Special equiform Smarandache curves in Minkowski space-time

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ABSTRACT

In this paper, we introduce special equiform Smarandache curves reference to the equiform Frenet frame of a curve ζ on a spacelike surface M in Minkowski 3-space E13. Also, we study the equiform Frenet invariants of the special equiform Smarandache curves in E13. Moreover, we give some properties to these curves when the curve ζ has constant curvature or it is a circular helix. Finally, we give an example to illustrate these curves.

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1. Introduction

A regular non-null curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [1]. Recently special Smarandache curves have been studied by some authors [2–5].

In this work, we study special equiform Smarandache curves with reference to the equiform Frenet frame of a curve ζ on a spacelike surface M in Minkowski 3-space E13. In Section 2, we clarify the basic conceptions of Minkowski 3-space E13 and give of equiform Frenet frame that will be used during this work. Section 3 is dedicated to the study of the special four equiform Smarandache curves, Tη, Tξ, ηξ and Tηξ-equiform Smarandache curves by being the connection with the first and second equiform curvature k1(θ), and k2(θ) of the equiform spacelike curve ζ in E13. Furthermore, we present some properties on the curves when the curve ζ has constant curvature or it is a circular helix. Finally, we give an example to clarify these curves. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2. Preliminaries

The Minkowski 3-space E13 is the Euclidean 3-space E3 provided with the metric

\[ g = -dz_1^2 + dz_2^2 + dz_3^2, \]

where (z1, z2, z3) is a rectangular coordinate system of E13. Any arbitrary vector v ∈ E13 can have one of three Lorentzian clause depicts: it can be timelike if \( g(v, v) < 0 \), spacelike if \( g(v, v) > 0 \) or \( v = 0 \), and lightlike if \( g(v, v) = 0 \) and \( v \neq 0 \). Similarly, any arbitrary curve \( ζ = ζ(s) \) can be timelike, spacelike or lightlike if all of its velocity vectors \( ζ'(s) \) are timelike, spacelike or lightlike, respectively.

Let \( ζ = ζ(s) \) be a regular non-null curve parametrized by arclength in E13 and \( \{t, n, b, κ, τ\} \) be its Frenet invariants where \( \{t, n, b\} \), \( κ \) and \( τ \) are the moving Frenet frame and the natural curvature functions respectively. If \( ζ \) is a spacelike curve with spacelike principal normal vector, then the Frenet formulas of the curve \( ζ \) can be given as [6–8]:

\[
\begin{pmatrix}
\dot{t}(s) \\
\dot{n}(s) \\
\dot{b}(s)
\end{pmatrix} =
\begin{pmatrix}
0 & κ(s) & 0 \\
-κ(s) & 0 & τ(s) \\
0 & τ(s) & 0
\end{pmatrix}
\begin{pmatrix}
t(s) \\
n(s) \\
b(s)
\end{pmatrix},
\]

where \( \frac{d}{ds} \), \( \dot{g}(t, t) = g(n, n) = -g(b, b) = 1 \), and \( \dot{g}(t, n) = \dot{g}(t, b) = 0 \).

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Definition 2.1. A surface $M$ in the Minkowski 3-space $E^3_1$ is said to be timelike, spacelike surface if, respectively the induced metric on the surface is a Lorentz metric, positive definite Riemannian metric. In other words, the normal vector on the timelike(spacelike) surface is a spacelike(timelike) vector \[8\].

Let $\zeta : I \to E^3_1$ be a spacelike curve in Minkowski space $E^3_1$. We define the equiparametric parameter of $\zeta$ by $\theta = f(x)$. Then, we have $\rho = \frac{ds}{d\theta}$ where $\rho = \frac{1}{k}$ is the radius of curvature of the curve $\zeta$. Let $\mathcal{F}$ be a homothety with the center in the origin and the coefficient $\mu$. If we put $\tilde{\zeta} = \mathcal{F}(\zeta)$, then it follows

$S = \mu s$ and $\tilde{\rho} = \mu \rho$,$$
$$
where $\tilde{s}$ is the arc-length parameter of $\tilde{\zeta}$ and $\tilde{\rho}$ the radius of curvature of this curve. Therefore, $\theta$ is an equiparametric parameter of $\xi$ \[9\]. From that point, we recall that $T, \eta, \xi$ be the moving equiform Frenet frame where $T(\theta) = \rho s(\theta)$, $\eta(\theta) = \rho n(\theta)$ and $\xi(\theta) = \rho b(\theta)$ are the equiparametric tangent, equiparametric principal normal vector and equiparametric binormal vector respectively. Additionally, the first and second equiparametric curvature of the curve $\zeta = \zeta(\theta)$ are defined by $k_1(\theta) = \frac{d\rho}{ds}$ and $k_2(\theta) = \frac{\rho'}{k}$. So, the moving equiparametric Frenet frame of $\zeta = \zeta(\theta)$ is given as \[10\]:

