ON THE SABBAN FRAME BELONGING TO INVOLUTE-EVOLUTE CURVES

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In this article, we investigate special Smarandache curves with regard to Sabban frame of involute curve. We created Sabban frame belonging to spherical indicatrix of involute curve. It was explained Smarandache curves position vector is consisted by Sabban vectors belonging to spherical indicatrix. Then, we calculated geodesic curvatures of this Smarandache curves. The results found for each curve was given depend on evolute curve. The example related to the subject were given and their figures were drawn with Mapple program.

Key Words: Geodesic curvature, Involute-evolute curves, Sabban frame, Smarandache curves

1. Introduction and Preliminaries

The involute of the curve is decent known by the mathematicians especially the differential geometry scientists. There are many essential consequences and properties of curves. Involute curves examined by some authors[3, 6]. Frenet vectors of a curve were taken as the position vector and the regular curve drawn by the new vector is identified. This curve were called the Smarandache curve [10]. Special Smarandache curves examined by certain writers [1, 2, 5, 7, 8, 9]. K. Taşköprü, M. Tosun studious particular Smarandache curves belonging to Sabban frame on S^2 [11]. Şenyurt and Çalışkan investigated particular Smarandache curves belonging to Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves [4]. Let $\alpha : I \to E^3$ be a unit speed curve and the quantities $\{T, N, B, \kappa, \tau\}$ are collectively Frenet-Serret apparatus of this curve. The Frenet formulae are also well known as, respectively [6]

$$T'(s) = \kappa(s)N(s), \quad N'(s) = -\kappa(s)T(s) + \tau(s)B(s), \quad B'(s) = -\tau(s)N(s). \tag{1.1}$$

Let $\alpha_1 : I \to E^3$ be the C^2-class differentiable curve and $\{T_1(s), N_1(s), B_1(s), \ \kappa_1(s), \tau_1(s)\}$ is Frenet- serret apparatus of the α_1 involute curve, then [3]

$$T_1 = N, \quad N_1 = -\cos\varphi T + \sin\varphi B, \quad B_1 = \sin\varphi T + \cos\varphi B. \tag{1.2}$$

where $\measuredangle(W, B) = \varphi$. For the curvatures and the torsions we have

$$\kappa_1 = \frac{\|W\|}{|c - s|\kappa}, \quad \tau_1 = \frac{(\kappa\tau' - \tau\kappa')}{\kappa\,|c - s|\,\|W\|^2}. \tag{1.3}$$

$$\sin\varphi_1 = \frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|^2}}, \ \cos\varphi_1 = \frac{\|W\|}{\sqrt{\phi'^2 + \|W\|^2}}, \varphi_1' = \left(\frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|^2}}\right)' \frac{\sqrt{\varphi'^2 + \|W\|^2}}{\|W\|} \tag{1.4}$$

Let $\gamma : I \to S^2$ be a unit speed spherical curve. We symbolize s as the arc-length parameter of γ. Let's give

$$\gamma(s) = \gamma(s), t(s) = \gamma'(s), d(s) = \gamma(s) \wedge t(s) \tag{1.5}$$

$\{\gamma(s), t(s), d(s)\}$ frame is denominated the Sabban frame of γ on S^2. The spherical Frenet formulae of γ is as follows

$$\gamma'(s) = t(s), \quad t'(s) = -\gamma(s) + \kappa_g(s)d(s), \quad d'(s) = -\kappa_g(s)t(s) \tag{1.6}$$

where κ_g is denominated the geodesic curvature of the curve γ on S^2 which is

$$\kappa_g(s) = \langle t'(s), d(s)\rangle \quad [11]. \tag{1.7}$$

2. On The Sabban Frame Belonging To İnvolute-Evolute Curves

In this section, we investigated special Smarandache curves such as created by Sabban frame, $\{T_1, T_{T_1}, T_1 \wedge T_{T_1}\}$, $\{N_1, T_{N_1}, N_1 \wedge T_{N_1}\}$ and $\{B_1, T_{B_1}, B_1 \wedge T_{B_1}\}$. We will find some results. These results will be expressed depending on the evolute curve. Let's find results on this Smarandache curves. Let $\alpha_{T_1}(s) = T_1(s)$, $\alpha_{N_1}(s) = N_1(s)$ and $\alpha_{B_1}(s) = B_1(s)$ be a regular spherical curves on S^2. The Sabban frames of spherical indicatrix belonging to involute curve are as follows:

$$T_1 = T_1, \quad T_{T_1} = N_1, \quad T_1 \wedge T_{T_1} = B_1, \tag{2.1}$$

$$N_1 = N_1, \quad T_{N_1} = -\cos\varphi_1 T_1 + \sin\varphi_1 B_1, \quad N_1 \wedge T_{N_1} = \sin\varphi_1 T_1 + \cos\varphi_1 B_1, \tag{2.2}$$

$$B_1 = B_1, \quad T_{B_1} = -N_1, \quad B_1 \wedge T_{B_1} = T_1. \tag{2.3}$$

From the equation (1.6), the spherical Frenet formulae of (T_1), (N_1) and (B_1) are as follows, respectively

$$T_1' = T_{T_1}, \quad T_{T_1}' = -T_1 + \frac{\tau_1}{\kappa_1} T_1 \wedge T_{T_1}, \quad (T_1 \wedge T_{T_1})' = -\frac{\tau_1}{\kappa_1} T_{T_1}, \tag{2.4}$$

$$N_1' = T_{N_1}, \quad T_{N_1}' = -N_1 + \frac{\varphi_1'}{\|W_1\|} N_1 \wedge T_{N_1}, \quad (N_1 \wedge T_{N_1})' = -\frac{\varphi_1'}{\|W_1\|} T_{N_1}, \tag{2.5}$$

$$B_1' = T_{B_1}, \quad T_{B_1}' = -B_1 + \frac{\kappa_1}{\tau_1} B_1 \wedge T_{B_1}, \quad (B_1 \wedge T_{B_1})' = -\frac{\kappa_1}{\tau_1} T_{B_1}. \tag{2.6}$$

