Accurate Independent Domination in Graphs

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Abstract: A dominating set $D$ of a graph $G = (V, E)$ is an independent dominating set, if the induced subgraph $(D)$ has no edges. An independent dominating set $D$ of $G$ is an accurate independent dominating set if $V - D$ has no independent dominating set of cardinality $|D|$. The accurate independent domination number $i_a(G)$ of $G$ is the minimum cardinality of an accurate independent dominating set of $G$. In this paper, we initiate a study of this new parameter and obtain some results concerning this parameter.

Key Words: Domination, independent domination number, accurate independent domination number, Smarandache $H$-dominating set.

AMS(2010): 05C69.

§1. Introduction

All graphs considered here are finite, nontrivial, undirected with no loops and multiple edges. For graph theoretic terminology we refer to Harary [1].

Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. Let $\Delta(G)(\delta(G))$ denote the maximum (minimum) degree and $\lceil x \rceil (\lfloor x \rfloor)$ the least (greatest) integer greater(less) than or equal to $x$. The neighborhood of a vertex $u$ is the set $N(u)$ consisting of all vertices $v$ which are adjacent with $u$. The closed neighborhood is $N[u] = N(u) \cup \{u\}$. A set of vertices in $G$ is independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of $G$ and is denoted by $\beta_o(G)$.

For any set $S$ of vertices of $G$, the induced subgraph $\langle S \rangle$ is maximal subgraph of $G$ with vertex set $S$.

The corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$. A wounded spider is the graph formed by subdividing at most $n - 1$ of the edges of a star $K_{1,n}$ for $n \geq 0$. Let $\Omega(G)$ be the set of all pendant vertices of $G$, that is, the set of vertices of degree 1. A vertex $v$ is called a support vertex if $v$ is neighbor of a pendant vertex and $d_G(v) > 1$. Denote by $X(G)$ the set of all support vertices in $G$, $M(G)$ be the set

$\text{Supported by University Grant Commission(UGC), New Delhi, India through UGC-SAP-DRS-III, 2016-2021: F.510/3/DRS-III/2016 (SAP-I).}$

$\text{Received January 11, 2018, Accepted May 25, 2018.}$
of vertices which are adjacent to support vertex and \( J(G) \) be the set of vertices which are not adjacent to a support vertex. The diameter \( \text{diam}(G) \) of a connected graph \( G \) is the maximum distance between two vertices of \( G \), that is \( \text{diam}(G) = \max_{u,v \in V(G)} d_G(u,v) \). A set \( B \subseteq V \) is a 2-packing if for each pair of vertices \( u, v \in B \), \( N_G[u] \cap N_G[v] = \emptyset \).

A proper coloring of a graph \( G = (V(G), E(G)) \) is a function from the vertices of the graph to a set of colors such that any two adjacent vertices have different colors. The chromatic number \( \chi(G) \) is the minimum number of colors needed in a proper coloring of a graph. A dominator coloring of a graph \( G \) is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number \( \chi_d(G) \) is the minimum number of color classes in a dominator coloring of a graph \( G \). This concept was introduced by R. Gera et al. [3].

A set \( D \) of vertices in a graph \( G = (V, E) \) is a dominating set of \( G \), if every vertex in \( V - D \) is adjacent to some vertex in \( D \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set. For a comprehensive survey of domination in graphs, see [4, 5, 7].

Generally, if \( \langle D \rangle \cong H \), such a dominating set \( D \) is called a Smarandache \( H \)-dominating set. A dominating set \( D \) of a graph \( G = (V, E) \) is an independent dominating set, if the induced subgraph \( \langle D \rangle \) has no edges, i.e., a Smarandache \( H \)-dominating set with \( E(H) = \emptyset \). The independent domination number \( i(G) \) is the minimum cardinality of an independent dominating set.

A dominating set \( D \) of \( G = (V, E) \) is an accurate dominating set if \( V - D \) has no dominating set of cardinality \( |D| \). The accurate domination number \( \gamma_a(G) \) of \( G \) is the minimum cardinality of an accurate dominating set. This concept was introduced by Kulli and Kattimani [6, 9].

An independent dominating set \( D \) of \( G \) is an accurate independent dominating set if \( V - D \) has no independent dominating set of cardinality \( |D| \). The accurate independent domination number \( i_a(G) \) of \( G \) is the minimum cardinality of an accurate independent dominating set of \( G \). This concept was introduced by Kulli [8].

