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Strong Domination Number of Some Cycle Related Graphs

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Abstract: Let $G = (V(G), E(G))$ be a graph and $u, v \in V(G)$. If $uv \in E(G)$ and $deg(u) \geq deg(v)$, then we say that u strongly dominates v or v weakly dominates u. A subset D of $V(G)$ is called a strong dominating set of G if every vertex $v \in V(G) - D$ is strongly dominated by some $u \in D$. The smallest cardinality of strong dominating set is called a strong domination number. In this paper we explore the concept of strong domination number and investigate strong domination number of some cycle related graphs.

Key Words: Dominating strong set, Smarandachely strong dominating set, strong domination number, $d-$balanced graph.

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§1. Introduction

In this paper we consider finite, undirected, connected and simple graph G. The vertex set and edge set of the graph G is denoted by $V(G)$ and $E(G)$ respectively. For any graph theoretic terminology and notations we rely upon Chartrand and Lesniak [2]. We denote the degree of a vertex v in a graph G by $deg(v)$. The maximum and minimum degree of the graph G is denoted by $\triangle(G)$ and $\delta(G)$ respectively.

A subset $D \subseteq V(G)$ is independent if no two vertices in D are adjacent. A set $D \subseteq V(G)$ of vertices in the graph G is called a dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. A dominating set D is a minimal dominating set if no proper subset $D^{'} \subset D$ is a dominating set. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set of the graph G. A detailed bibliography on the concept of domination can be found in Hedetniemi and Laskar [7] as well as Cockayne and Hedetniemi [3]. A dominating set $D \subseteq V(G)$ is called an independent dominating set if it is also an independent set. The minimum cardinality of an independent dominating set in G is called the independent domination number $i(G)$ of the graph G. For the better understanding of domination and its related concepts we refer to Haynes et al [6].

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We will give some definitions which are useful for the present work.

Definition 1.1([10]) *For graph G and $uv \in E(G)$, we say u strongly dominates v (v weakly dominates u) if $deg(u) \geq deg(v)$.*

Definition 1.2([10]) *A subset D is a strong(weak) dominating set $sd-set(wd-set)$ if every vertex $v \in V(G)-D$ is strongly(weakly) dominated by some u in D. The strong(weak) domination number $\gamma_{st}(G)(\gamma_w(G))$ is the minimum cardinality of a $sd-set(wd-set)$.*

Generally, for a subset $O \subset V(G)$ with $\langle O\rangle_G$ isomorphic to a special graph, for instance a tree, a subset D_S of $V(G)$ is a Smarandachely strong(weak) dominating set of G on O if every vertex $v \in V(G)-D-O$ is strongly(weakly) dominated by some vertex in D_S. Clearly, if $O=\emptyset$, D_S is nothing else but the strong dominating set of G.

The concepts of strong and weak domination were introduced by Sampathkumar and Pushpa Latha [10]. In the same paper they have defined the following concepts.

Definition 1.3 *The independent strong(weak) domination number $i_{st}(G)$ $(i_w(G))$ of the graph G is the minimum cardinality of a strongly(weakly) dominating set which is independent set.*

Definition 1.4 *Let $G=(V(G),E(G))$ be a graph and $D \subset V(G)$. Then D is s-full (w-full) if every $u \in D$ strongly (weakly) dominates some $v \in V(G)-D$.*

Definition 1.5 *A graph G is domination balanced (d-balanced) if there exists an sd-set D_1 and a wd-set D_2 such that $D_1 \cap D_2 = \phi$.*

Several results on the concepts of strong and weak domination have also been explored by Domke et al [4]. The bounds on strong domination number and the influence of special vertices on strong domination is discussed by Rautenbach [8,9] while Hattingh and Henning have investigated bounds on strong domination number of connected graphs in [5]. For regular graphs $\gamma_{st}=\gamma_w=\gamma$ as reported by Swaminathan and Thangaraju in [11]. Therefore we consider the graph G which is not regular.

§2. Main Results

We begin with propositions which are useful for further results.