Using the equation (1.7), the geodesic curvatures of (T_1), (N_1) and (B_1) are

$$\kappa_g^{T_1} = \frac{\tau_1}{\kappa_1}, \quad \kappa_g^{N_1} = \frac{\varphi_1'}{\|W_1\|}, \quad \kappa_g^{B_1} = \frac{\kappa_1}{\tau_1}. \tag{2.7}$$

Definition 2.1. Let (T_1) spherical curve be of α_1, T_1 and T_{T_1} be Sabban vectors of (T_1). In the fact β_1-Smarandache curve can be identified by

$$\beta_1(s) = \frac{1}{\sqrt{2}}\left(T_1 + T_{T_1}\right). \tag{2.8}$$

Theorem 2.1. The geodesic curvature according to β_1-Smarandache curve of the involute curve is

$$\kappa_g^{\beta_1} = \frac{\kappa_1^4}{\left(2\kappa_1^2 + \tau_1^2\right)^{\frac{5}{2}}}\left(\tau_1\lambda_1 + \tau_1\lambda_2 + 2\kappa_1\lambda_3\right), \tag{2.9}$$

where coefficients are

$$\lambda_1 = -2 - \left(\frac{\tau_1}{\kappa_1}\right)^2 + \left(\frac{\tau_1}{\kappa_1}\right)'\left(\frac{\tau_1}{\kappa_1}\right), \quad \lambda_2 = -2 - 3\left(\frac{\tau_1}{\kappa_1}\right)^2 - \left(\frac{\tau_1}{\kappa_1}\right)^4 - \left(\frac{\tau_1}{\kappa_1}\right)'\left(\frac{\tau_1}{\kappa_1}\right),$$
$$\lambda_3 = 2\left(\frac{\tau_1}{\kappa_1}\right) + \left(\frac{\tau_1}{\kappa_1}\right)^3 + \left(\frac{\tau_1}{\kappa_1}\right)' \tag{2.10}$$

Proof: $\beta_1(s)=\frac{1}{\sqrt{2}}\left(T_1+T_{T_1}\right)$ or from the equation (2.1), we can write

$$\beta_1(s)=\frac{1}{\sqrt{2}}\left(T_1+N_1\right). \tag{2.11}$$

If taken derivative of the equation (2.11), T_{β_1} vector is

$$T_{\beta_1}(s)=\frac{1}{\sqrt{2\kappa_1+\tau_1{}^2}}\left(-\kappa_1T_1+\kappa_1N_1+\tau_1B_1\right). \tag{2.12}$$

Considering the equations (2.11) and (2.12), we have

$$(\beta_1\wedge T_{\beta_1})(s)=\frac{1}{\sqrt{4\kappa_1{}^2+2\tau_1{}^2}}\left(\tau_1T_1-\tau_1N_1+2\kappa_1B_1\right). \tag{2.13}$$

Using the equation (2.12), $T_{\beta_1}{}'$ vector is

$$T_{\beta_1}{}'(s)=\frac{\sqrt{2}\kappa_1{}^4}{\left(2\kappa_1{}^2+\tau_1{}^2\right)^2}\left(\lambda_1T_1+\lambda_2N_1+\lambda_3B_1\right). \tag{2.14}$$

From the equation (2.13) and (2.14), $\kappa_g^{\beta_1}$ geodesic curvature of $\beta_1(s)$ is

$$\kappa_g^{\beta_1}=\frac{\kappa_1{}^4}{\left(2\kappa_1{}^2+\tau_1{}^2\right)^{\frac{5}{2}}}\left(\tau_1\lambda_1+\tau_1\lambda_2+2\kappa_1\lambda_3\right).$$

Corollary 2.1. The geodesic curvature belonging to β_1-Smarandache curve of the evolute curve is

$$\kappa_g^{\beta_1}=\frac{\|W\|^6}{(2\|W\|^2+\phi'^2)^{\frac{5}{2}}}(\phi'\overline{\lambda}_1+\phi'\overline{\lambda}_2+\frac{2}{\|W\|}\overline{\lambda}_3), \tag{2.15}$$

where coefficients are i

$$\begin{aligned}&\overline{\lambda}_1=-2-(\frac{\varphi'}{\|W\|})^2+(\frac{\varphi'}{\|W\|})'(\frac{\varphi'}{\|W\|}),\quad \overline{\lambda}_2=-2-3(\frac{\varphi'}{\|W\|})^2-(\frac{\varphi'}{\|W\|})^4-(\frac{\varphi'}{\|W\|})'(\frac{\varphi'}{\|W\|}),\\ &\overline{\lambda}_3=2(\frac{\varphi'}{\|W\|})+(\frac{\varphi'}{\|W\|})^3+(\frac{\varphi'}{\|W\|})'\end{aligned} \tag{2.16}$$

Proof: From the equations (1.2) and (2.11), we calculate

$$\beta_1(s)=\frac{1}{\sqrt{2}}(-\cos\varphi T+N+\sin\varphi B). \tag{2.17}$$

If taken derivative of the this expression, T_{β_1} vector is,

$$T_{\beta_1}=\frac{\varphi'\sin\varphi-\|W\|\cos\varphi}{\sqrt{2\|W\|^2+\varphi'^2}}T-\frac{\|W\|}{\sqrt{2\|W\|^2+\varphi'^2}}N+\frac{\varphi'\cos\varphi+\|W\|\sin\varphi}{\sqrt{2\|W\|^2+\varphi'^2}}B. \tag{2.18}$$

