On a dual of the Pseudo-Smarandache function

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1 Introduction

In paper [3] we have defined certain generalizations and extensions of the Smarandache function. Let \( f : \mathbb{N}^* \rightarrow \mathbb{N}^* \) be an arithmetic function with the following property: for each \( n \in \mathbb{N}^* \) there exists at least a \( k \in \mathbb{N}^* \) such that \( n \mid f(k) \). Let

\[
F_f : \mathbb{N}^* \rightarrow \mathbb{N}^* \text{ defined by } F_f(n) = \min\{k \in \mathbb{N}^* : n \mid f(k)\}. \quad (1)
\]

This function generalizes many particular functions. For \( f(k) = k! \) one gets the Smarandache function, while for \( f(k) = \frac{k(k+1)}{2} \) one has the Pseudo-Smarandache function \( Z \) (see [1], [4-5]). In the above paper [3] we have defined also dual arithmetic functions as follows: Let \( g : \mathbb{N}^* \rightarrow \mathbb{N}^* \) be a function having the property that for each \( n \geq 1 \) there exists at least a \( k \geq 1 \) such that \( g(k) \mid n \).

Let

\[
G_g(n) = \max\{k \in \mathbb{N}^* : g(k) \mid n\}. \quad (2)
\]

For \( g(k) = k! \) we obtain a dual of the Smarandache function. This particular function, denoted by us as \( S_e \) has been studied in the above paper. By putting \( g(k) = \frac{k(k+1)}{2} \) one obtains a dual of the Pseudo-Smarandache function. Let us denote this function, by analogy by \( Z_\ast \). Our aim is to study certain elementary properties of this arithmetic function.
2 The dual of yhe Pseudo-Smarandache function

Let

\[ Z_*(n) = \max \left\{ m \in \mathbb{N}^* : \frac{m(m+1)}{2} | n \right\}. \]  

(3)

Recall that

\[ Z(n) = \min \left\{ k \in \mathbb{N}^* : n | \frac{k(k+1)}{2} \right\}. \]  

(4)

First remark that

\[ Z_*(1) = 1 \quad \text{and} \quad Z_*(p) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \]  

(5)

where \( p \) is an arbitrary prime. Indeed, \( \frac{2 \cdot 3}{2} = 3 | 3 \) but \( \frac{m(m+1)}{2} | p \) for \( p \neq 3 \) is possible only for \( m = 1 \). More generally, let \( s \geq 1 \) be an integer, and \( p \) a prime. Then:

Proposition 1.

\[ Z_*(p^s) = \begin{cases} 2, & p = 3 \\ 1, & p \neq 3 \end{cases} \]  

(6)

Proof. Let \( \frac{m(m+1)}{2} | p^s \). If \( m = 2M \) then \( M(2M+1)|p^s \) is impossible for \( M > 1 \) since \( M \) and \( 2M + 1 \) are relatively prime. For \( M = 1 \) one has \( m = 2 \) and \( 3 | p^s \) only if \( p = 3 \). For \( m = 2M - 1 \) we get \( (2M - 1)M | p^s \), where for \( M > 1 \) we have \( (M, 2M - 1) = 1 \) as above, while for \( M = 1 \) we have \( m = 1 \).

The function \( Z_* \) can take large values too, since remark that for e.g. \( n \equiv 0(\text{mod}6) \) we have \( \frac{3 \cdot 4}{2} = 6 | n \), so \( Z_*(n) \geq 3 \). More generally, let \( a \) be a given positive integer and \( n \) selected such that \( n \equiv 0(\text{mod}(2a + 1)) \). Then

\[ Z_*(n) \geq 2a. \]  

(7)

Indeed, \( \frac{2a(2a+1)}{2} = a(2a+1)|n \) implies \( Z_*(n) \geq 2a \).

A similar situation is in

Proposition 2. Let \( q \) be a prime such that \( p = 2q - 1 \) is a prime, too. Then

\[ Z_*(pq) = p. \]  

(8)
Proof. \( \frac{p(p+1)}{2} = pq \) so clearly \( Z_p(pq) = p \).

Remark. Examples are \( Z_5(5 \cdot 3) = 5, Z_{13}(13 \cdot 7) = 13 \), etc. It is a difficult open problem that for infinitely many \( q \), the number \( p \) is prime, too (see e.g. [2]).