Proposition 2.1([10]) *For a graph G of order n, $\gamma \leq \gamma_{st} \leq n-\triangle(G)$.*

Proposition 2.2([1]) *For a nontrivial path P_n,*

$$\gamma_{st}(P_n)=\lceil \tfrac{n}{3}\rceil \text{ and } \gamma_w(P_n)=\begin{cases} \left\lceil \dfrac{n}{3}\right\rceil & if\ n \equiv 1(mod\ 3) \\ \left\lceil \dfrac{n}{3}\right\rceil+1 & otherwise \end{cases}$$

Proposition 2.3([1]) *For cycle C_n, $\gamma_{st}(C_n)=\gamma_w(C_n)=\gamma(C_n)=\lceil \frac{n}{3}\rceil$.*

Proposition 2.4([11]) *For any non regular graph G, $\gamma_{st}(G)+\triangle(G)=n$ and $\gamma_w(G)+\delta(G)=n$*

if and only if

(1) *for every vertex u of degree δ, $V(G) - N[u]$ is an independent set and every vertex in $N(u)$ is adjacent to every vertex in $V(G) - N(u)$.*

(2) *for every vertex v of degree $\triangle$, $V(G) - N[v]$ is independent, each vertex in $V(G) - N(v)$ is of degree $\geq \delta + 1$ and no vertex of $N(v)$ strongly dominates two or more vertices of $V(G) - N[v]$.*

Proposition 2.5([10]) *For a graph G, the following statements are equivalent.*

(1) *G is d-balanced;*

(3) *There exists an sd-set D which is s-full;*

(3) *There exists an wd-set D which is w-full.*

Theorem 2.6 *Let G be the graph of order n. If there exists a vertex u_1 with $deg(u_1) = \triangle$ and $deg(u_i) = m$, where $2 \leq i \leq n$ then, $\gamma_{st}(G) = \gamma(G)$.*

Proof Let G be the graph of order n and let u_1 be the vertex with $deg(u_1) = \triangle(G)$. The set $V(G) - N(u_1)$ contains the vertices of degree m. It is clear that the graph G contains two types of vertices: a vertex of degree $\triangle$ and remaining vertices of degree m. The vertex $u_1 \in \gamma_{st}$ - set as it is of maximum degree.

To prove the result we consider following two cases.

Case 1. $N[u_1] = V(G)$.

If $N[u_1] = V(G)$ implies that $\gamma(G) = 1$. Hence $deg(D) > deg(V(G) - D)$. Therefore $u_1 \in D$ strongly dominates $V(G) - D$. Thus $\gamma_{st}(G) = \gamma(G) = 1$.

Case 2. $N[u_1] \neq V(G)$.

Let us partition the vertex set $V(G)$ into V_1 and V_2. Now to construct a dominating set or a strong dominating set of minimum cardinality the vertex u_1 must belong to every strong dominating set. So let $N[u_1] \in V_1$ and remaining $n - \triangle - 1$ vertices are in V_2. Now the vertices in V_2 are of degree m. Thus the vertices V_2 forms a regular graph. For regular graphs $\gamma(G) = \gamma_{st}(G) = \gamma_w(G)$. Let k be the domination number of vertex set V_2. Therefore $\gamma(G) = \gamma_{st}(G) = \gamma_{st}(V_1) + \gamma_{st}(V_2) = 1 + k$.

In any case, if G contains a vertex of degree $\triangle(G)$ and remaining vertices of same degree m then $\gamma_{st}(G) = \gamma(G)$. □

Corollary 2.7 $\gamma(K_{1,n}) = \gamma_{st}(K_{1,n}) = i_{st}(K_{1,n}) = 1$.

Corollary 2.8 $\gamma(W_n) = \gamma_{st}(W_n) = i_{st}(W_n) = 1$.

