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Rough sets in Neutrosophic Approximation Space

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Abstract. A rough set is a formal approximation of a crisp set which gives lower and upper approximation of original set to deal with uncertainties. The concept of neutrosophic set is a mathematical tool for handling imprecise, indeterministic and inconsistent data. In this paper, we introduce the concepts of neutrosophic rough Sets and investigate some of its properties. Further as the characterisation of neutrosophic rough approximation operators, we introduce various notions of cut sets of neutrosophic rough sets.

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1. Introduction

Rough set theory [9], is an extension of set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Moreover, it is a mathematical tool for machine learning, information sciences and expert systems and successfully applied in data analysis and data mining. There are two basic elements in rough set theory, crisp set and equivalence relation, which constitute the mathematical basis of rough set.In classical rough set theory partition or equivalence relation is the basic concept. The theory of rough sets is based upon the classification mechanism, from which the classification can be viewed as an equivalence relation and knowledge blocks induced by it be a partition on universe. The basic idea of rough set is based upon the approximation of sets by a pair of sets known as the lower approximation and the upper approximation of a set . Any subset of a universe can be characterized by two definable or observable subsets called lower and upper approximations. Zadeh introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. Now fuzzy sets are combined with rough sets in a fruitful way and defined by rough fuzzy sets and fuzzy rough sets [6,7,8]. Atanassov[1] introduced the degree of nonmembership/falsehood (f) and defined the intuitionistic fuzzy sets. One of the interesting generalizations of the theory of fuzzy sets and intuitionistic fuzzy sets is the theory of neutrosophic sets introduced by F. Smarandache[11,12] which deals with the degree of indeterminacy/neutrality (i) as independent component. Neutrosophy is a branch of philosphy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. The idea of neutrosophy is applied in many fields in order to solve problems related to indeterminacy. Neutrosophic sets are described by three functions: Truth function, indeterminacy function and false function that are independently related. The theories of neutrosophic set have achieved great success in various areas such as medical diagnosis, database, topology, image processing, and decision making problem [4,5,18,19]. While the neutrosophic set is a powerful tool to deal with indeterminate and inconsistent data and the theory of rough sets is a powerful mathematical tool to deal with incompleteness. Neutrosophic sets and rough sets are two different topics, none conflicts the other. Recently many researchers had applied the notion of neutrosophic sets to relations, group theory, ring theory, Soft set theory and so on.

In this paper we combine the mathematical tools rough sets and neutrosophic sets and introduce a new class of rough sets in neutrosophic approximation space.First we review some basic notions related to rough sets and neutrosophic sets and then we construct the neutrosophic rough approximation operators and introduce neutrosophic rough sets and discuss some of their interesting properties.

2. Preliminaries

Definition 2.1 ([11])**.** A Neutrosophic set A on the universe of discourse X is defined as $A = \{\langle\ x, T_A(x), I_A(x), F_A(x)\rangle, x \in X\}$ where $T, F, I : X \longrightarrow [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$

Definition 2.2 ([11])**.** If $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle\ /x \in X\}$ and $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle\ /x \in X\}$ are any two neutrosophic sets of X then

(i) $A \subseteq B \ \Leftrightarrow T_A(x) \leq T_B(x);\ I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$

(ii) $A = B \ \Leftrightarrow T_A(x) = T_B(x);\ I_A(x) = I_B(x)$ and $F_A(x) = F_B(x)\ \forall x \in X$

(iii) $\sim A = \{\langle x, F_A(x), 1 - I_A(x), T_A(x)\rangle\ /x \in X\}$

(iv) $A \cap B = \{\langle x, T_{(A\cap B)}(x), \ \ I_{(A\cap B)}(x), F_{(A\cap B)}(x)\rangle/x \in X\}$ where
$T_{A\cap B}(x) = min\{T_A(x), T_B(x)\} \quad I_{A\cap B}(x) = min\{I_A(x), I_B(x)\}$
$F_{A\cap B}(x) = max\{F_A(x), F_B(x)\}$

$v)$ $A \cup B = \{\langle x, T_{(A\cup B)}(x), \ \ I_{(A\cup B)}(x), F_{(A\cup B)}(x)\rangle/x \in X\}$ where
$T_{A\cup B}(x) = max\{T_A(x), T_B(x)\} \quad I_{A\cup B}(x) = max\{I_A(x), I_B(x)\}$
$F_{A\cup B}(x) = min\{F_A(x), F_B(x)\}$

Definition 2.3 ([7])**.** Let $R \subseteq U \times U$ be a crisp binary relation on U. R is referred to as reflexive if $(x, x) \in R$ for all $x \in U$. R is referred to as symmetric if for all (x,y) $\in$ U, (x,y) $\in$ R implies (y,x) $\in$ R and R is referred to as transitive if for all x,y,z $\in$ U, (x,y) $\in$ R and (y,z) $\in$ R implies (x,z) $\in$ R.

Definition 2.4 ([7]). Let U be a non empty universe of discourse and $R \subseteq U \times U$, an arbitrary crisp relation on U. Then $xR = \{y \in U/(x,y) \in R\}, \in U$
where xR is called the R-after set of x (Bandler and kohout 1980) or successor neighbourhood of x with respect to R (Yao 1998 b). The pair (U,R) is called a crisp approximation space. For any $A \subseteq U$ the upper and lower approximation of A with respect to (U,R) denoted by $\overline{R}$ and $\underline{R}$ are respectively defined as follows
$\overline{R} = \{x \in U/xR \cap A \neq \varphi\}$
$\underline{R} = \{x \in U/xR \subseteq A\}$
The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as crisp rough set of A with respect to (U,R) and $\overline{R}, \underline{R} : \rho(U) \to \rho(U)$ are referred to upper and lower crisp approximation operator respectively.
The crisp approximation operator satisfies the following properties for all A, B $\in \rho(U)$

(L_1) $\underline{R}(A) = \overline{R'}(A')$	$(U_1)\overline{R} = \ \underline{R}(\ A)$
$(L_2)\underline{R}(U) = U$	$(U_2)\overline{R} \ \ \varphi = \varphi$
(L_3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$	$(U_3)\overline{R}(A \cap B) = \bar{R}(A) \cup \bar{R}(B)$
(L_4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$	$(U_4)A \subseteq B = \overline{R}(A) \subseteq \overline{R}(B)$
(L_5) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$	$(U_5)\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$

Properties $(L_1)$and(U_1) show that the approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other. Properties with the same number may be considered as a dual properties. If R is equivalence relation in U then the pair (U,R) is called a Pawlak approximation space and $(\underline{R}(A), \overline{R}(A))$ is a Pawlak rough set, in such a case the approximation operators have additional properties.

