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$\mathfrak{b}-$Smarandache m_1m_2 Curves of Biharmontic New Type $\mathfrak{b}-$Slant Helices According to Bishop Frame in the Sol Space $\boldsymbol{Sol}^3$

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Abstract: In this paper, we study $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ curves of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. We characterize the $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the $\boldsymbol{Sol}^3$.

Key Words: New type $\mathfrak{b}-$slant helix, Sol space, curvatures.

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§1. Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the *bitension field* of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organized as follows: Firstly, we study $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ curves of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. Secondly, we characterize the $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ curves in terms of their Bishop curvatures. Finally, we find explicit equations of $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ curves in the $\boldsymbol{Sol}^3$.

§2. Riemannian Structure of Sol Space $\boldsymbol{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\boldsymbol{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

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where (x, y, z) are the standard coordinates in $\mathbb{R}^3$ [11,12].

Note that the Sol metric can also be written as:

$$g_{\boldsymbol{Sol}^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i,$$

where

$$\omega^1 = e^z dx, \quad \omega^2 = e^{-z} dy, \quad \omega^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z} \frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^z \frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

Proposition 2.1 *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\boldsymbol{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i} \mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\boldsymbol{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x, y, z) &\to (x + c, y, z), \\ (x, y, z) &\to (x, y + c, z), \\ (x, y, z) &\to \left(e^{-c} x, e^c y, z + c\right). \end{aligned}$$

§3. Biharmonic New Type $\mathfrak{b}-$Slant Helices in Sol Space $\boldsymbol{Sol}^3$

Assume that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{t}} \mathbf{t} &= \kappa \mathbf{n}, \\ \nabla_{\mathbf{t}} \mathbf{n} &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \nabla_{\mathbf{t}} \mathbf{b} &= -\tau \mathbf{n}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion [14,15] and

$$\begin{aligned} g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) &= 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{n},\mathbf{n})=1,\ g_{\boldsymbol{Sol}^3}(\mathbf{b},\mathbf{b})=1, \\ g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{n}) &= g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{b})=g_{\boldsymbol{Sol}^3}(\mathbf{n},\mathbf{b})=0. \end{aligned} \tag{3.2}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, [1]. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1+k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= -k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= -k_2\mathbf{t}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) &= 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_1)=1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_2,\mathbf{m}_2)=1, \\ g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_1) &= g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_2)=g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_2)=0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\delta(s)=\arctan\frac{k_2}{k_1}$, $\tau(s)=\delta'(s)$ and $\kappa(s)=\sqrt{k_1^2+k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\delta(s), \\ k_2 &= \kappa(s)\sin\delta(s). \end{aligned}$$

The relation matrix may be expressed as

$$\begin{gathered} \mathbf{t}=\mathbf{t}, \\ \mathbf{n}=\cos\delta(s)\,\mathbf{m}_1+\sin\delta(s)\,\mathbf{m}_2, \\ \mathbf{b}=-\sin\delta(s)\,\mathbf{m}_1+\cos\delta(s)\,\mathbf{m}_2. \end{gathered}$$

On the other hand, using above equation we have

$$\begin{gathered} \mathbf{t}=\mathbf{t}, \\ \mathbf{m}_1=\cos\delta(s)\,\mathbf{n}-\sin\delta(s)\,\mathbf{b} \\ \mathbf{m}_2=\sin\delta(s)\,\mathbf{n}+\cos\delta(s)\,\mathbf{b}. \end{gathered}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1e_1+t^2e_2+t^3e_3, \\ \mathbf{m}_1 &= m_1^1\mathbf{e}_1+m_1^2\mathbf{e}_2+m_1^3\mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1\mathbf{e}_1+m_2^2\mathbf{e}_2+m_2^3\mathbf{e}_3. \end{aligned} \tag{3.5}$$

Theorem 3.1 *$\gamma: I\longrightarrow \boldsymbol{Sol}^3$ is a biharmonic curve according to Bishop frame if and only if*

$$\begin{aligned} k_1^2+k_2^2 &= \text{constant}\neq 0, \\ k_1''-\left[k_1^2+k_2^2\right]k_1 &= -k_1\left[2m_2^3-1\right]-2k_2m_1^3m_2^3, \\ k_2''-\left[k_1^2+k_2^2\right]k_2 &= 2k_1m_1^3m_2^3-k_2\left[2m_1^3-1\right]. \end{aligned} \tag{3.6}$$