For example, we consider the graph \( G \) in Figure 1. The accurate independent dominating sets are \( \{1, 2, 6, 7\} \) and \( \{1, 3, 6, 7\} \). Therefore \( i_a(G) = 4 \).

![Figure 1](image-url)

\[ G : \]

\[ 2 \]
\[ 1 \]
\[ 6 \]
\[ 7 \]
\[ 3 \]
\[ 4 \]
\[ 5 \]

§2. Results

Observation 2.1

1. Every accurate independent dominating set is independent and dominating. Hence it is a minimal dominating set.
2. Every minimal accurate independent dominating set is a maximal independent dominating set.

**Proposition 2.1** For any nontrivial connected graph \( G \), \( \gamma(G) \leq i_a(G) \).

*Proof* Clearly, every accurate independent dominating set of \( G \) is a dominating set of \( G \). Thus result holds. \( \Box \)

**Proposition 2.2** If \( G \) contains an isolated vertex, then every accurate dominating set is an accurate independent dominating set.

Now we obtain the exact values of \( i_a(G) \) for some standard class of graphs.

**Proposition 2.3** For graphs \( P_p, W_p \) and \( K_{m,n} \), there are

1. \( i_a(P_p) = \lceil p/3 \rceil \) if \( p \geq 3 \);
2. \( i_a(W_p) = 1 \) if \( p \geq 5 \);
3. \( i_a(K_{m,n}) = m \) for \( 1 \leq m < n \).

**Theorem 2.1** For any graph \( G \), \( i_a(G) \leq p - \gamma(G) \).

*Proof* Let \( D \) be a minimal dominating set of \( G \). Then there exist at least one accurate independent dominating set in \((V - D)\) and by proposition 2.1,

\[
i_a(G) \leq |V| - |D| \leq p - \gamma(G).
\]

Notice that the path \( P_4 \) achieves this bound. \( \Box \)

**Theorem 2.2** For any graph \( G \),

\[
\left\lceil \frac{p}{\Delta + 1} \right\rceil \leq i_a(G) \leq \left\lfloor \frac{p}{\Delta} / \Delta + 1 \right\rfloor
\]

and these bounds are sharp.

*Proof* It is known that \( p/\Delta + 1 \leq \gamma(G) \) and by proposition 2.1, we see that the lower bound holds. By Theorem 2.1,

\[
i_a(G) \leq p - \gamma(G),
\]

\[
\leq p - p/\Delta + 1
\]

\[
\leq p \Delta / \Delta + 1.
\]

Notice that the path \( P_p, p \geq 3 \) achieves the lower bound. This completes the proof. \( \Box \)

**Proposition 2.4** If \( G = K_{m_1,m_2,m_3,\ldots,m_r} \), \( r \geq 3 \), then

\[
i_a(G) = m_1 \text{ if } m_1 < m_2 < m_3 \cdots < m_r.
\]
Theorem 2.3 For any graph $G$ without isolated vertices $\gamma_a(G) \leq i_a(G)$ if $G \neq K_{m_1,m_2,m_3,\ldots,m_r}$, $r \geq 3$. Furthermore, the equality holds if $G = P_p (p \neq 4, p \geq 3)$, $W_p (p \geq 5)$ or $K_{m,n}$ for $1 \leq m < n$.

Proof Since we have $\gamma(G) \leq \gamma_a(G)$ and by Proposition 2.1, $\gamma_a(G) \leq i_a(G)$.

Let $\gamma_a(G) \leq i_a(G)$. If $G = K_{m_1,m_2,m_3,\ldots,m_r}$, $r \geq 3$ then by Proposition 2.4, $i_a(G) = m_1$ if $m_1 < m_2 < m_3 \cdots < m_r$ and also accurate domination number is $\lceil p/2 \rceil + 1$ i.e., $\gamma_a(G) = \lceil p/2 \rceil + 1 > m_1 = i_a(G)$, a contradiction. □

Corollary 2.1 For any graph $G$, $i_a(G) = \gamma(G)$ if $\text{diam}(G) = 2$.

Proposition 2.5 For any graph $G$ without isolated vertices $i(G) \leq i_a(G)$. Furthermore, the equality holds if $G = P_p (p \geq 3)$, $W_p (p \geq 5)$ or $K_{m,n}$ for $1 \leq m < n$.

Proof Every accurate independent dominating set is a independent dominating set. Thus result holds. □

Definition 2.1 The double star $S_{n,m}$ is the graph obtained by joining the centers of two stars $K_{1,n}$ and $K_{1,m}$ with an edge.