If made cross product from the equations (2.17) and (2.18), we have

$$\beta_1\wedge T_{\beta_1}=\frac{2\|W\|\sin\varphi+\varphi\cos\varphi}{\sqrt{4\|W\|^2+2\varphi'^2}}T-\frac{\phi'}{\sqrt{4\|W\|^2+2\varphi'^2}}N+\frac{2\|W\|\cos\varphi-\varphi'\sin\varphi}{\sqrt{4\|W\|^2+2\varphi'^2}}B. \tag{2.19}$$

From the equation (2.18), $T_{\beta_1}{}'$ vector is

$$T_{\beta_1}{}'=\frac{\|W\|^4(\overline{\lambda}_3\sin\varphi-\overline{\lambda}_2\cos\varphi)\sqrt{2}}{(2\|W\|^2+\varphi'^2)^2}T+\frac{\|W\|^4\overline{\lambda}_1\sqrt{2}}{(2\|W\|^2+\varphi'^2)^2}N+\frac{\|W\|^4(\overline{\lambda}_3\cos\varphi+\overline{\lambda}_2\sin\varphi)\sqrt{2}}{(2\|W\|^2+\varphi'^2)^2}B. \tag{2.20}$$

If made inner product from the equations (2.19) and (2.20), $\kappa_g^{\beta_1}$ geodesic curvature is found like equations (2.15).
The proofs of the subsequent theorems and corollaries belonging to β_2, β_3, β_{ς_1}, β_{ς_2}, β_{ς_3}, β_{ξ_1}, β_{ξ_2} and β_{ξ_3} - Smarandache curves will be similar to the theorem 2.1 and corollaries 2.1.

Definition 2.2. Let (T_1) spherical curve be of α_1, T_{T_1} and $T_1 \wedge T_{T_1}$ be Sabban vectors of (T_1). In the fact β_2 -Smarandache curve can be identified by

$$\beta_2(s) = \frac{1}{\sqrt{2}}(T_{T_1} + T_1 \wedge T_{T_1}). \tag{2.21}$$

Theorem 2.2. Let α_1 be involute of α. The geodesic curvature and belonging to $\beta_2(s) = \frac{1}{\sqrt{2}}(N_1 + B_1)$, β_2 -Smarandache curve of involute curve is,

$$\kappa_g^{\beta_2} = \frac{\kappa_1^4}{(\kappa_1^2 + 2\tau_1^2)^{\frac{5}{2}}}(2\tau_1\varepsilon_1 + \kappa_1(-\varepsilon_2 + \varepsilon_3)) \tag{2.22}$$

where coefficients are

$$\varepsilon_1 = \frac{\tau_1}{\kappa_1} + 2(\frac{\tau_1}{\kappa_1})^3 + 2(\frac{\tau_1}{\kappa_1})'(\frac{\tau_1}{\kappa_1}),\ \ \varepsilon_2 = -1 - 3(\frac{\tau_1}{\kappa_1})^2 - 2(\frac{\tau_1}{\kappa_1})^4 - (\frac{\tau_1}{\kappa_1})',\ \ \varepsilon_3 = -(\frac{\tau_1}{\kappa_1})^2 - 2(\frac{\tau_1}{\kappa_1})^4 + (\frac{\tau_1}{\kappa_1})'. \tag{2.23}$$

Corollary 2.2. Let α_1 be involute of α. The geodesic curvature belonging to $\beta_2 = \frac{1}{\sqrt{2}}((\sin\phi - \cos\phi)T + (\sin\phi + \cos\phi)B)$ β_2 -Smarandache curve of evolute curve is,

$$\kappa_g^{\beta_2} = \frac{\|W\|^4}{(\|W\|^2 + 2(\varphi')^2)^{\frac{5}{2}}}(2\varphi'\bar{\varepsilon}_1 + \|W\|(-\bar{\varepsilon}_2 + \bar{\varepsilon}_3)) \tag{2.24}$$

where coefficients are

$$\begin{aligned}\bar{\varepsilon}_1 &= (\frac{\varphi'}{\|W\|}) + (\frac{\varphi'}{\|W\|})^3 + 2(\frac{\varphi'}{\|W\|})'(\frac{\varphi'}{\|W\|}),\ \bar{\varepsilon}_2 = -1 - 3(\frac{\varphi'}{\|W\|})^2 - 2(\frac{\varphi'}{\|W\|})^4 - (\frac{\varphi'}{\|W\|})',\\ \bar{\varepsilon}_3 &= -(\frac{\varphi'}{\|W\|})^2 - 2(\frac{\varphi'}{\|W\|})^4 + (\frac{\varphi'}{\|W\|})'\end{aligned} \tag{2.25}$$

Definition 2.3. Let (T_1) spherical curve be of α_1, T_1, T_{T_1} and $T_1 \wedge T_{T_1}$ be Sabban vectors of (T_1). In the fact β_3 -Smarandache curve can be identified by

$$\beta_3(s) = \frac{1}{\sqrt{3}}(T_1 + T_{T_1} + T_1 \wedge T_{T_1}) \tag{2.26}$$