Definition 2.9 *One point union $C_n^{(k)}$ of k copies of cycle C_n is the graph obtained by taking v as a common vertex such that any two cycles $C_n^{(i)}$ and $C_n^{(j)}$ $(i \neq j)$ are edge disjoint and do not have any vertex in common except v.*

Corollary 2.10 $\gamma_{st}(C_n^{(k)}) = \gamma(C_n^{(k)}) = 1 + k\lceil \frac{n-3}{3} \rceil$, *for* $n \geq 3$.

Proof Let $v_1^p, v_2^p, \ldots v_n^p$ be the vertices of p^{th} copy of cycle C_n for $1 \leq p \leq k$, $k \in \mathbb{N}$ and

v be the common vertex in graph C_n^k such that $v = v_1^1 = v_1^2 = v_1^3 = \cdots = v_1^p$. Consequently $|V(C_n^k)| = kn - k + 1$.

The $deg(v) = 2k$ which is of maximum degree, then it must be in every dominating set D and the vertex v will dominate $2k + 1$ vertices.

Now to dominate the remaining k disconnected copies of path each of length $n - 3$ we require minimum $k\left\lceil \frac{n-3}{3} \right\rceil$ vertices.

This implies that $\gamma(C_n^k) \geq 1 + k\left\lceil \frac{n-3}{3} \right\rceil$. Let us partition the vertex set $V(C_n^k)$ into $V_1(C_n^k)$ and $V_2(C_n^k)$ such that $V(C_n^k) = V_1(C_n^k) \bigcup V_2(C_n^k)$ depending on the degree of vertices. Let $V_1(C_n^k)$ contain $N[v]$ which forms a star graph $K_{1,2k}$. Thus, from above Corollary 2.7 $\gamma(K_{1,n}) = 1$. Let $V_2(C_n^k)$ contain the remaining vertices, that is, $|V_2(C_n^k)| = |V(C_n^k)|$ - $|V_1(C_n^k)| = kn - k + 1$ $-(2k+1) = kn - 3k$ in k copies. Thus, in one copy there are $n - 3$ vertices which forms a path of order $n - 3$. Therefore, from above Proposition 2.2, $\gamma(P_{n-3}) = \left\lceil \frac{n-3}{3} \right\rceil$. For k copies of path the domination number is $\gamma[k(P_{n-3})] = k\left\lceil \frac{n-3}{3} \right\rceil$. Hence, $\gamma(C_n^k) = \gamma(K_{1,n}) + \gamma[k(P_{n-3})]$ $= 1 + k\left\lceil \frac{n-3}{3} \right\rceil$, for $n \geq 3$. Therefore D is a dominating set of minimum cardinality. Thus, D is also the strong dominating set of minimum cardinality. Therefore,

$$\gamma_{st}(C_n^{(k)}) = \gamma(C_n^{(k)}) = 1 + k\lceil \frac{n-3}{3} \rceil,$$

for $n \geq 3$. □

Definition 2.11 *Duplication of a vertex v_i by a new edge $e' = u'v'$ in a graph G results into a graph G' such that $N(u') = \{v_i, v'\}$ and $N(v') = \{v_i, u'\}$.*

Theorem 2.12 *If G' is the graph obtained by duplication of each vertex of graph G by a new edge then $\gamma_{st}(G') = \gamma(G') = n$.*

Proof Let $V(G)$ be the set of vertices and $E(G)$ be the set of edges for the graph G. Let us denote vertices of graph G by $u_1, u_2, u_3 \cdots, u_n$. Hence $|V(G)| = n$ and $|E(G)| = m$. Each vertex of G is duplicated by a new edge. Let us denote these new added vertices by $v_1, v_2, v_3 \cdots, v_n$ and $w_1, w_2, w_3 \cdots, w_n$ respectively. Hence, the obtained graph G' contains $3n$ vertices and $3n + m$ edges. Thus the degree of u_i $(1 \leq i \leq n)$ will increase by two and the degree of v_i and w_i $(1 \leq i \leq n)$ is two. The graph G' contains n vertex disjoint cycles of order 3. By Proposition 2.3, $\gamma_{st}(C_3) = 1$. Thus minimum n vertices are essential to strongly dominate n vertex disjoint cycles. Hence, $\gamma_{st}(G') \geq n$. Since u_i are the vertices of maximum degree, they must be in every strong dominating set. We claim that it is enough to take u_i in strong dominating set as the vertices v_i and w_i are adjacent to a common vertex u_i. Thus, $D = \{u_1, u_2, u_3 \cdots, u_n\}$ is the only strong dominating set with minimum cardinality. Hence,

