A LINEAR COMBINATION WITH
SMARANDACHE FUNCTION
TO OBTAIN THE IDENTITY

by

M. Andrei, I. Balăченю, C. Dumitrescu, E. Rădescu, N. Rădescu, V. Seleacu

In this paper we consider a numerical function $i_p: \mathbb{N}^* \rightarrow \mathbb{N}$ ($p$ is an arbitrary prime number) associated with a particular Smarandache Function $S_p: \mathbb{N}^* \rightarrow \mathbb{N}$ such that $(1/p)S_p(a) + i_p(a) = a$.

1. INTRODUCTION.

In [7] is defined a numerical function $S: \mathbb{N}^* \rightarrow \mathbb{N}$, $S(n)$ is the smallest integer such that $S(n)!$ is divisible by $n$. This function may be extended to all integers by defining $S(-n) = S(n)$.

If $a$ and $b$ are relatively prime then $S(a \cdot b) = \max\{S(a), S(b)\}$, and if $[a, b]$ is the least common multiple of $a$ and $b$ then $S([a \cdot b]) = \max\{S(a), S(b)\}$.

Suppose that $n = p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ is the factorization of $n$ into primes. In this case,

$$S(n) = \max\{S(p_i^{a_i}); i = 1, ..., r\} \quad (1)$$

Let $a_n(p) = \frac{(p^n - 1)}{(p - 1)}$ and $[p]$ be the generalized numerical scale generated by $(a_n(p))_{n \in \mathbb{N}}$ :

$$[p]: a_1(p), a_2(p), ..., a_n(p), ...$$

By $(p)$ we shall note the standard scale induced by the net $b_n(p) = p^n$ :

$$(p): 1, p, p^2, p^3, ..., p^n, ...$$

In [2] it is proved that

$$S(p^n) = p\left(\frac{a_n(p)}{[p]}\right) \quad (2)$$

That is the value of $S(p^n)$ is obtained multiplying by $p$ the number obtained writing the exponent $a$ in the generalized scale $[p]$ and "reading" it in the standard scale $(p)$.

Let us observe that the calculus in the generalized scale $[p]$ is different from the calculus in the standard scale $(p)$, because $a_n+1(p) = p a_n(p) + 1$ and $b_n+1(p) = p b_n(p)$ \hspace{1cm} (3)

We have also

$$a_m(p) \leq a \Leftrightarrow \frac{(p^m - 1)}{(p - 1)} \leq a \Leftrightarrow p^m \leq (p - 1) \cdot a + 1 \Leftrightarrow m \leq \log_p \left(\frac{(p - 1) \cdot a + 1}{p}\right)$$

so if

$$a_{[p]} = v_1 a_1(p) + v_{i-1} a_{i-1}(p) + \ldots + v_{i+1} a_{i+1}(p)$$

is the expression of $a$ in the scale $[p]$ then $t$ is the integer part of $\log_p \left(\frac{(p - 1) \cdot a + 1}{p}\right)$

$$t = \left[\log_p \left(\frac{(p - 1) \cdot a + 1}{p}\right)\right]$$

and the digit $v_i$ is obtained from $a = v_1 a_1(p) + r_{i+1}$.

In [1] it is proved that

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1 This paper has been presented at 26th Annual Iranian Math. Conference 28-31 March 1995 and is published in the Proceedings of Conference (437-439).
\[ S(p') = (p-1) \cdot a + \sigma_{[p]}(a) \]  \hspace{1cm} (4)

where \( \sigma_{[p]}(a) = \nu_1 + \nu_2 + \ldots + \nu_p \).

A Legendre formula asserts that

\[ a! = \prod_{p_i \in \sigma_p} \frac{a}{p_i} \]

where \( E_p(a) = \sum_{p_i \in \sigma_p} \frac{a}{p_i} \).

We have also that (5)

\[ E_p(a) = \frac{a - \sigma_{[p]}(a)}{p-1} \]

and (11) \( E_p(a) = \left( \frac{a}{p} \right)_{[p]} \).

In [1] is given also the following relation between the function \( E_p \) and the Smarandache function

\[ S(p') = \frac{(p-1)^2}{p} \left( E_p(a) + a \right) + \frac{p-1}{p} \sigma_{[p]}(a) + \sigma_{[p]}(a) \]

There exist a great number of problems concerning the Smarandache function. We present some of these problems.

P. Gronas finds the solution of the diophantine equation \( F_n(n) = n \), where \( F_n(n) = \sum_{d|n} S(d) \). The solution are \( n = 9 \), \( n = 16 \) or \( n = 24 \), or \( n = 2p \), where \( p \) is a prime number.