3. NEUTROSOPHIC ROUGH SETS

In this section, we introduce neutrosophic approximation space and neutrosophic approximation operators induced from the same. Further we define a new type of set called neutrosophic rough set and investigate some of its properties.

Definition 3.1. A constant neutrosophic set is defined by $\overline{\overline{(\alpha,\beta,\gamma)}} = \{\ \langle x, \alpha, \beta, \gamma\ \rangle / \ x \in U\}$ where $0 \leq \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta + \gamma \leq 3$. and

We introduce a special Neutrosophic set ly for y $\in$ U as follows

$$T_{1y}(x) = \begin{cases} 1, & if \quad x = y \\ 0, & if \quad x \neq y \end{cases}$$

$$T_{1(u-y)}(x) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y \end{cases}$$

$$I_{1y} = \begin{cases} 1, & if \quad x = y \\ 0, & if \quad x \neq y \end{cases}$$

$$I_{1(u-(y)}(x) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y \end{cases}$$

$$F_{1y}(x) = \begin{cases} 0, & if \quad x = y \\ 1, & if \quad x \neq y \end{cases}$$

$$F_{1(u-(y))} = \begin{cases} 1, & if \quad x = y \\ 0, & if \quad x \neq y \end{cases}$$

.

Definition 3.2. A neutrosophic relation on U is a neutrosophic subset
$R = \{\langle x, y\rangle, T_R(x, y), I_R(x, y), F_R(x, y)/x, y \in U\}$
$T_R : U \times U \longrightarrow [0, 1]$; $I_R : U \times U \longrightarrow [0, 1]$; $F_R : U \times U \longrightarrow [0, 1]$ satisfies
$0 \leq T_R(x, y) + I_R(x, y) + F_R(x, y) \leq 3$ for all $(x, y) \in U \times U$. We denote the family of all neutrosophic relation on U by N(U × U).

Definition 3.3. Let U be a non empty universe of discourse. For an arbitrary neutrosophic relation R over U × U the pair (U,R) is called neutrosophic approximation space. For any A ∈ N(U), we define the upper and lower approximations with respect to (U, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$ respectively.

$$\overline{R}(A) = \{\langle x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x)\rangle/x \in U\}$$

$$\underline{R}(A) = \{\langle x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x)\rangle/x \in U\}$$

where,

$$T_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\ T_R(x, y) \wedge T_A(y)\ ]$$

$$I_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\ I_R(x, y) \wedge I_A(y)\ ]$$

$$F_{\overline{R}(A)}(x) = \bigwedge_{y \in U} [\ F_R(x, y) \vee F_A(y)\ ]$$

$$T_{\underline{R}(A)}(x) = \bigwedge_{y \in U} [\ F_R(x, y) \vee T_A(y)\ ]$$

$$I_{\underline{R}(A)}(x) = \bigwedge_{y \in U} [\ 1 - I_R(x, y) \vee I_A(y)\ ]$$

$$F_{\underline{R}(A)}(x) = \bigvee_{y \in U} [\ T_R(x, y) \wedge F_A(y)\ ]$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called neutrosophic rough set of A with respect to (U,R) and $\underline{R}, \overline{R} : N(U) \longrightarrow N(U)$ are referred to as upper and lower neutrosophic rough approximation operators respectively.

Remark 3.4. If R is an intuitionistic fuzzy relation on U then (U,R) is a intutionistic fuzzy approximation space, neutrosophic rough operators are induced from a intuitionistic fuzzy approximation space that is

$$\overline{R}(A) = \{\langle x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x)\rangle/x \in U\}\ A \in N(U)$$

$$\underline{R}(A) = \{\langle x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x)\rangle/x \in U\}\ A \in N(U)$$

where,

$$T_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\ \mu_R(x, y) \wedge T_A(y)\ ]$$

$$I_{\overline{R}(A)}(x) = \bigvee_{y \in U} [\ 1 - (\mu_R(x, y) + \gamma_R(x, y))\ \wedge I_R(y)]$$

$$F_{\overline{R}(A)}(x) = \bigwedge_{y \in U} [\ \gamma_R(x, y) \vee F_A(y)\ ]$$

$$T_{\underline{R}(A)}(x) = \bigwedge_{y \in U} [\ \gamma_R(x, y) \vee T_A(y)\ ]$$

$I_{\underline{R}(A)}(x) = \bigwedge_{y\in U} [\ (\mu_R(x,y) + \gamma_R(x,y)) \vee I_A(y)\ ]$
$F_{\underline{R}(A)}(x) = \bigvee_{y\in U} [\ \mu_R(x,y) \wedge F_A(y)\ ]$

.

Remark 3.5. If R is a crisp binary relation on U and (U,R) is a crisp approximation space, then neutrosophic rough approximation operators are induced from crisp approximation space, such that $\forall A \in N(U)$

$$\overline{R}(A) = \{\langle x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x)\rangle / x \in U\}$$
$$\underline{R}(A) = \{\langle x, T_{\underline{R}(A)}(x), I_{\underline{R}(A)}(x), F_{\underline{R}(A)}(x)\rangle / U \in U\}$$

where,
$T_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} T_A(y) \quad I_{\overline{R}(A)}(x) = \bigvee_{y\in[x]_R} I_A(y) \quad F_{\overline{R}(A)}(x) = \bigwedge_{y\in[x]_R} F_A(y)$
$T_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} T_A(y) \quad I_{\underline{R}(A)}(x) = \bigwedge_{y\in[x]_R} I_A(y) \quad F_{\underline{R}(A)}(x) = \bigvee_{y\in[x]_R} F_A(y).$

Theorem 3.6. *Let (U,R) be a neutrosophic approximation space. Then the lower and upper neutrosophic rough approximation operators induced from (U,R) satisfy the following properties.* $\forall\ A, B \in N(U)$, $\forall\ \alpha, \beta, \gamma \in [0,1]$ *with* $\alpha + \beta + \gamma\ \leq\ 3$
$(FNL1)\underline{R}(A) =\sim \underline{R}(\sim A)$, $\qquad (FNU1)\overline{R}(A) = \overline{R}(A) =\sim \overline{R}(\sim A)$
$(FNL2)\underline{R}(A\cup(\overline{\overline{\alpha,\beta,\gamma}})) = \underline{R}(A)\cup(\overline{\overline{\alpha,\beta,\gamma}})$, $(FNU2)\overline{R}(A\cap(\overline{\overline{\alpha,\beta,\gamma}})) = \overline{R}(A)\cap(\overline{\overline{\alpha,\beta,\gamma}})$
$(FNL3)\underline{R}(A\cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\qquad (FNU3)\overline{R}(A\cup B) = \bar{R}(A) \cup \bar{R}(B)$
$(FNL4)A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$ $\qquad (FNU4)A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$
$(FNL5)\underline{R}(A\cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$ $\qquad (FNU5)\overline{R}(A\cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$
$(FNL6)R_1 \subseteq R_2 \Rightarrow \underline{R_1}(A) \supseteq \underline{R_2}(A)$ $\qquad (FNU6)R_1 \subseteq R_2 \Rightarrow \overline{R_1}(A) \subseteq \overline{R_2}(A)$