\[
\begin{pmatrix}
T'(	heta) \\
\eta'(	heta) \\
\xi'(	heta)
\end{pmatrix} = \begin{pmatrix} k_1(\theta) & 1 & 0 \\
-1 & k_1(\theta) & k_2(\theta) \\
0 & k_2(\theta) & k_1(\theta)
\end{pmatrix} \begin{pmatrix} T(\theta) \\
\eta(\theta) \\
\xi(\theta)
\end{pmatrix},
\]

where \[d \theta \frac{d}{d \theta} \mathcal{G}(T,T) = \mathcal{G}(\eta,\eta) = \mathcal{G}(\xi,\xi) = \rho^2\] and $\mathcal{G}(T,\eta) = \mathcal{G}(\eta,\xi) = \mathcal{G}(\xi,T) = 0$.

The pseudo-Riemannian sphere with center at the origin and of radius $r = 1$ in the Minkowski 3-space $E^3_1$ is a quadric defined by

$S^2_1 = \{ \bar{u} \in E^3_1 : -u_1^2 + u_2^2 + u_3^2 = 1 \}$.

Let $\zeta = \zeta(\theta)$ be a regular non-null curve parametrized by arc-length in Minkowski 3-space $E^3_1$ with its moving equiparametric Frenet frame $\{ T, \eta, \xi \}$. Then $T\eta$, $T\xi$ and $\eta\xi$-equiparametric Smarandache curves of $\zeta$ are defined, respectively as follows \[11\]:

$\lambda = \lambda(\theta^*) = \frac{1}{\sqrt{2}} (T(\theta^*) + \eta(\theta^*))$,

$\lambda = \lambda(\theta^*) = \frac{1}{\sqrt{2}} (T(\theta^*) + \xi(\theta^*))$,

$\lambda = \lambda(\theta^*) = \frac{1}{\sqrt{2}} (\eta(\theta^*) + \xi(\theta^*))$,

$\lambda = \lambda(\theta^*) = \frac{1}{\sqrt{2}} (T(\theta^*) + \eta(\theta^*) + \xi(\theta^*))$.

3. Special equiparametric Smarandache curves in $E^3_1$

In this section, we define the special equiparametric Smarandache curves reference to the equiparametric Frenet frame of a curve $\zeta$ in Minkowski 3-space $E^3_1$. Furthermore, we obtain the natural equiparametric curvature functions of the equiparametric Smarandache curves lying completely on pseudo-sphere $S^2_1$ and give some properties on the curves when the curve $\zeta$ has constant curvature or it is a circular helix

Definition 3.1. A curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another curve, is called a Smarandache curve.

As consequence with the above definition, we introduce a special form of the equiparametric Smarandache curves in $E^3_1$ in the following subsection

3.1. $T\eta$-equiparametric Smarandache curves in $E^3_1$

Definition 3.2. Let $\zeta = \zeta(\theta)$ be a regular equiform spacelike curve lying completely on a spacelike surface $M$ in $E^3_1$ with moving equiform Frenet frame $\{ T, \eta, \xi \}$. Then $T\eta$-equiparametric Smarandache curves are defined by

$\lambda = \lambda(\theta^*) = \frac{1}{\sqrt{2}} (T(\theta^*) + \eta(\theta^*))$.

(3)

Theorem 3.1. Let $\zeta = \zeta(\theta)$ be a spacelike curve with spacelike principal normal vector in $E^3_1$. If $\zeta$ is a circular helix with $k > 0$, then $T\eta$-equiparametric Smarandache curve is also circular helix and its the natural curvature functions are satisfied the following equation,

$\kappa_3(\theta^*) = \frac{\sqrt{2}}{\rho(k_3^2 - 2)} : k_3 \neq \pm \sqrt{2}$,

(4)

$\tau_3(\theta^*) = \frac{\sqrt{2}}{\rho^2} \frac{k_2(2k_2 + 1) - (k_2 + 1)(k_2 + 2)}{2(k_2^2 + 2)} : k_2 \neq \pm \sqrt{2}$.

