denoted C, is defined by:

- $\Phi \subset C$, $\bot \in C$ and $\top \in C$,
- If $\phi, \psi \in C$, then $\neg\phi \in C$, $\phi \wedge \psi \in C$, $\phi \vee \psi \in C$ and $\phi \rightarrow \psi \in C$.

The set of modal propositions, $M(\Phi)$ also denoted M, is constructed as follows:

- $C \subset M$,
- If $\phi \in M$, then $\Box\phi \in M$ and $\diamond\phi \in M$,
- If $\phi, \psi \in M$, then $\neg\phi \in M$, $\phi \wedge \psi \in M$, $\phi \vee \psi \in M$ and $\phi \rightarrow \psi \in M$.

The proposition $\Box\phi$ will mean that *the proposition ϕ is true for sure*. The proposition $\diamond\phi$ will mean that *the proposition ϕ is possibly true*.

In the sequel, the notation $\vdash \phi$ means that ϕ *is proved in* T. A proposition ϕ such that $\vdash \phi$ is also called an axiom. The notation $\phi \equiv \psi$ means both $\vdash \phi \rightarrow \psi$ and $\vdash \psi \rightarrow \phi$.

All axioms are defined recursively by assuming some deduction rules and initial axioms.

Modus Ponens (MP). For any proposition $\phi, \psi \in M$, such that $\vdash \phi$ and $\vdash \phi \rightarrow \psi$, it is deduced $\vdash \psi$.

Classical axioms. For any $\phi, \psi, \eta \in M$, it is assumed the axioms:

1. $\vdash \top$,
2. $\vdash \phi \rightarrow (\psi \rightarrow \phi)$,
3. $\vdash (\eta \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\eta \rightarrow \phi) \rightarrow (\eta \rightarrow \psi))$,
4. $\vdash (\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)$,
5. $\bot \equiv \neg\top$,
6. $\phi \rightarrow \psi \equiv \neg\phi \vee \psi$,
7. $\phi \wedge \psi \equiv \neg(\neg\phi \vee \neg\psi)$.

It is deduced from these axioms that:

- The relation $\vdash \phi \rightarrow \psi$ is a pre-order with a minimum $\bot$ and a maximum $\top$: $\bot$ is the strongest proposition, $\top$ is the weakest proposition,
- The relation $\equiv$ is an equivalence relation.

Modal axioms and rule. Let $\phi, \psi \in M$.

i. From $\vdash \phi$ is deduced $\vdash \Box\phi$; axioms are sure. *This does not mean* $\vdash \phi \rightarrow \Box\phi$ *which is false!*

ii. $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$; when the inference is sure and the premise is sure, the conclusion is sure,

iii. $\vdash \Box\phi \to \phi$; sure propositions are true,

iv. $\diamond\phi \equiv \neg\Box\neg\phi$; is unsure what cannot be false for sure.

It is deduced that the proposition $\Box\phi$ is stronger than ϕ which is stronger than $\diamond\phi$.

Notation. In the sequel, $\psi \leq \phi$ means $\vdash \psi \to \phi$, and $\psi < \phi$ means both $\psi \leq \phi$ and $\phi \not\equiv \psi$.

The logical operators are compatible with $\equiv$. Denote $\phi_{/\equiv} = \{\psi \in M / \psi \equiv \phi\}$, the class of equivalence of ϕ . Let $\phi, \psi \in M$, $\hat{\phi} \in \phi_{/\equiv}$ and $\hat{\psi} \in \psi_{/\equiv}$. Then holds:

- $\hat{\phi} \to \hat{\psi} \in (\phi \to \psi)_{/\equiv}$
- $\neg\hat{\phi} \in (\neg\phi)_{/\equiv}$
- $\hat{\phi} \wedge \hat{\psi} \in (\phi \wedge \psi)_{/\equiv}$
- $\hat{\phi} \vee \hat{\psi} \in (\phi \vee \psi)_{/\equiv}$
- $\Box\hat{\phi} \in (\Box\phi)_{/\equiv}$
- $\diamond\,\hat{\phi} \in (\diamond\phi)_{/\equiv}$

The logical operators over M are thus extended naturally to the classes of M by setting:

- $\phi_{/\equiv} \to \psi_{/\equiv} \triangleq (\phi \to \psi)_{/\equiv}$
- $\neg\phi_{/\equiv} \triangleq (\neg\phi)_{/\equiv}$
- $\phi_{/\equiv} \wedge \psi_{/\equiv} \triangleq (\phi \wedge \psi)_{/\equiv}$
- $\phi_{/\equiv} \vee \psi_{/\equiv} \triangleq (\phi \vee \psi)_{/\equiv}$
- $\Box\phi_{/\equiv} \triangleq (\Box\phi)_{/\equiv}$
- $\diamond\,\phi_{/\equiv} \triangleq (\diamond\phi)_{/\equiv}$

From now on, the class $\phi_{/\equiv}$ is simply denoted ϕ.

Hyper-power set. Construct the subset of classical propositions $F(\Phi)$ recursively by the properties $\Phi \subset F(\Phi)$ and $\forall\phi, \psi \in F(\Phi)$, $[\,\phi\wedge\psi \in F(\Phi)$ and $\phi\vee\psi \in F(\Phi)\,]$. The hyper-power of Φ , denoted $< \Phi >$, is the set of equivalence classes of $F(\Phi)$ according to the relation $\equiv$:

$$< \Phi >= F(\Phi)_{/\equiv} = \{\phi_{/\equiv} \,/\, \phi \in F(\Phi)\} \,.$$

8.6.1.1 Useful theorems

Let $\phi, \psi \in M$.

1. $\vdash (\Box\phi \wedge \Box\psi) \to \Box(\phi \wedge \psi)$ and $\vdash \Box(\phi \wedge \psi) \to (\Box\phi \wedge \Box\psi)$
2. $\vdash (\diamond\phi \vee \diamond\psi) \to \diamond(\phi \vee \psi)$ and $\vdash \diamond(\phi \vee \psi) \to (\diamond\phi \vee \diamond\psi)$
3. $\vdash (\Box\phi \vee \Box\psi) \to \Box(\phi \vee \psi)$ **but** $\nvdash \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$
4. $\vdash \diamond(\phi \wedge \psi) \to (\diamond\phi \wedge \diamond\psi)$ **but** $\nvdash (\diamond\phi \wedge \diamond\psi) \to \diamond(\phi \wedge \psi)$

Proof. Theorem 1 and theorem 2 are dual and thus equivalent (rules 7 and iv.). It is exactly the same thing for theorem 3 and theorem 4.

Proof of $\vdash (\Box\phi \wedge \Box\psi) \to \Box(\phi \wedge \psi)$.

Classical rules yield the axiom:

$\vdash \phi \to (\psi \to (\phi \wedge \psi))$

Rule i. implies then:

$\vdash \Box(\phi \to (\psi \to (\phi \wedge \psi)))$

Applying rule ii. twice, it is deduced:

$\vdash \Box\phi \to \Box(\psi \to (\phi \wedge \psi))$

$\vdash \Box\phi \to (\Box\psi \to \Box(\phi \wedge \psi))$

The proof is concluded by applying the classical rules.