Proof. We only prove properties of the lower neutrocphic rough approximation operator $\underline{R}(A)$. The upper rough neutrosophic approximation operator $\overline{R}(A)$ can be proved similarly.
(FNL1)By Definition 3.3 , we have
$\sim \underline{R}(\sim A) = \{\langle x, F_{\underline{R}(\sim A)}(X), 1 - I_{\underline{R}(\sim A)}(u), T_{\underline{R}(\sim A)}(u)\rangle / x \in U\}$
$= \{\langle x, \bigvee_{y\in U} \left[\ T_R(x,y) \wedge F_{(\sim A)}(y)\ \right], \bigwedge_{y\in U} \left[\ 1 - I_R(x,y) \vee I_{(\sim A)}(y)\ \right],$
$\bigwedge_{y\in U} \left[\ F_R(x,y) \vee T_{(\sim A)}(y)\ \right]\}$
$= \{\langle x, \bigvee_{y\in U} [\ T_R(x,y) \wedge T_A(y)\ ], \bigvee_{y\in U} [\ I_R(x,y) \wedge I_A(y)\ ], \bigwedge_{y\in U} [\ F_R(x,y) \vee F_A(y)\ ]\}$
$= \{\langle x, T_{\overline{R}(A)}(x), I_{\overline{R}(A)}(x), F_{\overline{R}(A)}(x)\rangle / x \in U\} = \overline{R}(A)$

(FNL2) $\underline{R}(A\cap B) = \{\langle x, T_{\underline{R}(A\cap B)}(x), I_{\underline{R}(A\cap B)}(x), F_{\underline{R}(A\cap B)}(x) | x \in U\rangle\}$
$= \{\langle x, \bigwedge_{y\in U} T_{(A\cap B)}(y), \bigwedge_{y\in U} I_{(A\cap B)}(y), \bigvee_{y\in U} F_{(A\cap B)}(y) | x \in U\rangle\}$
$= \{\langle x, \bigwedge_{y\in U} (T_A(y) \bigwedge T_B(y)), \bigwedge_{y\in U} (I_A(y) \bigwedge I_B(y)), \bigvee_{y\in U} (F_A(y) \bigvee F_B(y)) | x \in U\}$
$= \{\langle, T_{\underline{R}(A)}(x) \bigwedge T_{\underline{R}(B)}(x), I_{\underline{R}(A)}(x) \bigwedge I_{\underline{R}(B)}(x), F_{\underline{R}(A)}(x) \bigwedge F_{\underline{R}(B)}(x) | x \in U\rangle\}$
$= \underline{R}(A) \cap \underline{R}(B).$

(FNL3) It can be easily verified by definition of $\underline{R}(A)$.
(FNL4) It is straightforward.

Similarly we can prove the properties of the upper rough neutrosophic approximation operator. □

Remark 3.7. The properties (FNL1) and (FNU1) shows that neutrosophic rough approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other and the properties (FNL2) and (FNU2) imply, following properties $(FNL2)^{'}$ and $(FNU2)^{'}$
$(FNL2)^{'}$ $\underline{R}(U) = U$ $\quad (FNU2)^{'} = \overline{R}(\varphi) = \varphi$
.

Example 3.8. Let (U,R) be a FN approximation space where U $= \{x_1, x_2, x_3\}$ and $R \in FNR(U \times U)$ is defined as
$R = \{\langle (x_1, x_1)0.8, 0.7, 0.1\rangle \ \ \langle (x_1, x_2), 0.2, 0.5, 0.4\rangle \ \ \langle (x_1, x_3)0.6, 0.5, 0.7\rangle$

$$\langle (x_2, x_1)0.4, 0.6, 0.3\rangle \ \ \langle (x_2, x_2)0.7, 0.8, 0.1\rangle \ \ \langle (x_2, x_3)0.5, 0.3, 0.1\rangle$$

$$\langle (x_3, x_1)0.6, 0.2, 0.1\rangle \ \ \langle (x_3, x_2)0.7, 0.8, 0.1\rangle \ \ \langle (x_3, x_3)1, 0.9, 0.1\rangle\}$$

If a Fuzzy Neutrosophic set
$A = \{\langle x_1, 0.8, 0.9, 0.1\rangle \ \ \langle x_2, 0.5, 0.4, 0.3\rangle \ \ \langle x_3, 0.5, 0.4, 0.7\rangle\}$
we can calculate,

$$\overline{R}(A) = \{\langle x_1, 0.8, 0.7, 0.1\rangle \ \ \langle x_2, 0.7, 0.6, 0.3\rangle \ \ \langle x_3, 0.6, 0.4, 0.1\rangle\}$$

$$\underline{R}(A) = \{\langle x_1, 0.5, 0.5, 0.4\rangle \ \ \langle x_2, 0.5, 0.4, 0.3\rangle \ \ \langle x_3, 0.5, 0.5, 0.7\rangle\}$$

upper and lower approximations of A respectively.

Definition 3.9. Let $A \in N(U)$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma \leq 3$ and (α, β, γ) level set of A denoted by $A^{(\alpha\beta\gamma)}$ is defined as
$A^{(\alpha\beta\gamma)} = \{x \in U/T_A(x) \geq \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}$
We define

$A_\alpha = \{x \in U/T_A(x) \geq \alpha\}$ and $A_{\alpha+} = \{x \in U/T_A(x) > \alpha\}$ the α level cut and strong α level cut of truth value function generated by A.

$A\beta = \{x \in U/I_A(x) \geq \beta\} and \beta+ = \{x \in U/I_A(x) > \beta\}$ the β level cut and strong β level cut of indeterminacy function generated by A and

$A^\gamma = \{x \in U/F_A(x) \leq \gamma\}$ and $A^{\gamma+} = \{x \in U/F_A(x) < \gamma\}$ the γ level cut and strong γ level cut of false value function generated by A.