Proposition 2.6 For any graph $G$, $i_a(G) \leq \beta_o(G)$. Furthermore, the equality holds if $G = S_{n,m}$.

Proof Since every minimal accurate independent dominating set is a maximal independent dominating set. Thus result holds. □

Theorem 2.4 For any graph $G$, $i_a(G) \leq p - \alpha_0(G)$.

Proof Let $S$ be a vertex cover of $G$. Then $V - S$ is an accurate independent dominating set. Then $i_a(G) \leq |V - S| \leq p - \alpha_0(G)$. □

Corollary 2.2 For any graph $G$, $i_a(G) \leq p - \beta_0(G) + 2$.

Theorem 2.5 If $G$ is any nontrivial connected graph containing exactly one vertex of degree $\triangle(G) = p - 1$, then $\gamma(G) = i_a(G) = 1$.

Proof Let $G$ be any nontrivial connected graph containing exactly one vertex $v$ of degree $\deg(v) = p - 1$. Let $D$ be a minimal dominating set of $G$ containing vertex of degree $\deg(v) = p = 1$. Then $D$ is a minimum dominating set of $G$ i.e.,

$$|D| = \gamma(G) = 1. \quad (1)$$

Also $V - D$ has no dominating set of same cardinality $|D|$. Therefore,

$$|D| = i_a(G). \quad (2)$$

Hence, by (1) and (2) $\gamma(G) = i_a(G) = 1$. □
Theorem 2.6 If $G$ is a connected graph with $p$ vertices then $i_a(G) = p/2$ if and only if $G = H \circ K_1$, where $H$ is any nontrivial connected graph.

Proof Let $D$ be any minimal accurate independent dominating set with $|D| = p/2$. If $G \neq H \circ K_1$ then there exist at least one vertex $v_i \in V(G)$ which is neither a pendant vertex nor a support vertex. Then there exist a minimal accurate independent dominating set $D'$ containing $v_i$ such that

$$|D'| \leq |D| - \{v_i\} \leq p/2 - \{v_i\} \leq p/2 - 1,$$

which is a contradiction to minimality of $D$.

Conversely, let $l$ be the set of all pendant vertices in $G = H \circ K_1$ such that $|l| = p/2$. If $G = H \circ K_1$, then there exist a minimal accurate independent dominating set $D \subseteq V(G)$ containing all pendant vertices of $G$. Hence $|D| = |l| = p/2$.

Now we characterize the trees for which $i_a(T) = p - \Delta(T)$.

Theorem 2.7 For any tree $T$, $i_a(T) = p - \Delta(T)$ if and only if $T$ is a wounded spider and $T \neq K_1, K_{1,1}$.

Proof Suppose $T$ is wounded spider. Then it is easy to verify that $i_a(T) = p - \Delta(T)$.

Conversely, suppose $T$ is a tree with $i_a(T) = p - \Delta(T)$. Let $v$ be a vertex of maximum degree $\Delta(T)$ and $u$ be a vertex in $N(v)$ which has degree 1. If $T - N[v] = \phi$ then $T$ is the star $K_{1,n}, n \geq 2$. Thus $T$ is a double wounded spider. Assume now there is at least one vertex in $T - N[v]$. Let $S$ be a maximal independent set of $(T - N[v])$. Then either $S \cup \{v\}$ or $S \cup \{u\}$ is an accurate independent dominating set of $T$. Thus $p = i_a(T) + \Delta(T) \leq |S| + 1 + \Delta(T) \leq p$. This implies that $V - N(v)$ is an accurate independent dominating set. Furthermore, $N(v)$ is also an accurate independent dominating set.

The connectivity of $T$ implies that each vertex in $V - N[v]$ must be adjacent to at least one vertex in $N(v)$. Moreover if any vertex in $V - N[v]$ is adjacent to two or more vertices in $N(v)$, then a cycle is formed. Hence each vertex in $V - N[v]$ is adjacent to exactly one vertex in $N(v)$. To show that $\Delta(T) + 1$ vertices are necessary to dominate $T$, there must be at least one vertex in $N(v)$ which are not adjacent to any vertex in $V - N[v]$ and each vertex in $N(v)$ has either 0 or 1 neighbors in $V - N[v]$. Thus $T$ is a wounded spider.

Proposition 2.7 If $G$ is a path $P_p$, $p \geq 3$ then $\gamma(P_p) = i_a(P_p)$.

