ON THE SABBAN FRAME BELONGING TO INVOLUTE-EVOLUTE CURVES

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In this article, we investigate special Smarandache curves with regard to Sabban frame of involute curve. We created Sabban frame belonging to spherical indicatrix of involute curve. It was explained Smarandache curves position vector is consisted by Sabban vectors belonging to spherical indicatrix. Then, we calculated geodesic curvatures of this Smarandache curves. The results found for each curve was given depend on evolute curve. The example related to the subject were given and their figures were drawn with Mapple program.

Key Words: Geodesic curvature, Involute-evolute curves, Sabban frame, Smarandache curves

1. Introduction and Preliminaries

The involute of the curve is decent known by the mathematicians especially the differential geometry scientists. There are many essential consequences and properties of curves. Involute curves examined by some authors[3, 6]. Frenet vectors of a curve were taken as the position vector and the regular curve drawn by the new vector is identified. This curve were called the Smarandache curve [10]. Special Smarandache curves examined by certain writers [1, 2, 5, 7, 8, 9]. K. Taşköprü, M. Tosun studied particular Smarandache curves belonging to Sabban frame on $S^2$ [11]. Şenyurt and Çalışkan investigated particular Smarandache curves belonging to Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves [4].

Let $\alpha: I \rightarrow E^3$ be a unit speed curve and the quantities $\{T, N, B, \kappa, \tau\}$ are collectively Frenet-Serret apparatus of this curve. The Frenet formulae are also well known as, respectively [6]

$$T'(s) = \kappa(s)N(s), \quad N'(s) = -\kappa(s)T(s) + \tau(s)B(s), \quad B'(s) = -\tau(s)N(s). \quad (1.1)$$

Let $\alpha_1: I \rightarrow E^3$ be the $C^2$-class differentiable curve and $\{T_1(s), N_1(s), B_1(s), \kappa_1(s), \tau_1(s)\}$ is Frenet-Serret apparatus of the $\alpha_1$ involute curve, then [3]

$$T_1 = N, \quad N_1 = -\cos \varphi T + \sin \varphi B, \quad B_1 = \sin \varphi T + \cos \varphi B. \quad (1.2)$$

where $\Delta(W, B) = \varphi$. For the curvatures and the torsions we have

$$\kappa_1 = \frac{|W|}{c - s}, \quad \tau_1 = \frac{(\kappa \tau' - \tau \kappa')}{\kappa_1 |c - s| |W|^2}. \quad (1.3)$$

$$\sin \varphi_1 = \frac{\varphi'}{\sqrt{\varphi'^2 + |W|^2}}, \quad \cos \varphi_1 = \frac{|W|}{\sqrt{\varphi'^2 + |W|^2}}, \quad \varphi_1' = \frac{(\varphi')'}{\sqrt{\varphi'^2 + |W|^2}} \quad (1.4)$$

Let $\gamma: I \rightarrow S^2$ be a unit speed spherical curve. We symbolize $s$ as the arc-length parameter of $\gamma$. Let’s give

$$\gamma(s) = \gamma(s), t(s) = \gamma'(s), d(s) = \gamma(s) \wedge t(s) \quad (1.5)$$

$\{\gamma(s), t(s), d(s)\}$ frame is denominated the Sabban frame of $\gamma$ on $S^2$. The spherical Frenet formulae of $\gamma$ is as follows

$$\gamma'(s) = t(s), \quad t'(s) = -\gamma(s) + \kappa_s(s)d(s), \quad d'(s) = -\kappa_s(s)t(s) \quad (1.6)$$
where $\kappa_g$ is denominated the geodesic curvature of the curve $\gamma$ on $S^2$ which is
\[ \kappa_g(s) = \langle t'(s), d(s) \rangle \quad [11]. \]

2. On The Sabban Frame Belonging To Involute-Evolute Curves

In this section, we investigated special Smarandache curves such as created by Sabban frame, $\{T_1, T_{1T}, T_1 \wedge T_{1T}\}$, $\{N_1, T_{1N}, N_1 \wedge T_{1N}\}$ and $\{B_1, T_{1B}, B_1 \wedge T_{1B}\}$. We will find some results. These results will be expressed depending on the evolute curve. Let’s find results on this Smarandache curves. Let $\alpha_T(s) = T_1(s)$, $\alpha_N(s) = N_1(s)$ and $\alpha_B(s) = B_1(s)$ be a regular spherical curves on $S^2$. The Sabban frames of spherical indicatrix belonging to involute curve are as follows:
\[ T_1 = T_1, \quad T_{1T} = N_1, \quad T_1 \wedge T_{1T} = B_1, \quad (T_1)' = -\frac{1}{\kappa_1} T_1, \]
\[ N_1 = N_1, \quad N_{1N} = -\cos \phi T_1 + \sin \phi B_1, \quad N_1 \wedge N_{1N} = \sin \phi T_1 + \cos \phi B_1, \]
\[ B_1 = B_1, \quad B_{1B} = -N_1, \quad B_1 \wedge B_{1B} = T_1. \]