Similarly, we can define the level cuts sets, such as

$A^{(\alpha+,\beta+,\gamma+)} = \{x \in U/T_A(x) > \alpha, I_A(x) > \beta, F_A(x) < \gamma\}$ is $(\alpha+, \beta+, \gamma+)$,
$A^{(\alpha+,\beta,\gamma)} = \{x \in U/T_A(x) > \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}$ is $(\alpha+, \beta, \gamma)$,
$A^{(\alpha,\beta+,\gamma)} = \{x \in U/T_A(x) \geq \alpha, I_A(x) > \beta, F_A(x) \leq \gamma\}$ is $(\alpha, \beta+, \gamma)$ and
$A^{(\alpha,\beta,\gamma+)} = \{x \in U/T_A(x) \geq \alpha, I_A(x) \geq \beta, F_A(x) < \gamma\}$ is $(\alpha, \beta, \gamma+)$
level cut set of A respaectively. Like wise other level cuts can also be defined.

Theorem 3.10. *The level cut sets of neutrosophic sets satisfy the following properties:* $\forall\ A, B \in N(U)$, $\alpha, \beta, \gamma \in [0,1]$ *with* $\alpha+\beta+\gamma \leq 3$, $\alpha_1, \beta_1, \gamma_1 \in [0,1]$ *with* $\alpha_1+\beta_1+\gamma_1 \leq 3$ *and* $\alpha_2, \beta_2, \gamma_2 \in [0,1]$ *with* $\alpha_2+\beta_2+\gamma_2 \leq 3$

(1) $A^{(\alpha,\beta,\gamma)} = A_\alpha \bigcap A\beta \bigcap A^\gamma$

(2) $(A^{'})_\alpha = A^{'\alpha+} : (A^{'})\beta = A^{'}(1-\beta+); (A^{'})^\gamma = A^{'}_{\gamma+}$

(3) $\left(\bigcap\limits_{i\in J} A_i\right)_\alpha = \bigcap\limits_{i\in J}(A_i)_\alpha$

$$\left(\bigcap_{i\in J} A_i\right)\beta = \bigcap_{i\in J}(A_i)\beta$$

$$\left(\bigcap_{i\in J} A_i\right)^\gamma = \bigcap_{i\in J}(A_i)^\gamma$$

(4) $\left(\bigcup\limits_{i\in J} A_i\right)_\alpha = \bigcup\limits_{i\in J}(A_i)_\alpha$

$$\left(\bigcup_{i\in J} A_i\right)\beta = \bigcup_{i\in J}(A_i)\beta$$

$$\left(\bigcup_{i\in J} A_i\right)^\gamma = \bigcup_{i\in J}(A_i)^\gamma$$

(5) $\left(\bigcup\limits_{i\in J} A_i\right)^{(\alpha,\beta,\gamma)} \supseteq \bigcup\limits_{i\in J}(A_i)^{(\alpha,\beta,\gamma)}$

(6) $\left(\bigcap\limits_{i\in J} A_i\right)^{(\alpha,\beta,\gamma)} \supseteq \bigcap\limits_{i\in J}(A_i)^{(\alpha,\beta,\gamma)}$

(7) For $\alpha_1 \geq \alpha_2 \quad \beta_1 \geq \beta_2 \quad \gamma_1 \leq \gamma_2$
$A_{\alpha_1} \subseteq A_{\alpha_2}; \quad A\beta_1 \subseteq A\beta_2 \quad A^{\gamma_1} \subseteq A^{\gamma_2},\ A^{(\alpha_1,\beta_1,\gamma_1)} \subseteq A^{(\alpha_2,\beta_2,\gamma_2)}$

Proof. (1) and (3) follow directly from Definition 3.9

(2) Since $A^{'} = \{\langle x, F_A(x), 1-I_A(x), T_A(x)\rangle / x \in U\}$
$(A^{'})_\alpha = \{x \in U / F_A(x) \geq \alpha\}$
By definition,

$$A^{\alpha+} = \{x \in U / F_A(x) < \alpha\}$$
$$A^{'\alpha+} = \{x \in U / F_A(x) \geq \alpha\}$$
$$\Rightarrow (A)^{'}_\alpha = (A^{'\alpha+})$$

Similarly we can prove,
$(A^{'})\beta = (A^{'}(1-\beta+))$ and $(A^{'})^\gamma = (A^{'}_{\gamma+})$.

(4) $\bigcap_{i\in J} A_i = \left\{\langle x, \bigwedge_{i\in J} T_{A_i}(x), \bigwedge_{i\in J} I_{A_i}(x), \bigvee_{i\in J} F_{A_i}(x)\rangle / x \in U\right\}$

We have $\left(\bigcap_{i\in J} A_i\right)_\alpha = \left\{x \in U / \bigwedge_{i\in J} T_{A_i}(x) \geq \alpha\right\}$ $=\{x \in U/T_{A_i}(x) \geq \alpha\} = \bigcap_{i\in J} (A_i)_\alpha$

Similarly,

$\left(\bigcap_{i\in J} A_i\right)\beta = \left\{x \in U / \bigwedge_{i\in J} I_{A_i}(x) \geq \beta\right\} = \{x \in U/I_{A_i}(x) \geq \beta \forall i \in J\} = \bigcap_{i\in J} (A_i)\beta$

and

$\left(\bigcap_{i\in J} A_i\right)^\gamma = \left\{x \in U / \bigvee_{i\in J} F_{A_i}(x) \leq \gamma\right\} = \{x \in U/F_{A_i}(x) \leq \gamma \forall i \in J\} = \bigcap_{i\in J} (A_i)_\beta$

We can conclude

$\left(\bigcap_{i\in J} A_i\right)^{\alpha,\beta,\gamma} = \left(\bigcap_{i\in J} A_i\right)_\alpha \cap \left(\bigcap_{i\in J} A_i\right)\beta \cap \left(\bigcap_{i\in J} A_i\right)^\gamma = \bigcap_{i\in J} ((A_i)_\alpha \cap (A_i)\beta \cap (A_i)^\gamma)$

$= \bigcap_{i\in J} (A_i)^{(\alpha,\beta,\gamma)}$

(5) We know

$\bigcup_{i\in J} (A_i) = \left\{\langle x, \bigvee_{i\in J} T_{A_i}(x), \bigvee_{i\in J} I_{A_i}(x), \bigwedge_{i\in J} F_{A_i}(x)\rangle / x \in U\right\}$

$\left(\bigcup_{i\in J} A_i\right)_\alpha = \left\{x \in U / \bigvee_{i\in J} T_{A_i}(x) \geq \alpha\right\} = \left\{x \in U / \bigvee_{i\in J} T_{A_i}(x) \geq \alpha,\ \exists i \in J\right\} = \bigcup_{i\in J} (A_i)_\alpha$

$\left(\bigcup_{i\in J} A_i\right)\beta = \left\{x \in U / \bigvee_{i\in J} I_{A_i}(x) \geq \beta\right\} = \{x \in U/I_{A_i}(x) \geq \beta,\ \forall i \in J\} = \bigcup_{i\in J} (A_i)\beta$

$\left(\bigcup_{i\in J} A_i\right)^\gamma = \left\{x \in U / \bigwedge_{i\in J} F_{A_i}(x) \leq \gamma\right\} = \{x \in U/F_{A_i}(x) \leq \gamma,\ \forall i \in J\} = \bigcup_{i\in J} (A_i)^\gamma$

(6) For any $x \in A_\alpha$, according to Definition 3.9 we have for $T_A(x) \geq \alpha_1 \geq \alpha_2$, we obtain $A_{\alpha_1} \subseteq A_{\alpha_2}$.