$$\gamma_{st}(G') = \gamma(G') = n.$$

□

Theorem 2.13 *If G' is the graph obtained by duplication of each vertex of graph G by a new edge then G' is $d-$ balanced.*

Proof As argued in Theorem 2.12, $D = \{u_1, u_2, u_3 \cdots, u_n\}$ is the only strong dominating set. Hence it is the strong dominating set with minimum cardinality. The vertices u_i $(1 \leq i \leq n)$ strongly dominates v_i and w_i in $V(G') - D$ where $(1 \leq i \leq n)$. Thus, D is $s-$full. Hence from Proposition 2.5 G' is $d-$ balanced. □

Definition 2.14 *The switching of a vertex v of G means removing all the edges incident to v and adding edges joining to every vertex which is not adjacent to v in G. We denote the resultant graph by $\widetilde{G}$.*

Theorem 2.15 *If $\widetilde{C_n}$ is the graph obtained by switching of an arbitrary vertex v in cycle C_n, $(n > 3)$ then,*

$$\gamma_{st}(\widetilde{C_n}) = \begin{cases} 1 & if\ n = 4 \\ 2 & if\ n = 5 \\ 3 & if\ n \geq 6 \end{cases}$$

Proof Let $v_1, v_2, v_3 \ldots, v_n$ be the vertices of the cycle C_n. Without loss of generality we switch the vertex v_1 of C_n. We consider following cases to prove the theorem.

Case 1. $n = 4$.

The graph $\widetilde{C_4}$ is obtained by switching of vertex v_1 in cycle C_4 which is same as $K_{1,3}$. Hence $D = \{v_3\}$ is the only strong dominating set as discussed in Corollary 2.7. It is the only strong dominating set with minimum cardinality. Therefore the strong domination number $\gamma_{st}(\widetilde{C_4}) = 1$.

Case 2. $n = 5$.

The graph $\widetilde{C_5}$ obtained by switching of vertex v_1 in cycle C_5. The degree $deg(v_1) = 2$, $deg(v_2) = deg(v_5) = 1$ while $deg(v_3) = deg(v_4) = 3$. The vertex v_3 strongly dominates v_1, v_2 and v_4 along with itself. It is enough to take the vertex v_4 in the strong dominating set to strongly dominate the vertex v_5. Thus $D = \{v_3, v_4\}$ is the only strong dominating set with minimum cardinality. Hence arbitrary switching of a vertex of cycle C_5 results into $\gamma_{st}(\widetilde{C_5}) = 2$.

Case 3. $n \geq 6$.

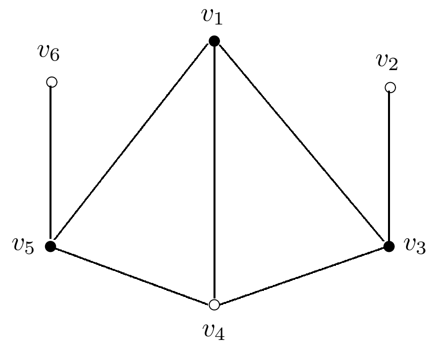
Let $\widetilde{C_n}$ be the graph obtained by switching of vertex v_1 in cycle C_n. The degree $deg(v_1) = n - 3$ while $deg(v_2) = deg(v_n) = 1$ and remaining $n - 3$ vertices are of degree three. Thus, $|V(\widetilde{C_n})| = n$. By the Proposition 2.1 $\gamma_{st}(\widetilde{C_n}) \leq n - \triangle(\widetilde{C_n}) = n - (n-3)$, implying $\gamma_s(\widetilde{C_n}) \leq 3$.