Proof of $\vdash \Box(\phi \wedge \psi) \to (\Box\phi \wedge \Box\psi)$.

Classical rules yield the axioms:

$\vdash (\phi \wedge \psi) \to \phi$ and $\vdash (\phi \wedge \psi) \to \psi$

Rule i. implies then:

$\vdash \Box((\phi \wedge \psi) \to \phi)$ and $\vdash \Box((\phi \wedge \psi) \to \psi)$

Applying rule ii., it is deduced:

$\vdash \Box(\phi \wedge \psi) \to \Box\phi$ and $\vdash \Box(\phi \wedge \psi) \to \Box\psi$

The proof is concluded by applying the classical rules.

Proof of $\vdash (\Box\phi \vee \Box\psi) \to \Box(\phi \vee \psi)$.

Classical rules yield the axioms:

$\vdash \phi \to (\phi \vee \psi)$ and $\vdash \psi \to (\phi \vee \psi)$

Rule i. implies then:

$\vdash \Box(\phi \to (\phi \vee \psi))$ and $\vdash \Box(\psi \to (\phi \vee \psi))$

Applying rule ii., it is deduced:

$\vdash \Box\phi \to \Box(\phi \vee \psi)$ and $\vdash \Box\psi \to \Box(\phi \vee \psi)$

The proof is concluded by applying the classical rules.

Why $\nvdash \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$?

To answer this question precisely, the Kripke semantic should be introduced. Such discussion is outside the scope of this paper. However, some practical considerations will clarify this assertion. When $\phi \vee \psi$ *is sure*, does that mean that ϕ *is sure* or ψ *is sure*? Not really since we know that ϕ or ψ is true, but *we do not know which one* is true. Moreover, it may happen that ϕ is true sometimes, while ψ is true the other times. As a conclusion, we are not sure of ϕ and are not sure of ψ.

This example is a counter-example of $\vdash \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$.

□□□

8.6.2 A multi-modal logic

Assume that several informations are obtained from different sources. Typically, these informations are modalities such as *"according to the source σ , the proposition ϕ is sure"* . Such a modality could be naturally denoted $\Box_\sigma\phi$ (a modality depending on σ) . A more readable notation $[\phi|\sigma] \stackrel{\Delta}{\equiv} \Box_\sigma\phi$ is prefered. Take note that $[\phi|\sigma]$ is not related to the Bayes inference $(\phi|\sigma)$! Now, the question arising is *how to combine these modalities?* For example, is it possible to deduce something from $[\phi_1|\sigma_1]\wedge[\phi_2|\sigma_2]$? Without any relations between *heterogeneous* modalities, it is not possible to answer this question. Such a relation, however, is foreseeable. Assume that the source τ involves the source σ , *ie.* $\tau\rightarrow\sigma$. Now assume that the proposition ϕ should be known from the source σ , *ie.* $[\phi|\sigma]$. Since τ involves σ , it is natural to state that ϕ should be known from the source τ , *ie.* $[\phi|\tau]$. This natural deduction could be formalized by the rule:

$$\vdash \tau\rightarrow\sigma \quad \text{implies} \quad \vdash [\phi|\sigma]\rightarrow[\phi|\tau]\ .$$

With this rule, it is now possible to define the logic.

The set of multi-modal propositions, $mM(\Phi)$ also denoted mM, is defined recursively:

- $C\subset mM$,
- If $\phi,\sigma\in mM$, then $[\phi|\sigma]\in mM$,
- If $\phi,\psi\in mM$, then $\neg\phi\in mM$, $\phi\wedge\psi\in mM$, $\phi\vee\psi\in mM$ and $\phi\rightarrow\psi\in mM$.

The multi-modal logic obeys to the following rules and axioms:

Modus Ponens.

Classical axioms. Axioms 1 to 7 ,

Modal axioms and rule. Let $\sigma,\tau,\phi,\psi\in mM$.

m.i. From $\vdash\phi$ is deduced $\vdash[\phi|\sigma]$: axioms are sure, according to any sources ,

m.ii. $\vdash[\phi\rightarrow\psi|\sigma]\rightarrow([\phi|\sigma]\rightarrow[\psi|\sigma])$. If a source of information asserts a proposition and recognizes a weaker proposition, then it asserts this weaker proposition ,

m.iii. $\vdash[\phi|\sigma]\rightarrow\phi$. *The sources of information always tell the truth.* If a source asserts a proposition, this proposition is actually true ,

m.iv. $\vdash\tau\rightarrow\sigma$ implies $\vdash[\phi|\sigma]\rightarrow[\phi|\tau]$. Knowledge increases with stronger sources of information.

The axiom m.iii. is questionable and may be changed. But the work presented in this paper is restricted to this axiom.

It is also possible to consider some exotic rules like $\phi \equiv [\phi|\bot]$, *ie.* a perfect source of information $\bot$ yields a perfect knowledge of the propositions ϕ. Similarly, the modality $[\phi|\top]$ could be interpreted as the proposition *"ϕ is an absolute truth"* or *"ϕ has a proof"*: one does not need any source of information to assert an absolute truth...

8.6.3 Some multi-modal theorems

8.6.3.1 Modal theorems map into multi-modal logic

Let $\mu \in M$ be a modal proposition. Let $\sigma \in mM$ be a multi-modal proposition. Let $\mu[\sigma] \in mM$ be the multi-modal proposition obtained by replacing $\Box$ by $[\cdot|\sigma]$ and $\diamond$ by $\neg[\neg \cdot |\sigma]$ in the proposition μ. Then $\vdash \mu$ implies $\vdash \mu[\sigma]$.

8.6.3.2 Useful multi-modal theorems

If the source σ asserts ϕ and the source τ asserts ψ, then the fused sources assert $\phi \wedge \psi$:

$$\vdash \big([\phi|\sigma] \wedge [\psi|\tau]\big) \rightarrow [\phi \wedge \psi|\sigma \wedge \tau]$$

Proof. From the axioms $\vdash (\sigma \wedge \tau) \rightarrow \sigma$ and $\vdash (\sigma \wedge \tau) \rightarrow \tau$, it is deduced:

$$\vdash [\phi|\sigma] \rightarrow [\phi|\sigma \wedge \tau] \,,$$

and

$$\vdash [\psi|\tau] \rightarrow [\psi|\sigma \wedge \tau] \,.$$

From the useful theorems proved for modal logic, it is deduced:

$$[\phi|\sigma \wedge \tau] \wedge [\psi|\sigma \wedge \tau] \equiv [\phi \wedge \psi|\sigma \wedge \tau] \,.$$

The proof is concluded by applying the classical rules.

□□□

If one of the sources σ or τ asserts ϕ, then the fused sources assert ϕ:

$$\vdash \big([\phi|\sigma] \vee [\phi|\tau]\big) \rightarrow [\phi|\sigma \wedge \tau] \,.$$

Proof. This results directly from $\vdash [\phi|\sigma] \rightarrow [\phi|\sigma \wedge \tau]$ and $\vdash [\phi|\tau] \rightarrow [\phi|\sigma \wedge \tau]$.