Proposition 3. For all \( n \geq 1 \) one has

\[ 1 \leq Z(n) \leq Z(n). \tag{9} \]

Proof. By (3) and (4) we can write \( \frac{m(m+1)}{2} | \frac{k(k+1)}{2} \), therefore \( m(m+1)|k(k+1) \).

If \( m > k \) then clearly \( m(m+1) > k(k+1) \), a contradiction.

Corollary. One has the following limits:

\[ \lim_{n \to \infty} Z(n) = 0, \quad \lim_{n \to \infty} \frac{Z(n)}{Z(n)} = 1. \tag{10} \]

Proof. Put \( n = p \) (prime) in the first relation. The first result follows by (6) for \( s = 1 \) and the well-known fact that \( Z(p) = p \). Then put \( n = \frac{a(a+1)}{2} \), when \( \frac{Z(n)}{Z(n)} = 1 \) and let \( a \to \infty \).

As we have seen,

\[ Z \left( \frac{a(a+1)}{2} \right) = Z \left( \frac{a(a+1)}{2} \right) = a. \]

Indeed, \( \frac{a(a+1)}{2} \left( \frac{k(k+1)}{2} \right) \) is true for \( k = a \) and is not true for any \( k < a \). In the same manner, \( \frac{m(m+1)}{2} | \frac{a(a+1)}{2} \) is valid for \( m = a \) but not for any \( m > a \). The following problem arises: What are the solutions of the equation \( Z(n) = Z(n) \)?

Proposition 4. All solutions of equation \( Z(n) = Z(n) \) can be written in the form

\[ n = \frac{r(r+1)}{2} \ (r \in \mathbb{N}^*). \]

Proof. Let \( Z(n) = Z(n) = t \). Then \( n | \frac{t(t+1)}{2} | n \) so \( \frac{t(t+1)}{2} = n \). This gives \( t^2 + t - 2n = 0 \) or \( (2t+1)^2 = 8n + 1 \), implying \( t = \frac{\sqrt{8n+1} - 1}{2} \), where \( 8n + 1 = m^2 \). Here \( m \) must be odd, let \( m = 2r + 1, \) so \( n = \frac{(m-1)(m+1)}{8} \) and \( t = \frac{m-1}{2} \). Then \( m-1 = 2r, m+1 = 2(r+1) \) and \( n = \frac{r(r+1)}{2} \).

Proposition 5. One has the following limits:

\[ \lim_{n \to \infty} \sqrt[\infty]{Z(n)} = \lim_{n \to \infty} \sqrt{Z(n)} = 1. \tag{11} \]
Proof. It is known that \( Z(n) \leq 2n - 1 \) with equality only for \( n = 2^k \) (see e.g. [5]). Therefore, from (9) we have

\[
1 \leq \sqrt[n]{Z_*(n)} \leq \sqrt[n]{Z(n)} \leq \sqrt[2n-1]{1},
\]

and by taking \( n \to \infty \) since \( \sqrt[2n-1]{1} \to 1 \), the above simple result follows.

As we have seen in (9), upper bounds for \( Z(n) \) give also upper bounds for \( Z_*(n) \). E.g. for \( n = \) odd, since \( Z(n) \leq n - 1 \), we get also \( Z_*(n) \leq n - 1 \). However, this upper bound is too large. The optimal one is given by:

**Proposition 6.**

\[
Z_*(n) \leq \frac{\sqrt[8n+1]{1} - 1}{2} \text{ for all } n. \tag{12}
\]

**Proof.** The definition (3) implies with \( Z_*(n) = m \) that \( \frac{m(m+1)}{2} \mid n \), so \( \frac{m(m+1)}{2} \leq n \), i.e. \( m^2 + m - 2n \leq 0 \). Resolving this inequality in the unknown \( m \), easily follows (12).