Theorem 2.3. Let α_1 be involute of α. The geodesic curvature belonging to $\beta_3(s) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1)$ β_3 -Smarandache curve of involute curve is,

$$\kappa_g^{\beta_3} = \frac{\kappa_1^4}{4\sqrt{2}(\kappa_1^2 + \kappa_1\tau_1 + \tau_1^2)^{\frac{5}{2}}}((2\tau_1 - \kappa_1)\varphi_1 - (\kappa_1 + \tau_1)\varphi_2 + (2\kappa_1 - \tau_1)\varphi_3) \tag{2.27}$$

where coefficients are

$$\phi_1 = -2 + 4(\frac{\tau_1}{\kappa_1}) - (\frac{\tau_1}{\kappa_1})^2 + 2(\frac{\tau_1}{\kappa_1})^3 + (\frac{\tau_1}{\kappa_1})'(2\frac{\tau_1}{\kappa_1} - 1),$$

$$\phi_2 = -2+2(\frac{\tau_1}{\kappa_1})-4(\frac{\tau_1}{\kappa_1})^2+2(\frac{\tau_1}{\kappa_1})^3-2(\frac{\tau_1}{\kappa_1})^4-(\frac{\tau_1}{\kappa_1})'(1+\frac{\tau_1}{\kappa_1}),$$

$$\phi_3 = 2(\frac{\tau_1}{\kappa_1})-4(\frac{\tau_1}{\kappa_1})^2+4(\frac{\tau_1}{\kappa_1})^3-2(\frac{\tau_1}{\kappa_1})^4+(\frac{\tau_1}{\kappa_1})'(2-\frac{\tau_1}{\kappa_1}) \tag{2.28}$$

Corollary 2.3. Let α_1 be involute of α. The geodesic curvature belonging to $\beta_3(s)=\frac{1}{\sqrt{3}}((\sin\varphi-\cos\varphi)T+N+(\sin\varphi+\cos\varphi)B)$, β_3-Smarandache curve of evolute curve is,

$$\kappa_g^{\beta_3} = \frac{\|W\|^4}{4\sqrt{2}(\|W\|^2+\varphi'\|W\|+\varphi'^2)^{\frac{5}{2}}}((2\varphi'-\|W\|)\bar{\phi}_1-(\|W\|+\varphi')\bar{\phi}_2+(2\|W\|-\varphi')\bar{\phi}_3) \tag{2.29}$$

where coefficients are

$$\bar{\phi}_1 = -2+4(\frac{\varphi'}{\|W\|})+4(\frac{\varphi'}{\|W\|})-(\frac{\varphi'}{\|W\|})^2+2(\frac{\varphi'}{\|W\|})^3+(\frac{\varphi'}{\|W\|})'(2\frac{\varphi'}{\|W\|-1})$$

$$\bar{\phi}_2 = -2+2(\frac{\varphi'}{\|W\|})-4(\frac{\varphi'}{\|W\|})^2+(\frac{\varphi'}{\|W\|})^3-2(\frac{\varphi'}{\|W\|})^4-(\frac{\varphi'}{\|W\|})'(1+\frac{\varphi'}{\|W\|})$$

$$\bar{\phi}_3 = 2(\frac{\varphi'}{\|W\|})-4(\frac{\varphi'}{\|W\|})^2+4(\frac{\varphi'}{\|W\|})^3-2(\frac{\varphi'}{\|W\|})^4+(\frac{\varphi'}{\|W\|})'(2-\frac{\varphi'}{\|W\|-1}) \tag{2.30}$$

Definition 2.4. Let (N_1) spherical curve be of α_1, N_1, T_{N_1} be Sabban vectors of (N_1). In the fact β_{ς_1}-Smarandache curve can be identified by

$$\beta_{\varsigma_1}(s)=\frac{1}{\sqrt{2}}(N_1+T_{N_1}) \tag{2.31}$$

Theorem 2.4. Let α_1 be involute of α. The geodesic curvature belonging to $\beta_{\varsigma_1}(s)=\frac{1}{\sqrt{2}}(-\cos\varphi_1 T_1+N_1+\sin\varphi_1 B_1)$ β_{ς_1}-Smarandache curve of involute curve is,

$$\kappa_g^{\beta_{\varsigma_1}} = \frac{\|W_1\|^4}{(2\|W_1\|^2+(\varphi_1')^2)^{\frac{5}{2}}}\left(\varphi_1'\chi_1-\varphi_1'\chi_2+2\|W_1\|\chi_3\right) \tag{2.32}$$

where coefficients are

$$\chi_1 = -2-(\frac{\varphi_1'}{\|W_1\|})^2+(\frac{\varphi_1'}{\|W_1\|})'(\frac{\varphi_1'}{\|W_1\|}),\ \ \chi_2 = -2-3(\frac{\varphi_1'}{\|W_1\|})^2-(\frac{\varphi_1'}{\|W_1\|})^4-(\frac{\varphi_1'}{\|W_1\|})'(\frac{\varphi_1'}{\|W_1\|}),$$

$$\chi_3 = 2(\frac{\varphi_1'}{\|W_1\|})+(\frac{\varphi_1'}{\|W_1\|})^3+(\frac{\varphi_1'}{\|W_1\|})' \tag{2.33}$$

Corollary 2.4. Let α_1 be involute of α. The geodesic curvature Sabban apparatus belonging to

$$\beta_{\varsigma_1}(s)=\frac{\varphi'\sin\varphi-\sqrt{\varphi'^2+\|W\|^2}\cos\varphi}{\sqrt{2\varphi'^2+2\|W\|^2}}T-\frac{\|W\|}{\sqrt{2\varphi'^2+2\|W\|^2}}N+\frac{\varphi'\cos\varphi+\sqrt{\varphi'^2+\|W\|^2}\sin\varphi}{\sqrt{2\varphi'^2+2\|W\|^2}}B$$