Proof. Let $\lambda = \lambda(\theta^*)$ be a $T\eta$-equiparametric Smarandache curves reference to the equiparametric spacelike curve $\zeta = \zeta(\theta)$. From Eq. (3) and using Eq. (2), we get

$\lambda(\theta^*) = \frac{d\theta}{d\theta^*} \frac{d\theta^*}{d\theta} = \frac{1}{\sqrt{2}} ((k_1 - 1)T(\theta) + (k_1 + 1)\eta(\theta) + k_2\xi(\theta))$,

hence

$T_3(\theta^*) = \frac{1}{\rho\sqrt{2k_1^2 - k_2^2 - 2}} ((k_1 - 1)T(\theta) + (k_1 + 1)\eta(\theta) + k_2\xi(\theta))$,

where

$\frac{d\theta}{d\theta^*} = \frac{\rho}{\sqrt{2}} \frac{\sqrt{2k_1^2 - k_2^2 - 2}}{k_3}$.

Now

$\frac{d\theta^*}{d\theta} = \frac{\rho^2\sqrt{2k_1^2 - k_2^2 - 2}}{\sqrt{2}}$.

(7)

where

$\lambda_1 = (k_1 - 1)(2k_1k_2 - k_2k_2') + (2k_2^2 - k_2^2 - 2)(k_1^2 - 2(k_1^2 + 2k_1^2) - 3k_1)$,

$\lambda_2 = (k_1 + 1)(2k_1k_2 - k_2k_2') + (2k_2^2 - k_2^2 - 2)(k_1^2 + k_1^2 + k_1^2 + k_1 - 2)$,

$\lambda_3 = k_2(2k_1k_1' - k_2k_2') + (2k_2^2 - k_2^2 - 2)(k_1^2 + 2k_1^2)$.

Then

$\kappa_3(\theta^*) = \frac{d_T}{d\theta^*} = \frac{\sqrt{2}}{\rho^2} \frac{\sqrt{2k_1^2 + k_2^2 - 2}}{k_3}$.

(8)

and

$N_3(\theta^*) = \frac{\lambda_1T(\theta) + \lambda_2\eta(\theta) + \lambda_3\xi(\theta)}{\rho\sqrt{2k_1^2 + k_2^2 - 2}}$.

Also

$B_3(\theta^*) = \frac{1}{\rho^2}(m_1T(\theta) + m_2\eta(\theta) + m_3\xi(\theta))$.

where
Now, from Eq. (5)
\[
\begin{equation}
\zeta''(\theta^*) = \frac{1}{\sqrt{2}} \left\{ (k_1' + k_2' - 2k_1 - 1)\eta(\theta) + [k_2' + 2k_1k_2 + 2k_2]\xi(\theta) \right\}.
\end{equation}
\]
and thus
\[
\zeta'''(\theta^*) = \frac{1}{\sqrt{2}} \left( \beta_1 T(\theta) + \beta_2 \eta(\theta) + \beta_3 \xi(\theta) \right),
\]
where
\[
\begin{align*}
\beta_1 &= k_1'' + 3k_1'(k_1 - 1) - k_2'(k_2 + 2), \\
\beta_2 &= k_2'' + k_2' + 3k_1k_2 + k_2' + 3k_1(k_1 - 1) + k_1'k_1 + 1, \\
\beta_3 &= k_2' - k_2'(k_1 - 1) - k_1k_2 + 2k_2.
\end{align*}
\]
Hence, we have
\[
\tau_3(\theta^*) = \frac{\sqrt{2}}{\rho \beta} \left( w_1 + w_2 + w_3 \right),
\]
where
\[
\begin{align*}
w_1 &= (k_1' + k_2') - (k_1'k_1 - k_2'(k_2 + 1) - k_2(k_2 - k_1^2 + k_2^2) - k_2 - 2k_2 + 2k_2'(k_2 + k_2') - 2k_2(k_2 - k_1 + 2), \\
w_2 &= (k_2'k_2 - k_2'(k_1 - k_2) + k_2(k_2 - k_1) - 2), \\
w_3 &= (k_1'k_1 - k_2'(k_1 - k_2) + k_1(k_1 - 1) + k_2(k_2 - k_1^2 + k_2^2) - k_2 - 2k_2 + 2k_2'(k_2 + k_2') - 2k_2(k_2 - k_1 + 2).
\end{align*}
\]
Now, if \( \kappa \) and \( \tau \) are non-zero constants, then the natural curvature functions \( \kappa_3, \tau_3 \) are also non-zero constants and satisfying Eq. (4) which means that the T\( \eta \)-equi-parallel Smarandache curve is circular helix.

