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Minimum Equitable Dominating Randic Energy of a Graph

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Abstract: In this paper, we introduce the minimum equitable dominating Randic energy of a graph and computed the minimum dominating Randic energy of graph. Also, established the upper and lower bounds for the minimum equitable dominating Randic energy of a graph.

Key Words: Minimum equitable dominating set, Smarandachely equitable dominating set, minimum equitable dominating Randic eigenvalues, minimum equitable dominating Randic energy.

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§1. Introduction

Let G be a simple, finite, undirected graph, The energy $E(G)$ is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. For more details on energy of graph see [5, 6].

The Randic matrix $R(G) = (R_{ij})_{n\times n}$ is given by [1-3].

$$R_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_i}} & \text{if } v_i \sim v_j, \\ 0 & otherwise \end{cases}$$

We can see lower and upper bounds on Randic energy in [1,2]. Some sharp upper bounds for Randic energy of graphs were obtain in [3].

§2. The Minimum Equitable Dominating Randic Energy of Graph

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$ and edge set E. A subset U of $V(G)$ is an equitable dominating set, if for every $v \in V(G) - U$ there exists a

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vertex $u \in U$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$, and a Smarandachely equitable dominating set is its contrary, i.e., $|deg(u) - deg(v)| \geq 1$ for such an edge uv, where $deg(x)$ denotes the degree of vertex x in $V(G)$. Any equitable dominating set with minimum cardinality is called a minimum equitable dominating set. Let E be a minimum equitable dominating set of a graph G. The minimum equitable dominating Randic matrix $R^E(G) = (R^E_{ij})_{n\times n}$ is given by

$$R^E_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_i}} & \text{if } v_i \sim v_j, \\ 1 & \text{if } i = j \text{ and } v_i \in E, \\ 0 & otherwise \end{cases}$$

The characteristic polynomial of $R^E(G)$ is denoted by $\phi^E_R(G, \lambda) = det(\lambda I - R^E(G))$. Since the minimum equitable dominating Randic Matrix is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 > \lambda_2 > \cdots \lambda_n$. The minimum equitable dominating Randic Energy is given by

$$RE_E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

Definition 2.1 *The spectrum of a graph G is the list of distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots \lambda_r$, with their multiplicities $m_1, m_2, \ldots, m_r$, and we write it as*

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

This paper is organized as follows. In the Section 3, we get some basic properties of minimum equitable dominating Randic energy of a graph. In the Section 4, minimum equitable dominating Randic energy of some standard graphs are obtained.

§3. Some Basic Properties of Minimum Equitable Dominating Randic Energy of a Graph

Let us consider

$$P = \sum_{i<j} \frac{1}{d_i d_j},$$

where $d_i d_j$ is the product of degrees of two vertices which are adjacent.

Proposition 3.1 *The first three coefficients of $\phi^E_R(G, \lambda)$ are given as follows:*

(i) $a_0 = 1$;

(ii) $a_1 = -|E|$;

(iii) $a_2 = |E|C_2 - P$.

Proof (i) From the definition $\Phi_R^E(G,\lambda) = det[\lambda I - R^E(G)]$, we get $a_0 = 1$.

(ii) The sum of determinants of all 1×1 principal submatrices of $R^E(G)$ is equal to the trace of $R^E(G) \Rightarrow a_1 = (-1)^1$ trace of $[R^E(G)] = -|E|$.

(iii)

$$\begin{aligned}(-1)^2 a_2 &= \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} a_{ji}a_{ij} \\ &= |E|C_2 - P.\end{aligned}$$

□

Proposition 3.2 *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the minimum equitable dominating Randic eigenvalues of $R^E(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = |E| + 2P.$$

Proof We know that

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j}(a_{ij})^2 + \sum_{i=1}^{n}(a_{ii})^2 \\ &= 2\sum_{i<j}(a_{ij})^2 + |E| \\ &= |E| + 2P.\end{aligned}$$

□

Theorem 3.3 *Let G be a graph with n vertices and Then*

$$RE^E(G) \le \sqrt{n(|E| + 2[P])}$$

where

$$P = \sum_{i<j} \frac{1}{d_i d_j}$$

for which $d_i d_j$ is the product of degrees of two vertices which are adjacent.