β_{ς_1}-Smarandache curve of evolute curve is,

$$\kappa_g^{\beta_{\varsigma_1}} = \frac{1}{(2+\eta^2)^{\frac{5}{2}}}(\eta\bar{\chi}_1-\eta\bar{\chi}_2+2\bar{\chi}_3) \tag{2.34}$$

where in the event of $\eta = \frac{\varphi_1'}{\|W_1\|} = (\frac{\varphi'}{\sqrt{\varphi'^2 + \|W\|^2}})' \cos\varphi(c-s)$ coefficients are

$$\overline{\chi}_1 = -2 - \eta^2 + \eta'\eta, \quad \overline{\chi}_2 = -2 - 3\eta^2 - \eta^4 - \eta'\eta, \quad \overline{\chi}_3 = 2\eta + \eta^3 + \eta'. \tag{2.35}$$

Definition 2.5. Let (N_1) spherical curve be of α_1, T_{N_1} and $N_1 \wedge T_{N_1}$ be Sabban vectors of (N_1). In the fact β_{ς_2} -Smarandache curve can be identified by

$$\beta_{\varsigma_2}(s) = \frac{1}{\sqrt{2}}(T_{N_1} + N_1 \wedge T_{N_1}) \tag{2.36}$$

Theorem 2.5. Let α_1 be involute of α. The geodesic curvature belonging to $\beta_{\varsigma_2}(s) = \frac{1}{\sqrt{2}}((\sin\varphi_1 - \cos\varphi_1)T_1 + (\sin\varphi_1 + \cos\varphi_1)B_1)$, β_{ς_2} -Smarandache curve of involute curve is ,

$$\kappa_g^{\beta_{\varsigma_2}} = \frac{\|W_1\|^4}{(\|W_1\|^2 + 2(\varphi_1')^2)^{\frac{5}{2}}}(2\varphi_1'\phi_1 - \|W_1\|\phi_2 + \|W_1\|\phi_3) \tag{2.37}$$

where coefficients are

$$\begin{cases} \phi_1 = -2 + 4(\frac{\tau_1}{\kappa_1}) - (\frac{\tau_1}{\kappa_1})^2 + 2(\frac{\tau_1}{\kappa_1})^3 + (\frac{\tau_1}{\kappa_1})'(2\frac{\tau_1}{\kappa_1} - 1), \\ \phi_2 = -2 + 2(\frac{\tau_1}{\kappa_1}) - 4(\frac{\tau_1}{\kappa_1})^2 + 2(\frac{\tau_1}{\kappa_1})^3 - 2(\frac{\tau_1}{\kappa_1})^4 - (\frac{\tau_1}{\kappa_1})'(1 + \frac{\tau_1}{\kappa_1}), \\ \phi_3 = 2(\frac{\tau_1}{\kappa_1}) - 4(\frac{\tau_1}{\kappa_1})^2 + 4(\frac{\tau_1}{\kappa_1})^3 - 2(\frac{\tau_1}{\kappa_1})^4 + (\frac{\tau_1}{\kappa_1})'(2 - \frac{\tau_1}{\kappa_1}) \end{cases} \tag{2.38}$$

Corollary 2.5 Let α_1 be involute of α. The geodesic curvature belonging to $\beta_{\varsigma_2}(s) = \frac{(\varphi' + \|W\|)\sin\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}}T + \frac{\varphi' - \|W\|}{\sqrt{2\varphi'^2 + 2\|W\|^2}}N + \frac{(\varphi' - \|W\|)\cos\varphi}{\sqrt{2\varphi'^2 + 2\|W\|^2}}B$, β_{ς_2} -Smarandache curve of evolute curve is,

$$\kappa_g^{\beta_{\varsigma_2}} = \frac{1}{(2 + \eta^2)^{\frac{5}{2}}}(2\eta\overline{\phi}_1 - \overline{\phi}_2 + \overline{\phi}_3) \tag{2.39}$$

where coefficients are

$$\overline{\phi}_1 = \eta + 2\eta^3 + 2\eta'\eta, \; \overline{\phi}_2 = -1 - 3\eta^2 - 2\eta^4 - \eta', \; \overline{\phi}_3 = -\eta^2 - 2\eta^4 + \eta'. \tag{2.40}$$

Definition 2.6. Let (N_1) spherical curve be of α_1, N_1, T_{N_1} and $N_1 \wedge T_{N_1}$ be Sabban vectors of (N_1). In the fact β_{ς_3} -Smarandache curve can be identified by

$$\beta_{\varsigma_3}(s) = \frac{1}{\sqrt{3}}(N_1 + T_{N_1} + N_1 \wedge T_{N_1}) \tag{2.41}$$

Theorem 2.6 Let α_1 be involute of α. The geodesic curvature belonging to $\beta_{\varsigma_3}(s) = \frac{1}{\sqrt{3}}((\sin\varphi_1 - \cos\varphi_1)T_1 + N_1 + (\sin\varphi_1 + \cos\varphi_1)B_1)$ β_{ς_3} -Smarandache curve of involute curve is,

$$\kappa_g^{\beta_{\varsigma_3}} = \frac{(2\frac{\varphi_1'}{\|W_1\|}-1)\rho_1 + (-1-\frac{\varphi_1'}{\|W_1\|})\rho_2 + (2-\frac{\varphi_1'}{\|W_1\|})\rho_3}{4\sqrt{2}(1-\frac{\varphi_1'}{\|W_1\|}+(\frac{\varphi_1'}{\|W_1\|})^2)^{\frac{5}{2}}} \tag{2.42}$$