T. Yau finds the triplets which verify the Fibonacci relationship \( S(n) = S(n+1) + S(n+2) \).

Checking the first 1200 numbers, he finds just two triplets which verify this relationship: \( (9,10,11) \) and \( (119,120,121) \). He can’t find theoretical proof.

The following conjecture that: “the equation \( S(n) = S(n+1) \), has no solution”, was not completely solved until now.

2. The Function \( i_p(a) \). In this section we shall note \( S(p') = S_p(a) \). From the Legendre formula it results (4) that

\[ S_p(a) = p(a - i_p(a)) \] with \( 0 \leq i_p(a) \leq \left\lfloor \frac{a-1}{p} \right\rfloor . \]  \hspace{1cm} (6)

That is we have

\[ \frac{1}{p} S_p(a) + i_p(a) = a \]  \hspace{1cm} (7)

and so for each function \( S_p \), there exists a function \( i_p \), such that we have the linear combination (7) to obtain the identity.

In the following we keep out some formulae for the calculus of \( i_p \). We shall obtain a duality relation between \( i_p \) and \( E_p \).

Let

\[ a_p = u_k p^{k-1} + u_{k-1} p^{k-2} + \ldots + u_1 p + u_0 . \]

Then
\[
\begin{align*}
  a &= (p-1) \left( \sum_{k=1}^{p-1} \left( \frac{p^k - 1}{p-1} + \frac{p^{k-1} - 1}{p-1} + \ldots + \frac{p^{k-1} - 1}{p-1} \right) + \sum_{i=1}^{n} \frac{u_i}{p} \right) + u_0 = \\
  &\quad \quad \left( p-1 \right) \left( \frac{a}{p} \right) + \sigma_p(a) = (p-1)E_p(a) + \sigma_p(a) \\
  \end{align*}
\]

From (4) it results

\[
  a = \frac{S_p(a) - \sigma_p(a)}{p-1}
\]

From (8) and (9) we deduce

\[
(p-1)E_p(a) + \sigma_p(a) = \frac{S_p(a) - \sigma_p(a)}{p-1}
\]

So,

\[
S_p(a) = (p-1)^2E_p(a) + (p-1)\sigma_p(a) + \sigma_p(a)
\]

From (4) and (7) it results

\[
i_p(a) = \frac{a - \sigma_p(a)}{p}
\]

and it is easy to observe a complementary with the equality (5).

Combining (5) and (11) it results

\[
i_p(a) = \frac{(p-1)E_p(a) + \sigma_p(a) - \sigma_p(a)}{p}
\]

From

\[
a = \sum_{i=1}^{n} u_i + v_r = \sum_{i=1}^{n} \frac{u_i}{p} + \left( \frac{a}{p} \right) \left( \frac{\sigma_p(a)}{p} \right)
\]

it results that

\[
a = \left( \frac{u_1 p^{i-1} + u_{i-1} p^{i-2} + \ldots + u_2 p + u_1}{p} \right) + \left( \frac{v_1}{p} \right) + \left( \frac{v_2}{p} \right) + \ldots + \left( \frac{v_r}{p} \right)
\]

because

\[
\left[ \frac{a}{p} \right] = \left[ \frac{v_1}{p} \right] + \left[ \frac{v_2}{p} \right] + \ldots + \left[ \frac{v_r}{p} \right] + \left[ \frac{\sigma_p(a)}{p} \right]
\]

we have \([n+x] = n + [x]\).

Then

\[
a = \left( \frac{a}{p} \right) + \left[ \frac{a}{p} \right] - \left[ \frac{\sigma_p(a)}{p} \right]
\]

or

\[
a = \frac{S_p(a)}{p} + \left[ \frac{a}{p} \right] - \left[ \frac{\sigma_p(a)}{p} \right]
\]

It results that
From (11) and (14) we obtain

\[ i_p(a) = p\left( a - \left\lfloor \frac{\sigma_p(a)}{p} \right\rfloor \right) \]  \hspace{1cm} (14)

From (11) and (14) we obtain

\[ i_p(a) = \left\lfloor \frac{a - \sigma_p(a)}{p} \right\rfloor \]  \hspace{1cm} (15)

It is know that there exists \( m, n \in \mathbb{N} \) such that the relation

\[ \frac{m - n}{p} = \left\lfloor \frac{m}{p} - \frac{n}{p} \right\rfloor \]  \hspace{1cm} (16)

is not verifies.

But if \( \frac{m - n}{p} \in \mathbb{N} \) then the relation (16) is satisfied.

From (11) and (15) it results

\[ \frac{a - \sigma_p(a)}{p} = \left\lfloor \frac{a}{p} - \frac{\sigma_p(a)}{p} \right\rfloor \]

This equality results also by the fact that \( i_p(a) \in \mathbb{N} \).