### 3.2. T\( \xi \)-equi-parallel Smarandache curves in \( E_1^3 \)

**Definition 3.3.** Let \( \xi = \xi(\theta) \) be a regular equi-parallel spacelike curve lying completely on a spacelike surface \( M \) in \( E_1^3 \) with moving equi-parallel Frenet frame \( (T, \eta, \xi) \). Then T\( \xi \)-equi-parallel Smarandache curves are defined by
\[
\zeta = \zeta(\theta^*) = \frac{1}{\sqrt{2}} (T(\theta) + \xi(\theta)).
\]

**Theorem 3.2.** Let \( \xi = \xi(\theta) \) be a spacelike curve with spacelike principal normal vector in \( E_1^3 \). If \( \xi \) is a circular helix with \( \kappa > 0 \), then T\( \xi \)-equi-parallel Smarandache curve is contained in a plane and its curvature is satisfied the following equation,
\[
\kappa_3(\theta^*) = \frac{\sqrt{2}(\tau_3(\theta^*) + (k_2 + 1)\eta(\theta) + k_1\xi(\theta))}{\sqrt{2} \rho (k_2 + 1)^2} : k_2 \neq -1.
\]

**Proof.** Let \( \zeta = \zeta(\theta^*) \) be a T\( \xi \)-equi-parallel Smarandache curves of \( \xi = \xi(\theta) \). Then from Eq. (10), we have
\[
\begin{equation}
\zeta''(\theta^*) = \frac{1}{\sqrt{2}} (k_1T(\theta) + (k_2 + 1)\eta(\theta) + k_1\xi(\theta)).
\end{equation}
\]
and hence
\[
\begin{equation}
\zeta'''(\theta^*) = \frac{1}{\sqrt{2}} \left( k_1T(\theta) + (k_2 + 1)\eta(\theta) + k_1\xi(\theta) \right).
\end{equation}
\]
where
\[
\begin{equation}
\frac{d\theta}{dT} = \frac{\sqrt{2}}{\rho (k_2 + 1)^2}.
\end{equation}
\]

Now
\[
\begin{equation}
\frac{d\theta}{d\eta} = \frac{\sqrt{2}}{\rho (k_2 + 1)^2} (\xi_1 T(\theta) + \xi_2 \eta(\theta) + \xi_3 \eta(\theta)),
\end{equation}
\]
where
\[
\begin{align*}
\xi_1 &= (k_2 + 1)(k_2' - 2k_1 - 1) - k_1k_2, \\
\xi_2 &= k_1k_2, \\
\xi_3 &= (k_2 + 1)(k_1' + k_2(k_2 + 1) + k_2^2 - k_1^2) - k_1k_2.
\end{align*}
\]
Then
\[
\begin{equation}
k_5(\theta^*) = \frac{\sqrt{2}(\xi_1^2 + \xi_2^2 - \xi_3^2)}{\rho (k_2 + 1)^2},
\end{equation}
and
\[
\begin{equation}
N_5(\theta^*) = \frac{\xi_1 T(\theta) + \xi_2 \eta(\theta) + \xi_3 \eta(\theta)}{\rho \sqrt{2}(k_2 + 1)^2}.
\end{equation}
\]
Also
\[
\begin{equation}
B_5(\theta^*) = \frac{1}{\rho (k_2 + 1)^2} \left\{ (k_2 + 1)\eta(\theta) + (k_2 + 2)\eta(\theta) + \xi(\theta) \right\}.
\end{equation}
\]
and
\[
\begin{equation}
\zeta'''(\theta^*) = \frac{1}{\sqrt{2}} \left( \delta_1 T(\theta) + \delta_2 \eta(\theta) + \delta_3 \xi(\theta) \right),
\end{equation}
where
\[
\begin{align*}
\delta_1 &= k_1' - k_2' + k_2 + k_1(k_1 - 1), \\
\delta_2 &= k_2' + k_1' + 3k_1k_2 + k_2' + 2k_2 + 2k_2', \\
\delta_3 &= k_1' + k_2' - k_2'(k_1 - 1) + k_1(k_1 - 1) + k_2(k_2 - k_1^2 + k_2^2) - k_2(k_2 - k_1 - 1) + k_2(k_2 - k_1 + 2).
\end{align*}
\]
Hence, we have
\[
\tau_5(\theta^*) = \frac{\sqrt{2}}{\rho \beta} \left( (k_1' + 2k_2 + 2k_2') \delta_1 T(\theta) + (k_2 + 2k_2 + 2k_1) \delta_2 \eta(\theta) + (k_1' + k_2 - k_1'k_1 + 1) \delta_3 \xi(\theta) \right).
\]

So, if \( \kappa \) and \( \tau \) are non-zero constants, then \( \kappa_5 \) is non-zero constant and satisfying Eq. (11), also \( \tau_5 = 0 \) which means that the T\( \xi \)-equi-parallel Smarandache curve is contained in a plane.