where coefficients are

$$\begin{cases} \rho_1 = -2+4(\frac{\varphi_1'}{\|W_1\|}) - (\frac{\varphi_1'}{\|W_1\|})^2 + 2(\frac{\varphi_1'}{\|W_1\|})^3 + (\frac{\varphi_1'}{\|W_1\|})'(2\frac{\varphi_1'}{\|W_1\|}-1) \\ \rho_2 = -2+2(\frac{\varphi_1'}{\|W_1\|}) - 4(\frac{\varphi_1'}{\|W_1\|})^2 + 2(\frac{\varphi_1'}{\|W_1\|})^3 - 2(\frac{\varphi_1'}{\|W_1\|})^4 - (\frac{\varphi_1'}{\|W_1\|})'(1+\frac{\varphi_1'}{\|W_1\|}) \\ \rho_3 = 2(\frac{\varphi_1'}{\|W_1\|}) - 4(\frac{\varphi_1'}{\|W_1\|})^2 + 4(\frac{\varphi_1'}{\|W_1\|})^3 - 2(\frac{\phi_1'}{\|W_1\|})^4 + (\frac{\varphi_1'}{\|W_1\|})'(2-\frac{\varphi_1'}{\|W_1\|}) \end{cases} \tag{2.43}$$

Corollary 2.6. Let α_1 be involute of α. The geodesic curvature belonging to

$$\beta_{\varsigma_3}(s) = \frac{(\varphi'+\|W\|)\sin\varphi - \sqrt{\varphi'^2+\|W\|^2}\cos\varphi}{\sqrt{3\phi'^2+3\|W\|^2}}T + \frac{\varphi'-\|W\|}{\sqrt{3\varphi'^2+3\|W\|^2}}N + \frac{(\varphi'+\|W\|)\cos\varphi + \sqrt{\varphi'^2+\|W\|^2}\sin\varphi}{\sqrt{3\varphi'^2+3\|W\|^2}}B$$

β_{ς_3} -Smarandache curve of evolute curve is,

$$+\frac{(\varphi'+\|W\|)\cos\varphi + \sqrt{\varphi'^2+\|W\|^2}\sin\varphi}{\sqrt{3\varphi'^2+3\|W\|^2}}B$$

$$\kappa_g^{\beta_{\varsigma_3}} = \frac{(2\eta-1)\overline{\rho}_1 + (-1-\eta)\overline{\rho}_2 + (2-\eta)\overline{\rho}_3}{4\sqrt{2}(1-\eta+\eta^2)^{\frac{5}{2}}} \tag{2.44}$$

where coefficients are

$$\begin{gathered} \overline{\rho}_1 = -2+4\eta-4\eta^2+2\eta^3+2\eta'(2\eta-1), \quad \overline{\rho}_2 = -2+2\eta-4\eta^2+2\eta^3-2\eta^4-\eta'(1+\eta) \\ \overline{\rho}_3 = 2\eta-4\eta^2+4\eta^3-2\eta^4+\eta'(2-\eta) \end{gathered} \tag{2.45}$$

Definition 2.7. Let (B_1) spherical curve be of α_1, B_1 and T_{B_1} be Sabban vectors of (B_1). In the fact β_{ξ_1} -Smarandache curve can be identified by

$$\beta_{\xi_1}(s) = \frac{1}{\sqrt{2}}(B_1 + T_{B_1}) \tag{2.46}$$

Theorem 2.7. Let α_1 be involute of α. The geodesic curvature belonging to $\beta_{\xi_1}(s) = \frac{1}{\sqrt{2}}(-N_1 + B_1)$ β_{ξ_1} -Smarandache curve of involute curve is,

$$\kappa_g^{\beta_{\xi_1}} = \frac{\tau_1^{\,4}}{(2\tau_1^{\,2}+\kappa_1^{\,2})^{\frac{5}{2}}}(\kappa_1\omega_1 - \kappa_1\omega_2 + 2\tau_1\omega_3) \tag{2.47}$$

where coefficients are

$$\omega_1 = -2-(\frac{\kappa_1}{\tau_1})^2+(\frac{\kappa_1}{\tau_1})'(\frac{\kappa_1}{\tau_1}),\ \ \omega_2 = -2-3(\frac{\kappa_1}{\tau_1})^2-(\frac{\kappa_1}{\tau_1})^4-(\frac{\kappa_1}{\tau_1})'(\frac{\kappa_1}{\tau_1})$$
$$\omega_3 = 2(\frac{\kappa_1}{\tau_1})+(\frac{\kappa_1}{\tau_1})^3+(\frac{\kappa_1}{\tau_1})' \tag{2.48}$$

Corollary 2.7. Let α_1 be involute of α. The geodesic curvature belonging to

$\beta_{\xi_1}(s)=\frac{1}{\sqrt{2}}((\cos\varphi+\sin\varphi)T+(\cos\varphi-\sin\varphi)B,$ β_{ξ_1} -Smarandache curve of evolute curve is,

$$\kappa_g^{\beta_{\xi_1}} = \frac{\varphi'^4}{(2\varphi'^2+\|W\|^2)^{\frac{5}{2}}}(\|W\|\left(\overline{\omega}_1-\overline{\omega}_2\right)+2\varphi'\overline{\omega}_3), \tag{2.49}$$

where coefficients are

$$\overline{\omega}_1 = -2-(\frac{\|W\|}{\varphi'})^2+(\frac{\|W\|}{\varphi'})'(\frac{\|W\|}{\varphi'}),\ \ \overline{\omega}_2 = -2-3(\frac{\|W\|}{\varphi'})^2-(\frac{\|W\|}{\varphi'})^4-(\frac{\|W\|}{\varphi'})'(\frac{\|W\|}{\varphi'})$$
$$\overline{\omega}_3 = 2(\frac{\|W\|}{\varphi'})+(\frac{\|W\|}{\varphi'})^3+2(\frac{\|W\|}{\varphi'})'. \tag{2.50}$$