From the equation (1.6), the spherical Frenet formulae of $\alpha_T(s)$, $\alpha_N(s)$ and $\alpha_B(s)$ are as follows,
\[ T' = T_{1T}, \quad T_{1T}' = -T_1 + \frac{\tau_1}{\kappa_1} T_1 \wedge T_{1T}, \quad (T_1 \wedge T_{1T})' = -\frac{1}{\kappa_1} T_1, \]
\[ N' = N_{1N}, \quad N_{1N}' = -N_1 + \frac{\varphi_1}{W_{11}} N_1 \wedge N_{1N}, \quad (N_1 \wedge N_{1N})' = -\frac{\varphi_1}{W_{11}} N_1, \]
\[ B' = B_{1B}, \quad B_{1B}' = -B_1 + \frac{\kappa_1}{\tau_1} B_1 \wedge B_{1B}, \quad (B_1 \wedge B_{1B})' = -\frac{\kappa_1}{\tau_1} B_1. \]

Using the equation (1.7), the geodesic curvatures of $\alpha_T(s)$, $\alpha_N(s)$ and $\alpha_B(s)$ are
\[ \kappa_g^T = \frac{\tau_1}{\kappa_1}, \quad \kappa_g^N = \frac{\varphi_1}{W_{11}}, \quad \kappa_g^B = \frac{\kappa_1}{\tau_1}. \]

Definition 2.1. Let $(T_1)$ spherical curve be of $\alpha_1$, $T_1$ and $T_{1T}$ be Sabban vectors of $(T_1)$. In the fact $\beta_1$-Smarandache curve can be identified by
\[ \beta_1(s) = \frac{1}{\sqrt{2}} \left( T_1 + T_{1T} \right). \]

Theorem 2.1. The geodesic curvature according to $\beta_1$-Smarandache curve of the involute curve is
\[ \kappa_g^\beta = \frac{\kappa_1^4}{\left( 2\kappa_1^2 + \tau_1^2 \right)^2} \left( \tau_1^3 \lambda_1 + \tau_1^2 \lambda_2 + 2\kappa_1 \lambda_3 \right), \]
where coefficients are
\[ \lambda_1 = -2 \left( \frac{\tau_1}{\kappa_1} \right)^2 + \left( \frac{\tau_1}{\kappa_1} \right)^4, \quad \lambda_2 = -3 \left( \frac{\tau_1}{\kappa_1} \right)^2 - \left( \frac{\tau_1}{\kappa_1} \right)^4 - \left( \frac{\tau_1}{\kappa_1} \right)^6, \]
\[ \lambda_3 = 2 \left( \frac{\tau_1}{\kappa_1} \right) + \left( \frac{\tau_1}{\kappa_1} \right)^2 + \left( \frac{\tau_1}{\kappa_1} \right)^4. \]
Proof: \( \beta_i(s) = \frac{1}{\sqrt{2}} \left( T_i + T_{i-1} \right) \) or from the equation (2.1), we can write

\[
\beta_i(s) = \frac{1}{\sqrt{2}} \left( T_i + N_i \right).
\]  

(2.11)

If taken derivative of the equation (2.11), \( T_{\beta_i} \) vector is

\[
T_{\beta_i}(s) = \frac{1}{\sqrt{2} \kappa_i + \tau_i} \left( -\kappa_i T_i + \kappa_i N_i + \tau_i B_i \right).
\]  

(2.12)

Considering the equations (2.11) and (2.12), we have

\[
(\beta_i \wedge T_{\beta_i})(s) = \frac{1}{\sqrt{4 \kappa_i^2 + 2 \tau_i^2}} \left( \tau_i T_i - \tau_i N_i + 2 \kappa_i B_i \right).
\]  

(2.13)

Using the equation (2.12), \( T_{\beta_i} \) vector is

\[
T_{\beta_i}(s) = \frac{\sqrt{2} \kappa_i^4}{\left( 2 \kappa_i^2 + \tau_i^2 \right)^2} \left( \lambda_1 T_i + \lambda_2 N_i + \lambda_3 B_i \right).
\]  

(2.14)

From the equation (2.13) and (2.14), \( \kappa_{\beta_i} \) geodesic curvature of \( \beta_i(s) \) is

\[
\kappa_{\beta_i} = \frac{\kappa_i^4}{\left( 2 \kappa_i^2 + \tau_i^2 \right)^2} \left( \tau_i \lambda_1 + \tau_i \lambda_2 + 2 \kappa_i \lambda_3 \right).
\]  

(2.15)

Corollary 2.1. The geodesic curvature belonging to \( \beta_i \)-Smarandache curve of the evolute curve is

\[
\kappa_{\beta_i} = \frac{\|W\|}{\left( 2 \|W\|^2 + \phi^2 \right)^2} \left( \phi' \lambda_1 + \phi' \lambda_2 + \frac{2}{\|W\|} \lambda_3 \right),
\]  

where coefficients are i

\[
\lambda_1 = -2 - \left( \frac{\phi'}{\|W\|} \right)^2 + \left( \frac{\phi'}{\|W\|} \right) \left( \frac{\phi'}{\|W\|} \right), \quad \lambda_2 = -2 - 3 \left( \frac{\phi'}{\|W\|} \right)^2 - \left( \frac{\phi'}{\|W\|} \right)^{\frac{3}{2}} - \left( \frac{\phi'}{\|W\|} \right)^2 \left( \frac{\phi'}{\|W\|} \right), \quad \lambda_3 = 2 \left( \frac{\phi'}{\|W\|} \right) + \left( \frac{\phi'}{\|W\|} \right)^3 + \left( \frac{\phi'}{\|W\|} \right)^4,
\]  