The degree $deg(v_1) = n-3$ which is of maximum degree, that is, v_1 must be in every strong dominating set and v_1 will strongly dominate $n-2$ vertices except the pendant vertices v_2 and v_n. Hence either these pendant vertices must be in every strong dominating set or the supporting vertices v_{n-1} and v_3. Thus, $D_1 = \{v_1, v_2, v_n\}$ or $D_2 = \{v_1, v_3, v_{n-1}\}$ or $D_3 = \{v_1, v_2, v_{n-1}\}$ or $D_4 = \{v_1, v_n, v_3\}$ are strong dominating sets with minimum cardinality. Therefore,

$$\gamma_{st}(\widetilde{C_n}) = 3.$$

for $n \geq 6$. □

Illustration 2.16 In Figure 2.1, the solid vertices are the elements of strong dominating sets of $\widetilde{C_6}$ as shown below.



$$\gamma(\widetilde{C_6}) = \gamma_{st}(\widetilde{C_6}) = 3$$

Figure 2.1

Corollary 2.17 *$\gamma_{st}(\widetilde{C_n}) = \gamma(\widetilde{C_n})$ for $n > 3$.*

Proof We continue with the terminology and notations used in Theorem 2.15 and consider the following cases to prove the corollary.

Case 1. $n = 4$.

As shown in Theorem 2.15, $D = \{v_3\}$ is the only strong dominating set with minimum cardinality which is also the dominating set of minimum cardinality. As discussed in Corollary 2.7, $\gamma_{st}(\widetilde{C_4}) = \gamma(\widetilde{C_4})$.

Case 2. $n = 5$.

As shown in Theorem 2.15, $D = \{v_3, v_4\}$ is the only strong dominating set with minimum cardinality which is also the dominating set of minimum cardinality. Hence $\gamma_{st}(\widetilde{C_5}) = \gamma(\widetilde{C_5})$.

Case 3. $n \geq 6$.

As shown in Theorem 2.15 we have obtained four possible strong dominating sets. The strong dominating sets $D_1 = \{v_1, v_2, v_n\}$ or $D_2 = \{v_1, v_3, v_{n-1}\}$ or $D_3 = \{v_1, v_2, v_{n-1}\}$ or $D_4 = \{v_1, v_n, v_3\}$ are strong dominating sets with minimum cardinality which are also the dominating set of minimum cardinality. Thus, $\gamma_{st}(\widetilde{C_n}) = \gamma(\widetilde{C_n})$, for $n \geq 6$. □

Theorem 2.18 *If $\widetilde{C_n}$ is the graph obtained by switching of an arbitrary vertex v in cycle C_n then,$\widetilde{C_n}$ $(n > 3)$ is $d-$ balanced.*

Proof We continue with the terminology and notations used in Theorem 2.15 and consider the following cases to prove the corollary.

Case 1. $n = 4$.

As discussed in Theorem 2.15 the set $D = \{v_3\}$ is the strong dominating with minimum cardinality. The set D is $s-$ full since the vertex v_3 strongly dominates remaining three vertices in $V(\widetilde{C_4}) - D$. Hence from Proposition 2.5 $\widetilde{C_4}$ is $d-$ balanced.

Case 2. $n = 5$.

As shown in Theorem 2.15, the set $D = \{v_3, v_4\}$ is a strong dominating set with minimum cardinality. The set $D = \{v_3, v_4\}$ is $s-$ full since v_i $(i = 3, 4)$ strongly dominates v_2, v_4 and v_5 in $V(\widetilde{C_5}) - D$. Hence from Proposition 2.5 $\widetilde{C_5}$ is $d-$ balanced.

Case 3. $n \geq 6$.