Inequality (12) cannot be improved since for \( n = \frac{p(p+1)}{2} \) (thus for infinitely many \( n \)) we have equality. Indeed,

\[
\left( \sqrt[\sqrt{8(p+1)p + 1}]{1} - 1 \right)/2 = \left( \sqrt[2]{4p(p+1) + 1} - 1 \right)/2 = [(2p + 1) - 1]/2 = p.
\]

**Corollary.**

\[
\lim_{n \to \infty} \frac{Z_*(n)}{\sqrt{n}} = 0, \quad \lim_{n \to \infty} \frac{Z_*(n)}{\sqrt{2n}} = \sqrt{2}. \tag{13}
\]

**Proof.** While the first limit is trivial (e.g. for \( n = \) prime), the second one is a consequence of (12). Indeed, (12) implies \( Z_*(n)/\sqrt{n} \leq \sqrt{2} \left( \sqrt{1 + \frac{1}{8n}} - \sqrt{1/8n} \right) \), i.e.

\[
\lim_{n \to \infty} \frac{Z_*(n)}{\sqrt{n}} \leq \sqrt{2}. \quad \text{But this upper limit is exact for } n = \frac{p(p+1)}{2} \quad (p \to \infty).
\]

Similar and other relations on the functions \( S \) and \( Z \) can be found in [4-5].

An inequality connecting \( S_*(ab) \) with \( S_*(a) \) and \( S_*(b) \) appears in [3]. A similar result holds for the functions \( Z \) and \( Z_*. \)

**Proposition 7.** For all \( a, b \geq 1 \) one has

\[
Z_*(ab) \geq \max\{Z_*(a), Z_*(b)\}, \tag{14}
\]
\[ Z(ab) \geq \max\{Z(a), Z(b)\} \geq \max\{Z_*(a), Z_*(b)\}. \] (15)

**Proof.** If \( m = Z_*(a) \), then \( \frac{m(m+1)}{2} | \alpha \). Since \( \alpha | ab \) for all \( b \geq 1 \), clearly \( \frac{m(m+1)}{2} | ab \), implying \( Z_*(ab) \geq m = Z_*(a) \). In the same manner, \( Z_*(ab) \geq Z_*(b) \), giving (14).

Let now \( k = Z(ab) \). Then, by (4) we can write \( ab|\frac{k(k+1)}{2} \). By \( \alpha | ab \) it results \( a|\frac{k(k+1)}{2} \), implying \( Z(a) \leq k = Z(ab) \). Analogously, \( Z(b) \leq Z(ab) \), which via (9) gives (15).

**Corollary.** \( Z_*(3^s \cdot p) \geq 2 \) for any integer \( s \geq 1 \) and any prime \( p \).

Indeed, by (14), \( Z_*(3^s \cdot p) \geq \max\{Z_*(3^s), Z(p)\} = \max\{2, 1\} = 2 \), by (6).

We now consider two irrational series.

**Proposition 8.** The series \( \sum_{n=1}^\infty \frac{Z_*(n)}{n!} \) and \( \sum_{n=1}^\infty \frac{(-1)^{n-1}Z_*(n)}{n!} \) are irrational.

**Proof.** For the first series we apply the following irrationality criterion ([6]). Let \((v_n)\) be a sequence of nonnegative integers such that

(i) \( v_n < n \) for all large \( n \);

(ii) \( v_n < n - 1 \) for infinitely many \( n \);

(iii) \( v_n > 0 \) for infinitely many \( n \).

Then \( \sum_{n=1}^\infty \frac{v_n}{n!} \) is irrational.

Let \( v_n = Z_*(n) \). Then, by (12) \( Z_*(n) < n - 1 \) follows from \( \frac{\sqrt{n+1} - 1}{2} < n - 1 \), i.e. (after some elementary fact, which we omit here) \( n > 3 \). Since \( Z_*(n) \geq 1 \), conditions (i)-(iii) are trivially satisfied.

For the second series we will apply a criterion from [7]:

Let \((a_k), (b_k)\) be sequences of positive integers such that

(i) \( k|a_1 a_2 \ldots a_k \);

(ii) \( \frac{b_{k+1}}{a_{k+1}} < b_k < a_k \) \( (k \geq k_0) \). Then \( \sum_{k=1}^\infty (-1)^{k-1} \frac{b_k}{a_1 a_2 \ldots a_k} \) is irrational.

Let \( a_k = k, b_k = Z_*(k) \). Then (i) is trivial, while (ii) is \( \frac{Z_*(k+1)}{k+1} < Z_*(k) < k \). Here \( Z_*(k) < k \) for \( k \geq 2 \). Further \( Z_*(k+1) < (k+1)Z_*(k) \) follows by \( 1 \leq Z_*(k) \) and \( Z_*(k+1) < k+1 \).
References