(2.16)

Proof: From the equations (1.2) and (2.11), we calculate

\[
\beta_i(s) = \frac{1}{\sqrt{2}} \left( -\cos \phi T + N + \sin \phi B \right).
\]  

(2.17)

If taken derivative of the this expression, \( T_{\beta_i} \) vector is,

\[
T_{\beta_i} = \frac{\phi' \sin \phi - \|W\| \cos \phi}{\sqrt{2 \|W\|^2 + \phi^2}} T - \frac{\|W\|}{\sqrt{2 \|W\|^2 + \phi^2}} N + \frac{\phi' \cos \phi + \|W\| \sin \phi}{\sqrt{2 \|W\|^2 + \phi^2}} B.
\]  

(2.18)

If made cross product from the equations (2.17) and (2.18), we have

\[
\beta_i \wedge T_{\beta_i} = \frac{2 \|W\| \sin \phi + \phi \cos \phi}{\sqrt{4 \|W\|^2 + 2 \phi^2}} T - \frac{\phi'}{\sqrt{4 \|W\|^2 + 2 \phi^2}} N + \frac{2 \|W\| \cos \phi - \phi' \sin \phi}{\sqrt{4 \|W\|^2 + 2 \phi^2}} B.
\]  

(2.19)

From the equation (2.18), \( T_{\beta_i} \) vector is

\[
T_{\beta_i}' = \left[ \frac{\bar{\lambda}_3 \sin \phi - \bar{\lambda}_2 \cos \phi}{\sqrt{2 \|W\|^2 + \phi^2}} T + \frac{\|W\| \bar{\lambda}_1 \sqrt{2}}{\sqrt{2 \|W\|^2 + \phi^2}} T \right] N + \frac{\|W\| \left( \bar{\lambda}_3 \cos \phi + \bar{\lambda}_2 \sin \phi \right) \sqrt{2}}{\sqrt{2 \|W\|^2 + \phi^2}} B.
\]  

(2.20)
If made inner product from the equations (2.19) and (2.20), \( \kappa^\beta_g \) geodesic curvature is found like equations (2.15).

The proofs of the subsequent theorems and corollaries belonging to \( \beta_2, \beta_3, \beta_{\xi_1}, \beta_{\xi_2}, \beta_{\xi_3}, \beta_{\eta_1}, \beta_{\eta_2} \) and \( \beta_{\xi_3} \) - Smarandache curves will be similar to the theorem 2.1 and corollaries 2.1.

**Definition 2.2.** Let \((T_1)\) spherical curve be of \( \alpha_1, T_{T_1}, T_{T_1} \wedge T_{T_1} \) be Sabban vectors of \((T_1)\). In the fact \( \beta_{\xi_2} \) - Smarandache curve can be identified by

\[
\beta_{\xi_2}(s) = \frac{1}{\sqrt{2}}(T_{T_1} + T_{T_{T_1}}).
\]  

(2.21)

**Theorem 2.2.** Let \( \alpha_1 \) be involute of \( \alpha \). The geodesic curvature and belonging to \( \beta_{\xi_2}(s) = \frac{1}{\sqrt{2}}(N_1 + B_1) \).

\( \beta_{\xi_2} \) - Smarandache curve of involute curve is,

\[
\kappa_g^{\beta_2} = \frac{K_1^4}{(K_1^2 + 2T_1^2)^2} (2T_1^2 + K_1 (-e_2 + e_3))
\]

where coefficients are

\[
e_1 = \frac{1}{K_1} + 2(\frac{T_1}{K_1})^3 + \frac{T_1}{K_1}(\frac{T_1}{K_1})',
\]

\[
e_2 = -1 - 3(\frac{T_1}{K_1})^2 - 2(\frac{T_1}{K_1})^4 - (\frac{T_1}{K_1})',
\]

\[
e_3 = -\frac{T_1}{K_1} - 2(\frac{T_1}{K_1})^4 + (\frac{T_1}{K_1})' \]  

(2.22)

**Corollary 2.2.** Let \( \alpha_1 \) be involute of \( \alpha \). The geodesic curvature belonging to \( \beta_{\xi_2} = \frac{1}{\sqrt{2}}((\sin \phi - \cos \phi)T + (\sin \phi + \cos \phi)B) \) \( \beta_{\xi_2} \) - Smarandache curve of evolute curve is,

\[
\kappa_g^{\beta_2} = \frac{\|W\|^4}{(\|W\|^2 + 2(\phi')^2)^2} (2\phi' e_1 + \|W\| (-e_2 + e_3))
\]

where coefficients are

\[
e_1 = \frac{\phi'}{\|W\|^2} + (\frac{\phi'}{\|W\|^2})^3 + 2(\frac{\phi'}{\|W\|^2})(\frac{\phi'}{\|W\|^2}),
\]

\[
e_2 = -1 - 3(\frac{\phi'}{\|W\|^2})^2 - 2(\frac{\phi'}{\|W\|^2})^4 - (\frac{\phi'}{\|W\|^2})',
\]

\[
e_3 = -\frac{\phi'}{\|W\|^2} - 2(\frac{\phi'}{\|W\|^2})^4 + (\frac{\phi'}{\|W\|^2})' \]  

(2.24)

**Definition 2.3.** Let \((T_1)\) spherical curve be of \( \alpha_1, T_{T_1}, T_{T_1} \wedge T_{T_1} \) and \( T_{T_1} \wedge T_{T_1} \) be Sabban vectors of \((T_1)\). In the fact \( \beta_{\xi_3} \) - Smarandache curve can be identified by

\[
\beta_{\xi_3}(s) = \frac{1}{\sqrt{3}}(T_{T_1} + T_{T_{T_1}} + T_{T_{T_{T_1}}}).
\]

(2.26)

**Theorem 2.3.** Let \( \alpha_1 \) be involute of \( \alpha \). The geodesic curvature belonging to \( \beta_{\xi_3} = \frac{1}{\sqrt{3}}(N_1 + B_1) \).

