MEAN VALUE OF THE ADDITIVE ANALOGUE OF SMARANDACHE FUNCTION

Yi Yuan and Zhang Wenpeng
Research Center for Basic Science, Xi’an Jiaotong University Xi’an, Shaanxi, P.R.China
yuanyi@mail.xjtu.edu.cn

Abstract
For any positive integer \( n \), let \( S(n) \) denotes the Smarandache function, then \( S(n) \) is defined the smallest \( m \in \mathbb{N}^+ \), where \( n|m! \). In this paper, we study the mean value properties of the additive analogue of \( S(n) \), and give an interesting mean value formula for it.

Keywords: Smarandache function; Additive Analogue; Mean Value formula.

§1. Introduction and results

For any positive integer \( n \), let \( S(n) \) denotes the Smarandache function, then \( S(n) \) is defined the smallest \( m \in \mathbb{N}^+ \), where \( n|m! \). In paper [2], Jozsef Sandor defined the following analogue of Smarandache function:

\[
S_1(x) = \min \{ m \in \mathbb{N} : x \leq m! \}, \quad x \in (1, \infty),
\]

which is defined on a subset of real numbers. Clearly \( S(x) = m \) if \( x \in \left( (m - 1)! , m! \right] \) for \( m \geq 2 \) (for \( m = 1 \) it is not defined, as \( 0! = 1! = 1! \)), therefore this function is defined for \( x > 1 \).

About the arithmetical properties of \( S(n) \), many people had studied it before (see reference [3]). But for the mean value problem of \( S_1(n) \), it seems that no one have studied it before. The main purpose of this paper is to study the mean value properties of \( S_1(n) \), and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. For any real number \( x \geq 2 \), we have the mean value formula

\[
\sum_{n \leq x} S_1(n) = \frac{x \ln x}{\ln \ln x} + O(x).
\]

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. That is,
Lemma. For any fixed positive integers $m$ and $n$, if $(m-1)! < n \leq m!$, then we have
\[ m = \frac{\ln n}{\ln \ln n} + O(1). \]

Proof. From $(m-1)! < n \leq m!$ and taking the logistic computation in the two sides of the inequality, we get
\[ \sum_{i=1}^{m-1} \ln i < \ln n \leq \sum_{i=1}^{m} \ln i. \] (2)

Using the Euler’s summation formula, then
\[ \sum_{i=1}^{m} \ln i = \int_{1}^{m} \ln t \, dt + \int_{1}^{m} (t - \lfloor t \rfloor) \ln t \, dt = m \ln m - m + O(\ln m) \] (3)
and
\[ \sum_{i=1}^{m-1} \ln i = \int_{1}^{m-1} \ln t \, dt + \int_{1}^{m-1} (t - \lfloor t \rfloor) \ln t \, dt = m \ln m - m + O(\ln m). \] (4)

Combining (2), (3) and (4), we can easily deduce that
\[ \ln n = m \ln m - m + O(\ln m). \] (5)

So
\[ m = \frac{\ln n}{\ln m - 1} + O(1). \] (6)

Similarly, we continue taking the logistic computation in two sides of (5), then we also have
\[ \ln m = \ln \ln n + O(\ln \ln m), \] (7)
and
\[ \ln \ln m = O(\ln \ln \ln n). \] (8)

Hence,
\[ m = \frac{\ln n}{\ln \ln n} + O(1). \]

This completes the proof of Lemma.

Now we use Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $s_1(n)$ and Lemma we have
\[ \sum_{n \leq x} S_1(n) = \sum_{\substack{n \leq x \\(m-1)! < n \leq m!}} m = \sum_{n \leq x} \left( \frac{\ln n}{\ln \ln n} + O(1) \right) = \sum_{n \leq x} \frac{\ln n}{\ln \ln n} + O(x). \] (9)
By the Euler’s summation formula, we deduce that

\[
\sum_{n \leq x} \frac{\ln n}{\ln \ln n} = \int_2^x \frac{\ln t}{\ln \ln t} dt + \int_2^x \left( \frac{\ln t}{\ln \ln t} \right) \ln t \ dt + \frac{\ln x}{\ln \ln x} (x - [x]) \quad (10)
\]

So, from (9) and (10) we have

\[
\sum_{n \leq x} S_1(n) = \frac{x \ln x}{\ln \ln x} + O(x).
\]

This completes the proof of Theorem.

References