□□□

The converse is not necessarily true:

$$\nvdash [\phi|\sigma \wedge \tau] \rightarrow \big([\phi|\sigma] \vee [\phi|\tau]\big) \,.$$

In fact, *when sensors are not independent and possibly interactive, it is possible that the fused sensor* $\sigma\wedge\tau$ *works better than* σ *and* τ *separately!* On the other hand, this converse property could be considered as a necessary condition for the sensor independence. This discussion leads to the introduction of a new axiom, *the independence axiom* m.indep.:

$$\text{m.indep.}\ \vdash [\phi|\sigma\wedge\tau] \to \big([\phi|\sigma]\vee[\phi|\tau]\big)\ .$$

8.6.4 Sensor fusion

8.6.4.1 The context

Two sensors, σ and τ, are generating informations about a set of atomic propositions Φ. More precisely, the sensors will measure *independently* the probability for each proposition $\phi\in<\Phi>$ to be sure. In this section, it is discussed about fusing these sources of information.

This problem is clearly embedded in the multi-modal logic formalism. In particular, the modality $[\cdot|\sigma]$ characterizes the knowledge of σ about the universe $<\Phi>$. More precisely, the proposition $[\phi|\sigma]$ explains if ϕ is sure according to σ or not. This knowledge is probabilistic: the working data are the probabilities $p([\phi|\sigma])$ and $p([\phi|\tau])$ for $\phi\in<\Phi>$. The problem setting is more formalized in the next section.

Notation. From now on, the notation $p[\phi|\sigma]$ is used instead of $p([\phi|\sigma])$. *Beware* that $p[\phi|\sigma]$ *is not* the conditional probability $p(\phi|\sigma)$!

8.6.4.2 Sensors model and problem setting

The set of multi-modal propositions, $mM(\Theta)$, is constructed from the set $\Theta=\Phi\cup\{\sigma,\tau\}$. The propositions σ and τ are referring to the two independent sensors. The proposition $\sigma\wedge\tau$ is referring to the fused sensor. It is assumed for sure that $\bigvee_{\phi\in\Phi}\phi$ is true:

$$\vdash\left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\top\right].$$

Consequently:

$$\left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\sigma\wedge\tau\right]\equiv\left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\sigma\right]\equiv\left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\tau\right]\equiv\left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\top\right]\equiv\top\,,$$

and:

$$p\left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\sigma\wedge\tau\right]=p\left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\sigma\right]=p\left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\tau\right]=1\ .$$

The sensors σ and τ are giving probabilistic informations about the certainty of the other propositions. More precisely, it is known:

$$p[\phi|\sigma]\quad\text{and}\quad p[\phi|\tau]\ ,\quad\text{for any }\phi\in<\Phi>\ .$$

Since the propositions σ and τ are referring to independent sensors, it is assumed that:

- The axiom m.indep. holds for σ and τ ,
- For any $\phi, \psi \in <\Phi>$, $p([\phi|\sigma] \wedge [\psi|\tau]) = p[\phi|\sigma]p[\psi|\tau]$.

A pertinent fused information is expected:

How to compute $p[\phi|\sigma\wedge\tau]$ for any $\phi \in <\Phi>$?

8.6.4.3 Constructing the fused belief

Defining tools. The useful propositions $\phi^{(\sigma)}$ are defined for any $\phi \in <\Phi>$:

$$\phi^{(\sigma)} \stackrel{\Delta}{\equiv} [\phi|\sigma] \wedge \neg\left(\bigvee_{\psi\in<\Phi>:\psi<\phi} [\psi|\sigma]\right) .$$

The same propositions $\phi^{(\tau)}$ are defined for τ :

$$\phi^{(\tau)} \stackrel{\Delta}{\equiv} [\phi|\tau] \wedge \neg\left(\bigvee_{\psi\in<\Phi>:\psi<\phi} [\psi|\tau]\right) .$$

Properties.

The propositions $\phi^{(\sigma)}$ are exclusive:

$$\phi^{(\sigma)} \wedge \psi^{(\sigma)} \equiv \bot , \quad \text{for any } \phi \not\equiv \psi .$$

Proof. Since $[\phi|\sigma] \wedge [\psi|\sigma] \equiv [\phi\wedge\psi|\sigma]$, it is deduced:

$$\phi^{(\sigma)} \wedge \psi^{(\sigma)} \equiv [\phi\wedge\psi|\sigma] \wedge \neg\left(\bigvee_{\eta:\eta<\phi} [\eta|\sigma]\right) \wedge \neg\left(\bigvee_{\eta:\eta<\psi} [\eta|\sigma]\right) .$$

It follows:

$$\phi^{(\sigma)} \wedge \psi^{(\sigma)} \equiv [\phi\wedge\psi|\sigma] \wedge \left(\bigwedge_{\eta:\eta<\phi} \neg[\eta|\sigma]\right) \wedge \left(\bigwedge_{\eta:\eta<\psi} \neg[\eta|\sigma]\right) .$$

Since $\phi\wedge\psi < \phi$ or $\phi\wedge\psi < \psi$ when $\phi \not\equiv \psi$, the property is deduced.

□□□

Lemma: $\displaystyle\bigvee_{\psi:\psi<\phi} [\psi|\sigma] \leq [\phi|\sigma]$.

Proof. The property $\psi < \phi$ implies successively $\vdash \psi \to \phi$, $\vdash [\psi \to \phi|\sigma]$ and $\vdash [\psi|\sigma] \to [\phi|\sigma]$. The lemma is then deduced.

□□□

The propositions $\phi^{(\sigma)}$ are exhaustive:

$$\bigvee_{\psi:\psi\leq\phi} \psi^{(\sigma)} \equiv [\phi|\sigma] , \quad \text{and in particular:} \quad \bigvee_{\phi\in<\Phi>} \phi^{(\sigma)} \equiv \top .$$

Proof. The proof is recursive. First, the smallest element of $<\Phi>$ is $\mu \equiv \bigwedge_{\phi\in<\Phi>}\phi$ and verifies:

$$\mu^{(\sigma)} \equiv [\mu|\sigma]\,.$$

Secondly:

$$\bigvee_{\psi:\psi\leq\phi}\psi^{(\sigma)} \equiv \phi^{(\sigma)} \vee \left(\bigvee_{\psi:\psi<\phi}\psi^{(\sigma)}\right) \equiv \phi^{(\sigma)} \vee \left(\bigvee_{\psi:\psi<\phi}\;\bigvee_{\eta:\eta\leq\psi}\eta^{(\sigma)}\right) \equiv \phi^{(\sigma)} \vee \left(\bigvee_{\psi:\psi<\phi}[\psi|\sigma]\right).$$

Since $\phi^{(\sigma)} \equiv [\phi|\sigma] \wedge \neg\left(\bigvee_{\psi:\psi<\phi}[\psi|\sigma]\right)$ and $\bigvee_{\psi:\psi<\phi}[\psi|\sigma] \leq [\phi|\sigma]$, it follows:

$$\bigvee_{\psi:\psi\leq\phi}\psi^{(\sigma)} \equiv [\phi|\sigma]\,.$$

The second part of the property results from:

$$\bigvee_{\phi\in<\Phi>}\phi^{(\sigma)} \equiv \bigvee_{\psi:\psi\leq\bigvee_{\phi:\phi\in\Phi}\phi}\psi^{(\sigma)} \quad\text{and}\quad \left[\bigvee_{\phi\in\Phi}\phi\,\middle|\,\sigma\right] \equiv \top\,.$$