Definition 2.8. Let (B_1) spherical curve be of α_1, T_{B_1} and $B_1\wedge T_{B_1}$ be Sabban vectors of (B_1). In the fact β_{ξ_2} -Smarandache curve can be identified by

$$\beta_{\xi_2}(s)=\frac{1}{\sqrt{3}}(T_{B_1}+B_1\wedge T_{B_1}) \tag{2.51}$$

Theorem 2.8. Let α_1 be involute of α. The geodesic curvature belonging to

$\beta_{\xi_2}(s)=\frac{1}{\sqrt{2}}(T_1-N_1),$ β_{ξ_2} -Smarandache curve of involute curve is,

$$\kappa_g^{\beta_{\xi_2}} = \frac{\tau_1^{\ 4}}{(\tau_1^{\ 2}+2\kappa_1^{\ 2})^{\frac{5}{2}}}(2\kappa_1\psi_1-\tau_1\psi_2+\tau_1\psi_3), \tag{2.52}$$

where coefficients are

$$\psi_1 = \frac{\kappa_1}{\tau_1}+2(\frac{\kappa_1}{\tau_1})^3+2(\frac{\kappa_1}{\tau_1})'(\frac{\kappa_1}{\tau_1}),\ \ \psi_2 = -1-3(\frac{\kappa_1}{\tau_1})^2-2(\frac{\kappa_1}{\tau_1})^4-(\frac{\kappa_1}{\tau_1})',$$
$$\psi_3 = -(\frac{\kappa_1}{\tau_1})^2-2(\frac{\kappa_1}{\tau_1})^4+(\frac{\kappa_1}{\tau_1})' \tag{2.53}$$

Corollary 2.8. Let α_1 be involute of α. The geodesic curvature curvature belonging to

$\beta_{\xi_2}(s)=\frac{1}{\sqrt{2}}(\cos\varphi T+N-\sin\varphi B),$ β_{ξ_2} -Smarandache curve of evolute curve is,

$$\kappa_g^{\beta_{\xi_2}} = \frac{\varphi'^4}{(\varphi'^2+2\|W\|^2)^{\frac{5}{2}}}(2\|W\|\overline{\psi}_1-\varphi'\overline{\psi}_2+\varphi'\overline{\psi}_3), \tag{2.54}$$

where coefficients are

$$\overline{\psi}_1=(\frac{\|W\|}{\varphi'})+(\frac{\|W\|}{\varphi'})^3+2(\frac{\|W\|}{\varphi'})'(\frac{\|W\|}{\varphi'}),\quad \overline{\psi}_2=-1-3(\frac{\|W\|}{\varphi'})^2-2(\frac{\|W\|}{\varphi'})^4-(\frac{\|W\|}{\varphi'})'$$
$$\overline{\psi}_3=-(\frac{\|W\|}{\varphi'})^2-2(\frac{\|W\|}{\varphi'})^4+(\frac{\|W\|}{\varphi'})' \tag{2.55}$$

Definition 2.9. Let (B_1) spherical curve be of α_1, B_1, T_{B_1} and $B_1\wedge T_{B_1}$ be Sabban vectors of (B_1) In the fact β_{ξ_3} -Smarandache curve can be identified by

$$\beta_{\xi_3}(s)=\frac{1}{\sqrt{3}}(B_1+T_{B_1}+B_1\wedge T_{B_1}) \tag{2.56}$$

Theorem 2.9. Let α_1 be involute of α. The geodesic curvature belonging to $\beta_{\xi_3}=\frac{1}{\sqrt{3}}(T_1-N_1+B_1)$, β_{ξ_3} -Smarandache curve of involute curve is,

$$\kappa_g^{\beta_{\xi_3}}=\frac{\tau_1^{\,4}}{4\sqrt{2}(\tau_1^{\,2}+\kappa_1\tau_1+\kappa_1^{\,2})^{\frac{5}{2}}}((2\kappa_1-\tau_1)\zeta_1-(\tau_1+\kappa_1)\zeta_2+(2\tau_1-\kappa_1)\zeta_3), \tag{2.57}$$

where coefficients are

$$\begin{cases}\zeta_1=-2+4(\frac{\kappa_1}{\tau_1})-(\frac{\kappa_1}{\tau_1})^2+2(\frac{\kappa_1}{\tau_1})^3+(\frac{\kappa_1}{\tau_1})'(2\frac{\kappa_1}{\tau_1}-1)\\ \zeta_2=-2+2(\frac{\kappa_1}{\tau_1})-4(\frac{\kappa_1}{\tau_1})^2+2(\frac{\kappa_1}{\tau_1})^3-2(\frac{\kappa_1}{\tau_1})^4-(\frac{\kappa_1}{\tau_1})'(1+\frac{\kappa_1}{\tau_1})\\ \zeta_3=2(\frac{\kappa_1}{\tau_1})-4(\frac{\kappa_1}{\tau_1})^2+4(\frac{\kappa_1}{\tau_1})^3-2(\frac{\kappa_1}{\tau_1})^4+(\frac{\kappa_1}{\tau_1})'(2-\frac{\kappa_1}{\tau_1})\end{cases} \tag{2.58}$$