We characterize the class of trees with equal domination and accurate independent domination number in the next section.

§3. Characterization of $(\gamma, i_a)$-Trees

For any graph theoretical parameter $\lambda$ and $\mu$, we define $G$ to be $(\lambda, \mu)$-graph if $\lambda(G) =$
\( \mu(G) \). Here we provide a constructive characterization of \((\gamma, i_a)\)-trees.

To characterize \((\gamma, i_a)\)-trees we introduce family \( \tau_1 \) of trees \( T = T_k \) that can be obtained as follows. If \( k \) is a positive integer, then \( T_{k+1} \) can be obtained recursively from \( T_k \) by the following operation.

**Operation** \( O \)  Attach a path \( P_3(x,y,z) \) and an edge \( mx \), where \( m \) is a support vertex of a tree \( T \).

\[
\tau = \{ T / \text{obtained from } P_5 \text{ by finite sequence of operations of } O \}
\]

![Tree T belonging to family \( \tau_1 \)](image)

**Observation** 3.1 If \( T \in \tau \), then

1. \( i_a(T) = \lceil p + 1/3 \rceil \);
2. \( X(T) \) is a minimal dominating set as well as a minimal accurate independent dominating set of \( T \);
3. \( (V - D) \) is totally disconnected.

**Corollary** 3.1 If tree \( T \) with \( p \geq 5 \) belongs to the family \( \tau \) then \( \gamma(T) = |X(T)| \) and \( i_a(T) = |X(T)| \).

**Lemma** 3.1 If a tree \( T \) belongs to the family \( \tau \) then \( T \) is a \((\gamma, i_a)\)-tree.

**Proof** If \( T = P_p \), \( p \geq 3 \) then from proposition 2.7 \( T \) is a \((\gamma, i_a)\)-tree. Now if \( T = P_p \), \( p \geq 3 \) then we proceed by induction on the number of operations \( n(T) \) required to construct the tree \( T \). If \( n(T) = 0 \) then \( T \in P_5 \) by proposition 2.7 \( T \) is a \((\gamma, i_a)\)-tree.

Assume now that \( T \) is a tree belonging to the family \( \tau \) with \( n(T) = k \), for some positive integer \( k \) and each tree \( T' \in \tau \) with \( n(T') < k \) and with \( V(T') \geq 5 \) is a \((\gamma, i_a)\)-tree in which \( X(T') \) is a minimal accurate independent dominating set of \( T' \). Then \( T \) can be obtained from a tree \( T' \) belonging to \( \tau \) by operation \( O \) where \( m \in V(T') - (M(T') - \Omega(T')) \) and we add
path \((x, y, z)\) and the edge \(mx\). Then \(z\) is a pendant vertex in \(T\) and \(y\) is a support vertex and \(x \in M(T)\). Thus \(S(T) = X(T') \cup \{y\}\) is a minimal accurate independent dominating set of \(T\). Therefore \(i_a(T) \geq |X(T)| = |X(T')| + 1\). Hence we conclude that \(i_a(T) = i_a(T') + 1\).

By the induction hypothesis and by observation 3.1(2) \(i_a(T') = \gamma(T') = |X(T')|\). In this way \(i_a(T) = |X(T)|\) and in particular \(i_a(T) = \gamma(T)\).

\[\square\]

**Lemma 3.2** If \(T\) is a \((\gamma, i_a)\) - tree, then \(T\) belongs to the family \(\tau\).

**Proof** If \(T\) is a path \(P_p\), \(p \geq 3\) then by proposition 2.7 \(T\) is a \((\gamma, i_a)\) - tree. It is easy to verify that the statement is true for all trees \(T\) with diameter less than or equal to 4. Hence we may assume that \(\text{diam}(T) \geq 4\). Let \(T\) be rooted at a support vertex \(m\) of a longest path \(P\). Let \(P\) be a \(m - z\) path and let \(y\) be the neighbor of \(z\). Further, let \(x\) be a vertex belongs to \(M(T)\). Let \(T\) be a \((\gamma, i_a)\)-tree. Now we proceed by induction on number of vertices \(|V(T)|\) of a \((\gamma, i_a)\) - tree. Let \(T\) be a \((\gamma, i_a)\)-tree and assume that the result holds good for all trees on \(V(T) - 1\) vertices. By observation 3.1(2) since \(T\) is \((\gamma, i_a)\)-tree it contains minimal accurate independent dominating set \(D\) that contains all support vertices of a tree. In particular \(\{m, y\} \subset D\) and the vertices \(x\) and \(z\) are independent in \((V - D)\).