From (2) and (11) or from (13) and (15) it results that

\[ i_p(a) = a - (a_{p|p}) \]  \hspace{1cm} (17)

From the condition on \( i_p \) in (6) it results that \( \Delta = \left\lfloor \frac{a - 1}{p} \right\rfloor - i_p(a) \geq 0 \).

To calculate the difference \( \Delta = \left\lfloor \frac{a - 1}{p} \right\rfloor - i_p(a) \) we observe that

\[ \Delta = \left\lfloor \frac{a - 1}{p} \right\rfloor - i_p(a) = \left\lfloor \frac{a - 1}{p} \right\rfloor - \left\lfloor \frac{a}{p} \right\rfloor + \left\lfloor \frac{\sigma_p(a)}{p} \right\rfloor \]  \hspace{1cm} (18)

For \( a \in kp + 1, kp + p - 1 \) we have \( \left\lfloor \frac{a - 1}{p} \right\rfloor = \left\lfloor \frac{a}{p} \right\rfloor \) so

\[ \Delta = \left\lfloor \frac{a - 1}{p} \right\rfloor - i_p(a) = \left\lfloor \frac{\sigma_p(a)}{p} \right\rfloor \]  \hspace{1cm} (19)

If \( a = kp \) then \( \left\lfloor \frac{a - 1}{p} \right\rfloor = \left\lfloor \frac{kp - 1}{p} \right\rfloor = \left\lfloor \frac{k - 1}{p} \right\rfloor = k - 1 \) and \( \left\lfloor \frac{a}{p} \right\rfloor = \left\lfloor \frac{k}{p} \right\rfloor = k \).

So, (18) becomes

\[ \Delta = \left\lfloor \frac{a - 1}{p} \right\rfloor - i_p(a) = \left\lfloor \frac{\sigma_p(a)}{p} \right\rfloor - 1 \]  \hspace{1cm} (20)

Analogously, if \( a = kp + p \), we have

\[ \left\lfloor \frac{a - 1}{p} \right\rfloor = \left\lfloor \frac{p(k + 1) - 1}{p} \right\rfloor = \left\lfloor \frac{k + 1 - 1}{p} \right\rfloor = k \) and \( \left\lfloor \frac{a}{p} \right\rfloor = \left\lfloor \frac{k + 1}{p} \right\rfloor = k + 1 \)

so, (18) has the form (20).

For any number \( a \), for which \( \Delta \) is given by (19) or by (20), we deduce that \( \Delta \) is maximum when \( \sigma_{p\mid p}(a) \) is maximum, so when

\[ a_{m} = \frac{(p - 1)(p - 1)\ldots(p - 1)}{p} \]  \hspace{1cm} (21)
That is

\[ a_M = (p-1)a_1(p) + (p-1)a_2(p) + \cdots + (p-1)a_t(p) + p = \]

\[ = (p-1)\left( \frac{p^1-1}{p-1} + \frac{p^2-1}{p-1} + \cdots + \frac{p^l-1}{p-1} \right) + p = \]

\[ = (p^1 + p^2 + \cdots + p^l + p) - (t-1) = pa_t(p) - (t-1) \]

It results that \( a_M \) is not multiple of \( p \) if and only if \( t-1 \) is not a multiple of \( p \).

In this case \( \sigma_{|p|}(a) = (t-1)(p-1) + p = pt - t + 1 \) and

\[ \Delta = \left[ \frac{\sigma_{|p|}(a)}{p} \right] = \left[ t - \frac{t-1}{p} \right] = t - \left[ \frac{t-1}{p} \right]. \]

So \( i_p(a_M) \geq \left[ \frac{a_M - 1}{p} \right] - t \) or \( i_p(a_M) \leq \left[ \frac{a_M - 1}{p} \right] - t - \left[ \frac{a_M - 1}{p} \right] \). If \( t-1 \in (kp, kp+p) \) then

\[ \left[ \frac{t-1}{p} \right] = k \] and \( k(p-1)+1 < \Delta(a_M) < k(p-1)+p+1 \) so \( \lim_{a_M \to \infty} \Delta(a_M) = \infty \).

We also observe that

\[ \left[ \frac{a_M - 1}{p} \right] = a_t(p) - \left[ \frac{t-1}{p} \right] = \left[ \frac{p^t - 1}{p-1} - \left[ \frac{p^{t+1} - 1}{p-1} - k \right] \right]. \]

Then if \( a_M \to \infty \) (as \( p^t \)), it results that \( \Delta(a_M) \to \infty \) (as \( x \)).

From \( \left[ \frac{a_M - 1}{p} \right] = a_t(p) - \left[ \frac{t-2}{p} \right] \) it results \( \lim_{a_M \to \infty} \frac{i_p(a)}{[a-1]p} = 1 \).

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**BIBLIOGRAPHY**