Similarly for $\beta_1 \geq \beta_2$ and $\gamma_1 \leq \gamma_2$ we obtain $A\beta_1 \subseteq A\beta_2$ and $A^{\gamma_1} \subseteq A^{\gamma_2}$.

Hence we have, $A^{(\alpha_1,\beta_1,\gamma_1)} \subseteq A^{(\alpha_2,\beta_2,\gamma_2)}$. . □

Corollary 3.11. *Assume that R is a neutrosophic relation in U,*

$R_\alpha = \{(x,y) \in U \times U/T_R(x,y) \geq \alpha\}$, $R_\alpha(x) = \{y \in U/T_R(x,y) \geq \alpha\}$,
$R_{\alpha+} = \{(x,y) \in U \times U/T_R(x,y) > \alpha\}$, $R_{\alpha+}(x) = \{y \in U/T_R(x,y) > \alpha\}$,
$R\beta = \{(x,y) \in U \times U/I_R(x,y) \geq \beta\}$, $R\beta(x) = \{y \in U/I_R(x,y) \geq \beta\}$,
$R\beta+ = \{(x,y) \in U \times U/I_R(x,y) > \beta\}$, $R\beta+(x) = \{y \in U/I_R(x,y) > \beta\}$,
$R^\gamma = \{(x,y) \in U \times U/F_R(x,y) \leq \gamma\}$, $R^\gamma(x) = \{y \in U/F_R(x,y) \leq \gamma\}$,
$R^{\gamma+} = \{(x,y) \in U \times U/F_R(x,y) < \gamma\}$, $R^{\gamma+}(x) = \{y \in U/F_R(x,y) < \alpha\}$,
$R^{(\alpha,\beta,\gamma)} = \{(x,y) \in U \times U/T_R(x,y) \geq \alpha, I_R(x,y) \geq \beta, F_R(x,y) \leq \gamma\}$,
$R^{(\alpha,\beta,\gamma)}(x) = \{y \in U/T_R(x,y) \geq \alpha, I_R(x,y) \geq \beta, F_R(x,y) \leq \gamma\}$
Then for all $R_\alpha, R_{\alpha+}, R\beta, R\beta+, R^\gamma, R^{\gamma+}, R^{(\alpha\beta\gamma)}$ are crisp relation in U and
1) If R is reflexive then the above level cuts are reflexive.
2) If R is symmetric then the above level cuts are symmetric.
3) If R is transitive then the above level cuts are transitive.

Proof. Since R is a crisp reflexive $\forall\ x \in U, \quad \alpha, \beta, \gamma \in [0,1]$
Take, $T_R(x,x) = 1 \quad I_R(x,x) = 1 \quad F_R(x,x) = 0 \quad \forall\ x \in U$
Now, we have R_α is a crisp binary relation in U and $x \in U$, $(x,x) \in R_\alpha$. Therefore R_α is reflexive. If R is symmetric then $\forall\ x, y \in U$, we have $(x,y) \in R_\alpha \Rightarrow (y,x) \in R_\alpha$. Therefore$R_\alpha$ is symmetry. Similarly we can prove $R\beta$ and R^γ are symmetric.
If R is transitive then $\forall x, y, z \in U$ and $\alpha, \beta, \gamma \in [0,1]$
$T_R(x,z) \geq T_R(x,y) \wedge T_R(y,z)$, $I_R(x,z) \geq I_R(x,y) \wedge I_R(y,z)$ and $F_R(x,z) \leq F_R(x,y) \vee F_R(y,z)$ for any $(x,y) \in R_\alpha$, $(y,z) \in R_\alpha$, $(x,y) \in R\beta$, $(y^{'}, z^{'}) \in R\beta$, $(x^{''}, y^{''}) \in R^\gamma$ and $(y^{''}, z^{''}) \in R^\gamma$
(ie) $T_R(x,y) \geq \alpha, \quad T_R(y,z) \geq \alpha \quad \Rightarrow T_R(x,z) \geq \alpha$
$I_R(x^{'}, y^{'}) \geq \beta, \quad I_R(y^{'}, z^{'}) \geq \beta \quad \Rightarrow I_R(x^{'}, z^{'}) \geq \beta$
$F_R(x^{''}, y^{''}) \leq \gamma, \quad F_R(y^{''}, z^{''}) \leq \gamma \quad \Rightarrow F_R(x^{''}, z^{''}) \leq \gamma$
Therefore $R_\alpha, R\beta, R^\gamma$ are transitive and hence $R^{(\alpha,\beta,\gamma)}$ is transitive.
Similarly we can prove other level cuts sets are transitive. □

Theorem 3.12. *Let (U,R) be a neutrosophic approximation space and $A \in N(U)$, then the upper neutrosophic approximation operator can be represented as follows $\forall\ x \in U$.*

1) $$T_{\overline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_\alpha(A_\alpha)(x)\right] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_\alpha(A_{\alpha+})(x)\right]$$
$$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_{\alpha+}(A_\alpha)(x)\right] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_{\alpha+}(A_{\alpha+})(x)\right]$$

2) $$I_{\overline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \bar{R}\alpha(A\alpha)(x)\right] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \bar{R}\alpha(A\alpha+)(x)\right]$$
$$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}\alpha+(A\alpha)(x)\right] = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}\alpha+(A\alpha+)(x)\right]$$

3) $$F_{\overline{R}(A)}(x) = \bigwedge_{\alpha\in[0,1]} \left[\alpha \vee \overline{R}^{\alpha}(A^\alpha)(x)\right] = \bigwedge_{\alpha\in[0,1]} \left[\alpha \vee \overline{R}^{\alpha}(A^{\alpha+})(x)\right]$$
$$= \bigwedge_{\alpha\in[0,1]} \left[\alpha \vee \overline{R}^{\alpha+}(A^\alpha)(x)\right] = \bigwedge_{\alpha\in[0,1]} \left[\alpha \vee \overline{R}^{\alpha+}(A^{\alpha+})(x)\right]$$

and more over for any $\alpha \in [0,1]$

4) $\left[\overline{R}(A)\right]_{\alpha+} \subseteq \bar{R}_{\alpha+}(A_{\alpha+}) \subseteq \bar{R}_\alpha(A_\alpha) \subseteq \left[\bar{R}(A)\right]_\alpha$