□□□

The propositions $\phi^{(\tau)}$ are also exclusive and exhaustive:

$$\phi^{(\tau)} \wedge \psi^{(\tau)} \equiv \bot\,,\quad \text{for any } \phi \not\equiv \psi\,,$$

and:

$$\bigvee_{\phi\in<\Phi>}\phi^{(\tau)} \equiv \top\,.$$

It is then deduced that *the propositions $\phi^{(\sigma)} \wedge \psi^{(\tau)}$ are exclusive and exhaustive:*[2]

$$\forall\phi_1,\psi_1,\phi_2,\psi_2 \in<\Phi>\,,\;\; (\phi_1,\psi_1)\not\equiv(\phi_2,\psi_2) \Longrightarrow \left(\phi_1^{(\sigma)}\wedge\psi_1^{(\tau)}\right)\wedge\left(\phi_2^{(\sigma)}\wedge\psi_2^{(\tau)}\right) \equiv \bot\,, \tag{8.9}$$

and:

$$\bigvee_{\phi,\psi\in<\Phi>}\left(\phi^{(\sigma)}\wedge\psi^{(\tau)}\right) \equiv \top\,. \tag{8.10}$$

From the properties (8.9) and (8.10) , it results that the set:

$$\Sigma = \left\{\phi^{(\sigma)}\wedge\psi^{(\tau)}\,\big/\,\phi,\psi\in<\Phi>\right\}$$

is *a partition* of $\top$. This property is particularly interesting, since it makes possible the computation of the probability of any proposition factorized within Σ:

$$\forall\,\Lambda\subset\Sigma\,,\; p\left(\bigvee_{\phi\in\Lambda}\phi\right) = \sum_{\phi\in\Lambda}p(\phi)\,. \tag{8.11}$$

[2]The notation $(\phi_1,\psi_1)\equiv(\phi_2,\psi_2)$ means $\phi_1\equiv\phi_2$ and $\psi_1\equiv\psi_2$.

Factorizing $\phi^{(\sigma\wedge\tau)}$. It has been shown that $\vdash ([\phi|\sigma]\wedge[\psi|\tau]) \rightarrow [\phi\wedge\psi|\sigma\wedge\tau]$. It follows:

$$\vdash \bigvee_{\phi\wedge\psi\leq\eta} ([\phi|\sigma]\wedge[\psi|\tau]) \rightarrow [\eta|\sigma\wedge\tau]\;, \tag{8.12}$$

The axiom m.indep. says $\vdash [\eta|\sigma\wedge\tau] \rightarrow \big([\eta|\sigma]\vee[\eta|\tau]\big)$. Since $\big[\bigvee_{\phi\in\Phi}\phi\big|\sigma\big] \equiv \big[\bigvee_{\phi\in\Phi}\phi\big|\tau\big] \equiv \top$, it is deduced:

$$\vdash [\eta|\sigma\wedge\tau] \rightarrow \left(\left([\eta|\sigma]\wedge\left[\bigvee_{\phi\in\Phi}\phi\middle|\tau\right]\right)\vee\left(\left[\bigvee_{\phi\in\Phi}\phi\middle|\sigma\right]\wedge[\eta|\tau]\right)\right).$$

At last

$$[\eta|\sigma\wedge\tau] \equiv \bigvee_{\phi\wedge\psi\leq\eta} ([\phi|\sigma]\wedge[\psi|\tau])\;. \tag{8.13}$$

It is then deduced:

$$\phi^{(\sigma\wedge\tau)} \equiv [\phi|\sigma\wedge\tau]\wedge\neg\left(\bigvee_{\psi<\phi}[\psi|\sigma\wedge\tau]\right) \equiv \left(\bigvee_{\eta\wedge\zeta\leq\phi}([\eta|\sigma]\wedge[\zeta|\tau])\right)\wedge\neg\left(\bigvee_{\psi<\phi}\;\bigvee_{\eta\wedge\zeta\leq\psi}([\eta|\sigma]\wedge[\zeta|\tau])\right).$$

Now:

$$\bigvee_{\eta\wedge\zeta\leq\phi}([\eta|\sigma]\wedge[\zeta|\tau]) \equiv \bigvee_{\eta\wedge\zeta\leq\phi}\left(\left(\bigvee_{\xi\leq\eta}\xi^{(\sigma)}\right)\wedge\left(\bigvee_{\chi\leq\zeta}\chi^{(\tau)}\right)\right)$$

$$\equiv \bigvee_{\eta\wedge\zeta\leq\phi}\;\bigvee_{\xi\leq\eta}\;\bigvee_{\chi\leq\zeta}\big(\xi^{(\sigma)}\wedge\chi^{(\tau)}\big) \equiv \bigvee_{\eta\wedge\zeta\leq\phi}\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big)\;.$$

At last:

$$\phi^{(\sigma\wedge\tau)} \equiv \left(\bigvee_{\eta\wedge\zeta\leq\phi}\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big)\right)\wedge\neg\left(\bigvee_{\psi<\phi}\;\bigvee_{\eta\wedge\zeta\leq\psi}\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big)\right)$$

$$\equiv \left(\bigvee_{\eta\wedge\zeta\leq\phi}\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big)\right)\wedge\neg\left(\bigvee_{\eta\wedge\zeta<\phi}\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big)\right).$$

Since Σ is a partition, it is deduced the final results:

$$\phi^{(\sigma\wedge\tau)} \equiv \bigvee_{\eta\wedge\zeta\equiv\phi}\eta^{(\sigma)}\wedge\zeta^{(\tau)} \tag{8.14}$$

and:

$$p\big(\phi^{(\sigma\wedge\tau)}\big) = \sum_{\eta\wedge\zeta\equiv\phi} p\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big)\;. \tag{8.15}$$

This last result is sufficient to derive $p[\phi|\sigma\wedge\tau]$, as soon as we are able to compute the probability over Σ. The next paragraph explains this computation.

The probability over Σ.

Computing $p(\phi^{(\sigma)})$ and $p(\phi^{(\tau)})$. These probabilities are computed recursively from $p[\phi|\sigma]$ and $p[\phi|\tau]$. More precisely, it is derived from the definition $\phi^{(\sigma)} \equiv [\phi|\sigma]\wedge\neg\Big(\bigvee_{\psi<\phi}[\psi|\sigma]\Big)$ and the property $[\phi|\sigma] \equiv \bigvee_{\psi\leq\phi}\psi^{(\sigma)}$ that:

$$\phi^{(\sigma)} \equiv [\phi|\sigma]\wedge\neg\left(\bigvee_{\psi<\phi}\psi^{(\sigma)}\right).$$

Since the propositions $\phi^{(\sigma)}$ are exclusive and $\bigvee_{\psi<\phi}\psi^{(\sigma)} < [\phi|\sigma]$, it follows:

$$p\big(\phi^{(\sigma)}\big) = p[\phi|\sigma] - \sum_{\psi<\phi} p\big(\psi^{(\sigma)}\big)\;. \tag{8.16}$$

This equation, related to the *Moebius transform*, is sufficient for computing $p\big(\phi^{(\sigma)}\big)$ recursively.