Theorem 3.2 *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix with constant slope. Then, the position vector of γ is*

$$\begin{aligned}\gamma(s) &= [\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\mathcal{S}_1\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]]+\mathcal{S}_4 e^{-\sin\mathcal{M}s+\mathcal{S}_3}]\mathbf{e}_1\\ &+[\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\mathcal{S}_1\sin[\mathcal{S}_1 s+\mathcal{S}_2]]+\mathcal{S}_5 e^{\sin\mathcal{M}s-\mathcal{S}_3}]\mathbf{e}_2\\ &+[-\sin\mathcal{M}s+\mathcal{S}_3]\mathbf{e}_3, \qquad (3.7)\end{aligned}$$

where $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\mathcal{S}_4,\mathcal{S}_5$ are constants of integration, [8].

We can use Mathematica in Theorem 3.4, yields

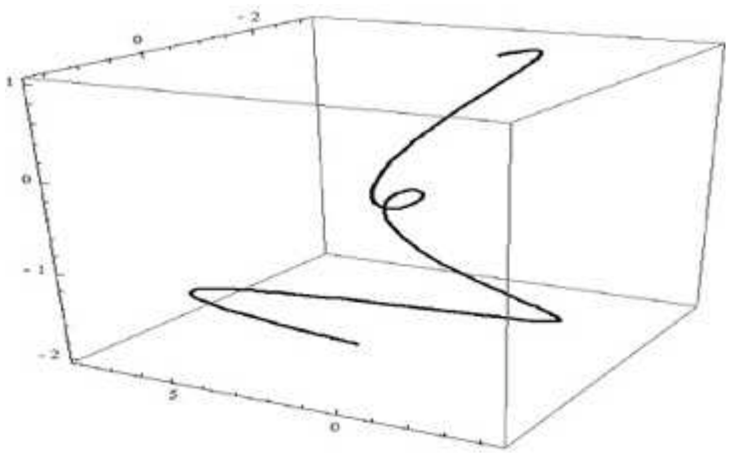

Fig.1

§4. $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ Curves of Biharmonic New Type $\mathfrak{b}-$Slant Helices in $\boldsymbol{Sol}^3$

To separate a Smarandache $\mathbf{m_1m_2}$ curve according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ curve.

Definition 4.1 *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix and $\{\mathbf{t},\mathbf{m_1},\mathbf{m_2}\}$ be its moving Bishop frame. $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ curves are defined by*

$$\gamma_{\mathbf{m_1m_2}} = \frac{1}{\sqrt{k_1^2+k_2^2}}\left(\mathbf{m_1}+\mathbf{m_2}\right). \qquad (4.1)$$

Theorem 4.2 *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the equation of $\mathfrak{b}-$Smarandache $\mathbf{m_1m_2}$ curves of biharmonic new type $\mathfrak{b}-$slant* *helix is given by*

$$\begin{aligned}\gamma_{\mathbf{m_1 m}_2}(s) &= \frac{1}{\sqrt{k_1^2+k_2^2}}[\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]+\cos[\mathcal{S}_1 s+\mathcal{S}_2]]\mathbf{e}_1 \\ &\quad +\frac{1}{\sqrt{k_1^2+k_2^2}}[\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]-\sin[\mathcal{S}_1 s+\mathcal{S}_2]]\mathbf{e}_2 \\ &\quad +\frac{1}{\sqrt{k_1^2+k_2^2}}[\cos\mathcal{M}]\mathbf{e}_3,\end{aligned} \tag{4.2}$$

where $\mathcal{C}_1, \mathcal{C}_2$ *are constants of integration.*

Proof Assume that γ is a non geodesic biharmonic new type $\mathfrak{b}-$slant helix according to Bishop frame.

From Theorem 3.2, we obtain

$$\mathbf{m}_2 = \sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_1+\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_2+\cos\mathcal{M}\mathbf{e}_3, \tag{4.3}$$

where $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{R}$.