### 3.3. \( \eta \xi \)-equi-parallel Smarandache curves in \( E_1^3 \)

**Definition 3.4.** Let \( \xi = \xi(\theta) \) be a regular equi-parallel spacelike curve lying completely on a spacelike surface \( M \) in \( E_1^3 \) with moving equi-parallel Frenet frame \( (T, \eta, \xi) \). Then \( \eta \xi \)-equi-parallel Smarandache curves are defined by
\[
\zeta = \zeta(\theta^*) = \frac{1}{\sqrt{2}} (\eta(\theta) + \xi(\theta)).
\]

**Theorem 3.3.** Let \( \xi = \xi(\theta) \) be a spacelike curve with spacelike principal normal vector in \( E_1^3 \). If \( \xi \) is a circular helix with \( \kappa > 0 \), then
\( \eta \xi \)-equiform Smarandache curve is also circular helix and its the natural curvature functions are satisfied the following equation,
\[
\kappa_3(\theta^*) = \sqrt{\frac{2}{\rho^2}} (k_2^2 - 1), \\
\tau_3(\theta^*) = \sqrt{\frac{2}{\rho^2}} k_2 : \quad k_2 \neq 0.
\]  
(18)

**Proof.** Let \( \gamma = \gamma(\theta^*) \) be a \( \eta \xi \)-equiform Smarandache curve of the curve \( \gamma = \gamma(\theta) \). From Eq. (17), we get
\[
\gamma'(\theta^*) = \frac{1}{\sqrt{2}} (-T(\theta) + (k_1 + k_2) \eta(\theta) + (k_1 + k_2) \xi(\theta)),
\]
(19) hence
\[
\gamma(\theta^*) = \frac{1}{\rho} (-T(\theta) + (k_1 + k_2) \eta(\theta) + (k_1 + k_2) \xi(\theta)),
\]
(20)
where
\[
\frac{d\theta^*}{d\theta} = \sqrt{\frac{2}{\rho^2}} \left( \gamma_1 T(\theta) + \gamma_2 \eta(\theta) + \gamma_3 \xi(\theta) \right),
\]
(21) Now
\[
d\theta^* = \frac{\sqrt{2}}{\rho^2} \left( \gamma_1 T(\theta) + \gamma_2 \eta(\theta) + \gamma_3 \xi(\theta) \right). \]
(22)

Then
\[
\kappa_3(\theta^*) = \sqrt{\frac{2(\gamma_1^2 + \gamma_2^2)}{\rho}}.
\]
(23)

\[
N_3(\theta^*) = \frac{\gamma_1 T(\theta) + \gamma_2 \eta(\theta) + \gamma_3 \xi(\theta)}{\rho \sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}}.
\]

Also
\[
B_3(\theta^*) = \frac{1}{\rho \sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}} \left[ \left( \gamma_2 - \gamma_1 \right)(k_1 + k_2) [T(\theta) \right.
\]
\[
+ \left[ \gamma_3 + \gamma_1 (k_1 + k_2) \right] \eta(\theta) - \left. \left[ \gamma_2 + \gamma_1 (k_1 + k_2) \right] \xi(\theta) \right].
\]
(24)

From Eq. (19), we have
\[
\gamma''(\theta^*) = \frac{1}{\sqrt{2}} \left[ -2(k_1 + k_2) T(\theta) + [k_1' + k_2' + (k_1 + k_2)^2 - 1] \eta(\theta) 
\]
\[
+ [k_1' + k_2' + (k_1 + k_2)^2] \xi(\theta) \right].
\]
(25)

and
\[
\gamma'''(\theta^*) = \frac{1}{\sqrt{2}} \left[ \omega_1 T(\theta) + \omega_2 \eta(\theta) + \omega_3 \xi(\theta) \right],
\]
(26)

where
\[
\omega_1 = \begin{cases} 3k_1'^2 + 2k_2'^2 + k_1(2k_1 + k_2) + (k_1 + k_2)^2 - 1, \\
3k_1'^2 + 2k_2'^2 + 3k_1 - k_2 + 3(k_1 + k_2)(k_1' + k_2') + (k_1 + k_2)^3,
\end{cases}
\]
(27)

\[
\omega_3 = k_1' + k_2' + 3(k_1 + k_2)(k_1' + k_2') + (k_1 + k_2)^3.
\]
(28)

Hence, we have
\[
\tau_3(\theta^*) = \sqrt{\frac{2}{\rho^2}} \left\{ \begin{array}{l}
\left[ k_1' + k_2' + (k_1 + k_2)^2 - 1 \right] \omega_3 + \omega_1 (k_1 + k_2) \right\} \\
\left\{ -\omega_2 (k_1 + k_2) (2k_1 + k_2) \right\},
\]
(29)

\[
T_3(\theta^*) = \frac{1}{\rho \sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}} \left[ \left[ \gamma_2 - \gamma_1 \right](k_1 + k_2) T(\theta) 
\right.
\]
\[
+ \left[ \gamma_3 + \gamma_1 (k_1 + k_2) \right] \eta(\theta) + \left( (k_1 + k_2) \right] \xi(\theta) \right].
\]
(30)