Deriving $p\big(\phi^{(\sigma\wedge\tau)}\big)$. First, it is shown recursively that:

$$p\big(\phi^{(\sigma)}\wedge\psi^{(\tau)}\big) = p\big(\phi^{(\sigma)}\big)p\big(\psi^{(\tau)}\big)\;. \tag{8.17}$$

Proof - step 1. For the smallest element $\mu\equiv\bigwedge_{\phi\in<\Phi>}\phi$, it happens that $\mu^{(\sigma)}\equiv[\mu|\sigma]$ and $\mu^{(\tau)}\equiv[\mu|\tau]$. Since $[\mu|\sigma]$ and $[\mu|\tau]$ are independent propositions, it follows then $p(\mu^{(\sigma)}\wedge\mu^{(\tau)}\big) = p\big(\mu^{(\sigma)}\big)p\big(\mu^{(\tau)}\big)$.

Proof - step 2. Being given $\phi,\psi\in<\Phi>$, assume $p\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big) = p\big(\eta^{(\sigma)}\big)p\big(\zeta^{(\tau)}\big)$ for any $\eta,\zeta\in<\Phi>$ such that ($\eta\leq\phi$ and $\zeta<\psi$) or ($\eta<\phi$ and $\zeta\leq\psi$). From $[\phi|\sigma]\equiv\bigvee_{\eta\leq\phi}\eta^{(\sigma)}$ and $[\psi|\tau]\equiv\bigvee_{\zeta\leq\psi}\zeta^{(\tau)}$ it is deduced:

$$[\phi|\sigma]\wedge[\psi|\tau]\equiv\left(\bigvee_{\eta\leq\phi}\eta^{(\sigma)}\right)\wedge\left(\bigvee_{\zeta\leq\psi}\zeta^{(\tau)}\right)\equiv\bigvee_{\eta\leq\phi}\bigvee_{\zeta\leq\psi}\Big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\Big)\;.$$

It follows:

$$\begin{cases} p\big([\phi|\sigma]\wedge[\psi|\tau]\big) = \displaystyle\sum_{\eta\leq\phi}\sum_{\zeta\leq\psi} p\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big) \\ p[\phi|\sigma] = \displaystyle\sum_{\eta\leq\phi} p\big(\eta^{(\sigma)}\big) \\ p[\psi|\tau] = \displaystyle\sum_{\zeta\leq\psi} p\big(\zeta^{(\tau)}\big) \end{cases}$$

Now, $[\phi|\sigma]$ and $[\psi|\tau]$ are independent and $p\big([\phi|\sigma]\wedge[\psi|\tau]\big) = p[\phi|\sigma]p[\psi|\tau]$. Then:

$$\sum_{\eta\leq\phi}\sum_{\zeta\leq\psi} p\big(\eta^{(\sigma)}\wedge\zeta^{(\tau)}\big) = \sum_{\eta\leq\phi}\sum_{\zeta\leq\psi} p\big(\eta^{(\sigma)}\big)p\big(\zeta^{(\tau)}\big)\;.$$

From the recursion assumption, it is deduced $p\big(\phi^{(\sigma)}\wedge\psi^{(\tau)}\big) = p\big(\phi^{(\sigma)}\big)p\big(\psi^{(\tau)}\big)$.

□□□

From the factorization (8.15), it is deduced:

$$p\big(\phi^{(\sigma\wedge\tau)}\big) = \sum_{\eta\wedge\zeta\equiv\phi} p\big(\eta^{(\sigma)}\big)p\big(\zeta^{(\tau)}\big) \tag{8.18}$$

This result relies strongly on the independence hypothesis about the sensors.

Back to $[\phi|\sigma\wedge\tau]$. Reminding that $[\phi|\sigma\wedge\tau]\equiv\bigvee_{\psi\leq\phi}\psi^{(\sigma\wedge\tau)}$, the fused probability $p[\phi|\sigma\wedge\tau]$ is deduced from $p\big(\psi^{(\sigma\wedge\tau)}\big)$ by means of the relation:

$$p[\phi|\sigma\wedge\tau] = \sum_{\psi\leq\phi} p\big(\psi^{(\sigma\wedge\tau)}\big)\;. \tag{8.19}$$

Conclusion. It is possible to derive an exact fused information $p[\phi|\sigma \wedge \tau]$, $\phi \in <\Phi>$ from the informations $p[\phi|\sigma]$, $\phi \in <\Phi>$ and $p[\phi|\tau]$, $\phi \in <\Phi>$ obtained from two *independent* sensors σ and τ . This derivation is done in 3 steps:

- Compute $p\big(\phi^{(\sigma)}\big)$ and $p\big(\phi^{(\tau)}\big)$ by means of (8.16) ,
- Compute $p\big(\phi^{(\sigma\wedge\tau)}\big)$ by means of (8.18) ,
- Compute $p[\phi|\sigma \wedge \tau]$ by means of (8.19) .

8.6.4.4 Link with the DSmT

It is noteworthy that the relation (8.18) looks strangely like the DSmT fusion rule (8.4) , although these two results have been obtained from quite different viewpoints. In fact the similarity is not just related to the fusion rule and the whole construction is identical. More precisely, let us now consider the problem from the DSmT viewpoint.

Let be defined for two sensors σ and τ the respective bba m_σ and m_τ over $<\Phi>$. The belief function associated to these two bba, denoted respectively Bel_σ and Bel_τ , are just verifying:

$$\mathrm{Bel}_\sigma(\phi) = \sum_{\psi\leq\phi} m_\sigma(\psi) \quad \text{and} \quad \mathrm{Bel}_\tau(\phi) = \sum_{\psi\leq\phi} m_\tau(\psi) \ .$$

Conversely, the bba m_σ is recovered by means of the recursion:

$$\forall\phi \in <\Phi>, \, m_\sigma(\phi) = \mathrm{Bel}_\sigma(\phi) - \sum_{\psi<\phi} m_\sigma(\psi) \ .$$

The fused bba $m_\sigma \oplus m_\tau$ is defined by:

$$m_\sigma \oplus m_\tau(\phi) = \sum_{\psi\wedge\eta\equiv\phi} m_\sigma(\psi) m_\tau(\eta) \, .$$

Now make the hypothesis that the probabilities $p[\phi|\sigma]$ and $p[\phi|\tau]$ are initialized for any $\phi \in <\Phi>$ by:

$$p[\phi|\sigma] = \mathrm{Bel}_\sigma(\phi) \quad \text{and} \quad p[\phi|\tau] = \mathrm{Bel}_\tau(\phi) \ .$$

Then, the following results are obviously obtained:

- $p\big(\phi^{(\sigma)}\big) = m_\sigma(\phi)$ and $p\big(\phi^{(\tau)}\big) = m_\tau(\phi)$,
- $p\big(\phi^{(\sigma\wedge\tau)}\big) = m_\sigma \oplus m_\tau(\phi)$,
- $p[\phi|\sigma \wedge \tau] = \mathrm{Bel}_\sigma \oplus \mathrm{Bel}_\tau(\phi)$, where $\mathrm{Bel}_\sigma \oplus \mathrm{Bel}_\tau$ is the belief function associated to $m_\sigma \oplus m_\tau$.