Using Bishop frame, we have

$$\mathbf{m}_1 = \cos[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_1-\sin[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_2. \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.1) we have (4.2), which completes the proof. □

In terms of Eqs. (2.1) and (4.2), we may give:

Corollary 4.3 *Let* $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ *be a unit speed non-geodesic biharmonic new type* $\mathfrak{b}-$*slant helix. Then, the parametric equations of* $\mathfrak{b}-$*Smarandache* $\mathbf{tm_1m}_2$ *curves of biharmonic new type* $\mathfrak{b}-$*slant helix are given by*

$$\begin{aligned}x_{\mathbf{tm_1m}_2}(s) &= \frac{e^{-\frac{1}{\sqrt{k_1^2+k_2^2}}[\cos\mathcal{M}]}}{\sqrt{k_1^2+k_2^2}}[\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]+\cos[\mathcal{S}_1 s+\mathcal{S}_2]], \\ y_{\mathbf{tm_1m}_2}(s) &= \frac{e^{\frac{1}{\sqrt{k_1^2+k_2^2}}[\cos\mathcal{M}]}}{\sqrt{k_1^2+k_2^2}}[\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]-\sin[\mathcal{S}_1 s+\mathcal{S}_2]], \\ z_{\mathbf{tm_1m}_2}(s) &= \frac{1}{\sqrt{k_1^2+k_2^2}}[\cos\mathcal{M}],\end{aligned} \tag{4.5}$$

where $\mathcal{S}_1, \mathcal{S}_2$ *are constants of integration.*

Proof Substituting (2.1) to (4.2), we have (4.5) as desired. □

We may use Mathematica in Corollary 4.3, yields

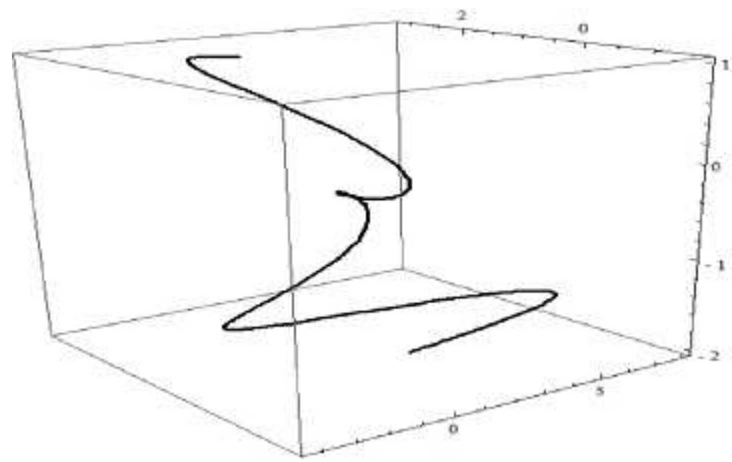

Fig.2

References

[1] L.R.Bishop, There is more than one way to frame a curve, *Amer. Math. Monthly*, 82 (3) (1975) 246-251.

[2] R.M.C.Bodduluri, B.Ravani, Geometric design and fabrication of developable surfaces, *ASME Adv. Design Autom.*, 2 (1992), 243-250.

[3] F.Dillen, W.Kuhnel, Ruled Weingarten surfaces in Minkowski 3-space, *Manuscripta Math.* 98 (1999), 307–320.

[4] I.Dimitric, Submanifolds of $\mathbb{E}^m$ with harmonic mean curvature vector, *Bull. Inst. Math. Acad. Sinica* 20 (1992), 53–65.

[5] J.Eells and L.Lemaire, A report on harmonic maps, *Bull. London Math. Soc.* 10 (1978), 1–68.

[6] J.Eells and J.H.Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* 86 (1964), 109–160.

[7] G.Y.Jiang, 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math. Ser.* A 7(2) (1986), 130–144.

[8] T.Körpınar and E.Turhan, Biharmonic new type b-slant helices according to Bishop frame in the sol space, (submitted).

[9] M.A.Lancret, Memoire sur les courbes ‘a double courbure, *Memoires presentes alInstitut* 1 (1806), 416-454.

[10] E.Loubeau and S.Montaldo, Biminimal immersions in space forms,preprint, 2004, *math.DG /0405320* v1.

[11] Y.Ou and Z.Wang, Linear Biharmonic Maps into Sol, Nil and Heisenberg Spaces, *Mediterr. J. Math.* 5 (2008), 379–394

[12] S.Rahmani, Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois, *Journal of Geometry and Physics*, 9 (1992), 295-302.

[13] D.J.Struik, *Lectures on Classical Differential Geometry,* Dover, New-York, 1988.

[14] E.Turhan and T. Körpınar, On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis^3, *Zeitschrift für Naturforschung A- A Journal of Physical Sciences*, 65a (2010), 641-648.

[15] E.Turhan and T.Körpınar, Parametric equations of general helices in the sol space $\mathfrak{Sol}^3$, *Bol. Soc. Paran. Mat.* 31 (1) (2013), 99–104.