\( \beta_{\xi_3} \) - Smarandache curve of involute curve is,

\[
\kappa_g^{\beta_3} = \frac{K_1^4}{4\sqrt{2}(K_1^2 + \kappa_1 \tau_1 + \tau_1^2)^2} ((2\tau_1 - \kappa_1)\phi_1 - (\kappa_1 + \tau_1)\phi_2 + (2\kappa_1 - \tau_1)\phi_3)
\]

where coefficients are

\[
\phi_1 = -2 + 4(\frac{\tau_1}{\kappa_1}) - 2(\frac{\tau_1}{\kappa_1})^3 + 2(\frac{\tau_1}{\kappa_1})^3 + (\frac{\tau_1}{\kappa_1})(2\frac{\tau_1}{\kappa_1} - 1),
\]

(2.27)
\[
\phi_2 = -2 + 2\left(\frac{\tau}{\kappa_1}\right) - 4\left(\frac{\tau}{\kappa_1}\right)^2 + 2\left(\frac{\tau}{\kappa_1}\right)^3 - 2\left(\frac{\tau}{\kappa_1}\right)^4 - \left(\frac{\tau}{\kappa_1}\right)(1 + \frac{1}{\kappa_1}),
\]
\[
\phi_3 = 2\left(\frac{\tau}{\kappa_1}\right) - 4\left(\frac{\tau}{\kappa_1}\right)^2 + 4\left(\frac{\tau}{\kappa_1}\right)^3 - 2\left(\frac{\tau}{\kappa_1}\right)^4 + \left(\frac{\tau}{\kappa_1}\right)(2 - \frac{1}{\kappa_1})
\]
(2.28)

**Corollary 2.3.** Let \(\alpha_i\) be involute of \(\alpha\). The geodesic curvature belonging to \(\beta_i(s) = \frac{1}{\sqrt{3}}\left((\sin \varphi - \cos \varphi)T + N + (\sin \varphi + \cos \varphi)B\right)\), \(\beta_i\)-Smarandache curve of evolute curve is, 
\[
\kappa^\beta_s = \frac{|W|^2}{4\sqrt{2}(|W|^2 + \varphi' |W|^2 + \varphi'^2)^2} ((2\varphi' - |W|\varphi'_1 - (|W| + \varphi')\varphi'_2 + (2|W| - \varphi')\varphi'_3)
\]
(2.29)

where coefficients are 
\[
\varphi'_1 = -2 + 4\left(\frac{\varphi'}{|W|}\right) + 4\left(\frac{\varphi'}{|W|}\right)^2 - \left(\frac{\varphi'}{|W|}\right)^2 + 2\left(\frac{\varphi'}{|W|}\right)^3 + \left(\frac{\varphi'}{|W|}\right)(2 - \frac{1}{|W|})
\]
\[
\varphi'_2 = -2 + 2\left(\frac{\varphi'}{|W|}\right) - 4\left(\frac{\varphi'}{|W|}\right)^2 + \left(\frac{\varphi'}{|W|}\right)^3 - 2\left(\frac{\varphi'}{|W|}\right)^4 + \left(\frac{\varphi'}{|W|}\right)(1 + \frac{1}{|W|})
\]
\[
\varphi'_3 = 2\left(\frac{\varphi'}{|W|}\right) - 4\left(\frac{\varphi'}{|W|}\right)^2 + 4\left(\frac{\varphi'}{|W|}\right)^3 - 2\left(\frac{\varphi'}{|W|}\right)^4 + \left(\frac{\varphi'}{|W|}\right)(2 - \frac{1}{|W|})
\]
(2.30)

**Definition 2.4.** Let \((N_1)\) spherical curve be of \(\alpha_i\), \(N_1\), \(T_{N_1}\) be Sabban vectors of \((N_1)\). In the fact \(\beta_{\gamma_i}\)-Smarandache curve can be identified by 
\[
\beta_{\gamma_i}(s) = \frac{1}{\sqrt{2}}(N_1 + T_{N_1})
\]
(2.31)