Let \(T' = T - (x, y, z)\). Then \(D - \{y\}\) is dominating set of \(T'\) and so \(\gamma(T') \leq \gamma(T) - 1\).

Any dominating set can be extended to a minimal accurate independent dominating set of \(T\) by adding to it the vertices \((x, y, z)\) and so \(i_a(T) \leq i_a(T') + 1\). Hence, \(i_a(T') \leq \gamma(T') \leq \gamma(T) + 1 \leq i_a(T) - 1 \leq i_a(T')\). Consequently, we must have equality throughout this inequality chain. In particular \(i_a(T') = \gamma(T')\) and \(i_a(T) = i_a(T') + 1\). By inductive hypothesis any minimal accurate independent dominating set of a tree \(T'\) can be extended to minimal accurate independent dominating set of a tree \(T\) by operation \(O\). Thus \(T \in \tau\).

\[\square\]

As an immediate consequence of lemmas 3.1 and 3.2, we have the following characterization of trees with equal domination and accurate independent domination number.

**Theorem 3.1** Let \(T\) be a tree. Then \(i_a(T) = \gamma(T)\) if and only if \(T \in \tau\).

\[\square\]

**§4. Accurate Independent Domination of Some Graph Families**

In this section accurate independent domination of \(\text{fan graph, double fan graph, helm graph and gear graph}\) are considered. We also obtain the corresponding relation between other dominating parameters and dominator coloring of the above graph families.

**Definition 4.1** A fan graph, denoted by \(F_n\) can be constructed by joining \(n\) copies of the cycle graph \(C_3\) with a common vertex.

**Observation 4.1** Let \(F_n\) be a fan. Then,

1. \(F_n\) is a planar undirected graph with \(2n + 1\) vertices and \(3n\) edges;
2. \(F_n\) has exactly one vertex with \(\Delta(F_n) = p - 1\);
3. \(\text{Diam}(F_n) = 2\).
Theorem 4.1([2]) For a fan graph $F_n, n \geq 2$, $\chi_d(F_n) = 3$.

Proposition 4.1 For a fan graph $F_n, n \geq 2$, $i_a(F_n) = 1$.

Proof By Observation 4.1(2) and Theorem 2.5 result holds. \hfill \Box

Proposition 4.2 For a fan graph $F_n, n \geq 2$, $i_a(F_n) < \chi_d(F_n)$.

Proof By Proposition 4.1 and Theorem 4.1, we know that $\chi_d(F_n) = 3$. This implies that $i_a(F_n) < \chi_d(F_n)$. \hfill \Box

Definition 4.2 A double fan graph, denoted by $F_{2,n}$, isomorphic to $P_n + 2K_1$.

Observation 4.2

1. $F_{2,n}$ is a planar undirected graph with $(n + 2)$ vertices and $(3n - 1)$ edges;
2. $\text{Diam}(G) = 2$.

Theorem 4.2([2]) For a double fan graph $F_{2,n}, n \geq 2$, $\chi_d(F_{2,n}) = 3$.

Theorem 4.3 For a double fan graph $F_{2,n}$, $n \geq 2$, $i_a(F_{2,2}) = 2$, $i_a(F_{2,3}) = 1$, $i_a(F_{2,5}) = 3$ and $i_a(F_{2,n}) = 2$ if $n \geq 7$.

Proof Our proof is divided into cases following.

Case 1. If $n = 2$ and $n \geq 7$, then $F_{2,n}$, $n \geq 2$ has only one accurate independent dominating set $D$ of $|D| = 2$. Hence, $i_a(F_{2,n}) = 2$.

Case 2. If $n = 3$, then $F_{2,3}$ has exactly one vertex of $\Delta(G) = p - 1$. Then by Theorem 2.5, $i_a(F_{2,n}) = 1$.

Case 3. If $n=5$ and $D$ be a independent dominating set of $G$ with $|D| = 2$, then $(V - D)$ also has an independent dominating set of cardinality 2. Hence $D$ is not accurate.

Let $D_1$ be a independent dominating set with $|D_1| = 3$, then $V - D_1$ has no independent dominating set of cardinality 3. Then $D_1$ is accurate. Hence, $i_a(F_{2,n}) = 3$.