From this discussion, *it seems natural to consider the probabilized multi-modal logic mM as a possible logical interpretation of DSmT.*

Evaluate the consequence of the independence axiom. By using the axiom m.indep. , it is possible to prove (8.13) . Otherwise, it is only possible to prove (8.12) , which means that possibly more belief is put on the smallest propositions, in comparison with the independent sensors case. Such a property expresses a better and more precise knowledge about the world. Then it appears, accordingly to the mM interpretation of DSmT, that *the fusion rule* $\oplus$ *is an optimal rule only for fusing independent and (strictly) reliable sensors*.

8.7 Logical interpretation of the Bayes inference

Notation. In the sequel, $\phi \leftrightarrow \psi$ is just an equivalent notation for $(\psi \rightarrow \phi) \wedge (\phi \rightarrow \psi)$.

General discussion. The Bayes inference explains the probability of a proposition ψ , while is known a proposition ϕ . This probability is expressed as follows by the quantity $p(\psi|\phi)$:

$$p(\phi \wedge \psi) = p(\phi)p(\psi|\phi) \ .$$

From this implicit and probabilistic definition, $(\psi|\phi)$ appears more like a mathematical artifice than an actual "logical" operator. However, $(\psi|\phi)$ has clearly a meta-logical meaning although it is intuitive and just implied: it characterizes the knowledge about ψ , when a prior information ϕ is known. In this section, we are trying to interpret the Bayes operator $(\ |\ )$ as a logical operator. The author admits that this viewpoint seems extremely suspicious: the Bayes inference implies a change of the probabilistic universe, and then a change of the truth values! It makes no sense to put at the same level a conditional probability with an unconditional probability! But in fact, there are logics which handle multiple truths: the modal logics, and more precisely, the multi-modal logics. However, the model we are defining here is quite different from the usual modal models.
From now on, we are assuming a same logic involving the whole operators, *ie.* $\wedge$, $\neg$, $\vee$, $\rightarrow$ *and* $(\ |\ )$, and a *same* probability function p defined over the resulting propositions.

When defining a logic, a first step is perhaps to enumerate the intuitive properties the new logic should have, and then derive new language and rules. Since a probability is based on a Boolean algebra, this logic will include the classical logic. A first question arises then: *is the Bayes inference* $(\ |\ )$ *the same inference than in classical logic ?* More precisely, do we have $(\psi|\phi) \equiv \phi \rightarrow \psi$? If our logical model is coherent with the probability, this should imply:

$$p(\psi|\phi) = p(\phi \rightarrow \psi) = p(\neg\phi \vee \psi) \ .$$

Applying the Bayes rule, it is deduced:

$$p(\phi \wedge \psi) = p(\phi)p(\neg\phi \vee \psi) = (p(\phi \wedge \psi) + p(\phi \wedge \neg\psi))(1 - p(\phi \wedge \neg\psi)) \ .$$

This is clearly false: *eg.* taking $p(\phi\wedge\neg\psi)=\frac{1}{4}$ and $p(\phi\wedge\psi)=\frac{1}{2}$ results in $\frac{1}{2}=\frac{9}{16}$! *The Bayes inference* $(\psi|\phi)$ *is not a classical inference.* Since it is a new kind of inference, we have to explain the meaning of this inference.

The Bayes inference seems to rely on the following principles:

- Any proposition ϕ induces a sub-universe, entirely characterized by the Bayes operator $(\cdot|\phi)$. For this reason, $(\cdot|\phi)$ could be seen as a conditional modality. But this modality possesses a strange quality: the implied sub-universe is essentially *classical*. From now on, $(\cdot|\phi)$ refers both to the modality and its induced sub-universe,
- The sub-universe $(\cdot|\top)$ is just the whole universe. The empty universe $(\cdot|\bot)$ is a singularity which cannot be manipulated,
- The sub-universe $(\cdot|\phi)$ is a projection of the sup-universe (which could be another sub-universe) into ϕ. In particular, the axioms of $(\cdot|\phi)$ result from the propositions which are axioms within the range ϕ in the sup-universe. Moreover, the modus ponens should work in the sub-universes,
- Any sub-proposition $(\psi|\phi)$ implies the infered proposition $\phi\to\psi$ in the sup-universe. This last point in not exactly the converse of the previous point. The previous point concerns axioms, while any possible propositions are considered here. This (modal-like) difference is necessary and makes the distinction between $(\ |\ )$ and $\to$,
- Since sub-universes are classical, the negation has a classical behavior: the double negation vanishes,
- The sub-universe of a sub-universe is the intersected sub-universe. For example, "considering blue animals within a universe of birds" means "considering blue birds".

In association with the Bayes inference is the notion of independence between propositions, described by the meta-operator $\times$, which is not an operator of the logic. More precisely, ψ is independent to ϕ, *ie.* $\psi\times\phi$, when it is equivalent to consider ψ within the sub-universe ϕ or within the sup-universe. Deciding whether this meta-operator is symmetric or not is probably another philosophical issue. In the sequel, this hypothesis is made possible in the axiomatization but is not required. Moreover, it seems reasonable that complementary propositions like ϕ and $\neg\phi$ cannot be independent unless $\phi\equiv\top$. In the following discussion, such a rule is proposed but not required.

8.7.1 Definitions

8.7.1.1 Bayesian modal language

The set of the *Bayesian* propositions bM is constructed recursively:

- $C\subset bM$,

- If $\phi, \psi \in bM$, then $(\psi|\phi) \in bM$,
- If $\phi, \psi \in bM$, then $\neg\phi \in bM$, $\phi \wedge \psi \in bM$, $\phi \vee \psi \in bM$ and $\phi \rightarrow \psi \in bM$.

8.7.1.2 Bayesian logical rules

The logic over bM obeys the following rules and axioms:

- Classical axioms and modus ponens,

b.i. $(\phi|\top) \equiv \phi$; *the sub-universe of* $\top$ *is of course the whole universe,*

b.ii. It is assumed $\nvdash \neg\phi$. Then, $\vdash \phi \rightarrow \psi$ implies $\vdash (\psi|\phi)$; *axioms within the range* ϕ *are axioms of the sub-universe* $(\cdot|\phi)$,

b.iii. It is assumed $\nvdash \neg\phi$. Then, $\vdash (\psi \rightarrow \eta|\phi) \rightarrow \big((\psi|\phi) \rightarrow (\eta|\phi)\big)$; *when both an inference and a premise are recognized true in a sub-universe, the conclusion also holds true in this sub-universe. This property allows the modus ponens within sub-universes,*

b.iv. It is assumed $\nvdash \neg\phi$. Then, $\vdash (\psi|\phi) \rightarrow (\phi \rightarrow \psi)$; *the modality* $(\cdot|\phi)$ *implies the truth within the range* ϕ*,*

b.v. It is assumed $\nvdash \neg\phi$. Then, $\neg(\neg\psi|\phi) \equiv (\psi|\phi)$; *there is no doubt within the modality* $(\cdot|\phi)$. *Sub-universes have a classical negation operator. However, truth may change depending on the proposition of reference* ϕ,