**Theorem 2.4.** Let \(\alpha_i\) be involute of \(\alpha\). The geodesic curvature belonging to \(\beta_{\gamma_i}(s) = \frac{1}{\sqrt{2}}(-\cos \varphi T_1 + N_1 + \sin \varphi B_1)\), \(\beta_{\gamma_i}\)-Smarandache curve of involute curve is, 
\[
\kappa^\beta_{\gamma_i} = \frac{|W|}{(2|W| + \varphi' |W|)^2} \left(\varphi'_1 \chi_1 - \varphi'_2 \chi_2 + 2|W| \chi_3\right)
\]
(2.32)

where coefficients are 
\[
\chi_1 = 2\left(-\frac{\varphi'_1 |W|}{|W|^2}\right)^2 + \left(\frac{\varphi'_1 |W|}{|W|^2}\right)^3 - \left(\frac{\varphi'_1 |W|}{|W|^2}\right)^4 + \left(\frac{\varphi'_1 |W|}{|W|^2}\right)^5
\]
\[
\chi_2 = 2\left(-\frac{\varphi'_2 |W|}{|W|^2}\right)^2 - \left(\frac{\varphi'_2 |W|}{|W|^2}\right)^3 + \left(\frac{\varphi'_2 |W|}{|W|^2}\right)^4 + \left(\frac{\varphi'_2 |W|}{|W|^2}\right)^5
\]
\[
\chi_3 = 2\left(-\frac{\varphi'_1 |W|}{|W|^2}\right)^2 + \left(\frac{\varphi'_1 |W|}{|W|^2}\right)^3 + \left(\frac{\varphi'_1 |W|}{|W|^2}\right)^4
\]
(2.33)

**Corollary 2.4.** Let \(\alpha_i\) be involute of \(\alpha\). The geodesic curvature Sabban apparatus belonging to 
\[
\beta_{\gamma_i}(s) = \frac{\varphi' \sin \varphi - \sqrt{\varphi'^2 + |W|^2} \cos \varphi T - \frac{|W|}{\sqrt{2\varphi'^2 + 2|W|^2}} N + \frac{\varphi' \cos \varphi + \sqrt{\varphi'^2 + |W|^2} \sin \varphi}{\sqrt{2\varphi'^2 + 2|W|^2}} B}{\sqrt{2\varphi'^2 + 2|W|^2}}
\]
\[
\beta_{\gamma_i}(s) = \frac{1}{2} (\eta \chi_1 - \eta \chi_2 + 2 \chi_3)
\]
(2.34)
where in the event of \( \eta = \frac{\phi'}{W_1} = \left(\frac{\phi'}{\sqrt{\phi'^2 + W_1^2}}\right)' \cos \phi(c - s) \) coefficients are
\[
\tilde{\zeta}_1 = -2 - \eta^2 + \eta' \eta, \quad \tilde{\zeta}_2 = -2 - 3 \eta^2 - \eta^3 - \eta' \eta, \quad \tilde{\zeta}_3 = 2 \eta + \eta^3 + \eta'.
\]

**Definition 2.5.** Let \((N_1)\) spherical curve be of \(\alpha_1\), \(T_{N_1}\) and \(N_1 \wedge T_{N_1}\) be Sabban vectors of \((N_1)\).

In the fact \(\beta_{z_2}\) -Smarandache curve can be identified by
\[
\beta_{z_2}(s) = \frac{1}{\sqrt{2}}(T_{N_1} + N_1 \wedge T_{N_1})
\]

**Theorem 2.5.** Let \(\alpha_1\) be involute of \(\alpha\). The geodesic curvature belonging to \(\beta_{z_2}\) -Smarandache curve of involute curve is,
\[
\kappa_{g_{z_2}} = \frac{\|W_1\|^2}{\left(\|W_1\|^2 + 2(\phi'_1)^2\right)^{\frac{3}{2}}}
\]
where coefficients are
\[
\phi_1 = -2 + 4(\frac{\tau}{\kappa_1}) - 3(\frac{\tau}{\kappa_1})^2 - 4(\frac{\tau}{\kappa_1})^3 + 2(\frac{\tau}{\kappa_1})^4 - (\frac{\tau}{\kappa_1}) (2 + \frac{\tau}{\kappa_1}) - 1,
\]
\[
\phi_2 = -2 + 2(\frac{\tau}{\kappa_1}) - 4(\frac{\tau}{\kappa_1})^2 - 2(\frac{\tau}{\kappa_1})^3 - (\frac{\tau}{\kappa_1}) (1 + \frac{\tau}{\kappa_1}) + \frac{\tau}{\kappa_1},
\]
\[
\phi_3 = 2(\frac{\tau}{\kappa_1}) - 4(\frac{\tau}{\kappa_1})^2 + 4(\frac{\tau}{\kappa_1})^3 - 2(\frac{\tau}{\kappa_1})^4 + (\frac{\tau}{\kappa_1}) (2 - \frac{\tau}{\kappa_1}).
\]

**Corollary 2.5** Let \(\alpha_1\) be involute of \(\alpha\). The geodesic curvature belonging to \(\beta_{z_2}\) -Smarandache curve of evolute curve is,
\[
\kappa_{g_{z_2}} = \frac{1}{(2 + \eta^2)^{\frac{3}{2}}}(2\eta \tilde{\phi}_1 - \tilde{\phi}_2 + \tilde{\phi}_3)
\]
where coefficients are
\[
\tilde{\phi}_1 = \eta + 2 \eta^3 + 3 \eta' \eta, \quad \tilde{\phi}_2 = -1 - 3 \eta^2 - 2 \eta^2 - \eta', \quad \tilde{\phi}_3 = -\eta^2 - 2 \eta^3 + \eta'.
\]

**Definition 2.6.** Let \((N_1)\) spherical curve be of \(\alpha_1\), \(N_1\), \(T_{N_1}\) and \(N_1 \wedge T_{N_1}\) be Sabban vectors of \((N_1)\). In the fact \(\beta_{z_3}\) -Smarandache curve can be identified by
\[
\beta_{z_3}(s) = \frac{1}{\sqrt{3}}(N_1 + T_{N_1} + N_1 \wedge T_{N_1})
\]