Corollary 2.9. Let α_1 be involute of α. The geodesic curvature belonging to $\beta_{\xi_3}(s)=\frac{1}{\sqrt{3}}((\sin\varphi+\cos\varphi)T+N+(\cos\varphi-\sin\varphi)B)$, β_{ξ_3} -Smarandache curve of evolute curve is,

$$\kappa_g^{\beta_{\xi_3}}=\frac{\varphi'^4}{4\sqrt{2}(\varphi'^5+\varphi'\|W\|+\|W\|^2)^{\frac{5}{2}}}((2\|W\|-\varphi')\overline{\zeta}_1-(\varphi'+\|W\|)\overline{\zeta}_2+(2\varphi'-\|W\|)\overline{\zeta}_3) \tag{2.59}$$

where coefficients are

$$\begin{cases}\overline{\zeta}_1=-2+4(\frac{\|W\|}{\varphi'})+4(\frac{\|W\|}{\varphi'})-(\frac{\|W\|}{\varphi'})^2+2(\frac{\|W\|}{\varphi'})^3+(\frac{\|W\|}{\varphi'})'(2\frac{\|W\|}{\varphi'}-1)\\ \overline{\zeta}_2=-2+2(\frac{\|W\|}{\varphi'})-4(\frac{\|W\|}{\varphi'})^2+(\frac{\|W\|}{\varphi'})^3-2(\frac{\|W\|}{\varphi'})^4-(\frac{\|W\|}{\varphi'})'(1+\frac{\|W\|}{\varphi'})\\ \overline{\zeta}_3=2(\frac{\|W\|}{\varphi'})-4(\frac{\|W\|}{\varphi'})^2+4(\frac{\|W\|}{\varphi'})^3-2(\frac{\|W\|}{\varphi'})^4+(\frac{\|W\|}{\varphi'})'(2-\frac{\|W\|}{\varphi'})\end{cases} \tag{2.60}$$

Example. Let us consider the unit speed evolute curve and involute curve, respectively

$$\alpha(s)=\left(\frac{2}{5}\sin(2t)-\frac{1}{40}\sin(8t),-\frac{2}{5}\cos(2t)+\frac{1}{40}\cos(8t),\frac{4}{15}\sin(3t)\right)$$
$$\alpha_1(s)=\left(\frac{2}{5}\sin(2s)-\frac{1}{40}\sin(8s)+\frac{4}{5}(1-s)\cos(5s),-\frac{2}{5},\right.$$

$$\cos(2s)+\frac{1}{40}\cos(8s)+\frac{4}{5}(1-s)\sin(5s),\frac{4}{15}\sin(3s)-\frac{3}{5}+\frac{3}{5}s\Bigg).$$

The Frenet vectors belonging to involute curve, α_1 are found as follows;

$$T_1=\left(\frac{4}{5}\cos(5s),\frac{4}{5}\sin(5s),-\frac{3}{5}\right)$$

$$N_1=\left(\left(\frac{1}{5}\cos(8s)-\frac{4}{5}\cos(2s)\right)\sin(3s)-\left(\frac{4}{5}\sin(2s)+\frac{1}{5}\sin(8s)\right)\cos(3s),\right.$$

$$\left.\left(-\frac{4}{5}\sin(2s)+\frac{1}{5}\sin(8s)\right)\sin(3s)+\cos(3s)\left(\frac{4}{5}\cos(2s)+\frac{1}{5}\cos(8s)\right),0\right),$$

$$B_1=\left(\cos(3s)\left(\frac{4}{5}\cos(2s)-\frac{1}{5}\cos(8s)\right)-\sin(3s)\left(\frac{4}{5}\sin(2s)+\frac{1}{5}\sin(8s)\right),\right.$$

$$\left.\cos(3s)\left(\frac{4}{5}\sin(2s)-\frac{1}{5}\sin(8s)\right)+\sin(3s)\left(\frac{4}{5}\cos(2s)+\frac{1}{5}\cos(8s)\right),\frac{4}{5}\right).$$

According to definitions, we reach specific Smarandache curves belonging to Sabban frame of this curve. β_1, β_2, β_3, β_{ς_1}, β_{ς_2}, β_{ς_3}, β_{ξ_1}, β_{ξ_2} and β_{ξ_3} (see Figure 1,2,3).

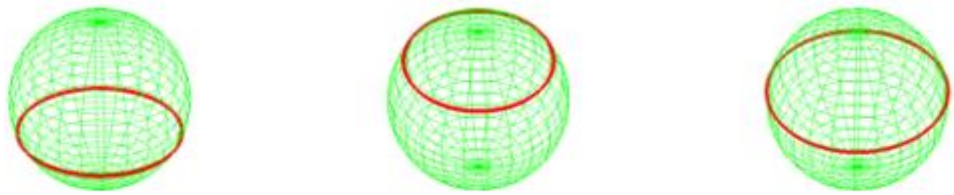

Figure 1: β_1-curve β_2-curve β_3-curve

Figure 2: β_{ς_1}-curve β_{ς_2}-curve β_{ς_3}- curve

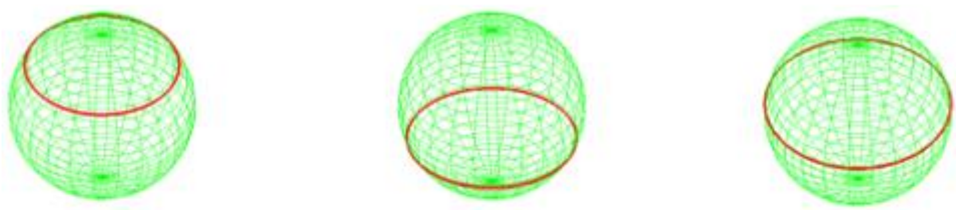

Figure 3: β_{ξ_1}-curve β_{ξ_2}-curve β_{ξ_3}- curve

3. Conclusion

In this article, we reviewed the well-known involute and evolute curves in the literature. We have created the Sabban frames on the unit sphere of the involute and evolute curves. We got Smarandache curves from the Sabban frame and calculated the geodesic curvature of these curves. Finally, we have given an example and have driven their shapes in the Mapple program.

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