5) $\left[\overline{R}(A)\right]\alpha+ \subseteq \overline{R}\alpha+(A\alpha+) \subseteq \bar{R}\alpha(A\alpha) \subseteq \left[\bar{R}(A)\right]\alpha$

6) $\left[\overline{R}(A)\right]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^{\alpha}}(A^\alpha) \subseteq \left[\overline{R}(A)\right]^\alpha$

7) $\left[\overline{R}(A)\right]_{\alpha+} \subseteq \overline{R}_{\alpha+}(A_{\alpha+}) \subseteq \overline{R}_\alpha(A_\alpha) \subseteq \left[\overline{R}(A)\right]_\alpha$

8) $\left[\overline{R}(A)\right]\alpha+ \subseteq \overline{R}\alpha+(A\alpha+) \subseteq \bar{R}\alpha(A\alpha) \subseteq \left[\bar{R}(A)\right]\alpha$

9) $\left[\overline{R}(A)\right]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^{\alpha}}(A^\alpha) \subseteq \left[\overline{R}(A)\right]^\alpha$

Proof. 1) For $x \in U$, we have
$$\bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{R}_\alpha(A_\alpha)(x)\right] = Sup\left\{\alpha \in [0,1]/x \in \overline{R}_\alpha(A_\alpha)\right\}$$

$= Sup\{\alpha \in [0,1]/R_{\alpha}(x) \cap A_{\alpha} \neq \varphi\}$

$= Sup\{\alpha \in [0,1]/\exists y \in U(y \in R_{\alpha}(x), y \in A_{\alpha})\}$

$= Sup\{\alpha \in [0,1]/\exists y \in U[T_R(x,y) \geq \alpha, T_A(y) \geq \alpha]\}$

$= \bigvee_{y \in U}[T_R(x,y) \wedge T_A(y)] = T_{\overline{R}(A)}(x)$

(2) $\bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{R}\alpha(A\alpha)(x)\right] = Sup\left\{\alpha \in [0,1]/x \in \overline{R}\alpha(A\alpha)\right\}$

$= Sup\{\alpha \in [0,1]/R\alpha(x) \cap A\alpha \neq \varphi\}$

$= Sup\{\alpha \in [0,1]/\exists y \in U(y \in R\alpha(x), y \in A\alpha)\}$

$= Sup\{\alpha \in [0,1]/\exists y \in U[I_R(x,y) \geq \alpha, I_A(y) \geq \alpha]\}$

$= \bigvee_{y \in U}[I_R(x,y) \wedge I_A(y)] = I_{\overline{R}(A)}(x)$

(3) $\bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{R}^{\alpha}(A^{\alpha})(x)\right] = inf\{\alpha \in [0,1]/R^{\alpha}(x) \cap A^{\alpha} \neq \varphi\}$

$= inf\{\alpha \in [0,1]/R^{\alpha}(x) \cap A^{\alpha} \neq \varphi\}$

$= inf\{\alpha \in [0,1]/\exists y \in U(y \in R^{\alpha}(x), y \in A^{\alpha})\}$

$= inf\{\alpha \in [0,1]/\exists y \in U[F_R(x,y) \leq \alpha, F_A(y) \leq \alpha]\}$

$= \bigwedge_{y \in U}[F_R(x,y) \vee F_A(y)] = F_{\overline{R}(A)}(x)$

Like wise we can conclude

$T_{\overline{R}(A)}(x) = \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{R}_{\alpha}(A_{\alpha+})(x)\right]$

$= \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{R}_{\alpha+}(A_{\alpha})(x)\right] = \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{R}_{\alpha+}(A_{\alpha+})(x)\right]$

$I_{\overline{R}(A)}(x) = \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{R}\alpha(A\alpha+)(x)\right]$

$= \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{R}\alpha+(A\alpha)(x)\right] = \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{R}\alpha+(A\alpha+)(x)\right]$

$F_{\overline{R}(A)}(x) = \bigwedge_{\alpha \in [0,1]} \left[\alpha \vee \overline{R}^{\alpha}(A^{\alpha+})(x)\right]$

$= \bigwedge_{\alpha \in [0,1]} \left[\alpha \vee \overline{R}^{\alpha+}(A^{\alpha})(x)\right] = \bigwedge_{\alpha \in [0,1]} \left[\alpha \vee \overline{R}^{\alpha+}(A^{\alpha+})(x)\right]$

(4) Since $\overline{R}_{\alpha+}(A_{\alpha+}) \subseteq \overline{R}_{\alpha+}(A_{\alpha}) \subseteq \overline{R_{\alpha}}(A_{\alpha})$

We prove only $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R}_{\alpha+}(A_{\alpha+})$ and $\bar{R}_{\alpha}(A_{\alpha}) \subseteq [R(A)]_{\alpha}$

For any $x \in [\overline{R}(A)]_{\alpha+}$, $T_{\overline{R}(A)} > \alpha \Rightarrow \bigvee_{y \in U}[T_R(x,y) \wedge T_A(y)] > \alpha \ \exists \ y' \in U \ni$ $T_R(x,y') \wedge T_R(y") > \alpha \Rightarrow y' \in R_{\alpha+}(x)$ and $y' \in A_{\alpha+} \Rightarrow R_{\alpha+}(x) \cap A_{\alpha+} \neq \varphi$

From the definition of upper crisp approximation operator we have $x \in \overline{R}_{\alpha+}(A_{\alpha+})$
Hence $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R}_{\alpha+}(A_{\alpha+})$
Next, to prove $\overline{R}_{\alpha}(A_{\alpha}) \subseteq [R(A)]_{\alpha}$
For any $x \in \overline{R}_{\alpha}(A_{\alpha}), R_{\alpha}(A_{\alpha})(x) = 1$, if $\exists\ \beta$, then $T_{\overline{R}(A)}(x) = \bigvee_{\beta \in [0,1]} \left[\beta \wedge \overline{R}_{\beta}(A_{\beta})(x)\right]$
$\geq \alpha \wedge \overline{R}_{\alpha}(A_{\alpha})(x) = \alpha$. We obtain $x \in [\overline{R}(A)]_{\alpha}$ $\overline{R}_{\alpha}(A_{\alpha}) \subseteq [\overline{R}(A)]_{\alpha}$

(5) Similar to (4) It is enough to prove $\overline{R\alpha+}(A\alpha+) \subseteq \overline{R\alpha+}(A\alpha) \subseteq \overline{R\alpha}(A\alpha)$
Hence we prove the following