Case 4. If $n=4$ and 6, there does not exist accurate independent dominating set. \hfill \Box

Proposition 4.3 For a double fan graph $F_{2,n}$, $n \geq 7$,

$\gamma(F_{2,n}) = i(F_{2,n}) = \gamma_a(F_{2,n}) = i_a(F_{2,n}) = 2$

Proof Let $F_{2,n}$, $n \geq 7$ be a Double fan graph. Then $2k_1$ forms a minimal dominating set of $F_{2,n}$ such that $\gamma(F_{2,n}) = 2$. Since this dominating set is independent and in $(V - D)$ there is no independent dominating set of cardinality 2 it is both independent and accurate independent dominating set. Also it is accurate dominating set. Hence,

$\gamma(F_{2,n}) = i(F_{2,n}) = \gamma_a(F_{2,n}) = i_a(F_{2,n}) = 2$. \hfill \Box
Proposition 4.4  For Double fan graph $F_{2,n}$, $n \geq 7$

\[ i_a(F_{2,n}) \leq \chi_d(F_{2,n}). \]

Proof  The proof follows by Theorems 4.2 and 4.3. \qed

Definition 4.3([1])  For $n \geq 4$, the wheel $W_n$ is defined to be the graph $W_n = C_{n-1} + K_1$. Also it is defined as $W_{1,n} = C_n + K_1$.

Definition 4.4  A helm $H_n$ is the graph obtained from $W_{1,n}$ by attaching a pendant edge at each vertex of the $n$-cycle.

Observation 4.3  A helm $H_n$ is a planar undirected graph with $(2n+1)$ vertices and $3n$ edges.

Theorem 4.4([2])  For Helm graph $H_n$, $n \geq 3$, $\chi_d(H_n) = n + 1$.

Proposition 4.5  For a helm graph $H_n$, $n \geq 3$, $i_a(H_n) = n$.

Proof  Let $H_n$, $n \geq 3$ be a helm graph. Then there exist a minimal independent dominating set $D$ with $|D| = n$ and $(V - D)$ has no independent dominating set of cardinality $n$. Hence $D$ is accurate. Therefore $i_a(H_n) = n$. \qed

Proposition 4.6  For a helm graph $H_n$, $n \geq 3$

\[ \gamma(H_n) = i(H_n) = \gamma_a(H_n) = i_a(H_n) = n. \]

Proposition 4.7  For a helm graph $H_n$, $n \geq 3$

\[ i_a(H_n) = \chi_d(H_n) - 1. \]

Proof  Applying Proposition 4.5, $i_a(H_n) = n + 1 - 1 = \chi_d(H_n) - 1$ by Theorem 4.4, $\chi_d(H_n) = n + 1$. Hence the proof. \qed

Definition 4.5  A gear graph $G_n$ also known as a bipartite wheel graph, is a wheel graph $W_{1,n}$ with a vertex added between each pair of adjacent vertices of the outer cycle.

Observation 4.4  A gear graph $G_n$ is a planar undirected graph with $2n + 1$ vertices and $3n$ edges.

Theorem 4.5([2])  For a gear graph $G_n$, $n \geq 3$,

\[ \chi_d(G_n) = \lceil 2n/3 \rceil + 2. \]
Theorem 4.6 For a gear graph $G_n$, $n \geq 3$, $i_a(G_n) = n$.

Proof It is clear from the definition of gear graph $G_n$ is obtained from wheel graph $W_{1,n}$ with a vertex added between each pair of adjacent vertices of the outer cycle of wheel graph $W_{1,n}$. These $n$ vertices forms an independent dominating set in $G_n$ such that $(V - D)$ has no independent dominating set of cardinality $n$. Therefore, the set $D$ with cardinality $n$ is accurate independent dominating set of $G_n$. Therefore $i_a(G_n) = n$. \(\square\)

Corollary 4.1 For any gear graph $G_n$, $n \geq 3$, $\gamma(G_n) = i(G_n) = n - 1$.

Proposition 4.8 For a gear graph $G_n$, $n \geq 3$,

$$i_a(G_n) = \gamma_a(G_n).$$

Proposition 4.9 For a graph $G_n$, $n \geq 3$,

$$i_a(G_n) = \gamma(G_n) + 1 = i(G_n) + 1.$$

Proof Applying Theorem 4.6 and Corollary 4.1, we know that $i_a(G_n) = n = n - 1 + 1 = \gamma(G_n) + 1 = i(G_n) + 1$. \(\square\)

References