In Theorem 2.15 we have obtained the strong dominating set $D_2 = \{v_1, v_3, v_{n-1}\}$ of minimum cardinality. The set D_2 is $s-$ full as v_1, v_3 and v_{n-1} strongly dominates remaining vertices in $V(\widetilde{C_n}) - D_2$. Thus from Proposition 2.5 $\widetilde{C_n}$ $(n \geq 6)$ is $d-$ balanced. □

Definition 2.19 *The book B_m is a graph $S_m \times P_2$ where $S_m = K_{1,m}$.*

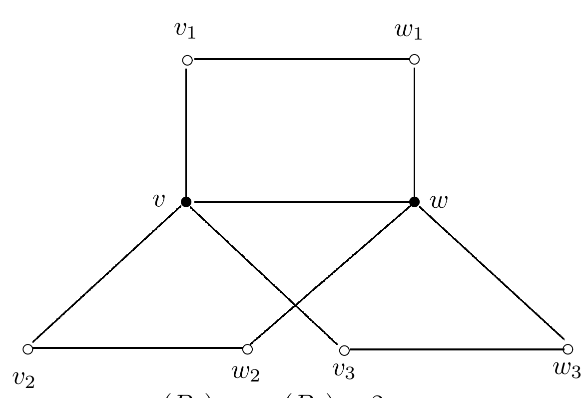
Theorem 2.20 $\gamma_{st}(B_m) = 2$ *for* $m \geq 3$.

Proof Let S_m be the graph with vertices $u, u_1, u_2, u_3 \cdots, u_m$ where u is the vertex of degree m and $u_1, u_2, u_3 \cdots, u_m$ are pendant vertices. Let P_2 be the path with vertices a_1 and a_2. We consider $v = (u, a_1), v_1 = (u_1, a_1), v_2 = (u_2, a_1) \cdots, v_m = (u_m, a_1)$ and $w = (u, a_2), w_1 = (u_1, a_2), w_2 = (u_2, a_2) \cdots, w_m = (u_m, a_2)$. Hence $|V(B_m)| = 2m + 2$.

In B_m there is no vertex with degree $2m + 1$, implying that $\gamma(B_m) > 1$. The $deg(v) = deg(w) = m + 1$ are the vertices of maximum degree. Let us partition the vertex set $V(B_m)$ into V_1 and V_2 such that $V(B_m) = V_1 \bigcup V_2$. Let $N[v] \in V_1$ and $N[w] \in V_2$. Then in both the partitions a star graph $K_{1,m}$ is formed. Thus from above Corollary 2.7, $\gamma(K_{1,m}) = \gamma_{st}(K_{1,m}) = 1$. Thus, it is enough to take v and w in strong dominating set as it strongly dominates $2m + 2$ vertices. Therefore $D = \{v, w\}$ is the strong dominating set with minimum cardinality. Hence,

$$\gamma_{st}(B_m) = 2 \quad \text{if} \quad m \geq 3.$$ □

Illustration 2.21 In Figure 2.2, the solid vertices are the elements of strong dominating set of B_3 as shown below.



$\gamma(B_3) = \gamma_{st}(B_3) = 2$

Figure 2.2

Corollary 2.22 $\gamma_{st}(B_m) = \gamma(B_m)$ *for* $m \geq 3$.

Proof As shown in Theorem 2.20 we have obtained the strong dominating set $D = \{v, w\}$. The set D also forms the dominating set of minimum cardinality. Thus, $\gamma_{st}(B_m) = \gamma(B_m)$, for $n \geq 3$. □

Theorem 2.23 *The book graph B_m is $d-$balanced.*

Proof In Theorem 2.20 we have obtained the strong dominating set $D = \{v, w\}$ of minimum cardinality. The vertex v strongly dominates $v_1, v_2 \cdots, v_m$ while the vertex w strongly dominate $w_1, w_2 \cdots, w_m$ in $V(B_m) - D$ respectively. Hence D is $s-$full set. Hence from Proposition 2.5 the book graph B_m is $d-$balanced. □

§3. Concluding Remarks

The strong domination in graph is a variant of domination. The strong domination number of various graphs are known. We have investigated the strong domination number of some graphs obtained from C_n by means of some graph operations. This work can be applied to rearrange the existing security network in the case of high alert situation and to beef up the surveillance.

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