$i)[\overline{R}(A)]\alpha+ \subseteq \overline{R\alpha+}(A\alpha+)$
$ii)\overline{R\alpha}(A\alpha) \subseteq [\overline{R}(A)]\alpha$
i) For $x \in [\overline{R}(A)]\alpha+, I_{\overline{R}(A)}(x) > \alpha \Rightarrow \bigvee_{y \in U} [I_R(x,y) \wedge I_A(y)] > \alpha$
$\exists\ \ y^{'} \in U \ni I_R(x,y^{'}) \wedge I_A(y^{'}) > \alpha$
(ie) $I_R(x,y^{'})\alpha$ and $I_A(y^{'})\alpha \Rightarrow y^{'} \in R\alpha+(x)$ and $y^{'} \in A\alpha+$
$y^{'} \in R(x) \cap A\alpha+ \Rightarrow R\alpha+(x) \cap A\alpha+ \neq \varphi$

By the definition of crisp approximation operator we have $x \in \overline{R\alpha+}(A\alpha+)$ therefore $[\overline{R}(A)]\alpha+ \subseteq \overline{R\alpha+}(A\alpha+)$. Next for any $x \in \overline{R\alpha}(A\alpha)$, $\overline{R}\alpha(A\alpha)(x) = 1$.
If there exists β then $T_{\overline{R}(A)}(x) = \bigvee_{\beta \in [0,1]} \left[\beta \wedge \overline{R}_{\beta}(A_{\beta})(x)\right] \geq \alpha \wedge \overline{R}\alpha(A\alpha)(x) = \alpha$
We obtain $x \in [\overline{R}(A)]\alpha$ therefore $\overline{R\alpha}(A\alpha) \subseteq [\overline{R}(A)]\alpha$

(6) The proof of (6) is similar to (4) and (5) we need to prove only

$$[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \text{ and } \overline{R^{\alpha}}(A^{\alpha}) \subseteq [\overline{R}(A)]^{\alpha}.$$

For any $x \in [\overline{R}(A)]^{\alpha+}, F_{\overline{R}(A)}(x) < \alpha$ (ie) $\bigwedge_{y \in U} [F_R(x,y) \vee F_A(y)] < \alpha$ and $\exists\ \ y^{'} \in U \ni$ $F_R(x,Y^{'}) \vee F_A(y^{'}) < \alpha$. Hence $F_R(x,y^{'}) < \alpha, T_A(y^{'}) < \alpha$ (ie) $y^{'} \in R^{\alpha+}(x)$ and $y^{'} \in A^{\alpha+}$. $R^{\alpha+}(x) \cap A^{\alpha+} \neq \phi$ therefore, $x \in \overline{R^{\alpha+}}(A^{\alpha+})$ and $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+})$.
Next for any $x \in \overline{R^{\alpha}}(A^{\alpha})$ note $\overline{R^{\alpha}}(A^{\alpha})(x) = 1$ then we have
$F_{\overline{R}(A)}(x) = \bigwedge_{\beta \in [0,1]} \left[\beta \vee \overline{R^{\beta}}(A^{\beta})(x)\right] \leq \alpha \vee \overline{R^{\alpha}}(A^{\alpha})(x) = \alpha.$

Thus $x \in [\overline{R}(A)]^{\alpha}$. Hence $\overline{R^{\alpha}}(A^{\alpha}) \subseteq [\overline{R}(A)]^{\alpha}$.
The proof of (7), (8), (9) can be obtained similar to (4), (5), (6). □

Theorem 3.13. *Let (U,R) be neutrosophic approximation space and $A \in N(U)$ then $\forall x \in U$*

(1)$T_{\underline{R}(A)}(x) = \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - \underline{R^{\alpha}}(A_{\alpha+})(x))] = \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - R^{\alpha}(A_{\alpha})(x))]$
$\bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - \underline{R^{\alpha+}}(A_{\alpha+})(x))] = \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - R^{\alpha+}(A_{\alpha})(x))]$
(2) $I_{\underline{R}(A)}(x) = \bigwedge_{\alpha \in [0,1]} \left[\alpha \vee (1 - \underline{R(1-\alpha)}(A\alpha+)(x))\right] = \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - R(1-\alpha)(A\alpha)(x))]$
$\bigwedge_{\alpha \in [0,1]} \left[\alpha \vee (1 - \underline{R(1-\alpha+)}(A\alpha+)(x))\right] = \bigwedge_{\alpha \in [0,1]} [\alpha \vee (1 - R(1-\alpha+)(A\alpha)(x))]$

(3) $F_{\underline{R}(A)}(x) = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge (1 - \underline{R_\alpha}(A^{\alpha+})(x))\right] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - R_\alpha(A^\alpha)(x))]$
$\bigvee_{\alpha\in[0,1]} \left[\alpha \wedge (1 - \underline{R_{\alpha+}}(A^{\alpha+})(x))\right] = \bigvee_{\alpha\in[0,1]} [\alpha \wedge (1 - R_{\alpha+}(A^\alpha)(x))]$
and for $\alpha \in [0,1]$

(4)$[\underline{R}(A)]_{\alpha+} \subseteq \underline{R}^\alpha(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_\alpha) \subseteq [\underline{R}(A)]_\alpha$

(5)$[\underline{R}(A)]\,\alpha+ \subseteq R1-\alpha(A\alpha+) \subseteq R1-\alpha+(A\alpha+) \subseteq R1-\alpha+(A\alpha) \subseteq [\underline{R}(A)]\alpha$

(6)$[\underline{R}(A)]^{\alpha+} \subseteq \underline{R}_\alpha(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^\alpha) \subseteq [\underline{R}(A)]^\alpha$

(7)$[\underline{R}(A)]_\alpha \subseteq \underline{R}^\alpha(A_{\alpha+}) \subseteq \underline{R^\alpha}(A_\alpha) \subseteq \underline{R^{\alpha+}}(A_\alpha) \subseteq [\underline{R}(A)]_\alpha$

(8)$[\underline{R}(A)]\,\alpha+ \subseteq \underline{R1-\alpha}(A\alpha+) \subseteq \underline{R\alpha}(A\alpha) \subseteq \underline{R(1-\alpha+)}(A\alpha) \subseteq [\underline{R}(A)]^\alpha$

(9)$[\underline{R}(A)]^{\alpha+} \subseteq \underline{R}_\alpha(A^{\alpha+}) \subseteq \underline{R_\alpha}(A^\alpha) \subseteq \underline{R_{\alpha+}}(A^\alpha) \subseteq [\underline{R}(A)]^\alpha$