Then, if \( \kappa \) and \( \tau \) are non-zero constants, then the natural curvature functions \( \kappa_3, \tau_3 \) are also non-zero constants and satisfying Eq. (18) which means that the \( \eta \xi \)-equiform Smarandache curve is circular helix. □

3.4. \( \eta \xi \)-equiform Smarandache curves in \( E_2^3 \)

**Definition 3.5.** Let \( \xi = \xi(\theta) \) be a regular equiform spacelike curve lying completely on a spacelike surface \( M \) in \( E_2^3 \) with moving equiform Frenet frame \( (T, \eta, \xi) \). Then \( \eta \xi \)-equiform Smarandache curves are defined by
\[
\gamma = \gamma(\theta^*) = \frac{1}{\sqrt{2}} (T(\theta) + \eta(\theta) + \xi(\theta)).
\]
(31)

**Theorem 3.4.** Let \( \xi = \xi(\theta) \) be a spacelike curve with spacelike principal normal vector in \( E_2^3 \). If \( \xi \) is a circular helix with \( \kappa > 0 \), then \( \eta \xi \)-equiform Smarandache curve is also circular helix and its the natural curvature functions are satisfied the following equation,
\[
\kappa_3(\theta^*) = \sqrt{\frac{3}{2}} \left( \frac{1}{1 - k_1^2} \right) : \quad |k_1| < 1,
\]
(32)
\[
\tau_3(\theta^*) = 3 \left( \frac{2}{3} k_2 (k_2 + 1) \right)^2 : \quad k_2 
\]
(33)

**Proof.** Let \( \gamma = \gamma(\theta^*) \) be a \( \eta \xi \)-equiform Smarandache curves of the curve \( \gamma = \gamma(\theta) \). Then from Eq. (24), we get
\[
\gamma'(\theta^*) = \frac{1}{\sqrt{3}} ((k_1 - 1) T(\theta) + (k_1 + k_2 + 1) \eta(\theta) + (k_1 + k_2) \xi(\theta)).
\]
(34)

\[
T_3(\theta^*) = \frac{1}{\rho \sqrt{k_1^2 + 2k_2 + 2}} ((k_1 - 1) T(\theta) + (k_1 + k_2 + 1) \eta(\theta)
\]
\[
+ (k_1 + k_2) \xi(\theta)).
\]
(35)

Now
\[
d\theta^* = \frac{\rho \sqrt{k_1^2 + 2k_2 + 2}}{\sqrt{3}}.
\]
(36)

Then
\[
\kappa_3(\theta^*) = \sqrt{\frac{3}{2}} \xi(\theta)
\]
(37)

and
\[
N_3(\theta^*) = \frac{1}{\rho \sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}} \left[ \left[ \chi_1 + \chi_2 \eta(\theta) + \chi_3 \xi(\theta) \right]
\]
(38)

where
\[
\left\{ \begin{array}{l}
\chi_1 = (k_1 - 1)(k_1 k_1' + k_2') - (k_2 + 1)(k_1^2 + 2k_2 + 2), \\
\chi_2 = (k_1 k_1' + k_2')(k_1 + k_2) (k_1 + k_2 - 1) \\
\chi_3 = (k_1 k_1' + k_2') (k_1 + k_2 + 1) + (k_1 + k_2) (k_1 k_2'),
\end{array} \right.
\]
(39)

Then
\[
\kappa_3(\theta^*) = \sqrt{\frac{3}{2}} \xi(\theta)
\]
(40)

and
\[
N_3(\theta^*) = \frac{1}{\rho \sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}} \left[ \left[ \chi_1 + \chi_2 \eta(\theta) + \chi_3 \xi(\theta) \right]
\]
(41)

Also
\[
B_3(\theta^*) = \frac{1}{\rho \sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}} 
\]
(42)

× \left\{ \left[ \chi_1 + \chi_2 \eta(\theta) + \chi_3 \xi(\theta) \right]
(43)

× \left\{ \left[ \chi_1 + \chi_2 \eta(\theta) + \chi_3 \xi(\theta) \right]
(44)

× \left\{ \left[ \chi_1 + \chi_2 \eta(\theta) + \chi_3 \xi(\theta) \right]
(45)