Proof Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of $R^E(G)$. Now by Cauchy - Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Let $a_i = 1$, $b_i =| \lambda_i |$. Then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

Thus,

$$[RE^E]^2 \leq n(|E| + 2P),$$

which implies that

$$[RE^E] \leq \sqrt{n(|E| + 2P)},$$

i.e., the upper bound. □

Theorem 3.4 *Let G be a graph with n vertices. If $R=$ det $R^E(G)$, then*

$$RE^E(G) \geq \sqrt{(|E| + 2P) + n(n-1)R^{\frac{2}{n}}}.$$

Proof By definition,

$$\begin{aligned}
\left(RE^E(G)\right)^2 &= \left(\sum_{i=1}^{n} | \lambda_i |\right)^2 \\
&= \sum_{i=1}^{n} | \lambda_i | \sum_{j=1}^{n} | \lambda_j | \\
&= \left(\sum_{i=1}^{n} | \lambda_i |^2\right) + \sum_{i \neq j} | \lambda_i || \lambda_j | .
\end{aligned}$$

Using arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} | \lambda_i || \lambda_j | \quad \geq \quad \left(\prod_{i \neq j} | \lambda_i || \lambda_j |\right)^{\frac{1}{n(n-1)}}.$$

Therefore,

$$\begin{aligned}
[RE^E(G)]^2 &\geq \sum_{i=1}^{n} | \lambda_i |^2 + n(n-1)\left(\prod_{i \neq j} | \lambda_i || \lambda_j |\right)^{\frac{1}{n(n-1)}} \\
&\geq \sum_{i=1}^{n} | \lambda_i |^2 + n(n-1)\left(\prod_{i=1}^{n} | \lambda_i |^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
&= \sum_{i=1}^{n} | \lambda_i |^2 + n(n-1)R^{\frac{2}{n}} \\
&= (|E| + 2P) + n(n-1)R^{\frac{2}{n}}.
\end{aligned}$$

Thus,

$$RE^E(G) \geq \sqrt{(|E| + 2P) + n(n-1)R^{\frac{2}{n}}}. \qquad \square$$

§4. Minimum Equitable Dominating Randic Energy of Some Standard Graphs

Theorem 4.1 *The minimum equitable dominating Randic energy of a complete graph K_n is $RE^E(K_n) = \frac{3n-5}{n-1}$.*

Proof Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The minimum equitable dominating set $= E = \{v_1\}$. The minimum equitable dominating Randic matrix is

$$R^E(K_n) = \begin{bmatrix} 1 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & 0 & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & 0 \end{bmatrix}.$$

The characteristic equation is

$$\left(\lambda + \frac{1}{n-1}\right)^{n-2} \left(\lambda^2 - \frac{2n-3}{n-1}\lambda + \frac{n-3}{n-1}\right) = 0$$

and the spectrum is $Spec_R^E(K_n) = \begin{pmatrix} \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{-1}{n-1} \\ 1 & 1 & n-2 \end{pmatrix}$.

Therefore, $RE^E(K_n) = \dfrac{3n-5}{n-1}$. $\square$

Theorem 4.2 *The minimum equitable dominating Randic energy of star graph $K_{1,n-1}$ is*

$$RE^E(K_{1,n-1}) = \sqrt{5}.$$

Proof Let $K_{1,n-1}$ be the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$. Here v_0 be the center. The minimum equitable dominating set $= E = V(G)$. The minimum equitable dominating Randic matrix is

$$R^E(K_{1,n-1}) = \begin{bmatrix} 1 & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \ldots & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} \\ \frac{1}{\sqrt{n-1}} & 1 & 0 & \ldots & 0 & 0 \\ \frac{1}{\sqrt{n-1}} & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & \ldots & 1 & 0 \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

The characteristic equation is

$$\lambda(\lambda - 1)^{n-2}[\lambda - 2] = 0$$

spectrum is $Spec_R^E(K_{1,n-1}) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & n-2 & 1 \end{pmatrix}$.