b.vi. It is assumed $\nvdash \neg(\phi \wedge \psi)$.[3] Then, $\big((\eta|\psi)\big|\phi\big) \equiv (\eta|\psi \wedge \phi)$; *the sub-universe* $(\cdot|\psi)$ *of a sub-universe* $(\cdot|\phi)$ *is the intersected sub-universe* $(\cdot|\phi \wedge \psi)$,

b.vii. $\psi \times \phi$ means $\vdash (\psi|\phi) \leftrightarrow \psi$; *a proposition* ψ *is independent to a proposition* ϕ *when it makes no difference to observe it in the sub-universe* $(\cdot|\phi)$ *or not,*

b.viii. (optional) $\psi \times \phi$ implies $\phi \times \psi$; *the independence relation is symmetric,*

b.ix. (optional) Assuming $\phi \times \psi$ and $\vdash \phi \vee \psi$ implies $\vdash \phi$ or $\vdash \psi$; *this uncommon logical rule explains that complementary and non trivial propositions cannot be independent.* EG. *to an extreme degree,* ϕ *and* $\neg\phi$ *are strictly complementary and at the same time are not independent unless* $\phi \equiv \top$ *or* $\phi \equiv \bot$.

These axioms leave the modality $(\cdot|\bot)$ undefined, by requiring the condition $\nvdash \neg\phi$ for any deduction on the sub-universe $(\cdot|\phi)$. In fact, the modality $(\cdot|\bot)$ is a singularity which cannot be defined according to the common axioms and rules. Otherwise, it would be deduced from $\vdash \bot \rightarrow \phi$ that $\vdash (\phi|\bot)$; this last deduction working for any ϕ would contradict the negation rule $\neg(\neg\phi|\bot) \equiv (\phi|\bot)$. Nevertheless, the axioms **b.vii.** and **b.viii.** induces a definition of $\times$ for any pair of propositions, except $(\bot, \bot)$.

[3] It will be proved that the hypothesis $\nvdash \neg(\phi \wedge \psi)$ implies the hypotheses $\nvdash \neg\phi$ and $\nvdash (\neg\psi|\phi)$.

8.7.2 Properties

8.7.2.1 Probability over bM

A probability p over bM is defined according to the definition of section 8.4. In particular, since the meta-operator $\times$ characterizes an independence between propositions, it is naturally hypothesized that:

$$\phi \times \psi \quad \text{implies} \quad p(\phi \wedge \psi) = p(\phi)p(\psi)\ .$$

8.7.2.2 Useful theorems

Sub-universes are classical. It is assumed $\nvdash \neg\phi$. Then:

- $(\neg\psi|\phi) \equiv \neg(\psi|\phi)$,
- $(\psi \wedge \eta|\phi) \equiv (\psi|\phi) \wedge (\eta|\phi)$,
- $(\psi \vee \eta|\phi) \equiv (\psi|\phi) \vee (\eta|\phi)$,
- $(\psi \rightarrow \eta|\phi) \equiv (\psi|\phi) \rightarrow (\eta|\phi)$.

Proof. The first theorem is a consequence of axiom **b.v.**

From axiom **b.iii.**, it is deduced $\vdash (\neg\psi \vee \eta|\phi) \rightarrow \big(\neg(\psi|\phi) \vee (\eta|\phi)\big)$. Applying the first theorem, it is deduced $\vdash (\neg\psi \vee \eta|\phi) \rightarrow \big((\neg\psi|\phi) \vee (\eta|\phi)\big)$. At last:

$$\vdash (\psi \vee \eta|\phi) \rightarrow \big((\psi|\phi) \vee (\eta|\phi)\big)\ . \tag{8.20}$$

It is deduced $\vdash \neg\big((\psi|\phi) \vee (\eta|\phi)\big) \rightarrow \neg(\psi \vee \eta|\phi)$ and, by applying the first theorem,

$$\vdash \big((\neg\psi|\phi) \wedge (\neg\eta|\phi)\big) \rightarrow (\neg\psi \wedge \neg\eta|\phi)\ .$$

At last:

$$\vdash \big((\psi|\phi) \wedge (\eta|\phi)\big) \rightarrow (\psi \wedge \eta|\phi)\ .$$

Now, it is deduced from $\vdash \phi \rightarrow \big((\psi \wedge \eta) \rightarrow \psi\big)$ that:

$$\vdash \big((\psi \wedge \eta) \rightarrow \psi\big|\phi\big)\ .$$

By applying the axiom **b.iii.**:

$$\vdash (\psi \wedge \eta|\phi) \rightarrow (\psi|\phi)\ .$$

It is similarly proved that $\vdash (\psi \wedge \eta|\phi) \rightarrow (\eta|\phi)$ and finally:

$$\vdash (\psi \wedge \eta|\phi) \rightarrow \big((\psi|\phi) \wedge (\eta|\phi)\big)\ .$$

The second theorem is then proved.

Third theorem is a consequence of the first and second theorem.

Last theorem is a consequence of the first and third theorem.

□□□

Inference property. It is assumed $\nvdash \neg\phi$. Then $(\psi|\phi) \wedge \phi \equiv \phi \wedge \psi$. In particular, the hypothesis $\nvdash \neg(\phi \wedge \psi)$ implies the hypotheses $\nvdash \neg\phi$ and $\nvdash (\neg\psi|\phi)$.

Proof. From **b.iv.** it comes $\vdash (\psi|\phi) \rightarrow (\phi \rightarrow \psi)$. Then $\vdash \neg(\phi \rightarrow \psi) \rightarrow \neg(\psi|\phi)$ and $\vdash (\phi\wedge\neg\psi) \rightarrow (\neg\psi|\phi)$.

It follows $\vdash (\phi \wedge \psi) \rightarrow (\psi|\phi)$ and finally:

$$\vdash (\phi \wedge \psi) \rightarrow \big((\psi|\phi) \wedge \phi\big) \ .$$

The converse is more simple. From $\vdash (\psi|\phi) \rightarrow (\phi \rightarrow \psi)$, it follows:

$$\vdash \big((\psi|\phi) \wedge \phi\big) \rightarrow \big((\phi \rightarrow \psi) \wedge \phi\big) \ .$$

Since $(\phi \rightarrow \psi) \wedge \phi \equiv \phi \wedge \psi$, the converse is proved.

□□□

Intra-independence. It is assumed $\nvdash \neg\phi$. Then $(\eta|\phi) \times (\psi|\phi)$ is equivalently defined by the property $\vdash \big((\eta|\psi) \leftrightarrow \eta\big|\phi\big)$.

Proof.

$$\big((\eta|\psi) \leftrightarrow \eta\big|\phi\big) \equiv \big((\eta|\psi)\big|\phi\big) \leftrightarrow (\eta|\phi) \equiv (\eta|\phi \wedge \psi) \leftrightarrow (\eta|\phi)$$

$$\equiv \big(\eta\big|\phi \wedge (\psi|\phi)\big) \leftrightarrow (\eta|\phi) \equiv \big((\eta|\phi)\big|(\psi|\phi)\big) \leftrightarrow (\eta|\phi) \ .$$

□□□

Independence invariant. $\psi \times \phi$ implies $\neg\psi \times \phi$.