× \left\{ \left[ \chi_1 + \chi_2 \eta(\theta) + \chi_3 \xi(\theta) \right]
(46)
From Eq. (26), we have

\[ z''(\theta^*) = \frac{1}{\sqrt{3}} \left[ (k_1^2 - k_1 - k_2 - 1)\tau(\theta) + [k_1' + k_2' + 2k_1 + (k_1 + k_2)^2 - 1]\eta(\theta) + [k_1' + k_2' + k_2 + (k_1 + k_2)^2]\xi(\theta) \right], \]

and thus

\[ z'''(\theta^*) = \frac{1}{\sqrt{3}} (\phi_1 T(\theta) + \phi_2 \eta(\theta) + \phi_3 \xi(\theta)). \]

where

\[
\begin{align*}
\phi_1 &= 2k_1'(k_1 + 1) - 2k_2' + k_1(k_2^2 - k_1 - k_2) - 2 + 1, \\
\phi_2 &= k_1'' + k_2'' + 2k_1' + (k_1 + k_2)^2 - 1 \left[ \phi_3(k_1 - 1) - \phi_1(k_1 + k_2) \right] + k_1(2k_1 + k_2 - 1) + k_1 + k_2^2 - 1, \\
\phi_3 &= k_1'' + k_2'' + k_1' + k_2 + (3k_1 - 1) + 3(k_1 + k_2)(k_1' + k_2') + (k_1 + k_2)^3.
\end{align*}
\]

Hence, we have

\[ \tau_3(\theta^*) = \frac{\sqrt{3}}{\rho^2} \left[ v_1 + v_2 + v_3 \right], \]

where

\[
\begin{align*}
v_1 &= (k_1^2 - k_1 - k_2 - 1)\phi_3(k_1 + k_2 - 1), \\
v_2 &= k_1' + k_2' + 2k_1 + (k_1 + k_2)^2 - 1 \left[ \phi_3(k_1 - 1) - \phi_1(k_1 + k_2) \right], \\
v_3 &= k_1' + k_2' + k_2 + (k_1 + k_2)^2 \left[ \phi_3(k_1 + k_2 - 1) - \phi_2(k_1 + 1) \right], \\
q_1 &= (k_1 + k_2)(2k_1 - k_2 - 1) - [k_1' + k_2' + 2k_1 + (k_1 + k_2)^2], \\
q_2 &= -(k_1 + k_2) + k_1 + k_2 + 1 \left[ k_1' + k_2' + 2k_1 + (k_1 + k_2)^2 \right], \\
q_3 &= (k_1 + 1) \left[ k_1' + k_2' + 2k_1 + (k_1 + k_2)^2 \right] + k_1(2k_1 - 3) + (k + 1 + k_2 + 1) \left[ 2(k_2 + 1) - k_1(k_2 + 1) \right] + 1.
\end{align*}
\]

Now, if \( k \) and \( \tau \) are non-zero constants, then the natural curvature functions \( k_3, \tau_3 \) are also non-zero constants and satisfying Eq. (25) which means that the \( T^\eta \)-equiform Smarandache curve is circular helix. \( \square \)

4. Example

Let \( \zeta(s) = \left( \sqrt{3}s, s\sin(\sqrt{3}\ln s), s\cos(\sqrt{3}\ln s) \right) \) be a unit speed spacelike curve parametrized by arc-length \( s \) with spacelike principal normal vector in \( E_1^3 \) (see Fig. 1). Then it is easy to show that

\[
\begin{align*}
t(s) &= \left( \sqrt{3}, \sin(\sqrt{3}\ln s) + \sqrt{3}\cos(\sqrt{3}\ln s), \\
&\cos(\sqrt{3}\ln s) - \sqrt{3}\sin(\sqrt{3}\ln s) \right), \\
n(s) &= \left( \frac{1}{\sqrt{3}}, 0, \cos(\sqrt{3}\ln s) - \sqrt{3}\sin(\sqrt{3}\ln s) \right), \\
-k(s) &= \left( \frac{1}{\sqrt{3}}, 0, \sin(\sqrt{3}\ln s) - \sqrt{3}\cos(\sqrt{3}\ln s) \right), \\
\kappa &= \frac{2\sqrt{3}}{s}, \\
\rho &= \frac{1}{2\sqrt{3}}, \\
k_1 &= \frac{1}{2\sqrt{3}}, \\
b(s) &= \left( 2, \frac{\sqrt{3}}{2}, \sin(\sqrt{3}\ln s) + \frac{1}{2}\cos(\sqrt{3}\ln s) \right), \\
\frac{\sqrt{3}}{2}\cos(\sqrt{3}\ln s) - \frac{1}{2}\sin(\sqrt{3}\ln s), \\
\tau &= \frac{3}{s}, \\
k_2 &= \frac{\sqrt{3}}{2},
\end{align*}
\]
Hence, the equiform parameter is $\theta = \int \kappa \, ds = 2\sqrt{3}s + c$. Here we take $c = 0$, then we have $s = e^{\theta/\sqrt{3}}$ and $\rho = e^{\theta/(2\sqrt{3})}$. So the equiform spacelike curve $\zeta$ is defined as (see Fig. 2)