**Theorem 2.6** Let \(\alpha_1\) be involute of \(\alpha\). The geodesic curvature belonging to \(\beta_{z_3}\) -Smarandache curve of involute curve is,
\[
\kappa_{g_{3}} = \frac{(2 \frac{\varphi'}{W} - 1) \rho_1 + (-1 - \frac{\varphi'}{W}) \rho_2 + (2 - \frac{\varphi'}{W}) \rho_3}{4\sqrt{2}(1 - \frac{\varphi'}{W}^2 + (\frac{\varphi'}{W})^2)^{\frac{5}{2}}}
\]

where coefficients are

\[
\begin{align*}
\rho_1 &= -2 + 4(\frac{\varphi'}{W} - \frac{\varphi'}{W})^2 + 2(\frac{\varphi'}{W})^3 + (\frac{\varphi'}{W})(2 \frac{\varphi'}{W} - 1) \\
\rho_2 &= -2 + 2(\frac{\varphi'}{W} - 4(\frac{\varphi'}{W})^2 + 2(\frac{\varphi'}{W})^3 - 2(\frac{\varphi'}{W})^4 - (\frac{\varphi'}{W})(1 + \frac{\varphi'}{W}) \\
\rho_3 &= 2(\frac{\varphi'}{W} - 4(\frac{\varphi'}{W})^2 + 4(\frac{\varphi'}{W})^3 - 2(\frac{\varphi'}{W})^4 + (\frac{\varphi'}{W})(2 - \frac{\varphi'}{W})
\end{align*}
\]

**Corollary 2.6.** Let \( \alpha \) be involute of \( \alpha \). The geodesic curvature belonging to \( \beta_{s_{33}}(s) \) is

\[
\beta_{s_{33}}(s) = \frac{(\varphi' + ||W||) \sin \varphi - \sqrt{3\varphi'^2 + ||W||^2} \cos \varphi}{\sqrt{3\varphi'^2 + 3||W||^2}} + \frac{\varphi' - ||W||}{\sqrt{3\varphi'^2 + 3||W||^2}} N + \frac{(\varphi' + ||W||) \cos \varphi + \sqrt{3\varphi'^2 + ||W||^2} \sin \varphi}{\sqrt{3\varphi'^2 + 3||W||^2}} B
\]

\[
\kappa_{g_{s_{3}}} = \frac{(2\eta - 1) \bar{\rho}_1 + ((-1 - \eta) \bar{\rho}_2 + (2 - \eta) \bar{\rho}_3}{4\sqrt{2}(1 - \eta + \eta^2)^{\frac{5}{2}}}
\]

where coefficients are

\[
\begin{align*}
\bar{\rho}_1 &= -2 + 4\eta - 4\eta^2 + 2\eta^3 + 2\eta'(2\eta - 1) \\
\bar{\rho}_2 &= -2 + 2\eta - 4\eta^2 + 2\eta^3 - 2\eta^4 - \eta'(1 + \eta) \\
\bar{\rho}_3 &= 2\eta - 4\eta^2 + 4\eta^3 - 2\eta^4 + \eta'(2 - \eta)
\end{align*}
\]

**Definition 2.7.** Let \( (B_1) \) spherical curve be of \( \alpha \), \( B \) and \( T_\eta \) be Sabban vectors of \( (B_1) \). In the fact \( \beta_{s_{3}} \)-Smarandache curve can be identified by

\[
\beta_{s_{3}}(s) = \frac{1}{\sqrt{2}} (B_1 + T_\eta)
\]

**Theorem 2.7.** Let \( \alpha \) be involute of \( \alpha \). The geodesic curvature belonging to \( \beta_{s_{1}}(s) = \frac{1}{\sqrt{2}} (-N_1 + B_1) \)

\[
\beta_{s_{1}}(s) - \text{Smarandache curve of involute curve is,}
\]

\[
\kappa_{g_{s_{1}}} = \frac{\tau_1^4}{(2\tau_1^2 + \kappa_1^2)^{\frac{5}{2}}} (\kappa_1 \omega_1 - \kappa_1 \omega_2 + 2 \tau_1 \omega_3)
\]

where coefficients are
\[
\omega_1 = -2 - \left( \frac{K}{\tau_1} \right)^2 + \left( \frac{K}{\tau_1} \right)' \left( \frac{K}{\tau_1} \right), \quad \omega_2 = -2 - 3 \left( \frac{K}{\tau_1} \right)^2 - \left( \frac{K}{\tau_1} \right)' \left( \frac{K}{\tau_1} \right)
\]
\[
\omega_3 = 2 \left( \frac{K}{\tau_1} \right) + \left( \frac{K}{\tau_1} \right)^3 + \left( \frac{K}{\tau_1} \right)'
\]
\[ (2.48) \]