Therefore, $RE^E(K_{1,n-1}) = n$. □

Theorem 4.3 *The minimum equitable dominating Randic energy of Crown graph S_n^0 is*

$$RE^E(S_n^0) = \frac{(4n-7) + \sqrt{4n^2 - 8n + 5}}{n-1}.$$

Proof Let S_n^0 be a crown graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and minimum dominating set $= E = \{u_1, v_1\}$. The minimum equitable dominating Randic matrix is

$$R^E(S_n^0) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & \frac{1}{n-1} \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{n-1} & 0 & \ldots & \frac{1}{n-1} & \frac{1}{n-1} \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 0 & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & \frac{1}{n-1} & 0 \\ 0 & \frac{1}{n-1} & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 1 & 0 & \ldots & 0 & 0 \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & 0 & \ldots & \frac{1}{n-1} & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\left(\lambda+\frac{1}{n-1}\right)^{n-2}\left(\lambda-\frac{1}{n-1}\right)^{n-2}\left(\lambda^2-\frac{1}{n-1}\lambda-1\right)\left(\lambda^2-\frac{2n-3}{n-1}\lambda+\frac{n-3}{n-1}\right)=0$$

spectrum is $Spec_R^E(S_n^0)$
$$=\begin{pmatrix} \frac{(2n-3)+\sqrt{4n-3}}{2(n-1)} & \frac{1+\sqrt{4n^2-8n+5}}{2(n-1)} & \frac{(2n-3)-\sqrt{4n-3}}{2(n-1)} & \frac{1}{n-1} & \frac{-1}{n-1} & \frac{1-\sqrt{4n^2-8n+5}}{2(n-1)} \\ 1 & 1 & 1 & n-2 & n-2 & 1 \end{pmatrix}.$$

Therefore, $RE^E(S_n^0)=\dfrac{(4n-7)+\sqrt{4n^2-8n+5}}{n-1}$. □

Theorem 4.4 *The minimum equitable dominating Randic energy of complete bipartite graph $K_{n,n}$ of order $2n$ with vertex set $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n\}$ is*

$$RE^E(K_{n,n})=\frac{2\sqrt{n-1}}{\sqrt{n}}+2.$$

Proof Let $K_{n,n}$ be the complete bipartite graph of order $2n$ with vertex set $\{u_1,u_2,\cdots,u_n, v_1,v_2,\cdots,v_n\}$. The minimum equitable dominating set $=E=\{u_1,v_1\}$ with a minimum equitable dominating Randic matrix

$$R^E(K_{n,n})=\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & 1 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\lambda^{2n-4}(\lambda^2-\frac{n-1}{n})[\lambda^2-2\lambda+\frac{n-1}{n}]=0$$

Hence, spectrum is

$$Spec_R^E(K_{n,n})=\begin{pmatrix} 1+\sqrt{\frac{1}{n}} & \frac{\sqrt{n-1}}{\sqrt{n}} & 1-\sqrt{\frac{1}{n}} & 0 & -\frac{\sqrt{n-1}}{\sqrt{n}} \\ 1 & 1 & 1 & 2n-4 & 1 \end{pmatrix}.$$

Therefore, $RE^E(K_{n,n})=\dfrac{2\sqrt{n-1}}{\sqrt{n}}+2$. □

Theorem 4.5 *The minimum equitable dominating Randic energy of cocktail party graph $K_{n\times 2}$ is*

$$RE^E(K_{n\times 2}) = \frac{4n-6}{n-1}.$$

Proof Let $K_{n\times 2}$ be a Cocktail party graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. The minimum equitable dominating set $= E = \{u_1, v_1\}$ with a minimum equitable dominating Randic matrix

$$R^E(K_{n\times 2}) = \begin{bmatrix} 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 \end{bmatrix}.$$

The characteristic equation is

$$\lambda^{n-1}\left(\lambda + \frac{1}{n-1}\right)^{n-2}(\lambda - 1)[\lambda^2 - \frac{2n-3}{n-1}\lambda + \frac{n-3}{n-1}] = 0$$

Hence, spectrum is

$$Spec_R^E(K_{n\times 2}) = \begin{pmatrix} \frac{2n-3+\sqrt{4n-3}}{2(n-1)} & 1 & \frac{2n-3-\sqrt{4n-3}}{2(n-1)} & 0 & \frac{-1}{n-1} \\ 1 & 1 & 1 & n-1 & n-2 \end{pmatrix}.$$

Therefore, $RE^E(K_{n\times 2}) = \dfrac{4n-6}{n-1}$. □

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