$$\zeta(\theta) = \left( \sqrt{3} e^{\theta/2\sqrt{3}}, e^{\theta/2\sqrt{3}} \sin \left( \frac{\theta}{3} \right), e^{\theta/2\sqrt{3}} \cos \left( \frac{\theta}{3} \right) \right).$$

It easy to show that

$$T(\theta) = \frac{e^{\theta/2\sqrt{3}}}{2} \left( 1, \frac{1}{\sqrt{3}} \sin \left( \frac{\theta}{2} \right) + \cos \left( \frac{\theta}{2} \right), \frac{1}{\sqrt{3}} \cos \left( \frac{\theta}{2} \right) - \sin \left( \frac{\theta}{2} \right) \right).$$

It is clear that $T$ is an equiform spacelike vector. Also

$$\eta(\theta) = \frac{e^{\theta/2\sqrt{3}}}{4} \left( 0, \frac{1}{\sqrt{3}} \cos \left( \frac{\theta}{2} \right) - \sin \left( \frac{\theta}{2} \right), \frac{1}{\sqrt{3}} \sin \left( \frac{\theta}{2} \right) - \cos \left( \frac{\theta}{2} \right) \right),$$

and

$$\xi(\theta) = \frac{e^{\theta/2\sqrt{3}}}{4} \left( \frac{4}{\sqrt{3}} \sin \left( \frac{\theta}{2} \right) + \sqrt{3} \cos \left( \frac{\theta}{2} \right), \cos \left( \frac{\theta}{2} \right) - \sqrt{3} \sin \left( \frac{\theta}{2} \right) \right).$$

Then $\eta$ is an equiform spacelike vector and $\xi$ is an equiform timelike vector.

The $T\eta$-equiform Smarandache curve $\zeta(\theta^*)$ of the curve $\zeta(\theta)$ is given by (see Fig. 3)

$$\zeta(\theta^*) = \frac{\sqrt{6} e^{\theta/2\sqrt{3}}}{24} \left( 2\sqrt{3}, (2\sqrt{3} + 1) \cos \left( \frac{\theta}{2} \right) \right)$$

$$+ (2 - \sqrt{3}) \sin \left( \frac{\theta}{2} \right), (2 - \sqrt{3}) \cos \left( \frac{\theta}{2} \right) - (2\sqrt{3} + 1) \sin \left( \frac{\theta}{2} \right) \right).$$

The $T\xi$-equiform Smarandache curve $\zeta(\theta^*)$ of the curve $\zeta(\theta)$ is given by (see Fig. 4)

$$\zeta(\theta^*) = \frac{\sqrt{6} e^{\theta/2\sqrt{3}}}{24} \left( 2(2 + \sqrt{3}), (2 + \sqrt{3}) \sin \left( \frac{\theta}{2} \right) \right)$$

$$+ (2\sqrt{3} + 3) \cos \left( \frac{\theta}{2} \right), (2 + \sqrt{3}) \cos \left( \frac{\theta}{2} \right) - (2\sqrt{3} + 3) \sin \left( \frac{\theta}{2} \right) \right).$$

The $\eta\xi$-equiform Smarandache curve $\zeta(\theta^*)$ of the curve $\zeta(\theta)$ is given by (see Fig. 5)

$$\zeta(\theta^*) = \frac{\sqrt{6} e^{\theta/2\sqrt{3}}}{6} \left( 1, \cos \left( \frac{\theta}{2} \right), -\sin \left( \frac{\theta}{2} \right) \right).$$

The $T\eta\xi$-equiform Smarandache curve $\zeta(\theta^*)$ of the curve $\zeta(\theta)$ is given by (see Fig. 6)
Fig. 6. The $T\xi$-equiform Smarandache curve $\Im(\theta^*)$ on $S^2_1$.

$$\Im(\theta^*) = \frac{\sqrt{3} e^{\theta/2} \pi}{18} \left( 2 + \sqrt{3} \sin\left(\frac{\theta}{2}\right) + (2 + \sqrt{3}) \cos\left(\frac{\theta}{2}\right) \right)$$

$$- (2 + \sqrt{3}) \sin\left(\frac{\theta}{2}\right).$$

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