**Corollary 2.7.** Let \( \alpha \) be involute of \( \alpha \). The geodesic curvature belonging to
\[
\beta_{\xi}(s) = \frac{1}{\sqrt{2}} ((\cos \varphi + \sin \varphi) T + (\cos \varphi - \sin \varphi) B, \quad \beta_{\xi} - \text{Smarandache curve of evolute curve is,}
\]
\[
\kappa_{g}^{\xi} = \frac{\varphi'^4}{(2\varphi'^2 + \|W\|^2)^{\frac{3}{2}}} \left( \|W\| (\overline{\omega}_1 - \overline{\omega}_2) + 2\varphi' \overline{\omega}_3 \right),
\]
where coefficients are
\[
\overline{\omega}_1 = -2 - \left( \frac{\|W\|^2}{\varphi'} \right)^2 + \left( \frac{\|W\|^2}{\varphi'} \right)(\frac{\|W\|^2}{\varphi'}), \quad \overline{\omega}_2 = -2 - 3 \left( \frac{\|W\|^2}{\varphi'} \right)^2 - \left( \frac{\|W\|^2}{\varphi'} \right)' - \left( \frac{\|W\|^2}{\varphi'} \right)'
\]
\[
\overline{\omega}_3 = 2 \left( \frac{\|W\|^2}{\varphi'} \right) + \left( \frac{\|W\|^2}{\varphi'} \right)^3 + 2\left( \frac{\|W\|^2}{\varphi'} \right)'.
\]
\[ (2.49) \]

**Definition 2.8.** Let \((B_1)\) spherical curve be of \( \alpha \), \( T_B \) and \( B_1 \wedge T_B \) be Sabban vectors of \((B_1)\). In the fact \( \beta_{\xi} - \text{Smarandache curve can be identified by}
\]
\[
\beta_{\xi}(s) = \frac{1}{\sqrt{3}} (T_{B_1} + B_1 \wedge T_{B_1})
\]
\[ (2.50) \]

**Theorem 2.8.** Let \( \alpha \) be involute of \( \alpha \). The geodesic curvature belonging to
\[
\beta_{\xi}(s) = \frac{1}{\sqrt{2}} (T_{1} - N_{1}), \quad \beta_{\xi} - \text{Smarandache curve of involute curve is,}
\]
\[
\kappa_{g}^{\xi} = \frac{\tau_1^4}{(\tau_1^2 + 2\kappa_1^2)^{\frac{3}{2}}} \left( 2\kappa_1 \psi_1 - \tau_1 \psi_2 + \tau_1 \psi_3 \right),
\]
where coefficients are
\[
\psi_1 = \frac{\kappa}{\tau_1} + 2 \left( \frac{\kappa}{\tau_1} \right)^3 + 2 \left( \frac{\kappa}{\tau_1} \right) \left( \frac{\kappa}{\tau_1} \right)', \quad \psi_2 = -1 - 3 \left( \frac{\kappa}{\tau_1} \right)^2 - 2 \left( \frac{\kappa}{\tau_1} \right)' - \left( \frac{\kappa}{\tau_1} \right)',
\]
\[
\psi_3 = -\left( \frac{\kappa}{\tau_1} \right)^2 - 2 \left( \frac{\kappa}{\tau_1} \right)^4 + \left( \frac{\kappa}{\tau_1} \right)'.
\]
\[ (2.51) \]

**Corollary 2.8.** Let \( \alpha \) be involute of \( \alpha \). The geodesic curvature belonging to
\[
\beta_{\xi}(s) = \frac{1}{\sqrt{2}} (\cos \varphi T + N - \sin \varphi B), \quad \beta_{\xi} - \text{Smarandache curve of evolute curve is,}
\]
\[
\kappa_{g}^{\xi} = \frac{\varphi'^4}{(\varphi'^2 + \|W\|^2)^{\frac{3}{2}}} \left( 2\|W\| \overline{\psi}_1 - \varphi' \overline{\psi}_2 + \varphi' \overline{\psi}_3 \right),
\]
where coefficients are
\[ \bar{\psi}_1 = (\frac{||W||}{\phi'})^3 + 2(\frac{||W||}{\phi'})^2(\frac{||W||}{\phi'}), \quad \bar{\psi}_2 = -1 - 3(\frac{||W||}{\phi'})^2 - 2(\frac{||W||}{\phi'}) \] (2.55)

\[ \bar{\psi}_3 = -\frac{||W||}{\phi'}^2 - 2(\frac{||W||}{\phi'})^4 + (\frac{||W||}{\phi'}) \]

**Definition 2.9.** Let \((B_1)\) spherical curve be of \(\alpha_1, B_1, T_{B_1}\) and \(B_1 \wedge T_{B_1}\) be Sabban vectors of \((B_1)\).

In the fact \(B_1\) -Smarandache curve can be identified by

\[ \beta_{z_3}(s) = \frac{1}{\sqrt{3}}(B_1 + T_{B_1} + B_1 \wedge T_{B_1}) \] (2.56)

**Theorem 2.9.** Let \(\alpha_1\) be involute of \(\alpha\). The geodesic curvature belonging to \(\beta_{z_3}\) is

\[ \kappa_g = \frac{\tau^4}{4\sqrt{2}(\tau_1^2 + \kappa_1 \tau + \kappa_1^2)} \]

where coefficients are

\[ \zeta_1 = -2 + 4(\frac{\kappa}{\tau_1}) - (\frac{\kappa}{\tau_1})^2 + 2(\frac{\kappa}{\tau_1})^3 + (\frac{\kappa}{\tau_1})^4(\frac{2}{\tau_1} - 1) \]

\[ \zeta_2 = -2 + 2(\frac{\kappa}{\tau_1}) - 4(\frac{\kappa}{\tau_1})^3 - 2(\frac{\kappa}{\tau_1})^4(1 + \frac{\kappa}{\tau_1}) \] (2.58)

\[ \zeta_3 = 2(\frac{\kappa}{\tau_1}) - 4(\frac{\kappa}{\tau_1})^3 + 4(\frac{\kappa}{\tau_1})^4 - 2(\frac{\kappa}{\tau_1})^5(2 - \frac{\kappa}{\tau_1}) \]