Proof.

$$(\neg\psi|\phi) \leftrightarrow \neg\psi \equiv \neg(\psi|\phi) \leftrightarrow \neg\psi \equiv (\psi|\phi) \leftrightarrow \psi \ .$$

□□□

Inter-independence. It is assumed $\nvdash \neg\phi$. Then $(\psi|\phi) \times \phi$.

Proof. From axiom **b.vi.**:

$$((\psi|\phi)|\phi) \equiv (\psi|\phi \wedge \phi) \equiv (\psi|\phi) \ .$$

It is deduced $(\psi|\phi) \times \phi$.

□□□

Corollary: assuming the rules b.viii. and b.ix., the hypotheses $\nvdash \neg\phi$ and $\nvdash (\neg\psi|\phi)$ imply the hypothesis $\nvdash \neg(\phi \wedge \psi)$.

Proof. Assume $\vdash \neg(\phi \wedge \psi)$. Then $\vdash \neg\big(\phi \wedge (\psi|\phi)\big)$ and $\vdash \neg\phi \vee \neg(\psi|\phi)$. Since $(\neg\psi|\phi) \times \phi$, it follows $\phi \times (\neg\psi|\phi)$ from rule **b.viii.** And then $\neg\phi \times \neg(\psi|\phi)$. Now, applying the rule **b.ix.** to $\vdash \neg\phi \vee \neg(\psi|\phi)$, it is deduced $\vdash \neg\phi$ or $\vdash \neg(\psi|\phi)$.

□□□

A proposition is true in its proper sub-universe. It is assumed $\nvdash \neg\phi$. Then $\vdash (\phi|\phi)$.

Proof. Obvious from $\vdash \phi \rightarrow \phi$.

□□□

Narcissist independence. It is assumed $\nvdash \neg\phi$. Then, $\phi \times \phi$ implies $\vdash \phi$ and conversely . In particular, $\phi \times \phi$ implies $\phi \equiv \top$.

Proof.

$$(\phi|\phi) \leftrightarrow \phi \equiv \top \leftrightarrow \phi \equiv \phi \,.$$

□□□

Non transitivity (modus barbara fails). It is assumed $\nvdash \neg\phi$ and $\nvdash \neg\psi$. Then

$$\nvdash (\psi|\phi) \rightarrow \big((\eta|\psi) \rightarrow (\eta|\phi)\big) \,.$$

Proof. The choice $\psi \equiv \top$, $\eta \equiv \neg\phi$ and $\phi \not\equiv \top$ is a counter example:

$$(\top|\phi) \rightarrow \big((\neg\phi|\top) \rightarrow (\neg\phi|\phi)\big) \equiv \top \rightarrow (\neg\phi \rightarrow \bot) \equiv \phi \,.$$

□□□

8.7.2.3 Axioms and rules extend to sub-universes

Assume $\nvdash \neg\phi$. The rules and axioms of bM extend on the sub-universe $(\cdot|\phi)$:

- $\vdash \psi$ implies $\vdash (\psi|\phi)$,
- It is assumed $\nvdash \neg(\phi \wedge \psi)$. Then $\vdash (\psi \rightarrow \eta|\phi)$ implies $\vdash \big((\eta|\psi)|\phi\big)$,
- It is assumed $\nvdash \neg(\phi \wedge \psi)$. Then $\vdash \big((\eta \rightarrow \zeta|\psi)\big|\phi\big) \rightarrow \big((\eta|\psi) \rightarrow (\zeta|\psi)\big|\phi\big)$,
- It is assumed $\nvdash \neg(\phi \wedge \psi)$. Then $\vdash \big((\eta|\psi)\big|\phi\big) \rightarrow (\psi \rightarrow \eta|\phi)$.

Proof. $\vdash \psi$ implies $\vdash \phi \rightarrow \psi$ and then $\vdash (\psi|\phi)$. *First point is then proved.*

It is successively implied from $\vdash (\psi \rightarrow \eta|\phi)$:

$\vdash (\psi|\phi) \rightarrow (\eta|\phi)$,

$\vdash \big((\eta|\phi)\big|(\psi|\phi)\big)$,

$\vdash \big(\eta\big|\phi \wedge (\psi|\phi)\big)$,

$\vdash (\eta|\phi \wedge \psi)$,

$\vdash \big((\eta|\psi)\big|\phi\big)$.

Second point is then proved.

By applying axiom **b.iii.** and first point, it comes:

$$\vdash \Big((\eta \to \zeta|\psi) \to \big((\eta|\psi) \to (\zeta|\psi) \big) \Big| \phi \Big) .$$

It follows:

$$\vdash \big((\eta \to \zeta|\psi) \big| \phi \big) \to \big((\eta|\psi) \to (\zeta|\psi) \big| \phi \big) .$$

Third point is proved.

By applying axiom **b.iv.** and first point, it comes:

$$\vdash \big((\eta|\psi) \to (\psi \to \eta) \big| \phi \big) .$$

It follows:

$$\vdash \big((\eta|\psi) \big| \phi \big) \to (\psi \to \eta|\phi) .$$

Fourth point is proved.

□□□

8.7.2.4 Bayes inference

It is assumed $\nvdash \neg\phi$. Define $p(\psi|\phi)$ as an abbreviation for $p\big((\psi|\phi)\big)$. Then:

$$p(\psi|\phi)p(\phi) = p(\phi \wedge \psi) .$$

Proof. This result is implied by the theorems $(\psi|\phi) \wedge \phi \equiv \phi \wedge \psi$ and $(\psi|\phi) \times \phi$.

□□□

8.7.2.5 Conclusion

Finally, the Bayes inference has been recovered from our axiomatization of the operator $(\cdot|\cdot)$. Although this result needs more investigation, in particular for the justification of the coherence of bM , it appears that the Bayesian inference could be interpreted logically as a manner to handle the knowledges. A similar result has been obtained for the fusion rule of DSmT. At last, it seems possible to conjecture that logics and probability could be mixed in order to derive many other belief rules or inferences.

8.8 Conclusion

In this contribution, it has been shown that DSmT was interpretable in the paradigm of probabilized multi-modal logic. This logical characterization has made apparent the true necessity of an independence hypothesis about the sensors, when applying the $\oplus$ fusion rule. Moreover, it is expected that our work has given some clarifications about the semantic associated with the conjunctive rule of DSmT.

A similar logical interpretation of the Bayes inference has been constructed, although this preliminary work should be improved. At last, it seems possible to handle probabilized logics as a relatively general framework for manipulating non deterministic informations. This is perhaps a generic method for constructing new customized belief theories. The principle is first to construct a logic well adapted to the problem, second to probabilize this logic, and third to derive the implied new belief theory (and forget then the mother logic!) :

Classical Logic	Probabilized	Probability
↓	$\Longrightarrow$	↓
New Logic	propositions	New Belief Theory

It seems obviously that there could be many theories and rules for manipulating non deterministic informations. This is not a new result and I feel necessary to refer to the works of *Sombo, Lefèvre, De Brucq and al.* [6, 4, 7] , which have investigated such questions.

At last, a common framework for both DSmT and Bayesian inference could be certainly derived by fusing the logics mM and bM .

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8.9 References

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