Proof. (1) and (2). For any $x\epsilon U$, by the duality of upper and lower crisp approximation operators and in terms of Theorem , we have
$T_{\overline{R}(A)}(x) = \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge \overline{R_\alpha}(A_\alpha)(x)]$ $I_{\overline{R}(A)}(x) = \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge \overline{R\alpha}(A_\alpha)(x)]$
$F_{\overline{R}(A)}(x) = \bigwedge_{\alpha\epsilon[0,1]} [\alpha \vee \overline{R^\alpha}(A_\alpha)(x)]$, then $T_{\overline{R}(\sim A)}(x) = \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge \overline{R_\alpha}(\sim A_\alpha)(x)]$
$= \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge \overline{R_\alpha}(\sim A^{\alpha+})(x)]$
$= \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge \underline{\sim R_\alpha}(A^{\alpha+})(x)]$
$= \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge (1 - \underline{R_\alpha}(\sim A^{\alpha+})(x))]$
$I_{\overline{R}(\sim A)}(x) = \bigvee \alpha\epsilon[0,1][\alpha \wedge \overline{R\alpha}(\sim A_\alpha)(x)]$
$= \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge \overline{R\alpha}(\sim A1-\alpha+)(x)]$
$= \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge \underline{\sim R\alpha}(A1-\alpha+)(x)]$
$= \bigvee_{\alpha\epsilon[0,1]} [\alpha \wedge (1 - \underline{R\alpha}(\sim A1-\alpha+)(x))]$
$F_{\overline{R}(\sim A)}(x) = \bigwedge_{\alpha\epsilon[0,1]} [\alpha \vee \overline{R^\alpha}(\sim A^\alpha)(x)]$
$= \bigwedge_{\alpha\epsilon[0,1]} [\alpha \vee \overline{R^\alpha}(\sim A_{\alpha+})(x)]$
$= \bigwedge_{\alpha\epsilon[0,1]} [\alpha \vee \underline{\sim R^\alpha}(A_{\alpha+})(x)]$
$= \bigwedge_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R^\alpha}(\sim A_{\alpha+})(x))]$
Thus by fixing $\underline{R}(A) = \sim \overline{R}(\sim A)$,
we conclude $T_{\underline{R}(A)}(x) = T_{\overline{R}(\sim A)}(x) = \bigwedge_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R_\alpha}(\sim A^{\alpha+})(x))]$

$I_{\underline{R}(A)}(x) = I_{\overline{R}(\sim A)}(x) = \bigwedge\limits_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R\alpha}(\sim A1 - \alpha+)(x))$
$F_{\underline{R}(A)}(x) = F_{\overline{R}(\sim A)}(x) = \bigvee\limits_{\alpha\epsilon[0,1]} [\alpha \wedge (1 - \underline{R^{\alpha}}(\sim A_{\alpha+})(x))]$
Likewise, we can prove
$T_{\underline{R}(A)}(x) = T_{\overline{R}(\sim A)}(x) = \bigwedge\limits_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R_{\alpha}}(\sim A^{\alpha})(x))] = T_{\overline{R}(\sim A)}(x) = \bigwedge\limits_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R_{\alpha+}}(\sim A^{\alpha+})(x))] = T_{\overline{R}(\sim A)}(x) = \bigwedge\limits_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R_{\alpha+}}(\sim A^{\alpha})(x))]$
$I_{\underline{R}(A)}(x) = I_{\overline{R}(\sim A)}(x) = \bigwedge\limits_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R\alpha}(\sim A1 - \alpha)(x)) = I_{\overline{R}(\sim A)}(x) = \bigwedge\limits_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R\alpha+}(\sim A1 - \alpha+)(x)) = I_{\overline{R}(\sim A)}(x) = \bigwedge\limits_{\alpha\epsilon[0,1]} [\alpha \vee (1 - \underline{R\alpha+}(\sim A1 - \alpha)(x))$
$F_{\underline{R}(A)}(x) = F_{\overline{R}(\sim A)}(x) = \bigvee\limits_{\alpha\epsilon[0,1]} [\alpha \wedge (1 - \underline{R^{\alpha}}(\sim A_{\alpha})(x))] = F_{\overline{R}(\sim A)}(x) = \bigvee\limits_{\alpha\epsilon[0,1]} [\alpha \wedge (1 - \underline{R^{\alpha+}}(\sim A_{\alpha+})(x))] = F_{\overline{R}(\sim A)}(x) = \bigvee\limits_{\alpha\epsilon[0,1]} [\alpha \wedge (1 - \underline{R^{\alpha+}}(\sim A_{\alpha})(x))]$
It is easy to prove that $\underline{R^{\alpha}}A_{\alpha+} \subseteq \underline{R^{\alpha+}}A_{\alpha+} \subseteq \underline{R^{\alpha+}}A_{\alpha}$. Now we tend to prove that $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^{\alpha}}A_{\alpha+}$ and $\underline{R^{\alpha+}}A_{\alpha} \subseteq [\underline{R}(A)]_{\alpha}$ For any $x\epsilon\underline{R^{\alpha}}A_{\alpha+}$, we have $T_R(A)(x) > \alpha$ then we have $\bigwedge\limits_{y\in U}$ [$F_R(x,y) \vee T_A(y)$] $> \alpha$ then [$F_R(x,y) \vee T_A(y)$] $> \alpha$ for any $y\epsilon U$, that is if $F_R(x,y) \leq \alpha$, then $T_A(y) > \alpha$.
Alternatively, for any $y\epsilon U$, if $y\epsilon R^{\alpha\delta}(x)$, then $y\epsilon A_{a+}$. Therefore, $R^{\alpha}(x) \subseteq A_{\alpha+}$, then by the definition of lower approximation operator we have $x\epsilon\underline{R^{\alpha}}(A_{\alpha+})$. Thus we conclude $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^{\alpha}}A_{\alpha+}$ Also, for any $x\epsilon\underline{R^{\alpha+}}(A_{\alpha})$, we have $\underline{R^{\alpha+}}(A_{\alpha} = 1$. Then,

$$\begin{aligned} T_{\underline{R}}(x) &= \bigwedge_{\alpha'\epsilon[0,1]} [\alpha' \vee \underline{R^{\alpha'+}}(A_{\alpha'})(x)] \\ &= \bigvee_{\alpha'\epsilon[0,1]} [\alpha' \wedge \underline{R^{\alpha'+}}(A_{\alpha'})(x)] \\ &\geq \alpha \wedge \underline{R^{\alpha+}}(A_{\alpha})(x) = \alpha \end{aligned}$$

Hence $x\epsilon\,[\underline{R}(A)]_{\alpha}$ and $\underline{R^{\alpha+}}A_{\alpha} \subseteq [\underline{R}(A)]_{\alpha}$.
Similarly we can prove (5) and (6) and hence (7), (8) and (9) can be concluded. □

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