**Corollary 2.9.** Let \(\alpha_1\) be involute of \(\alpha\). The geodesic curvature belonging to \(\beta_{z_3}\) is

\[ \beta_{z_3}(s) = \frac{1}{\sqrt{3}}(\sin \phi + \cos \phi)T + N + (\cos \phi - \sin \phi)B \]

\[ \kappa_g = \frac{\phi'^4}{4\sqrt{2}(\phi'^3 + \phi')||W||^2} \]

where coefficients are

\[ \tilde{\zeta}_1 = -2 + 4(\frac{||W||}{\phi'}) + 4(\frac{||W||}{\phi'})^2(\frac{||W||}{\phi'})^3 + 2(\frac{||W||}{\phi'})^4(\frac{2||W||}{\phi'} - 1) \]

\[ \tilde{\zeta}_2 = -2 + 2(\frac{||W||}{\phi'}) - 4(\frac{||W||}{\phi'})^3 - 2(\frac{||W||}{\phi'})^4 - (\frac{||W||}{\phi'})^5(1 + \frac{||W||}{\phi'}) \] (2.60)

\[ \tilde{\zeta}_3 = 2(\frac{||W||}{\phi'}) - 4(\frac{||W||}{\phi'})^3 + 4(\frac{||W||}{\phi'})^4 - 2(\frac{||W||}{\phi'})^5 + (\frac{||W||}{\phi'})^6(2 - \frac{||W||}{\phi'}) \]

**Example.** Let us consider the unit speed evolute curve and involute curve, respectively

\[ \alpha(s) = \left( \frac{2}{5} \sin(2t) - \frac{1}{40} \sin(8t), -\frac{2}{5} \cos(2t) + \frac{1}{40} \cos(8t), \frac{4}{15} \sin(3t) \right) \]

\[ \alpha_1(s) = \left( \frac{2}{5} \sin(2s) - \frac{1}{20} \sin(8s) + \frac{4}{5} (1-s) \cos(5s), -\frac{2}{5} \right), \]

\[ \alpha_2(s) = \left( \frac{2}{5} \sin(2s) - \frac{1}{20} \sin(8s) + \frac{4}{5} (1-s) \cos(5s), -\frac{2}{5} \right), \]
\[
\cos(2s) + \frac{1}{40}\cos(8s) + \frac{4}{5}(1-s)\sin(5s), \frac{4}{15}\sin(3s) - \frac{3}{5} + \frac{3}{5}s.
\]

The Frenet vectors belonging to involute curve, \(\alpha_1\) are found as follows;

\[
T_1 = \left(\frac{4}{5}\cos(5s), \frac{4}{5}\sin(5s), -\frac{3}{5}\right)
\]

\[
N_1 = \left(\left[\frac{1}{5}\cos(8s) - \frac{4}{5}\cos(2s)\right]\sin(3s) - \left[\frac{4}{5}\sin(2s) + \frac{1}{5}\sin(8s)\right]\cos(3s),
\left[\frac{4}{5}\sin(2s) + \frac{1}{5}\sin(8s)\right]\sin(3s) + \cos(3s)\left(\frac{4}{5}\cos(2s) + \frac{1}{5}\cos(8s)\right), 0\right),
\]

\[
B_1 = \left(\cos(3s)\left(\frac{4}{5}\cos(2s) - \frac{1}{5}\cos(8s)\right) - \sin(3s)\left(\frac{4}{5}\sin(2s) + \frac{1}{5}\sin(8s)\right),
\cos(3s)\left(\frac{4}{5}\sin(2s) - \frac{1}{5}\sin(8s)\right) + \sin(3s)\left(\frac{4}{5}\cos(2s) + \frac{1}{5}\cos(8s)\right), \frac{4}{5}\right).
\]

According to definitions, we reach specific Smarandache curves belonging to Sabban frame of this curve. \(\beta_1, \beta_2, \beta_3, \beta_{s_1}, \beta_{s_2}, \beta_{s_3}, \beta_{s_4}, \beta_{s_5}\) and \(\beta_{s_6}\) (see Figure 1,2,3).

**Figure 1:** \(\beta_1\)-curve  \(\beta_2\)-curve  \(\beta_3\)-curve

**Figure 2:** \(\beta_{s_1}\)-curve  \(\beta_{s_2}\)-curve  \(\beta_{s_3}\)-curve

**Figure 3:** \(\beta_{s_4}\)-curve  \(\beta_{s_5}\)-curve  \(\beta_{s_6}\)-curve

3. **Conclusion**

In this article, we reviewed the well-known involute and evolute curves in the literature. We have created the Sabban frames on the unit sphere of the involute and evolute curves. We got Smarandache curves from the Sabban frame and calculated the geodesic curvature of these curves. Finally, we have given an example and have driven their shapes in the Mapple program.
References


